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Casimir energy–momentum tensor for a brane in de Sitter spacetime

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Abstract

Vacuum expectation values of the energy–momentum tensor for a conformally coupled scalar field are investigated in de Sitter (dS) spacetime in presence of a curved brane on which the field obeys the Robin boundary condition with coordinate dependent coefficients. To generate the corresponding vacuum densities we use the conformal relation between dS and Rindler spacetimes and the results previously obtained by one of the authors for the Rindler counterpart. The resulting energy–momentum tensor is non-diagonal and induces anisotropic vacuum stresses. The asymptotic behaviour of this tensor is investigated near the dS horizon and the boundary.

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1. Introduction

de Sitter (dS) spacetime is the maximally symmetric solution of Einstein's equation with a positive cosmological constant. Recent astronomical observations of supernovae and cosmic microwave background [1] indicate that the universe is accelerating and can be well approximated by a world with a positive cosmological constant. If the universe would accelerate indefinitely, the standard cosmology leads to an asymptotic dS universe. de Sitter spacetime plays an important role in the inflationary scenario, where an exponentially expanding approximately dS spacetime is employed to solve a number of problems in standard cosmology. The quantum field theory on dS spacetime is also of considerable interest. In particular, the inhomogeneities generated by fluctuations of a quantum field during inflation provide an attractive mechanism for the structure formation in the universe. Another motivation for investigations of dS based quantum theories is related to the recently proposed holographic duality between quantum gravity on dS spacetime and a quantum field theory living on boundary identified with the timelike infinity of dS spacetime [2].

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The one of most striking macroscopic manifestations of quantum properties is the Casimir effect. The presence of reflecting boundaries alters the zero-point modes of a quantized field, and results in the shifts in the vacuum expectation values of quantities quadratic in the field, such as the energy density and stresses. In particular, vacuum forces arise acting on constraining boundaries. The particular features of these forces depend on the nature of the quantum field, the type of spacetime manifold, the boundary geometries and the specific boundary conditions imposed on the field. Since the original work by Casimir in 1948 [3] many theoretical and experimental works have been done on this problem (see, e.g., [4–13] and references therein). Many different approaches have been used: mode summation method, Green function formalism, multiple scattering expansions, heat-kernel series, zeta function regularization technique, etc. Recently new methods are developed for the Casimir energy calculations in given background fields [14–16].

The Casimir effect can be viewed as a polarization of vacuum by boundary conditions. The interaction of fluctuating quantum fields with background gravitational fields give rise to another type of vacuum polarization (see, for instance, [17,18]). Here we will study an example where both types of polarizations are present. Namely, we evaluate the vacuum expectation values for the energy–momentum tensor of a conformally coupled scalar field on background of $(D + 1)$ -dimensional dS spacetime when a curved brane is present (for investigations of the Casimir energy in braneworld models with dS branes see Refs. [19–21]). As a brane we take D -dimensional hypersurface which is the conformal image of a plate moving with constant proper acceleration in the Rindler spacetime. We will assume that the field is prepared in the state conformally related to the Fulling–Rindler vacuum in the Rindler spacetime. To generate the vacuum expectation values in dS bulk, we use the conformal relation between dS and Rindler spacetimes and the results from [22] for the corresponding Rindler problem with mixed boundary conditions. Previously this method has been used in [23] to derive the vacuum stress on parallel plates for a scalar field with Dirichlet boundary conditions in de Sitter spacetime and in Ref. [24] to investigate the vacuum characteristics of the Casimir configuration on background of conformally flat brane-world geometries for massless scalar field with Robin boundary conditions on plates.

The present Letter is organized as follows. In the next section the geometry of our problem and the conformal relation between dS and Rindler spacetimes are discussed. The results are presented for the vacuum expectation values of the energy–momentum tensor for a scalar field induced by a plate uniformly accelerated through the Fulling–Rindler vacuum. In Section 3, by using the formula relating the renormalized energy–momentum tensors for conformally related problems in combination with the appropriate coordinate transformation, we derive expressions for the vacuum energy–momentum tensor in dS space. The main results are rementioned and summarized in Section 4.

2. Conformal relation between dS and Rindler problems

Consider a conformally coupled massless scalar field $\varphi(x)$ satisfying the equation

$$(\nabla_l \nabla^l + \zeta R)\varphi(x) = 0, \quad \zeta = \frac{D-1}{4D}, \quad (1)$$

on background of a $(D + 1)$ -dimensional dS spacetime. In Eq. (1), ∇_l is the operator of the covariant derivative, and R is the Ricci scalar for the corresponding metric g_{ik} . In static coordinates $x^i = (t, r, \theta, \theta_2, \dots, \theta_{D-2}, \phi)$ dS metric has the form

$$ds_{\text{dS}}^2 = g_{ik} dx^i dx^k = \left(1 - \frac{r^2}{\alpha^2}\right) dt^2 - \frac{dr^2}{1 - r^2/\alpha^2} - r^2 d\Omega_{D-1}^2, \quad (2)$$

where $d\Omega_{D-1}^2$ is the line element on the $(D - 1)$ -dimensional unit sphere in Euclidean space, and the parameter α defines the dS curvature radius. Note that $R = D(D - 1)/\alpha^2$. We will assume that the field satisfies the mixed

boundary condition

$$(A + Bn^l \nabla_l)\varphi(x) = 0, \quad x \in S, \tag{3}$$

on the hypersurface S , n^l is the normal to this surface, $n_l n^l = -1$ (the form of the hypersurface will be specified below, see Eq. (8)). The results in the following will depend on the ratio of Robin coefficients A and B . However, to keep the transition to the Dirichlet and Neumann cases transparent we will use the form (3). Our main interest in the present Letter is to investigate the vacuum expectation value (VEV) of the energy–momentum tensor for the field $\varphi(x)$ induced by the hypersurface S . The presence of boundaries modifies the spectrum of the zero-point fluctuations compared to the case without boundaries and results in the shift in the VEV’s of physical quantities, such as vacuum energy density and stresses. This is the well-known Casimir effect.

To make maximum use of the flat spacetime calculations, first of all let us present the dS line element in the form conformally related to the Rindler metric. With this aim we make the coordinate transformation $x^i \rightarrow x'^i = (\tau, \xi, \mathbf{x}')$, $\mathbf{x}' = (x'^2, \dots, x'^D)$ (see Ref. [17] for the case $D = 3$)

$$\begin{aligned} \tau &= \frac{t}{\alpha}, & \xi &= \frac{\sqrt{\alpha^2 - r^2}}{\Omega}, & x'^2 &= \frac{r}{\Omega} \sin \theta \cos \theta_2, \dots, \\ x'^{D-2} &= \frac{r}{\Omega} \sin \theta \sin \theta_2 \dots \sin \theta_{D-3} \cos \theta_{D-2}, \\ x'^{D-1} &= \frac{r}{\Omega} \sin \theta \sin \theta_2 \dots \sin \theta_{D-2} \cos \phi, & x'^D &= \frac{r}{\Omega} \sin \theta \sin \theta_2 \dots \sin \theta_{D-2} \sin \phi, \end{aligned} \tag{4}$$

with the notation

$$\Omega = 1 - \frac{r}{\alpha} \cos \theta. \tag{5}$$

Under this coordinate transformation the dS line element takes the form

$$ds_{\text{dS}}^2 = g'_{ik} dx'^i dx'^k = \Omega^2 (\xi^2 d\tau^2 - d\xi^2 - d\mathbf{x}'^2). \tag{6}$$

In this form the dS metric is manifestly conformally related to the Rindler spacetime with the line element ds_{R}^2 :

$$ds_{\text{dS}}^2 = \Omega^2 ds_{\text{R}}^2, \quad ds_{\text{R}}^2 = g_{ik}^{\text{R}} dx'^i dx'^k = \xi^2 d\tau^2 - d\xi^2 - d\mathbf{x}'^2, \quad g'_{ik} = \Omega^2 g_{ik}^{\text{R}}. \tag{7}$$

By using the standard transformation formula for the vacuum expectation values of the energy–momentum tensor in conformally related problems (see, for instance, [17]), we can generate the results for dS spacetime from the corresponding results for the Rindler spacetime. In this Letter as a Rindler counterpart we will take the vacuum energy–momentum tensor induced by an infinite plate moving by uniform proper acceleration through the Fulling–Rindler vacuum. We will assume that the plate is located in the right Rindler wedge and has the coordinate $\xi = a$. Observe that in coordinates x^i the boundary $\xi = a$ is presented by the hypersurface

$$\sqrt{\alpha^2 - r^2} = a \left(1 - \frac{r}{\alpha} \cos \theta \right) \tag{8}$$

in dS spacetime. As a boundary S in Eq. (3) we will take this hypersurface. In Fig. 1 we have plotted the section of dS spacetime for fixed $(t, \theta_2, \dots, \theta_{D-2}, \phi)$. The corresponding surface is embedded into 3D Euclidean space with coordinates $(x, y, z) = ((r/\alpha) \cos \theta, (r/\alpha) \sin \theta, z)$ and is defined by the equation

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0. \tag{9}$$

In coordinates (x, y, z) the boundary (8) is defined by the intersection of the surface (9) with the cylinder

$$\left(x - \frac{a_\alpha^2}{1 + a_\alpha^2} \right)^2 + \frac{y^2}{1 + a_\alpha^2} = \frac{1}{(1 + a_\alpha^2)^2}, \quad a_\alpha = \frac{a}{\alpha}. \tag{10}$$

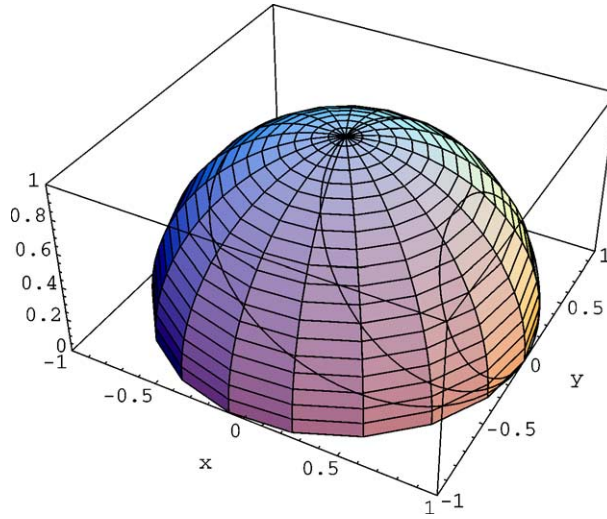


Fig. 1. The section of dS spacetime for fixed $(t, \theta_2, \dots, \theta_{D-2}, \phi)$ embedded into 3D Euclidean space with coordinates $(x, y, z) = ((r/\alpha) \cos \theta, (r/\alpha) \sin \theta, z)$ and described by Eq. (9). The curves on this surface correspond to the boundary (8) with $a_\alpha = 0.5, 1, 2$.

The corresponding curves on the surface (9) are plotted in Fig. 1 for values $a_\alpha = 0.5, 1, 2$. In the limit $a_\alpha \rightarrow 0$ the curves tend to the dS horizon presented by the circle $(r/\alpha = 1, z = 0)$. For all values of a_α the hypersurfaces (10) touch the dS horizon at $(r, \theta) = (\alpha, 0)$.

The expectation values of the energy–momentum tensor induced by the presence of an infinite plane boundary moving with uniform acceleration through the Fulling–Rindler vacuum were investigated by Candelas and Deutsch [25] for a conformally coupled 4D Dirichlet and Neumann massless scalar and electromagnetic fields. In this Letter the region of the right Rindler wedge to the right of the barrier is considered. In Ref. [22] we have investigated the Wightman function and the VEV of the energy–momentum tensor for a massive scalar field with general curvature coupling parameter, satisfying the Robin boundary conditions on an infinite plane in an arbitrary number of spacetime dimensions. Both regions, including the one between the barrier and Rindler horizon, are considered. Recently, the total Casimir energy in this problem is investigated [26] by using the zeta function regularization technique. The expectation values of the energy–momentum tensor for a scalar field $\varphi_R(x')$ in the Fulling–Rindler vacuum can be presented in the form of the sum

$$\langle 0_R | T_i^k [g_{lm}^R, \varphi_R] | 0_R \rangle = \langle \tilde{0}_R | T_i^k [g_{lm}^R, \varphi_R] | \tilde{0}_R \rangle + \langle T_i^k [g_{lm}^R, \varphi_R] \rangle^{(b)}, \tag{11}$$

where $|0_R\rangle$ and $|\tilde{0}_R\rangle$ are the amplitudes for the vacuum states in the Rindler space in presence and absence of the plate, respectively, and $\langle T_i^k [g_{lm}^R, \varphi_R] \rangle^{(b)}$ is the part of the vacuum energy–momentum tensor induced by the plate. Note that the state $|\tilde{0}_R\rangle$ corresponds to the standard Fulling–Rindler vacuum. In the case of a conformally coupled massless scalar field, for the part without boundaries one has (see Ref. [25] for the case $D = 3$ and Ref. [22] for an arbitrary D)

$$\langle \tilde{0}_R | T_i^k [g_{lm}^R, \varphi_R] | \tilde{0}_R \rangle = \frac{a_D \xi^{-D-1}}{2^{D-1} \pi^{D/2} \Gamma(D/2)} \text{diag} \left(-1, \frac{1}{D}, \dots, \frac{1}{D} \right), \tag{12}$$

with the notation

$$a_D = \int_0^\infty \frac{\omega^D d\omega}{e^{2\pi\omega} + (-1)^D} \prod_{l=1}^{l_m} \left[\left(\frac{D-1-2l}{2\omega} \right)^2 + 1 \right], \tag{13}$$

where $l_m = D/2 - 1$ for even $D > 2$ and $l_m = (D - 1)/2$ for odd $D > 1$, and the value for the product over l is equal to 1 for $D = 1, 2, 3$. For a scalar field $\varphi_R(x')$, satisfying the mixed boundary condition

$$(A_R + B_R n_R^l \nabla_l) \varphi_R(x') = 0, \quad \xi = a, \quad n_R^l = \delta_1^l, \tag{14}$$

with constants A_R, B_R , the boundary induced part in the region $\xi > a$ is defined by the formula [22]

$$\langle T_i^k [g_{lm}^R, \varphi_R] \rangle^{(b)} = \frac{-\delta_i^k}{2^{D-2} \pi^{(D+1)/2} \Gamma((D-1)/2)} \int_0^\infty dk k^{D-1} \int_0^\infty d\omega \frac{\bar{I}_\omega(ka)}{\bar{K}_\omega(ka)} F^{(i)} [K_\omega(k\xi)], \tag{15}$$

where the functions $F^{(i)}[g(z)]$ for $i = 0, 1$ have the form

$$\begin{aligned} F^{(0)}[g(z)] &= \left(\frac{1}{2} - 2\zeta\right) \left(\frac{dg(z)}{dz}\right)^2 + \frac{\zeta}{z} \frac{d}{dz} g^2(z) + \left[\frac{1}{2} - 2\zeta + \left(\frac{1}{2} + 2\zeta\right) \frac{\omega^2}{z^2}\right] g^2(z), \\ F^{(1)}[g(z)] &= -\frac{1}{2} \left(\frac{dg(z)}{dz}\right)^2 - \frac{\zeta}{z} \frac{d}{dz} g^2(z) + \frac{1}{2} \left(1 - \frac{\omega^2}{z^2}\right) g^2(z), \end{aligned} \tag{16}$$

and the functions $F^{(i)}[g(z)]$ for $i = 2, \dots, D$ are determined by the zero trace condition for the energy–momentum tensor,

$$F^{(i)}[g(z)] = -\frac{1}{D-1} \{F^{(0)}[g(z)] + F^{(1)}[g(z)]\}, \quad i = 2, \dots, D. \tag{17}$$

In Eq. (15), $I_\omega(z)$ and $K_\omega(z)$ are the Bessel modified functions and for a given function $f(z)$ we use the notation

$$\bar{f}(z) = A_R f(z) + B_R z f'(z). \tag{18}$$

The expression for the boundary part of the vacuum energy–momentum tensor in the region $\xi < a$ is obtained from formula (15) by replacements $I_\omega \rightarrow K_\omega, K_\omega \rightarrow I_\omega$.

3. Vacuum energy–momentum tensor in dS bulk

To find the VEV's induced by the surface (8) in dS spacetime, first we will consider the corresponding quantities in the coordinates (τ, ξ, \mathbf{x}') with metric (6). These quantities can be found from the corresponding results in the Rindler spacetime by using the standard transformation formula for the conformally related problems [17]:

$$\langle 0_{\text{dS}} | T_i^k [g'_{lm}, \varphi] | 0_{\text{dS}} \rangle = \Omega^{-D-1} \langle 0_{\text{R}} | T_i^k [g_{lm}^R, \varphi_R] | 0_{\text{R}} \rangle + \langle T_i^k [g'_{lm}, \varphi] \rangle^{(an)}, \tag{19}$$

where the second summand on the right is determined by the trace anomaly and is related to the divergent part of the corresponding effective action:

$$\langle T_i^k [g'_{lm}, \varphi] \rangle^{(an)} = 2g'^{kl} \frac{\delta}{\delta g'^{il}(x)} W_{\text{div}} [g'_{mn}, \varphi]. \tag{20}$$

Note that in odd spacetime dimensions the conformal anomaly is absent and the corresponding anomaly part vanishes:

$$\langle T_i^k [g'_{lm}, \varphi] \rangle^{(an)} = 0 \quad \text{for even } D. \tag{21}$$

For an odd number of spatial dimensions the anomaly part in dS spacetime has the form

$$\langle T_i^k [g'_{lm}, \varphi] \rangle^{(an)} = \frac{b_D}{\alpha^{D+1}} \delta_k^i, \tag{22}$$

with the numerical coefficient b_D . In $D = 3$ one has $b_3 = 1/960\pi^2$ [17].

The formulae given above allow us to present the dS VEV's in the decomposed form similar to Eq. (11):

$$\langle 0_{\text{dS}} | T_i^k [g'_{lm}, \varphi] | 0_{\text{dS}} \rangle = \langle \tilde{0}_{\text{dS}} | T_i^k [g'_{lm}, \varphi] | \tilde{0}_{\text{dS}} \rangle + \langle T_i^k [g'_{lm}, \varphi] \rangle^{(b)}, \quad (23)$$

where $\langle \tilde{0}_{\text{dS}} | T_i^k [g'_{lm}, \varphi] | \tilde{0}_{\text{dS}} \rangle$ is the expectation value in dS spacetime without boundaries and the part $\langle T_i^k [g'_{lm}, \varphi] \rangle^{(b)}$ is induced by the hypersurface (8). Conformally transforming the Rindler results one finds

$$\langle \tilde{0}_{\text{dS}} | T_i^k [g'_{lm}, \varphi] | \tilde{0}_{\text{dS}} \rangle = \Omega^{-D-1} \langle \tilde{0}_{\text{R}} | T_i^k [g_{lm}^{\text{R}}, \varphi_{\text{R}}] | \tilde{0}_{\text{R}} \rangle + \langle T_i^k [g'_{lm}, \varphi] \rangle^{(an)}, \quad (24)$$

$$\langle T_i^k [g'_{lm}, \varphi] \rangle^{(b)} = \Omega^{-D-1} \langle T_i^k [g_{lm}^{\text{R}}, \varphi_{\text{R}}] \rangle^{(b)}. \quad (25)$$

Under the conformal transformation $g'_{ik} = \Omega^2 g_{ik}^{\text{R}}$, the φ_{R} field will change by the rule

$$\varphi(x') = \Omega^{(1-D)/2} \varphi_{\text{R}}(x'), \quad (26)$$

where the conformal factor is given by expression (5). Now by comparing boundary conditions (14) and (3) and taking into account Eq. (26), one obtains the relation between the coefficients in the boundary conditions:

$$A = \frac{1}{\Omega} \left(A_{\text{R}} + \frac{D-1}{2} B_{\text{R}} n^l \nabla_l \Omega \right), \quad B = B_{\text{R}}, \quad x \in S. \quad (27)$$

To evaluate the expression $n^l \nabla_l \Omega$ we need the components of the normal to S in coordinates x^i . They can be found by transforming the components $n'^l = \delta^l_1 / \Omega$ in coordinates x'^i :

$$n^l = \left(0, \frac{a}{\alpha} \left(\cos \theta - \frac{r}{\alpha} \right), -\frac{a}{\alpha r} \sin \theta, 0, \dots, 0 \right). \quad (28)$$

Now it can be easily seen that $n^l \nabla_l \Omega = -\sqrt{\alpha^2 - r^2} / \alpha^2$ and, hence, the relation between the Robin coefficients in the Rindler and dS problems takes the form

$$A = \frac{a A_{\text{R}}}{\sqrt{\alpha^2 - r^2}} - \frac{D-1}{2} \frac{a B_{\text{R}}}{\alpha^2}, \quad B = B_{\text{R}}. \quad (29)$$

Note that the Robin coefficient A depends on the point of the hypersurface.

The VEV's of the energy–momentum tensor in coordinates x^i with line element (2) are obtained from expressions (24) and (25) by the standard coordinate transformation formulae. As before, we will present the corresponding components in the form of the sum of purely dS and boundary induced parts:

$$\langle 0_{\text{dS}} | T_i^k [g_{lm}, \varphi] | 0_{\text{dS}} \rangle = \langle \tilde{0}_{\text{dS}} | T_i^k [g_{lm}, \varphi] | \tilde{0}_{\text{dS}} \rangle + \langle T_i^k [g_{lm}, \varphi] \rangle^{(b)}. \quad (30)$$

By using relations (4) between the coordinates, for the purely dS part one finds

$$\langle \tilde{0}_{\text{dS}} | T_i^k [g_{lm}, \varphi] | \tilde{0}_{\text{dS}} \rangle = \frac{(\alpha^2 - r^2)^{-(D+1)/2} a_D}{2^{D-1} \pi^{D/2} \Gamma(D/2)} \text{diag} \left(-1, \frac{1}{D}, \dots, \frac{1}{D} \right) + \frac{b_D}{\alpha^{D+1}} \delta_i^k. \quad (31)$$

This formula generalizes the result for $D = 3$ given, for instance, in Ref. [17]. As for the boundary induced energy–momentum tensor the spatial part is anisotropic, the corresponding part in coordinates x^i is more complicated:

$$\langle T_i^k [g_{lm}, \varphi] \rangle^{(b)} = \Omega^{-D-1} \langle T_i^k [g_{lm}^{\text{R}}, \varphi_{\text{R}}] \rangle^{(b)}, \quad i, k = 0, 3, \dots, D, \quad (32)$$

$$\langle T_1^1 [g_{lm}, \varphi] \rangle^{(b)} = \frac{(\cos \theta - r/\alpha)^2}{\Omega^{D+3}} \langle T_1^1 [g_{lm}^{\text{R}}, \varphi_{\text{R}}] \rangle^{(b)} + \frac{1 - r^2/\alpha^2}{\Omega^{D+3}} \sin^2 \theta \langle T_2^2 [g_{lm}^{\text{R}}, \varphi_{\text{R}}] \rangle^{(b)}, \quad (33)$$

$$\langle T_1^2 [g_{lm}, \varphi] \rangle^{(b)} = \frac{(r/\alpha - \cos \theta) \sin \theta}{r \Omega^{D+3}} \{ \langle T_1^1 [g_{lm}^{\text{R}}, \varphi_{\text{R}}] \rangle^{(b)} - \langle T_2^2 [g_{lm}^{\text{R}}, \varphi_{\text{R}}] \rangle^{(b)} \}, \quad (34)$$

$$\langle T_2^2 [g_{lm}, \varphi] \rangle^{(b)} = \frac{1 - r^2/\alpha^2}{\Omega^{D+3}} \sin^2 \theta \langle T_1^1 [g_{lm}^{\text{R}}, \varphi_{\text{R}}] \rangle^{(b)} + \frac{(r/\alpha - \cos \theta)^2}{\Omega^{D+3}} \langle T_2^2 [g_{lm}^{\text{R}}, \varphi_{\text{R}}] \rangle^{(b)}, \quad (35)$$

where the expressions for the components of the boundary induced energy–momentum tensor in the Rindler spacetime are given by formulae (15)–(17) in the region $\xi > a$ and by similar formulae with the replacements $I_\omega \rightarrow K_\omega$ and $K_\omega \rightarrow I_\omega$ in the region $\xi < a$. In these expressions, ξ has to be substituted from (4). As we see the resulting energy–momentum tensor is non-diagonal. It follows from (28) that the induced metric on the brane is also non-diagonal.

Now we turn to the investigation for the limiting cases of the general formulae for the vacuum energy–momentum tensor. First of all let us consider the near horizon limit, $r \rightarrow \alpha$, for a fixed $\theta \neq 0$. In this limit one has $\xi \rightarrow 0$ and we can use the results from Ref. [22] for this limit of the Rindler part. As a result we obtain

$$\langle T_0^0[g_{lm}, \varphi] \rangle^{(b)} = -\frac{(D-1)B_0 \sin^{2(1-D)}(\theta/2)}{4^D \pi^{(D+1)/2} D \Gamma((D-1)/2) a^{D-1} (\alpha^2 - r^2) \ln^2(\sqrt{\alpha^2 - r^2}/a)}, \quad (36)$$

$$\langle T_1^1[g_{lm}, \varphi] \rangle^{(b)} = -\langle T_0^0[g_{lm}, \varphi] \rangle^{(b)}, \quad (37)$$

$$\langle T_2^1[g_{lm}, \varphi] \rangle^{(b)} = \frac{1}{\alpha} \cot \frac{\theta}{2} \langle T_1^1[g_{lm}, \varphi] \rangle^{(b)}, \quad (38)$$

$$\langle T_3^3[g_{lm}, \varphi] \rangle^{(b)} = \frac{2 \langle T_1^1[g_{lm}, \varphi] \rangle^{(b)}}{(D-1) \ln(\sqrt{\alpha^2 - r^2}/a)}, \quad (39)$$

where

$$B_0 = \int_0^\infty dy y^{D-2} \frac{\bar{K}_0(y)}{\bar{I}_0(y)}. \quad (40)$$

As we see the boundary part is divergent at the dS horizon. Recall that near the horizon the purely dS part behaves as $(\alpha^2 - r^2)^{-(D+1)/2}$ and, therefore, in this limit the total vacuum energy–momentum tensor is dominated by this part.

The boundary induced parts (32)–(35) diverge on the boundary, corresponding to the limit $\xi \rightarrow a$. In this limit, by taking into account that $\langle T_1^1[g_{Rlm}, \varphi_R] \rangle^{(b)} \sim (\xi/a - 1) \langle T_2^2[g_{Rlm}, \varphi_R] \rangle^{(b)}$, for $r \neq \alpha$ we can omit the terms containing $\langle T_1^1[g_{Rlm}, \varphi_R] \rangle^{(b)}$ and obtain the following relations between the boundary induced components:

$$\langle T_3^3 \rangle^{(b)} \sim -\frac{\langle T_0^0 \rangle^{(b)}}{D-1} \sim \frac{\alpha^2 \langle T_1^1 \rangle^{(b)}}{a^2 \sin^2 \theta} \sim \frac{r \Omega^2 \langle T_2^1 \rangle^{(b)}}{\sin \theta (\cos \theta - r/\alpha)} \sim \frac{\Omega^2 \langle T_2^2 \rangle^{(b)}}{(\cos \theta - r/\alpha)^2}, \quad (41)$$

where $\langle T_0^0 \rangle^{(b)} \sim (\xi - a)^{1-D}$ and we can substitute in the coefficients of these relations $\cos \theta = (\alpha/r)(1 - \sqrt{\alpha^2 - r^2}/a)$. Near the point $(r, \theta) = (\alpha, 0)$, where the boundary touches the horizon, the horizon and boundary divergences are mixed: in the coefficients of Eqs. (32)–(35) one has $\Omega \rightarrow 0$ and from the Rindler parts factors $(\xi - a)^{1-D}$ come.

In the discussion above we have considered the vacuum energy–momentum tensor of the bulk. For a scalar field on manifolds with boundaries in addition to the bulk part the energy–momentum tensor contains a contribution located on the boundary. For arbitrary bulk and boundary geometries the expression of the surface energy–momentum tensor is given in Ref. [27]. Special cases of flat, spherical and cylindrical boundaries in the Minkowski background are considered in Refs. [28–30]. In the case of a conformally coupled scalar field the transformation formula for the surface energy–momentum tensor under the conformal rescaling of the metric is the same as that for the volume part. For our problem in this Letter, the surface energy–momentum tensor is obtained from the corresponding Rindler counterpart by a way similar to that described above. The expression for the latter is given in Ref. [27].

4. Conclusion

In the present Letter we have investigated the Casimir densities in dS spacetime for a conformally coupled massless scalar field which satisfies the Robin boundary condition (3) on a hypersurface described by Eq. (8). The coefficients in the boundary condition are given by relations (29) with constants A_R and B_R and, in general, depend on the point of the hypersurface. The latter is the conformal image of the flat boundary moving by uniform proper acceleration in the Minkowski spacetime. We have assumed that the field in dS spacetime is in the state conformally related to the Fulling–Rindler vacuum. The energy–momentum tensor in dS spacetime is generated from the corresponding results in the Rindler spacetime by using the standard formula for the energy–momentum tensors in conformally related problems in combination with the appropriate coordinate transformation. The Rindler energy–momentum tensor is taken from Ref. [22], where the general case of the curvature coupling parameter is considered. The VEV of the energy–momentum tensor for a brane in dS spacetime consists of two parts given in Eq. (30). The first one corresponds to the purely dS contribution when the boundary is absent. It is determined by formula (31), where the second term on the right is due to the trace anomaly and is zero for odd spacetime dimensions. The second part in the vacuum energy–momentum tensor is due to the imposition of boundary conditions on the fluctuating quantum field. The corresponding components are related to the vacuum energy–momentum tensor in the Rindler spacetime by Eqs. (32)–(35) and the Rindler tensor in the region $\xi > a$ is given by formulae (15)–(17). The results for the region $\xi < a$ are obtained from these formulae by replacements $I_\omega \rightarrow K_\omega$, $K_\omega \rightarrow I_\omega$. Unlike to the purely dS part, the boundary induced part of the energy–momentum tensor is non-diagonal and depends on both dS static coordinates r and θ . At the dS horizon both parts in the vacuum energy–momentum tensor diverge, with the leading divergence $(\alpha^2 - r^2)^{-(D+1)/2}$, coming from the purely dS part. Another type of divergence arise on the brane, where the boundary induced part dominates. Near the points where the brane touches the horizon, the divergences are mixed and are stronger. Note that in this Letter we have considered vacuum densities which are finite away from the brane and dS horizon. As it has been mentioned in Ref. [15], the same results will be obtained in the model where instead of externally imposed boundary condition the fluctuating field is coupled to a smooth background potential that implements the boundary condition in a certain limit.

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References

- [1] A.G. Riess, et al., *Astron. J.* 116 (1998) 1009;
S. Perlmutter, et al., *Astrophys. J.* 517 (1999) 565;
P. de Bernardis, et al., *Nature* 404 (2000) 955;
C.L. Bennett, et al., *Astrophys. J. Suppl.* 148 (2003) 1;
M. Tegmark, et al., *astro-ph/0310723*.
- [2] A. Strominger, *JHEP* 0110 (2001) 034;
A. Strominger, *JHEP* 0110 (2001) 049.
- [3] H.B.G. Casimir, *Proc. Kon. Nederl. Akad. Wetensch.* 51 (1948) 793.
- [4] V.M. Mostepanenko, N.N. Trunov, *The Casimir Effect and Its Applications*, Clarendon, Oxford, 1997.
- [5] G. Plunien, B. Muller, W. Greiner, *Phys. Rep.* 134 (1986) 87.
- [6] S.K. Lamoreaux, *Am. J. Phys.* 67 (1999) 850.
- [7] M. Bordag (Ed.), *The Casimir Effect. 50 Years Later*, World Scientific, Singapore, 1999.
- [8] M. Bordag, U. Mohidden, V.M. Mostepanenko, *Phys. Rep.* 353 (2001) 1.
- [9] K. Kirsten, *Spectral Functions in Mathematics and Physics*, CRC Press, Boca Raton, FL, 2001.
- [10] M. Bordag (Ed.), *Proceedings of the Fifth Workshop on Quantum Field Theory under the Influence of External Conditions*, *Int. J. Mod. Phys. A* 17 (6–7) (2002).

- [11] K.A. Milton, *The Casimir Effect: Physical Manifestation of Zero-Point Energy*, World Scientific, Singapore, 2002.
- [12] E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko, S. Zerbini, *Zeta Regularization Techniques with Applications*, World Scientific, Singapore, 1994.
- [13] E. Elizalde, *Ten Physical Applications of Spectral Zeta Functions*, in: *Lecture Notes in Physics*, Springer Verlag, Berlin, 1995.
- [14] M. Bordag, K. Kirsten, *Phys. Rev. D* 53 (1996) 5753;
M. Bordag, K. Kirsten, *Phys. Rev. D* 60 (1999) 105019.
- [15] N. Graham, R.L. Jaffe, V. Khemani, M. Quandt, M. Scandurra, H. Weigel, *Nucl. Phys. B* 645 (2002) 49;
N. Graham, R.L. Jaffe, H. Weigel, *Int. J. Mod. Phys. A* 17 (2002) 846;
N. Graham, R.L. Jaffe, V. Khemani, M. Quandt, M. Scandurra, H. Weigel, *Phys. Lett. B* 572 (2003) 196.
- [16] H. Gies, K. Langfeld, L. Moyaerts, *JHEP* 0306 (2003) 018.
- [17] N.D. Birrel, P.C.W. Davies, *Quantum Fields in Curved Space*, Cambridge Univ. Press, Cambridge, 1982.
- [18] A.A. Grib, S.G. Mamayev, V.M. Mostepanenko, *Vacuum Quantum Effects in Strong Fields*, Friedmann Laboratory, St. Petersburg, 1994.
- [19] S. Nojiri, S.D. Odintsov, S. Zerbini, *Class. Quantum Grav.* 17 (2000) 4855.
- [20] W. Naylor, M. Sasaki, *Phys. Lett. B* 542 (2002) 289.
- [21] E. Elizalde, S. Nojiri, S.D. Odintsov, S. Ogushi, *Phys. Rev. D* 67 (2003) 063515.
- [22] A.A. Saharian, *Class. Quantum Grav.* 19 (2002) 5039.
- [23] M.R. Setare, R. Mansouri, *Class. Quantum Grav.* 18 (2001) 2659.
- [24] A.A. Saharian, M.R. Setare, *Phys. Lett. B* 552 (2003) 119.
- [25] P. Candelas, D. Deutsch, *Proc. R. Soc. London, Ser. A* 354 (1977) 79.
- [26] A.A. Saharian, R.S. Davtyan, A.H. Yeranyan, hep-th/0307163.
- [27] A.A. Saharian, hep-th/0308108.
- [28] A. Romeo, A.A. Saharian, *J. Phys. A* 35 (2002) 1297.
- [29] A.A. Saharian, *Phys. Rev. D* 63 (2001) 125007.
- [30] A. Romeo, A.A. Saharian, *Phys. Rev. D* 63 (2001) 105019.