Algebraic decision trees and Euler characteristics*

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Abstract

For any set \( S \subseteq \mathbb{R}^n \), let \( \chi(S) \) denote its Euler characteristic. In this paper, we show that any algebraic computation tree or fixed-degree algebraic decision tree must have height \( \Omega(\log |\chi(S)| - cn) \) for deciding the membership question of a compact semi-algebraic set \( S \). This extends a result in Björner et al. (1992), where it was shown that any linear decision tree for deciding the membership question of a closed polyhedron \( S \) must have height greater than or equal to \( \log_3 |\chi(S)| \).

1. Introduction

Many problems in geometry and combinatorial optimization can be viewed as solving membership problems for sets \( S \subseteq \mathbb{R}^n \): given an input \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), decide whether \( x \in S \). For example, element distinctness, the problem of deciding whether \( n \) real numbers are all distinct, is the membership problem for \( S = \{(x_1, x_2, \ldots, x_n) | x_i \neq x_j \ \forall i \neq j\} \); points collinearity, the problem of deciding whether any three of \( n \) given points \((x_i, y_i) (1 \leq i \leq n)\) lie on the same line, is the membership problem for \( S \), where \( S \subseteq \mathbb{R}^2 \) is the set of all \((x_1, y_1, \ldots, x_n, y_n)\) satisfying \( x_i y_j + x_j y_k + x_k y_i - x_j y_i - x_k y_j - x_i y_k = 0 \) for some \( i < j < k \); triangle inequalities problem (raised by Papadimitriou [20]), which is a version of the problem of finding shortest paths, is the membership problem for \( S = \{(x_{i,j}) | x_{i,j} \geq 0, x_{i,j} + x_{j,k} \leq x_{i,k} \ \forall 1 \leq i, j, k \leq n\} \subseteq \mathbb{R}^3 \).

Two familiar complexity models for membership problems are the fixed-degree algebraic tree model and the algebraic computation tree model (see e.g. [22, 23, 1]). Let \( C_d(S) \) and \( C(S) \) denote the complexities in the degree-\( d \) algebraic tree and the algebraic computation tree models.

Much work has been done in the search for connections between the computational complexity of the membership problem for \( S \) and the geometric or topological

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properties of $S$, from which hopefully useful lower bounds may be derived. It is known
that, when $S$ is a convex polytope, $\Omega(\log(\phi_s(S)))$ tests are needed in the linear decision
tree model, where $\phi_s(S)$ is the number of $s$-dimensional faces of $S$; the $s=0$ case was
proved in Morávek [15], and the general case was proved in Kalinová [9] and in Yao
and Rivest [27] (also see Morávek and Puklák [16]). For any $S$, let $\beta_0(S)$ denote
the number of connected components of $S$. Dobkin and Lipton [5] showed that
$\Omega(\log(\beta_0(S)))$ tests are needed in the linear decision tree model. Using the Ole-
inik–Petrovsky–Milnor–Thom bound [18,19,14,24] on algebraic sets, Steele and
Yao [22] showed that $C_d(S) + n \log C_d(S) \geq \Omega(\log(\beta_0(S)))$ for any fixed $d$. Ben-Or [1]
improved these bounds, showing that $C_d(S)$ (for fixed $d$) and $C(S)$ are both at least
$\Omega(\log(\beta_0(S)))$.

Recently, Björner et al. [3] found another link, proving that any linear decision
tree for deciding the membership question of a closed polyhedron $S$ must use at least
$\log_3 |\chi(S)|$ tests, where $\chi(S)$ is the Euler characteristic of $S$. This result offered an
approach for proving new lower bounds to some geometric problems. For example, it
was used in [3] to show that $\Omega(n \log(n/k))$ linear tests are needed to solve the "k-equal
problem" the problem of determining whether there are $k$ identical elements out of
$n$ given input real numbers.

In this paper, we show that this connection between $\chi(S)$ and the computational
complexity of $S$ also exists for models much more general than linear decision trees
(thus solving an open problem posed in [3]). We prove that any algebraic computa-
tion tree or fixed-degree algebraic decision tree must have height $\Omega(\log|\chi(S)| - cn)$ for
deciding the membership question of any compact semi-algebraic set $S$. Precisely, we
will prove that for some constants $a_d, a$, the following two theorems are true.

**Theorem A.** Let $d > 0$ be any fixed integer. Then, for any compact semi-algebraic set
$S \subseteq \mathbb{R}^n$, we have $C_d(S) \geq \Omega(\log_2 |\chi(S)|) - a_d n$.

**Theorem B.** Let $S \subseteq \mathbb{R}^n$ be a compact semi-algebraic set. Then $C(S) \geq \Omega(\log_2 |\chi(S)|) - a n$.

In fact, we will derive a lower bound valid for any (compact and noncompact)
semi-algebraic set $S$. We introduce a new quantity $\tilde{\chi}(S)$, which equals $\chi(S)$ when $S$ is
compact. Theorems A and B are immediate consequences of Theorems 6 and 7 in
Section 5.

We remark that there are other algebraic complexity models studied in the litera-
ture (e.g. [13,21,23]), in addition to the fixed-degree algebraic trees and algebraic
computation trees discussed here.

In Section 2 we review standard definitions and some needed background material.
In Section 3, the "modified" Euler characteristic $\tilde{\chi}(S)$ is defined. In Section 4, prop-
ties of $\tilde{\chi}(S)$ are studied, which are then used in Section 5 to prove the main theorems.
Section 6 applies the theorems to the complexity of the $k$-equal problem in the
algebraic computation tree model. Section 7 discusses extensions and open problems.
2. Preliminaries

We review in this section some standard facts in topology and fix the notations. The proof of Lemma 1 will need some additional concepts not discussed here.

Throughout this paper, \( n > 0, m, r, s \geq 0 \) denote integers. For any set \( S \subseteq \mathbb{R}^n \), \( \overline{S} \) is the closure of \( S \); define \( \partial S = \overline{S} - S \). (The quantity \( \partial S \) is not the same as the boundary of \( S \) in standard point-set topology, which is defined as \( S - \text{Int}(S) \), although they coincide when \( S \) is an open set. For instance, if \( S \subseteq \mathbb{R}^3 \) is the 2-dimensional disk \( \{(x, y, 0) \mid x^2 + y^2 < 1\} \), then \( \partial S \) is the circle \( \{(x, y, 0) \mid x^2 + y^2 = 1\} \), while \( S - \text{Int}(S) \) is the closed disk \( \{(x, y, 0) \mid x^2 + y^2 \leq 1\} \).) When \( S \) is considered as a topological space, it is understood that the topology is the subspace topology induced by \( \mathbb{R}^n \). All polynomials refer to polynomials with real coefficients.

For any topological space \( S \) and integer \( i \geq 0 \), let \( \beta_i(S) \) denote the \( i \)th Betti number, which is the rank of the \( i \)th singular homology group (over integer coefficients). The Euler characteristic \( \chi(S) \) is defined to be \( \sum_{i \geq 0} (-1)^i \beta_i(S) \), when the Betti numbers are finite and are nonzero only for a finite number of \( i \). It is known that, if \( S, T \) are topological spaces homeomorphic to, respectively, the unit closed ball and the unit sphere in \( \mathbb{R}^n \), then \( \chi(S) = 1 \) and \( \chi(T) = 1 + (-1)^{n+1} \).

An algebraic set in \( \mathbb{R}^n \) is the set of points \( \bar{x} \in \mathbb{R}^n \) satisfying a finite set of polynomial equations. The following result is well known.

Oleinik–Petrovsky–Milnor–Thom Bound \([18, 19, 14, 24]\). Let \( S = \{\bar{x} \mid f_i(\bar{x}) = 0, \ 1 \leq i \leq r\} \subseteq \mathbb{R}^n \). Then all the Betti numbers of \( S \) are finite, and \( \sum_{i \geq 0} \beta_i(S) \leq d(2d - 1)^{n-1} \), where \( d \) is the maximum degree of any \( f_i \).

As an immediate corollary, the above bound gives \( |\chi(S)| \leq d(2d - 1)^{n-1} \).

A set \( S \subseteq \mathbb{R}^n \) is a semi-algebraic set if \( S \) can be generated from sets of the form \( \{x \mid f(\bar{x}) > 0\} \) (with \( f \) being polynomials in \( \mathbb{R}^n \)) through complementation, finite union and finite intersection. It is well known that \( S \) is a semi-algebraic if and only if there exist a finite number of polynomials \( f_{ij}, g_{ij} \) on \( \mathbb{R}^n \) such that \( S = \bigcup_i S_i \), where \( S_i \) is the set of all \( \bar{x} \in \mathbb{R}^n \) satisfying \( f_{ij}(\bar{x}) = 0, g_{ij}(\bar{x}) > 0 \) for all \( j \).

Clearly, the class of semi-algebraic sets in \( \mathbb{R}^n \) is closed under complementation, finite union, and finite intersection. It is also known that if \( S \subseteq \mathbb{R}^n \) is semi-algebraic, then \( \overline{S} \) is semi-algebraic, and hence so is \( \partial S \).

Let \( B \subseteq \mathbb{R}^n \) be a semi-algebraic set, and \( f : B \to \mathbb{R}^m \) be a continuous map. We say \( f \) is a semi-algebraic map if the graph \( \{(\bar{x}, f(\bar{x})) \mid \bar{x} \in B\} \) is a semi-algebraic set in \( \mathbb{R}^{n+m} \). It is known (see e.g. \([7]\)) that the image of a semi-algebraic set under a semi-algebraic map must be semi-algebraic. In particular, the projection of a semi-algebraic set \( S \) on coordinates \( i_1, i_2, \ldots, i_m \), i.e. the set \( \{(x_{i_1}, x_{i_2}, \ldots, x_{i_m}) \mid (x_1, x_2, \ldots, x_n) \in S\} \), is semi-algebraic. Also, the composition of two semi-algebraic maps, when defined, is again a semi-algebraic map. Two semi-algebraic sets are homeomorphic semi-algebraically if there is a homeomorphism \( f \) between them and \( f \) is a semi-algebraic map; note that this is an equivalence relation. More information about semi-algebraic sets can be found in, for example, \([4, 7]\).
An open k-simplex $\Delta$ in $\mathbb{R}^n$, where $0 \leq k \leq n$, is a set of the form \( \{ \bar{x} \mid \bar{x} = \sum_{0 \leq i \leq k} t_i \bar{p}_i, \quad t_i > 0, \quad \sum_{0 \leq i \leq k} t_i = 1 \} \), where $\bar{p}_i$ are $k + 1$ affinely independent points in $\mathbb{R}^n$; we denote this set by $[\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_k]$. The dimension of the simplex, written as $\dim(\Delta)$, is defined to be $k$. Let $0 \leq i_0, i_1, \ldots, i_j \leq k$ be distinct integers, where $0 \leq j \leq k$. Then the $j$-simplex $[\bar{p}_{i_0}, \bar{p}_{i_1}, \ldots, \bar{p}_{i_j}]$ is a face of $[\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_k]$. Note that, if $k > 0$, $\partial \Delta$ is the union of all its faces of dimensions (strictly) less than $k$.

An open simplicial decomposition of set $S \subseteq \mathbb{R}^n$ is a finite family $H$ of disjoint open simplices such that $S = \bigcup_{\Delta \in H} \Delta$, and that $H$ contains all faces of all members of $H$. We list some elementary properties that will be useful later. Let $H$ be an open simplicial decomposition of $S \subseteq \mathbb{R}^n$. Then $\chi(S) = \sum_{\Delta \in H} (-1)^{\dim(\Delta)}$. Suppose $H' \subseteq H$, and $S' = \bigcup_{\Delta \in H'} \Delta$. Then $S'$ is a closed set if and only if $H'$ is an open simplicial decomposition of $S'$. See [2, 6, 17] for more discussions on simplices and other background information in topology.

Let $S_1, S_2, \ldots, S_m$ be sets in $\mathbb{R}^n$. A triangulation for $S_1, S_2, \ldots, S_m$ is a pair $(\mathcal{D}, h)$, where $\mathcal{D} = \{ D_\alpha \mid \alpha \in \mathcal{A} \}$ is an open simplicial decomposition of the set $E = \bigcup_{\alpha \in \mathcal{A}} D_\alpha \subseteq \mathbb{R}^n$ and $h : E \to \mathbb{R}^n$ is a homeomorphism between $E$ and $h(E)$, such that the following is true: for any $1 \leq i \leq m$, there exists a subset $A_i \subseteq \mathcal{A}$ satisfying $S_i = \bigcup_{\alpha \in A_i} h(D_\alpha)$. The existence of triangulations are known for several classes of well-behaved sets (see e.g. [10, 11, 7}). We call $(\mathcal{D}, h)$ a semi-algebraic triangulation if $h$ is a semi-algebraic map; it implies that $h(D_\alpha)$ is semi-algebraic for each $\alpha \in \mathcal{A}$. We will need the following result.

The Triangulation Theorem (Hironaka [7]). Let $S_1, S_2, \ldots, S_m$ be a finite family of bounded semi-algebraic sets in $\mathbb{R}^n$. Then there exists a semi-algebraic triangulation for $S_1, S_2, \ldots, S_m$.

Finally, we consider the inverse stereographic mapping $\varphi_n : \mathbb{R}^n \to \mathbb{R}^{n+1}$, given by

$$\varphi_n(x_1, x_2, \ldots, x_n) = (x_1(1 - y_{n+1}), x_2(1 - y_{n+1}), \ldots, x_n(1 - y_{n+1}), y_{n+1})$$

where $y_{n+1}$ is the unique solution to the system of constraints $0 < y_{n+1} < 1$ and $\sum_{1 \leq i \leq n} x_i^2(1 - y_{n+1})^2 + (y_{n+1} - 2/3)^2 = 1/9$. Let $J_{n+1} \subseteq \mathbb{R}^{n+1}$ denote the sphere centered at $(0, 0, \ldots, 0, 2/3)$ with radius $1/3$. Then $\varphi_n$ is a homeomorphism from $\mathbb{R}^n$ onto $J_{n+1} - \{(0, 0, \ldots, 0, 1)\}$. Geometrically, if we identify $\bar{x} = (x_1, x_2, \ldots, x_n)$ with the point $(x_1, x_2, \ldots, x_n, 0)$ and draw a line in $\mathbb{R}^{n+1}$ between it and the north pole of $J_{n+1}$, then $\varphi_n(\bar{x})$ is the unique point where the line intersects $J_{n+1}$.

For any $S \subseteq \mathbb{R}^n$, we call $\varphi_n(S)$ the inverse stereographic image of $S$. The mapping $\varphi_n$ is clearly a semi-algebraic map. Hence, if $S$ is a semi-algebraic set in $\mathbb{R}^n$, then $\varphi_n(S)$ is a semi-algebraic set in $\mathbb{R}^{n+1}$. We will write $\varphi$ instead of $\varphi_n$ when the dimension $n$ is clear from the context.

3. A modified Euler characteristic

Let $S \subseteq \mathbb{R}^n$ be a semi-algebraic set. We will introduce an integer-valued quantity $\tilde{\chi}(S)$, which is closely related to the Euler characteristic $\chi(S)$. 
We first consider the case when $S$ is bounded. Let $(\mathcal{D}, h)$ be any semi-algebraic triangulation for $S$. (By the Triangulation Theorem, such triangulations exist.) Assume $\mathcal{D} = \{D_\alpha | \alpha \in A\}$, and $S = \bigcup_{\alpha \in A} h(D_\alpha)$, where $B \subseteq A$. Define $\chi(S; \mathcal{D}, h)$ to be $\sum_{\alpha \in B} (-1)^{\dim(D_\alpha)}$. Note that $\chi(S; \mathcal{D}, h) = 0$ if $S$ is the empty set.

**Lemma 1.** Let $S \subseteq \mathbb{R}^n$ be a bounded set, homeomorphic semi-algebraically to an open $k$-simplex where $0 < k < n$. Then $\chi(S; \mathcal{D}, h) = (-1)^k$ for any semi-algebraic triangulation $(\mathcal{D}, h)$ for $S$.

**Proof.** As mentioned in Section 2, the proof of this lemma involves additional topological concepts. See [6, 25] for terminology and other background information.

We need to prove

$$\sum_{\alpha \in B} (-1)^{\dim(D_\alpha)} = (-1)^k. \quad (1)$$

If $k = 0$, then $|B| = 1$ and Eq. (1) is clearly true. If $k = 1$, then $\{D_\alpha | \alpha \in B\}$ consists of $m$ open 1-simplices and $m - 1$ open 0-simplices for some $m \geq 1$. Again (1) is true.

Let $k > 1$. For each $0 \leq j \leq k$, let $t_j$ denote the number of $\alpha \in B$ with $\dim(D_\alpha) = j$. Let $\Delta$ be some open $k$-simplex, and $g : S \to \Delta$ be a semi-algebraic homeomorphism. Then $\xi = g \circ h$ is a semi-algebraic homeomorphism from $\bigcup_{\alpha \in B} D_\alpha$ to $\Delta$. Clearly, $\Delta$ is the disjoint union of $\{D_\alpha, \alpha \in B\}$.

Now, $\tilde{\Delta}$ is the disjoint union of $\Delta$ and $\partial \Delta$; the latter can be regarded topologically as the union of a point and an open $(k - 1)$-dimensional simplex. This suggests that we can try to construct $\tilde{\Delta}$ as a spherical complex in the natural way, using $t_0 + 1$ 0-dimensional cells, $t_{k-1} + 1$ $(k - 1)$-dimensional cells, and $t_j$ $j$-dimensional cells, $j \neq 0, k - 1$. To ensure that this is possible, it is sufficient to show that, for each $\alpha \in B$ with $j = \dim(D_\alpha) > 0$, $\partial(\xi(D_\alpha))$ is the image of a continuous mapping from a $(j - 1)$-dimensional sphere. We leave the proof of this technical result to the Appendix.

From the theory of spherical complex, we have

$$\chi(\tilde{\Delta}) = (t_0 + 1) + (t_{k-1} + 1) \cdot (-1)^{k-1} + \sum_{j \neq 0, k-1} t_j \cdot (-1)^j$$

$$= 1 + (-1)^{k-1} + \sum_{\alpha \in B} (-1)^{\dim(D_\alpha)}. \quad (2)$$

Since $\tilde{\Delta}$ is homeomorphic to a $k$-dimensional ball, we have $\chi(\tilde{\Delta}) = 1$. Thus, (1) follows from (2). □

**Theorem 1.** Let $S \subseteq \mathbb{R}^n$ be a bounded semi-algebraic set. Then $\chi(S; \mathcal{D}, h)$ is independent of the choice of $(\mathcal{D}, h)$.

**Proof.** Clearly, we can assume that $S$ is not empty. Let $(\mathcal{D}, h)$ and $(\mathcal{D}', h')$ be two semi-algebraic triangulations for $S$. Let $\mathcal{D} = \{D_\alpha | \alpha \in A\}$, $\mathcal{D}' = \{D_\beta | \beta \in A'\}$, and $S = \bigcup_{\alpha \in B} h(D_\alpha) = \bigcup_{\beta \in B'} h'(D_\beta)$. \[3\]
We need to show that $\mathcal{X}(S; \mathcal{D}, h) = \mathcal{X}(S; \mathcal{D}', h')$, i.e.
\[
\sum_{\alpha \in B} (-1)^{\dim(D_\alpha)} = \sum_{\beta \in B'} (-1)^{\dim(D_\beta)}.
\] (4)

Consider the family $\mathcal{F}$ of bounded semi-algebraic sets $\{h(D_\alpha) | \alpha \in A\} \cup \{h'(D_\beta) | \beta \in A'\}$. By the Triangulation Theorem, there exists a semi-algebraic triangulation $(\mathcal{D}'', h'')$ for $\mathcal{F}$. Let $\mathcal{D}'' = \{D''_\gamma | \gamma \in A''\}$. Then, for each $\alpha \in B$, there exists $A''_\alpha \subseteq A''$ such that
\[
h(D_\alpha) = \bigcup_{\gamma \in A''_\alpha} h''(D''_\gamma).
\] (5)

Also, for each $\beta \in B'$, there exists $A''_\beta \subseteq A''$ such that
\[
h(D'_\beta) = \bigcup_{\gamma \in A''_\beta} h''(D''_\gamma).
\] (6)

Let $\alpha \in B$. Applying Lemma 1 to $h(D_\alpha)$, we have $\mathcal{X}(h(D_\alpha); \mathcal{D}'', h'') = (-1)^{\dim(D_\alpha)}$. Thus, (5) gives
\[
\sum_{\gamma \in A''_\alpha} (-1)^{\dim(D''_\gamma)} = (-1)^{\dim(D_\alpha)}.
\]

It follows that
\[
\sum_{\alpha \in B} (-1)^{\dim(D_\alpha)} = \sum_{\alpha \in B} \sum_{\gamma \in A''_\alpha} (-1)^{\dim(D''_\gamma)} = \sum_{\gamma \in \bigcup_{\alpha \in B} A''_\gamma} (-1)^{\dim(D''_\gamma)}.
\] (7)

Similarly, we obtain from (6)
\[
\sum_{\beta \in B'} (-1)^{\dim(D'_\beta)} = \sum_{\gamma \in \bigcup_{\beta \in B'} A''_\gamma} (-1)^{\dim(D''_\gamma)}.
\] (8)

Now, from (3), (5) and (6), we have $S = \bigcup_{\alpha \in B} \bigcup_{\gamma \in A''_\alpha} h''(D''_\gamma) = \bigcup_{\beta \in B'} \bigcup_{\gamma \in A''_\beta} h''(D''_\gamma)$. This implies that
\[
\bigcup_{\alpha \in B} A''_\alpha = \bigcup_{\beta \in B'} A''_\beta.
\] (9)

From (7)–(9), we obtain (4). This proves Theorem 1. \qed

Definition 1. For any bounded semi-algebraic set $S \subseteq \mathbb{R}^n$, let $\mathcal{X}(S) = \mathcal{X}(S; \mathcal{D}, h)$, where $(\mathcal{D}, h)$ is any semi-algebraic triangulation for $S$.

In view of Theorem 1 and the Triangulation Theorem, the above $\mathcal{X}(S)$ is well-defined. For unbounded set, we will define it to be the $\mathcal{X}$ of its inverse stereographic image.

Definition 2. For any unbounded semi-algebraic set $S \subseteq \mathbb{R}^n$, define $\mathcal{X}(S)$ to be $\mathcal{X}(\varphi(S))$.

We note that, if $S$ is any bounded semi-algebraic set, the equality $\mathcal{X}(S) = \mathcal{X}(\varphi(S))$ is also true. In fact, if $(\mathcal{D}, h)$ is a semi-algebraic triangulation for $S$, with $\mathcal{D} = \{D_\alpha | \alpha \in A\}$
and $S = \bigcup_{a \in B} h(D_a)$, then $(\mathcal{D}', h')$ is a semi-algebraic triangulation for $\varphi(S)$, with $\varphi' = \{D_a | x \in A\}$, $h' = \varphi \circ h$, and $\varphi(S) = \bigcup_{a \in B} h'(D_a)$; it follows that $\chi(S) = \sum_{a \in B} (-1)^{\dim(D_a)} = \chi(\varphi(S))$.

4. Properties of $\chi$

Theorem 2. If $S_1, S_2, \ldots, S_m$ are disjoint semi-algebraic sets in $\mathbb{R}^n$, then $\chi(\bigcup_{1 \leq i \leq m} S_i)$ is equal to $\sum_{1 \leq i \leq m} \chi(S_i)$.

Proof. First we prove the theorem when all $S_i$ are bounded. By the Triangulation Theorem, there exists a semi-algebraic triangulation $(\mathcal{D}, h)$ for $S_1, S_2, \ldots, S_m$. Let $\varphi = \{D_a | x \in A\}$ and $S_i = \bigcup_{a \in A_i} h(D_a), 1 \leq i \leq m$. By definition,

$$\chi(S_i) = \sum_{a \in A_i} (-1)^{\dim(D_a)}.$$

Observing that all $A_i$ are disjoint, and that

$$\bigcup_{1 \leq i \leq m} S_i = \bigcup_{a \in \bigcup_{1 \leq i \leq m} A_i} h(D_a),$$

we have

$$\chi\left(\bigcup_{1 \leq i \leq m} S_i\right) = \sum_{a \in \bigcup_{1 \leq i \leq m} A_i} (-1)^{\dim(D_a)}.$$

It follows that $\chi(\bigcup_{1 \leq i \leq m} S_i) = \sum_{1 \leq i \leq m} \chi(S_i).

We now turn to the general case when some of the $S_i$ may be unbounded. As their inverse stereographic images $\varphi(S_i)$ are disjoint bounded semi-algebraic sets, by applying the result we just derived, we obtain $\chi(\varphi(\bigcup_{1 \leq i \leq m} S_i)) = \sum_{1 \leq i \leq m} \chi(\varphi(S_i))$. But the $\chi$ of any semi-algebraic set is the same as the $\chi$ of its inverse stereographic image. This proves $\chi(\bigcup_{1 \leq i \leq m} S_i) = \sum_{1 \leq i \leq m} \chi(S_i)$. $\square$

Theorem 3. If $S \subseteq \mathbb{R}^n$ is a compact semi-algebraic set, then $\chi(S) = \chi(S)$.

Proof. By assumption, $S$ is bounded. By the Triangulation Theorem, there exists a semi-algebraic triangulation $(\mathcal{D}, h)$ for $S$, with $\varphi = \{D_a | x \in A\}$ and $S = \bigcup_{a \in B} h(D_a)$. By definition,

$$\chi(S) = \sum_{a \in B} (-1)^{\dim(D_a)}.$$

Let $E = \bigcup_{a \in B} D_a$. Then $S = h(E)$. As $S$ is compact, $E$ is compact and thus closed, which implies that $\{D_a | x \in B\}$ must be an open simplicial decomposition of $E$. Hence $\chi(E) = \sum_{a \in B} (-1)^{\dim(D_a)}$. Since $S$ is homeomorphic to $E$, this means $\chi(S) = \sum_{a \in B} (-1)^{\dim(D_a)}$. This proves $\chi(S) = \chi(S)$. $\square$

Theorem 4. Let $X \subseteq \mathbb{R}^n$ be a semi-algebraic set, and $\psi: X \to \mathbb{R}^n$ be a semi-algebraic map. Then $\chi(X) = \chi(S)$, where $S = \{((\tilde{x}, \psi(\tilde{x}))) | \tilde{x} \in X\}$. 
Proof. Let $\mathcal{D}, h$ be a semi-algebraic triangulation for $\varphi_n(X)$, with $\mathcal{D} = \{ D_\alpha | \alpha \in A \}$ and $\varphi_n(X) = \bigcup_{\alpha \in B} h(D_\alpha)$. Then

$$\chi(X) = \sum_{\alpha \in B} (-1)^{\dim(D_\alpha)}. \quad (10)$$

Let $X_\alpha = (\varphi_n^{-1} \circ h)(D_\alpha)$. Then each $X_\alpha$ is a semi-algebraic set, and $\varphi_n^{-1} \circ h$ is a semi-algebraic homeomorphism between $X_\alpha$ and $D_\alpha$. It is also clear that $X$ is the disjoint union of $X_\alpha$, $\alpha \in B$.

Let $S_\alpha = \{ (\vec{x}, \psi(\vec{x})) | \vec{x} \in X_\alpha \}$. Then each $S_\alpha$ is a semi-algebraic set, as $S_\alpha$ can be written as $S_\alpha \cap \{ (\vec{x}, \vec{y}) | \vec{x} \in X_\alpha \}$. Also, $S$ is the disjoint union of $S_\alpha$, $\alpha \in B$, since $X$ is the disjoint union of $X_\alpha$, $\alpha \in B$. By Theorem 2, we have

$$\chi(S) = \sum_{\alpha \in B} \chi(S_\alpha).$$

However, each $S_\alpha$ is homeomorphic semi-algebraically to $X_\alpha$ and hence to $D_\alpha$. Thus, the bounded semi-algebraic set $\varphi(S_\alpha)$ is homeomorphic semi-algebraically to $D_\alpha$. By Lemma 1, $\chi(\varphi(S_\alpha)) = (-1)^{\dim(D_\alpha)}$. It follows that $\chi(S_\alpha) = (-1)^{\dim(D_\alpha)}$, and $\chi(S) = \sum_{\alpha \in B} (-1)^{\dim(D_\alpha)}$. Comparing this equation with (10), we obtain $\chi(X) = \chi(S)$. □

For any set $Y$, let $\nu_Y$ be the characteristic function of $Y$, i.e., $\nu_Y(y) = 1$ if $y \in Y$ and 0 otherwise. We need the following lemma, which can be regarded as an extension of Theorem 2.

Lemma 2. Let $S, S_1, S_2, \ldots, S_m$ be semi-algebraic sets in $\mathbb{R}^n$. If $v_S = \sum_{1 \leq i \leq m} w_i v_{S_i}$, where $w_i$ are real numbers, then $\chi(S) = \sum_{1 \leq i \leq m} w_i \chi(S_i)$.

Proof. First we prove the lemma when all $S_i$ are bounded. By the Triangulation Theorem, there exists a semi-algebraic triangulation $(\mathcal{D}, h)$ for $S, S_1, S_2, \ldots, S_m$, with $\mathcal{D} = \{ D_\alpha | \alpha \in A \}$. Let $S = \bigcup_{\alpha \in A} a_\alpha h(D_\alpha)$, and $S_i = \bigcup_{\alpha \in A} a_{i, \alpha} h(D_\alpha)$, $1 \leq i \leq m$, where $a_\alpha, a_{i, \alpha} \in \{0, 1\}$. Clearly, if $\vec{x} \in h(D_\alpha)$, then $v_S(\vec{x}) = a_\alpha$ and $v_{S_i}(\vec{x}) = a_{i, \alpha}$. It follows from $v_S = \sum_{1 \leq i \leq m} w_i v_{S_i}$, that $a_\alpha = \sum_{1 \leq i \leq m} w_i a_{i, \alpha}$ for all $\alpha \in A$. This leads to

$$\chi(S) = \sum_{\alpha \in A} a_\alpha (-1)^{\dim(D_\alpha)}$$

$$= \sum_{\alpha \in A} \sum_{1 \leq i \leq m} w_i a_{i, \alpha} (-1)^{\dim(D_\alpha)}$$

$$= \sum_{1 \leq i \leq m} w_i \chi(S_i).$$

This proves the lemma when all the $S_i$'s are bounded.

For the general case, one can apply the above conclusion to obtain $\chi(\varphi(S)) = \sum_{1 \leq i \leq m} w_i \chi(\varphi(S_i))$. As $\chi(S) = \chi(\varphi(S))$ and $\chi(S_i) = \chi(\varphi(S_i))$, this implies immediately $\chi(S) = \sum_{1 \leq i \leq m} w_i \chi(S_i)$. □
Theorem 5. Let \( S = \{ \bar{x} : f_i(\bar{x}) = 0, g_j(\bar{x}) = 0, 1 \leq i \leq r, 1 \leq j \leq s \} \subseteq \mathbb{R}^n \), where \( f_i, g_j \) are polynomials. Then \( |\hat{\chi}(S)| \leq d(2d-1)^{n+s+2} \), where \( d = \max_{i,j}\{\deg(f_i), \deg(g_j)\} \).

Proof. Let \( S' = \varphi(S) \subseteq \mathbb{R}^{n+1} \). Then \( S' \) is bounded and semi-algebraic. Furthermore, \( \hat{\chi}(S') = \hat{\chi}(S) \). To prove the theorem, it suffices to prove \( \hat{\chi}(S') \leq d(2d-1)^{n+s+2} \).

We first write down the explicit constraints defining \( S' \). For \( 1 \leq i \leq r \), let \( F_i(\bar{y}) = (1-y_{n+1})^d f_i(1/(1-y_{n+1}), y_1/(1-y_{n+1}), \ldots, y_n/(1-y_{n+1})) \), where \( \bar{y} = (y_1, y_2, \ldots, y_n) \); for \( 1 \leq j \leq s \), let \( G_j(\bar{y}) = (1-y_{n+1})^d g_j(1/(1-y_{n+1}), y_2/(1-y_{n+1}), \ldots, y_n/(1-y_{n+1})) \). Let us further define \( F_{r+1}(\bar{y}) = \sum_{1 \leq i \leq n} y_i^2 + (y_{n+1} - 2/3)^2 - 1/9 \), \( G_{s+1}(\bar{y}) = y_{n+1} - 1 \), and \( G_{s+2}(\bar{y}) = -y_{n+1} \). Note that all \( F_i, G_j \) are polynomials of degrees no more than \( d \).

From the definitions of \( S' \) and \( \varphi \), it is easy to see that \( S' \) is the set of \( \bar{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^{n+1} \) satisfying \( F_i(\bar{y}) = 0, 1 \leq i \leq r+1 \) and \( G_j(\bar{y}) = 0, 1 \leq j \leq s+2 \).

Let \( W = \{(\bar{y}, \bar{u}) : F_i(\bar{y}) = 0, 1 \leq i \leq r+1, G_j(\bar{y}) + u_j = 0, 1 \leq j \leq s+2 \} \subseteq \mathbb{R}^{n+3} \). Let \( \mathcal{S} = \{1, -1\}^{s+2} \). For each \( \bar{s} = (s_1, s_2, \ldots, s_{s+2}) \in \mathcal{S} \), let \( W_{\bar{s}} = W \cap \{(\bar{y}, \bar{u}) \mid \forall j, u_j > 0 \} \). Then \( Q = \bigcup_{\mathcal{S}} W_{\bar{s}} \) is the disjoint union of all the \( W_{\bar{s}} \).

Clearly, \( Q \) and \( W_{\bar{s}} \) are semi-algebraic sets. By Theorem 2, we have

\[
\hat{\chi}(Q) = \sum_{\mathcal{S}} \hat{\chi}(W_{\bar{s}}). \tag{11}
\]

We intend to utilize Eq. (11) to obtain the desired upper bound on \( |\hat{\chi}(S')| \). The plan is to show \( \hat{\chi}(S') = \hat{\chi}(W_{\bar{s}}) \) by using Theorem 4, to get an upper bound on \( |\hat{\chi}(Q)| \) by using the Oleinik–Petrovsky–Milnor–Thorn Bound, and finally to substitute these relations into (11).

For each \( \bar{s} \in \mathcal{S} \), let \( \psi_{\bar{s}}(\bar{x}) = (s_1(-G_1(\bar{y})), s_2(-G_2(\bar{y})), \ldots, s_{s+2}(-G_{s+2}(\bar{y}))) \). Clearly, \( \psi_{\bar{s}} : S' \to \mathbb{R}^{s+2} \) is continuous over \( S' \). Now note that \( \{(\bar{y}, \psi_{\bar{s}}(\bar{y})) \mid \bar{y} \in S'\} \) is the same as \( W_{\bar{s}} \), which is by definition a semi-algebraic set. Thus \( \psi_{\bar{s}} \) is a semi-algebraic map, and we can apply Theorem 4 to obtain

\[
\hat{\chi}(S') = \hat{\chi}(W_{\bar{s}}). \tag{12}
\]

We now estimate \( |\hat{\chi}(Q)| \). For each subset \( L \subseteq \{1, 2, \ldots, s+2\} \), let \( Q_L = W \cap \{(\bar{y}, \bar{u}) \mid u_i = 0 \forall i \in L\} \). In particular, \( Q_\emptyset = W \). By inclusion–exclusion, we have for the characteristic functions

\[
v_Q = \sum_{L} (-1)^{|L|} v_{Q_L}.
\]

By Lemma 2, this leads to

\[
\hat{\chi}(Q) = \sum_{L} (-1)^{|L|} \hat{\chi}(Q_L).
\]

Now as each \( Q_L \) is clearly a compact algebraic set, we have, by Theorem 3, \( \hat{\chi}(Q_L) = \chi(Q_L) \). It follows that

\[
\hat{\chi}(Q) = \sum_{L} (-1)^{|L|} \chi(Q_L). \tag{13}
\]
For each $L$, we have from the Oleinik–Petrovsky–Milnor–Thom Bound,

$$|\chi(Q_L)| \leq \sum_{i \geq 0} \beta_i(Q_L)$$

$$\leq d(2d-1)^s+s+2. \quad (14)$$

From (13) and (14), we obtain

$$|\mathcal{C}(Q)| \leq 2s^2d(2d-1)^s+s+2. \quad (15)$$

It follows from (11), (12) and (15) that

$$|\mathcal{C}(S')| \leq d(2d-1)^s+s+2,$$

which completes the proof of Theorem 5. \qed

5. Algebraic decision trees

Let $S \subseteq \mathbb{R}^n$. The membership problem for $S$ is: given input $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, decide whether $\mathbf{x} \in S$. We consider the complexity of membership problems in two familiar decision tree models (see e.g. [1,22]): the fixed-degree algebraic tree model and the algebraic computation tree model. In both models, an algorithm is a decision tree whose nodes perform certain algebraic tests, and each of whose leaves contains either a YES or NO answer to the membership question for the input.

In a $d$th degree algebraic tree, $d \geq 1$, each internal node performs a test of the form $f(\mathbf{x}) : 0$, where $f$ is any polynomial of degree no more than $d$. In an algebraic computation tree, an internal node is either an arithmetic node or a branching node. Each arithmetic node $v$ performs an arithmetic assignment $z \leftarrow z' \text{ op } z''$, where the variable $z$ is a programming variable created at $v$, the operation $\text{op} \in \{+, -, *, /\}$, and $z', z''$ are either real constants, or the input variables $x_i$, or programming variables created at other arithmetic nodes along the path from the root to $v$. Furthermore, if the assignment operation “op” is division “/”, then $z''$ must not take on 0 value for any input leading to $v$. Each branching node $v$ performs a branching test $z' : 0$, where $z'$ is either an input variable $x_i$ or a programming variable created at an arithmetic node along the path from the root to $v$. Let $C_d(S)$, $C(S)$ be the minimum height of any $d$th degree algebraic tree and that of any algebraic computation tree for solving the membership problem for $S$. Note that $C_d(S)$ may be infinite even for semi-algebraic $S$.

Let $\lambda_{1,d} = \frac{1}{(\log_2 3 + \log_2 (2d-1))}$, $\lambda_{2,d} = (\log_2 (2d-1))/(\log_2 3 + \log_2 (2d-1))$, and $\lambda_{3,d} = (\log_2 d + 2 \log_2 (2d-1))/(\log_2 3 + \log_2 (2d-1))$.

Theorem 6. Let $S \subseteq \mathbb{R}^n$ be a semi-algebraic set and $d \geq 1$. Then $C_d(S) \geq \lambda_{1,d} \log_2 |\mathcal{C}(S)| - \lambda_{2,d} n - \lambda_{3,d}$. 

Corollary. For any semi-algebraic \( S, S' \subseteq \mathbb{R}^n \), \( C_d(S) \geq \lambda_{1,d} \log_2 |\hat{\chi}(S \cap S')| - \lambda_{2,d} n - \lambda_{3,d} - C_d(S') \).

Proof. If \( S \) is either \( \mathbb{R}^n \) or the empty set, the theorem is clearly true, since \( \hat{\chi}(\mathbb{R}^n) = (-1)^n \) and \( \hat{\chi}(\emptyset) = 0 \). We can thus assume that \( S \neq \mathbb{R}^n, \emptyset \), which implies \( C_d(S) > 0 \).

Let \( T \) be any \( d \)-th degree algebraic tree which solves the membership problem for \( S \). Let \( m \) denote the height of \( T \). We will show \( m \geq \lambda_{1,d} \log_2 |\hat{\chi}(S)| - \lambda_{2,d} n - \lambda_{3,d} \).

Let \( \mathcal{L} \) be the set of all YES leaves of \( T \). Clearly, \( \mathcal{L} \) is not empty. For each \( l \in \mathcal{L} \), let \( V_l \) be the set of all inputs \( \bar{x} \in \mathbb{R}^n \) reaching \( l \). Clearly, \( V_l \) are semi-algebraic sets, and \( S \) is the disjoint union of \( V_l, l \in \mathcal{L} \). By Theorem 2,

\[
\hat{\chi}(S) = \sum_{l \in \mathcal{L}} \hat{\chi}(V_l). \tag{16}
\]

Let \( l \in \mathcal{L} \). Then \( V_l \) can be written in the form \( \{ \bar{x} | f_1(\bar{x}) = 0, \ldots, f_r(\bar{x}) = 0, g_1(\bar{x}) < 0, \ldots, g_s(\bar{x}) < 0 \} \), where \( r + s \leq m \), and \( f_i, g_j \) are polynomials of degree at most \( d \). By Theorem 5,

\[
|\hat{\chi}(V_l)| \leq d(2d - 1)^{n+r+s} + 2 \\
\leq d(2d - 1)^{n+m+2} + 2. \tag{17}
\]

From (16) and (17), we obtain

\[
d(2d - 1)^{n+m+2} |\mathcal{L}| \geq |\hat{\chi}(S)|.
\]

Using the fact \( 3^m \geq |\mathcal{L}| \), we obtain \( m \geq \lambda_{1,d} \log_2 |\hat{\chi}(S)| - \lambda_{2,d} n - \lambda_{3,d} \). This proves Theorem 6. We obtain the corollary immediately by applying the theorem to \( S \cap S' \). \( \square \)

Remark. This generalizes Theorem 3.1 in [3] for linear decision trees (the case \( d = 1 \)).

Let \( c_1 = 1/(2 \log_2 3), c_2 = 1/2, \) and \( c_3 = (1 + 2 \log_2 3)/(2 \log_2 3) \).

Theorem 7. Let \( S \subseteq \mathbb{R}^n \) be a semi-algebraic set. Then \( C(S) \geq c_1 \log_2 |\hat{\chi}(S)| - c_2 n - c_3 \).

Corollary. For any semi-algebraic \( S, S' \subseteq \mathbb{R}^n \), \( C(S) \geq c_1 \log_2 |\hat{\chi}(S \cap S')| - c_2 n - c_3 - C(S') \).

Proof. As argued in the proof of the preceding theorem, we can assume without loss of generality that \( S \neq \mathbb{R}^n, \emptyset \). Let \( T \) be any algebraic computation tree which solves the membership problem for \( S \). Let \( m \) denote the height of \( T \). We will show \( m \geq c_1 \log_2 |\hat{\chi}(S)| - c_2 n - c_3 \).

Let \( \mathcal{L} \) be the set of all YES leaves of \( T \). Clearly, \( \mathcal{L} \) is not empty. For each \( l \in \mathcal{L} \), let \( V_l \) be the set of all inputs \( \bar{x} \in \mathbb{R}^n \) reaching \( l \). As shown in [1], \( V_l \) can be expressed as the
projections of algebraic sets and thus are semi-algebraic sets. Since $S$ is the disjoint union of all $V_i, i \in \mathcal{L}$, we have by Theorem 2,

$$\chi(S) = \sum_{i \in \mathcal{L}} \chi(V_i).$$

(18)

Let $i \in \mathcal{L}$. We will prove that

$$|\chi(V_i)| \leq 2 \cdot 3^n + m + 2.$$  

(19)

Let $v_0 = \text{root}, v_1, \ldots, v_s = l$ be the sequence of nodes along the path $\zeta$ from the root to $l$. Clearly, $s \leq m$. Let $v_i, i \in I$, be the set of arithmetic nodes along $\zeta$, and $I' = \{0, 1, \ldots, s-1\} - I$. Let $z_i, i \in I$, be the programming variables created at nodes $v_i$. For the rest of this section, we use the notation $\mathbf{z}$ to denote the $|I|$-component vector $(z_i | i \in I)$.

We associate with each $v_i, 0 \leq i \leq s-1$, a polynomial constraint in variables $x_1, x_2, \ldots, x_n, z_j, i \in I$. To simplify notations, let us agree that, for $j \in \{1, \ldots, n\}$, the symbol $z_j$ stands for $x_j$.

Case A: $i \in I'$. If the branch taken from $v_i$ to $v_{i+1}$ is labeled by $z_j \text{ rel}_i 0$, where rel$_i \in \{<, =, >\}$, then we associate with $v_i$ the inequality $f_i(\mathbf{z}, \mathbf{z}) \text{ rel}_i 0$, where $f_i$ is the linear polynomial $z_j$.

Case B: $i \in I$. Suppose that the arithmetic assignment performed at $v_i$ is $r_i z_j$. Then $I_i$ is one of the following: $z_j + z_k$, $z_j - z_k$, $z_j \times z_k$, $z_j / z_k$, $t_i + z_k$, $t_i - z_k$, $t_i \times z_k$, $t_i / z_k$, where $-n \leq j, k \leq 0$ and $t_i$ is some nonzero real constant. It is also required that, if $I_i$ is either $z_j / z_k$ or $c_i / z_k$, then $z_k$ cannot be evaluated to 0 for any input reaching $v_i$. We associate with $v_i$ the polynomial constraint $f_i(\mathbf{z}, \mathbf{z}) = 0$, where $f_i$ is defined as follows:

(a) if $I_i$ is $z_j \text{ op} z_k$ with op $\in \{+, -, \times\}$, then $f_i$ is $z_i - (z_j \text{ op} z_k)$;
(b) if $I_i$ is $z_j / z_k$, then $f_i$ is $z_i z_k - z_j$;  
(c) if $I_i$ is $t_i \text{ op} z_k$ with op $\in \{+, -, \times\}$, then $f_i$ is $z_i - (t_i \text{ op} z_k)$;  
(d) if $I_i$ is $t_i / z_k$, then $f_i$ is $z_i z_k - t_i$.

Note that in all cases the degrees of $f_i$ are at most 2.

For $i \in I$, let $a_i(\mathbf{x})$ be the value assigned to $z_i$ when the input is $\mathbf{x} \in V_i$. Let $M \subseteq \mathbb{R}^{|I'|}\times\mathbb{R}^n$ be the set of all $(\mathbf{x}, a_i(\mathbf{x}) | i \in I)$ with $\mathbf{x} \in V_i$.

Fact 1. $M$ can be described as the set of all $(\mathbf{x}, \mathbf{z}) \in \mathbb{R}^{|I'|}\times\mathbb{R}^n$ satisfying $f_i(\mathbf{x}, \mathbf{z}) = 0$ and $f_j(\mathbf{x}, \mathbf{z}) \text{ rel}_j 0$ for $i \in I, j \in I'$.

Fact 2. For each $i \in I$, $a_i$ is continuous over $V_i$.

Fact 1 is obvious from the definitions. Fact 2 can be obtained by a straightforward induction on $i$. (Actually, $a_i$ are rational functions, although we do not need this fact.)

By Fact 1, $M$ is defined by linear and quadratic constraints in $\mathbb{R}^{|I'|}\times\mathbb{R}^n$, with no more than $|I'|$ strict inequalities. We have from Theorem 5

$$|\chi(M)| \leq 2 \cdot 3^n + |I'| + |I'| + 2$$

$$\leq 2 \cdot 3^n + m + 2.$$  

(20)
Let $\psi : \mathbb{R}^n \to \mathbb{R}^{1|I}$ be the mapping $\psi(\bar{x}) = (a_i(\bar{x}) \mid i \in I)$. Clearly, $M = \{(\bar{x}, \psi(\bar{x})) \mid \bar{x} \in V_i\}$. Now, $M$ is semi-algebraic by Fact 1, and $\psi$ is continuous on $V_i$ by Fact 2. Thus $\psi$ is a semi-algebraic map, and by Theorem 4, we have

$$\hat{\chi}(V_i) = \hat{\chi}(M).$$

(21)

From (20) and (21), we obtain (19).

From (18) and (19), we obtain

$$|\hat{\chi}(S)| \leq 2 \cdot 3^n + 2m^2 + 2,$$

from which we obtain immediately $m \geq c_1 \log_2 |\hat{\chi}(S)| - c_2 n - c_3$. This proves Theorem 7. The corollary follows immediately. \(\square\)

6. The $k$-equal problem

In [3], it was shown that the "$k$-equal problem" has complexity $\theta(n \log(2n/k))$ in the linear decision tree model. We now apply Theorems 7 to determine its complexity in the algebraic computation tree model.

Let $V_{n,k} \subseteq \mathbb{R}^n$ denote the set of all $(x_1, x_2, \ldots, x_n)$ such that $x_{i_1} = x_{i_2} = \cdots = x_{i_k}$ for some $k$ distinct $i_1, i_2, \ldots, i_k$.

**Theorem 8.** Let $2 \leq k \leq n$. Then $C(V_{n,k}) = \theta(n \log(2n/k))$.

**Proof.** The proof follows the same outline as that for the linear decision trees in [3]. It was shown in [3] that there is a decision tree of height $O(n \log(2n/k))$ using comparisons of the form $x_i - x_j$. It follows that $C(V_{n,k}) \leq O(n \log(2n/k))$.

For the lower bounds, for any $1 \leq k \leq m$, let $\mathcal{B}_{m,k}$ denote the set of all hyperplanes of the form $\{(x_1, x_2, \ldots, x_m) \mid x_{i_1} = \cdots = x_{i_k}\}$, where $1 \leq i_1 < \cdots < i_k \leq m$, and let $K_{m,k} = \bigcap_{B \in \mathcal{B}_{m,k}} B$; clearly $K_{m,k}$ is just the set of all $(x, x, \ldots, x)$. Let $H_{m,k}$ be a hyperplane in $\mathbb{R}^m$ defined by $\sum a_i x_i = 1$, such that the translated hyperplane $\sum a_i x_i = 0$ contains $K_{m,k}$ but not any other $\bigcap_{B \in \mathcal{B}} B \neq K_{m,k}$, where $\mathcal{B} \subseteq \mathcal{B}_{m,k}$; such $H_{m,k}$ clearly exist. Let $c_{m,k} > 0$ be a sufficiently large constant that $H_{m,k}$ intersects every $\bigcap_{B \in \mathcal{B}} B \neq K_{m,k}$, where $\mathcal{B} \subseteq \mathcal{B}_{m,k}$, at least at some points inside the open cube $(-c_{m,k}, c_{m,k})^m$. Let $W_{m,k} = H_{m,k} \cap (-c_{m,k}, c_{m,k})^m$. It was proved in [3] that, if $1 \leq k \leq n/2$, there exists an $m$ such that $n - k + 1 \leq m \leq n$ and $|\hat{\chi}(W_{m,k} \cap W_{m,k})| \geq (m - 1)! k^{-m - 1} - 1$.

From the corollary to Theorem 7, we have

$$C(V_{m,k}) \geq c_1 \log_3 |\hat{\chi}(W_{m,k} \cap W_{m,k})| - c_2 m - c_3 - C(W_{m,k})$$

$$\geq c_1 \log_3 |\hat{\chi}(V_{m,k} \cap W_{m,k})| - c_2 m - c_3 - (6m + 1)$$

$$\geq c_1 \log_3 ((m - 1)! k^{-m - 1}) - (c_2 + 6)m - c_3 - 4$$

$$\geq c_1 \log_3 ((m - 1)/ke)^{-m - 1} - (c_2 + 6)m - c_3 - 4$$

$$\geq c_1 (n - k) \log_3 ((n - k)/k) - (3c_1 + c_2 + 6)n - c_3 - 4.$$  

(22)
Now, \( C(V_{n,k}) \geq C(V_{m,k}) - 4n \), since we can convert an algebraic computation tree \( T \) for \( V_{n,k} \) into one for \( V_{m,k} \) by first computing in \( 4n \) instructions \( z = \max_{1 \leq i \leq m} |x_i| \) and \( jz \) for all \( m+1 \leq j \leq n \), and then use \( T \) with every occurrence of \( x_j \) replaced by \( jz \) for \( m+1 \leq j \leq n \). Thus, (22) implies that there exists a constant \( 0 < \varepsilon < 1 \) such that, for \( 2 \leq k \leq en \), \( C(V_{n,k}) \geq \Omega(n \log(2n/k)) \).

It remains to show that the lower bound holds for \( k \geq en \). It is sufficient to prove that, in this case, \( C(V_{n,k}) \geq \Omega(n) \). We can assume that \( n \geq 4/e \). Let \( T \) be an algebraic computation tree for solving the membership problem of \( V_{n,k} \). Assume that the height of \( T \) is less than \( (k-1)/2 \). We will derive a contradiction. Consider the input \( \bar{x} = (x_1, x_2, \ldots, x_k) \) with \( x_1 = x_2 = \cdots = x_k = 0 \) and \( x_j = j \) for all other \( j \). This input must reach a leaf \( l \) accepting the input. Since each arithmetic or branching instruction involves at most two variables, there must be some \( x_{i_1}, \ldots, x_{i_k} \) not involved in any instruction along the path from the root to \( l \). If we change the input \( \bar{x} \) by making \( x_{i} = i \), this new input must still reach \( l \). This is a contradiction, as the modified input is no longer in the set \( V_{n,k} \) and should not be accepted. This proves that the height of the tree is at least \( (k-1)/2 \), which is \( \Omega(n) \). 

For \( d \)th degree algebraic trees, the situation is more complex. Let \( d \geq 2 \) be fixed. It is clear that \( C_d(V_{n,k}) \leq O((n \log(2n/k))) \). An argument similar to the one in the proof of Theorem 8 gives, for some \( 0 < \varepsilon < 1 \), \( C_d(V_{n,k}) \geq \Omega((n \log(2n/k))) \) for \( 2 \leq k \leq n \). However, it is no longer true that \( C_d(V_{n,k}) \geq \Omega(n) \) for all \( \varepsilon n \leq k \leq n \). We show below that in fact the complexity can be improved to \( o(n) \) when \( k \) is close to \( n \).

For \( 1 \leq j \leq n \) and \( A \subseteq \{1, 2, \ldots, n\} \), let \( I(j, A) \) denote the quadratic query \( \sum_{a \in A} (x_j - x_a)^2 = 0 \). Clearly, the answer is "-" if and only if all the \( x_a \) for \( a \in A \) have the common values \( x_j \). Note that \( j \) may be in \( A \). This type of queries can be useful in obtaining a better upper bound for the \( k \)-equal problem, as will be seen in the proof of Theorem 9 below.

Before stating and proving the theorem, we derive a useful combinatorial result. Let \( n > 0 \) and \( 0 \leq m \leq n \). An \((n, m)\)-tree \((W, L, w)\) is a triplet, where \( W \) is a nonempty rooted binary tree, \( L \subseteq L \) with \( L \) being the set of leaves of \( W \), and \( w \) is a mapping from \( L \) to the positive integers, in addition, the following conditions are satisfied: no two leaves in \( L - L_v \) can be siblings, \( \sum_{l \in L} w(l) = n \) and \( |L| = m \). For each internal node \( v \) of \( W \), let \( w(v) \) denote the sum of \( w(l) \) over all the descendent leaves \( l \) of \( v \). We say that \( W \) is balanced, if \( w(v) \) is always equal to either \( \lceil w(v)/2 \rceil \) or \( \lfloor w(v)/2 \rfloor \) whenever \( u \) is a child of \( v \). Let \( h(n, m) \) be the maximum number of leaves for any balanced \((n, m)\)-tree. Clearly, \( h(n, 0) = 1 \) for all \( n > 0 \).

**Lemma 3.** For all \( 1 \leq m \leq n \),

\[
h(n, m) \leq m + \log_2 \left( \frac{2^\lceil \log_2 n \rceil}{m} \right).
\]

**Proof.** Let \( h(n, m) \) be defined as \(-\infty\) if \( m > n \). We prove Lemma 3 by induction on the values of \( t = n + m \). If \( t = 2 \), the lemma is true since \( n = m = 1 \) and \( h(1, 1) = 1 \). Let
\(n + m = t > 2\), and assume that the lemma has been proved for all lesser values of \(t\).

Clearly, \(n \geq 2\). Observe that the following is true:

\[
h(n, m) \leq \max \{ h(\lceil n/2 \rceil, m_1) + h(\lfloor n/2 \rfloor, m - m_1) \mid 0 \leq m_1 \leq m \}.
\]

It is straightforward to verify that the lemma is true for the current \((n, m)\), using the above inequality and the induction hypothesis. \(\square\)

**Theorem 9.** Let \(d \geq 2\) be any fixed integer. For \(2 \leq k \leq n\),

\[
C_d(V_{n,k}) = O\left( n \log \frac{n}{k-1} + (n-k) \log \frac{n}{n-k+1} \right).
\]

**Proof.** For \(2 \leq k \leq 1 + 3n/4\), the theorem is true since \(C_d(V_{n,k}) \leq C_1(V_{n,k}) = O(n \log(2n/k))\). For \(k = n\), \(C_d(V_{n,n}) = 1\) since one query \(I(1, A)\) is sufficient to determine the membership problem for \(V_{n,n}\), where \(A = \{1, 2, \ldots, n\}\). The theorem is true in this case as \(n \log(n/(k-1)) = \Theta(1)\). Thus, we can assume that \(1 + 3n/4 < k \leq n - 1\). Clearly, there can be only one common value for any \(k\) identical elements (if they exist).

The algorithm works in two phases. In the first phase, we find in \(O(n-k)\) linear queries a median element of \(\{x_i \mid 1 \leq i \leq 2(n - k) + 1\}\); suppose it is \(x_j\). Note that if the input is in the set \(V_{n,k}\), then \(x_j\) is equal to the common value of any \(k\) identical elements. In the second phase, we identify in groups elements which are equal to \(x_j\). At any time, we maintain a partition \(\mathcal{P}\) of \(\{1, 2, \ldots, n\} - \{j\}\), with each member of \(\mathcal{P}\) colored in either red or green. If there are more than \(n-k\) red members, we halt immediately and declare that the input is not in \(V_{n,k}\). If the green members contain in total at least \(k-1\) integers in \(\{1, 2, \ldots, n\}\), we halt immediately and declare that the input in \(V_{n,k}\). Initially, we ask a query \(I(j, A)\) where \(A = \{1, 2, \ldots, n\} - \{j\}\), and form the partition \(\mathcal{P} = \{A\}\). Color \(A\) green if the answer to the query is \("=\)" and red otherwise. In a general step of the algorithm, choose any red member \(A \in \mathcal{P}\) with \(|A| > 1\), and write \(A = A_1 \cup A_2\), with \(|A_1| = \lceil |A|/2 \rceil\) and \(|A_1| = \lfloor |A|/2 \rfloor\); note that such \(A\) exists, as otherwise the algorithm should have halted. For each \(s \in \{1, 2\}\), ask the query \(I(j, A_s)\); color \(A_s\) green if the answer is \("=\)" and red otherwise. Now, replace \(A\) by \(A_1\) and \(A_2\) in the partition \(\mathcal{P}\).

The execution of the algorithm for any input can be regarded as the generation of some balanced \((n-1, m)\)-tree. Starting from a single leaf of weight \(n-1\) colored either as green or red dependent on the query result, each step changes a red leaf into an internal node and attaches two children leaves endowed with almost equal weights and marked with the proper colors; note that at least one of the two children must acquire a red color. The algorithm halts when the current tree either has too many \((\geq n-k+1)\) red leaves or the nonred leaves have too large \((\geq k-1)\) a combined weight.
The above algorithm halts after a finite number of steps for any input, since the partition \( \mathcal{P} \) gets refined at every step. To see that the algorithm gives the correct output, observe that each red member in \( \mathcal{P} \) contains at least some \( i \) with \( x_i \neq x_j \), and that each green member \( A \) satisfies the condition \( x_i = x_j \) for all \( i \in A \), as can be easily verified by induction. Thus, when the algorithm halts, either there are more than \( n-k \) elements \( x_i \) (with each red member \( A \) contributing at least one such \( i \)) not equal to \( x_j \), or there are at least \( k \) \( x_i \) with \( x_i - x_j \) (with \( i \) contributed from all the green members). In either case, we have enough evidence to determine whether the input belongs to \( V_{n,k} \).

It remains to show that the maximum number of steps taken for any input in phase 2 is \( O((n-k)\log(n/(n-k+1))) \). Observe that the algorithm given has the property that it halts at the earlier possible time, in the sense that if it halted before making the final step, then the algorithm would give the wrong answer for some inputs. It follows that, for the purpose of bounding the maximum number of steps, we can assume that the inputs are restricted to points \( x \) in \( V_{n,k} \). For any such input, the partition \( \mathcal{P} \) contains no more than \( n-k \) "red" members when the algorithm halts. The connection with balanced \((n,m)\)-trees implies that the number of steps taken is no greater than \( h(n-1,m) \) for some \( 0 < m < n-k \), and thus by Lemma 3, less than \( O(m + \log_2(\binom{m}{n}^2)) = O((n-k)\log(n/(n-k+1))) \).

It would be of interest to determine whether the bound given in Theorem 9 is in fact also a lower bound.

7. Concluding remarks

One can augment the algebraic computation tree model, by allowing assignments of the form \( z \leftarrow z^{'1/b} \) for positive integers \( b \leq \gamma \), where \( \gamma > 0 \) is some fixed integer, and \( z' \) are either input variables \( x_j \) or previously computed programming variables. (This augmented model was employed in [1] as the algebraic computation tree model.) Theorem 7 remains true with constants \( c_i \) now dependent also on the value \( \gamma \). The proof is an extension of the proof of Theorem 7, using \( z^b - z' = 0 \) as the polynomial constraints for nodes with assignments \( z \leftarrow z^{'1/b} \).

Many interesting questions remain to be answered. What can be said when the inputs are restricted to be integers? (Some restrictions on \( S \) have to be added in this case. See [8,12,26] for related work.) Can one obtain nontrivial lower bounds for problems other than the "\( k \)-equal problem" by estimating the Euler characteristics? (For example, consider the following decision problem: given \( n(n-1)/2 \) real numbers \( x_{i,j} \) where \( 1 \leq i < j \leq n \), decide whether there exist a size-\( k \) \( V \subseteq \{1, 2, \ldots, n\} \) such that \( x_i,j \) are identical for all \( i, j \in V \).) We remark that the present approach does not give any strong lower bound to points collinearity. In general, for any set \( S \subseteq \mathbb{R}^n \) defined by \( m \) polynomial equalities and inequalities with degree less than \( k \), the sum of the Betti numbers of \( S \) is no greater than \( (2+km)(1+km)^n \) by a theorem in [14]. If \( k \) is a constant and \( m \) is bounded by a polynomial of \( n \), the lower bound derived on the complexity by the results in this paper cannot exceed \( cn \log n \).
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Appendix. A fact needed in Lemma 1

In this appendix, we prove the following Fact which was used in the proof of Lemma 1.

Fact. Let $E \subseteq \mathbb{R}^n$ be a bounded set semi-algebraically homeomorphic to an open $j$-dimensional simplex where $j > 0$. Then $\partial E$ is the image of a continuous mapping from the $(j-1)$-dimensional unit sphere.

We introduce some notations. Let $d(x,y)$ denote the Euclidean distance between any two points $x,y$ in $\mathbb{R}^n$, and $d(x,Y) = \inf \{d(x,y) \mid y \in Y\}$ for any $Y \subseteq \mathbb{R}^n$. Let $\xi : F \to E$ be a semi-algebraic homeomorphism, where $F \subseteq \mathbb{R}^n$ is an open $j$-dimensional simplex. Without loss of generality, we can assume that the origin of $\mathbb{R}^n$ is in $F$. For any $a > 0$ and $L \subseteq \mathbb{R}^n$, let $aL = \{ax \mid x \in L\}$. Our goal is to show that there exists a continuous mapping from $a(\partial F)$ onto $\partial E$ for some $0 < a < 1$. This implies Fact, as $a(\partial F)$ is homeomorphic to the $(j-1)$-dimensional unit sphere.

We first prove that $\partial E$ is a closed set. For any point $p \in E$ or $p \in \mathbb{R}^n - \overline{E}$, it is easy to see that there exists an open neighborhood of $p$ which is disjoint from $\partial E$. This proves that the set $\mathbb{R}^n - \partial E = E \cup (\mathbb{R}^n - \overline{E})$ is open, and hence $\partial E$ is a closed set.

By the Triangulation Theorem, there exists a semi-algebraic triangulation $(\mathcal{D}, h)$ for $E$ and $\partial E$, with $\mathcal{D} = \{D_a \mid a \in A\}$, $\overline{E} = \bigcup_{a \in B} h(D_a)$, and $\partial E = \bigcup_{a \in C} h(D_a)$, where $C \subseteq B \subseteq A$. Thus, $\bigcup_{a \in B} D_a$ is a simplicial complex, and since $\partial E$ is closed, $h^{-1}(\partial E) = \bigcup_{a \in C} D_a$ is closed and hence a subcomplex.

Thus, $\overline{E}$ is a finite CW-complex, and $\partial E$ is a subcomplex of $\overline{E}$. It is well known ([25, p. 65, Theorem 2.19]) that any subcomplex $T$ of a finite CW-complex is a retract of some compact neighborhood of $T$ in the CW-complex. Let $\psi : \overline{E} \cap W \to \partial E$, where $W \subseteq \mathbb{R}^n$ is an open set containing $\partial E$, be the retraction as constructed in the proof of Theorem 2.19 in [25]. An examination of the construction in [25] shows that, for each $z \in \partial E$, there exists a point $v_z \in E \cap W$ and a path $P_z$ in $E \cap W$ connecting $z$ and $v_z$ such that $\psi(v_z) = z$ and $\psi(v) = z$ for every point $v$ on the path.

For each $v \in E$, let $t(v)$ be the unique $0 < \lambda < 1$ such that $v \in \xi(\lambda \partial F)$. The mapping $t : E \to [0, 1]$ is obviously continuous. Let $z \in \partial E$. By standard continuity arguments, for each $t(v_z) < s < 1$, there exists a point $v \in P_z$ with $t(v) = s$.

Let $b = \sup \{t(v) \mid v \in E \cap (\mathbb{R}^n - W)\}$. As $E \cap (\mathbb{R}^n - W) = \overline{E} \cap (\mathbb{R}^n - W)$ is compact, there is a point $v$ in the set achieving $t(v) = b$. This implies $0 < b < 1$ since $v$ is in $E$.

Let $a = (1 + b)/2$, then $b < a < 1$. Note that for every $x \in a \partial F$, we have $t(\xi(x)) = a > b$, and hence $\xi(x) \not\in \mathbb{R}^n - W$; thus, $\xi(x) \in E \cap W$. Define a continuous mapping $g : a \partial F \to \partial E$ by $g(x) = \psi(\xi(x))$. 
We can finish the proof of Fact by showing that $g$ is an onto mapping. Let $z \in \partial E$. Let $v \in P_z$ be such that $t(v) = a$. Then $g(x) = z$ where $x = \xi^{-1}(v) \in a \partial F$. This completes the proof.

References