Formulas for the expansion of the plethysms $s_2[s_{(a,b)}]$ and $s_2[s_{(nk)}]$

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Abstract

Building on the ideas of Carbonara, Remmel, and Yang who, recently, gave explicit formulas for the Schur function expansion of the plethysms $s_2[s_{(a,b)}]$ and $s_{12}[s_1]$ where $/\lambda$ is a hook shape, we derive explicit formulas for the Schur function expansion of the plethysms $s_2[s_{(a,b)}]$ and $s_{12}[s_1]$ where $/\lambda$ has two rows or two columns. These formulas generalize classical formulas of Littlewood for the Schur function expansions of $s_2[s_{(a,b)}]$ and $s_{12}[s_1]$. We also derive explicit formulas for $s_2[s_{(n,k)}]$ and $s_{12}[s_{(n,k)}]$ where $\lambda = (n^k)$ for some $k$ and $n$. © 1998 Elsevier Science B.V. All rights reserved

0. Introduction

One of the fundamental open problems in the representation theory of the symmetric group is to find a combinatorial rule to expand the plethysm of two Schur functions $s_\mu[s_\nu]$ as a sum of Schur functions. Plethysm is a type of product of Schur function which was first introduced by Littlewood [7]. Let $A_n$ denote the space of homogeneous symmetric polynomials of degree $n$. Then given symmetric polynomials with integer coefficients, $P \in A_n$ and $Q \in A_m$, we can formally define the plethysm $P[Q]$ as follows. First, write $Q = \sum_{\alpha} a_\alpha x^\alpha$, where $a_\alpha$ is an integer, and if $\alpha = (\alpha_1, \alpha_2, \ldots)$, then $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots$. Then define

$$e_\mu^Q(x) = \prod_{\alpha} (1 + tx^\alpha)^{a_\alpha} |_r.$$  

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and for \( \lambda = (\lambda_1, \ldots, \lambda_k) \),
\[
e_{\lambda}(x) = e_{\lambda_1}(x) \cdots e_{\lambda_k}(x).
\] (2)

Here, given a series \( f(t) \), \( f(t)|_r \) denotes the coefficients of \( t^r \) in \( f(t) \). Next, since \( P \) is symmetric, we can express \( P \) in terms of the elementary symmetric functions \( e_{\lambda}(x) \),
\[
P(x) = \sum_{\lambda \vdash n} c_{\lambda} e_{\lambda}(x).
\] (3)

Then by definition,
\[
P[Q] = \sum_{\lambda \vdash n} c_{\lambda} e_{\lambda}(x).
\] (4)

It is easy to see that if \( Q = \sum_a x_a^a \) and \( a_a \) is nonnegative for all \( a \), then \( P[Q] \) is nothing but the symmetric polynomial which results by substituting the monomials of \( Q \) for the variables of \( P \). The basic problem of plethysm is to find the coefficients \( a_{\lambda,\mu,\nu} \), where
\[
s_{\lambda}[s_{\mu}] = \sum a_{\lambda,\mu,\nu} s_{\nu}.
\] (5)

It is known that \( a_{\lambda,\mu,\nu} \) are nonnegative integers. In fact, the plethysm of two Schur functions is the Frobenius image of a character of the symmetric group. See \([6,9]\) for details.

The problem of computing the coefficients \( a_{\lambda,\mu,\nu} \) is one of the main open problems in the theory of symmetric functions. Combinatorial algorithms for the expansions of \( s_{\lambda}[s_{\mu}] \) are desirable, since this fundamental operation occurs in several areas, including representation theory, particle physics and atomic spectroscopy (see \([14]\)); more recent references can be found in Egecioglu–Remmel \([5]\). There is no explicit formula for the Schur function expansion of \( s_{\lambda}[s_{\mu}] \) for arbitrary \( \lambda \) and \( \mu \). Indeed, there is not even a direct combinatorial interpretation for the coefficient of \( s_{\lambda} \) in \( s_{\lambda}[s_{\mu}] \) \( \langle s_{\mu}[s_{\lambda}], s_{\lambda} \rangle \). Recently, Carré and Leclerc \([3]\) gave a very elegant combinatorial interpretation for the coefficient \( \langle s_{\mu}[s_{\lambda}], s_{\lambda} \rangle \) where \( \mu \) is a partition of 2, but even their combinatorial interpretation does not immediately give explicit formulas for the Schur function expansions of \( s_2[s_{\lambda}] \) and \( s_{12}[s_{\lambda}] \). Littlewood \([8]\) did provide explicit formulas for the Schur function expansion of \( s_2[s_{\lambda}] \) and \( s_{12}[s_{\lambda}] \). Carbonara et al. \([1]\) extended Littlewood's formulas by giving explicit formulas for the Schur function expansions of \( s_2[s_{\lambda}] \) and \( s_{12}[s_{\lambda}] \) where \( \lambda \) is of hook shape, i.e., where \( \mu \) is of the form \( (1^k, \ell) \) with \( 1 \leq \ell \). In this paper, we extend Littlewood's formula in two different ways. Namely, we provide explicit formulas for the Schur function expansion of \( s_2[s_{\lambda}] \) and \( s_{12}[s_{\lambda}] \) where \( \lambda \) is either a rectangle shape \( (nk) \), a two row shape \( (a,b) \), or a two column shape \( (1^a,2^b) \).

The approach we use to calculate \( s_2[s_{\lambda}] \) or \( s_{12}[s_{\lambda}] \) is the following. First, we expand \( s_2 \) and \( s_{12} \) in terms of the power sum symmetric functions.
\[
s_2 = \frac{1}{2}(p_1^2 + p_2) \quad \text{and} \quad s_{12} = \frac{1}{2}(p_1^2 - p_2).
\] (6)
However, $p_1[s_3] = s_3$ so that $p_1^2[s_3] = s_3^2$. Thus,

$$s_2[s_3] = \frac{1}{2}(s_3^2 + p_2[s_3]),$$

$$s_1^2[s_3] = \frac{1}{2}(s_3^2 - p_2[s_3]).$$

(7)

Now, in the special cases we are interested in, we can compute $s_2^2$ without too much difficulty via the version of the Littlewood–Richardson rule due to Remmel and Whitney [10]. We can compute $p_2[s_3]$ via the SXP-algorithm of Chen, Garsia, and Remmel. In the special cases we consider, $p_2[s_3]$ is multiplicity free, i.e., $\langle p_2[s_3], s_\lambda \rangle$ is either 1, −1, or 0 for all $\lambda$ which means that if $\langle s_2^2[s_3], s_\lambda \rangle$ is even, then $\langle s_2[s_3], s_\lambda \rangle = \langle s_1^2[s_3], s_\lambda \rangle = d_\lambda/2$ and if $\langle s_2^2[s_3], s_\lambda \rangle$ is odd, then $\{\langle s_2[s_3], s_\lambda \rangle, \langle s_1^2[s_3], s_\lambda \rangle\} = \{(d_\lambda + 1)/2, (d_\lambda - 1)/2\}$.

Thus, if $d_\lambda$ is odd, then we need only determine the sign of $(p_2[s_3], s_\lambda)$ to determine $(s_2[s_3], s_\lambda)$ and $(s_1^2[s_3], s_\lambda)$.

We should note that one could also approach the problem of finding explicit formulas for $s_2[s(n, n')], s_1^2[s(n, n')]$ for $\mu$ in a partition of 2 by using the combinatorial interpretation of the coefficients $a_{s_\lambda} = \langle s_2[s_\lambda], s_\mu \rangle$ and $b_{s_\lambda} = \langle s_1^2[s_\lambda], s_\mu \rangle$ of Carre and Leclerc [3]. If $\lambda = (\lambda_1, \ldots, \lambda_m)$, Carre and Leclerc show that $a_{s_\lambda}$ and $b_{s_\lambda}$ count the number of fillings of shape $(2\lambda_1, 2\lambda_2, 2\lambda_3, \ldots, 2\lambda_m, 2\lambda_m, 2\lambda_m)$ with dominoes that have certain properties. Such fillings are called Yamanouchi domino tableaux by Carre and Leclerc. We were not able to derive the formulas for $s_2[s(n, n')]$ and $s_1^2[s(n, n')]$ via this approach. However, one can derive the formulas for $s_2[s(n, n')]$ and $s_1^2[s(n, n')]$ that are equivalent to ours via this approach. The formulas for $s_2[s(n, n')]$ and $s_1^2[s(n, n')]$ derived via this approach have roughly the same form as our formulas, and the amount of work required to derive them is only a little less than is required via our derivation. However, we get formulas for $\langle s_2[s(n, n')], s_\lambda \rangle$ and $\langle p_2[s(n, n')], s_\lambda \rangle$ directly, which are useful in trying to find explicit formulas for computing the Schur function expansion of $s_\mu[s(n, n')]$ where $\mu$ is a partition of 3. Explicit formulas for the Schur function expansion of $s_\mu[s(n, n')]$ where $\mu$ is a partition of 3 exist, see [13] or [4], so it seems feasible to be able to extend these formulas to the case of computing the Schur function expansion of $s_\mu[s(n, n')]$ and possibly $s_\mu[s(n, n')]$ for $\mu$ a partition of 3. This will be pursued in later papers. Of course, using the Carre–Leclerc approach, we can also obtain formulas for $\langle s_2[s(n, n')], s_\lambda \rangle$ and $\langle p_2[s(n, n')], s_\lambda \rangle$ from the formulas for $\langle s_2[s_\lambda], s_\lambda \rangle$ and $\langle p_2[s_\lambda], s_\lambda \rangle$.

The outline of this paper is as follows. In Section 1, we present the skew Schur function expansion rule of Remmel and Whitney [10] and the SXP algorithm of Chen et al. [4]. In Section 2, we derive our explicit formulas for the Schur function expansions of $s_2[s(n, n')]$ and $s_1^2[s(n, n')]$. In Section 3, using the Remmel–Whitney algorithm, we expand the product $s_{(a, b)} s_{(a, b)}$. In Section 4, using the correspondence given by Carbonara–Remmel–Yang in [2] between circle diagrams of a partition $\lambda$ used in the SXP-algorithm of [4] and certain special rim hook tabloids of shape $\lambda$, we expand the plethysms $p_2[s_{(a, b)}] = s_{(a, b)}[p_2]$ of a Schur function $s_{(a, b)}$ and the power symmetric function $p_2$ as a sum of Schur functions. Then combining together the two
expansions according to (7), we will get our explicit formulas for $s_2[s_\lambda]$ and $s_{11}[s_\lambda]$ where $\lambda$ has either two rows or two columns.

1. Preliminaries, algorithms, and circle diagrams

The main purpose of this section is to present the skew Schur function expansion rule of Remmel and Whitney and the SXP algorithm of Chen, Garsia, and Remmel to compute $p_k[s_\mu]$. One of the things that we need to compute in the SXP algorithm is the expansion as a sum of Schur functions of the product of several Schur functions. That is, we need to compute the coefficients

$$g^K_{I_1,\ldots,I_k} = \langle s_{I_1} \cdots s_{I_k}, s_K \rangle,$$

where $I_1,\ldots,I_k$ and $K$ are partitions.

Given a partition $I$, let $F_I$ denote its Ferrers diagram. Let $I \ast J$ denote the skew diagram that results from $F_I$ and $F_J$ by placing $F_I$ on top of $F_J$ so that the start of the top row of $F_J$ is just below the end of the bottom row of $F_I$. For example, see Fig. 1.

Then clearly $s_{I \ast J} = s_I s_J$ so that computing the $g^K_{I,J}$'s is just a special case of expanding an arbitrary skew Schur function as a sum of Schur functions. Moreover, it should be clear that the problem of expanding an arbitrary product of Schur functions or skew Schur functions corresponds to expanding a single skew Schur function as a sum of Schur functions. For example, $s_{(2,3)} \cdot s_{(1,2)} \cdot s_{(4,4,4)/(1,2)}$ is equal to the skew Schur function whose Ferrer's diagram is depicted in Fig. 2.

Such expansions can be computed via the following version of the Littlewood-Richardson rule due to Remmel and Whitney (see [10]).

**Skew Schur Function Expansion Rule.** To compute $s_{J/I} = \sum_K g^K_{J/I} s_K$.

1. Form the reverse lexicographic filling of $J/I$, $rl(J/I)$, which is the filling of $F_{J/I}$ which starts at the bottom right corner of $F_{J/I}$ and fills in the integers $1,2,\ldots,n = |J/I|$ in order from right to left and bottom to top. For example, see Fig. 3.

$$F_{(1,2) \ast (2,3)} =$$

Fig. 1.
(2) We say a standard tableau $T$ is $(J/I)$-compatible if

(a) whenever $i + 1$ is immediately to the left of $i$ in $\text{rl}(J/I)$, then in $T$, $i + 1$ occurs to the southeast of $i$ in the sense that the cell of $T$ which contains $i + 1$ is strictly to the right and weakly below the cell of $T$ which contains $i$.

(b) whenever $y$ is immediately above $x$ in $\text{rl}(J/I)$, then in $T$, $y$ occurs to the northwest of $x$ in the sense that the cell of $T$ which contains $y$ is strictly above and weakly to the left of the cell of $T$ which contains $x$.

Then $g_{J/I}^K$ is the number of $(J/I)$-compatible tableaux of shape $K$.

It is good to visualize the condition $i + 1$ southeast of $i$ and $y$ northwest of $x$, respectively, by the patterns $i_{i+1} y_x$.

Thus for our example above, conditions (a) and (b) may be summarized by the patterns

$$2_3, 4_5,$$
$$2_1, 4_3, 6_5.$$ 

This given, the collection of $(J/I)$-compatible tableaux can easily be constructed by adding squares labeled $1, 2, \ldots, n$ in succession, always maintaining standardness and obeying each time conditions (a) and (b). In our example, one is naturally led to the tree in Fig. 4 for constructing the $(J/I)$-compatible tableaux.
Having constructed the tree, one can easily read off the expansion of $s_{J/I}$ as

$$s_{J/I} = \sum_{T(J/I)\text{-compatible}} S_{ab(T)}.$$  \hfill (9)

Thus, for our example

$$s_{(1,2,3,3)/(1,2)} = s_{(2,2,3)} + s_{(1,2,3)} + 2s_{(1,2,3)} + s_{(3)}.$$  

Next, we present the SXP-algorithm of Chen, Garsia and Remmel (see [4]) to compute $s_{\mu}[p_k]$. They show that if $\mu$ is a partition of $n$, then

$$s_{\mu}[p_k] = \sum_{|l_0| + \cdots + |l_{k-1}| = n} c_{l_0,\ldots,l_{k-1}}^\mu \cdot SS_{l_0,\ldots,l_{k-1}}(x)$$  \hfill (10)

where

(a) the sum is to be carried out over all $k$-tuples of partitions $l_0,\ldots,l_{k-1}$ whose diagrams are contained in $\mu$ and whose sum of parts add up to $n$,

(b) we have

$$c_{l_0,\ldots,l_{k-1}}^\mu = \langle s_{l_0} \cdots s_{l_{k-1}}, s_{\mu} \rangle$$  \hfill (11)

and

(c) the expression $SS_{l_0,\ldots,l_{k-1}}(x)$ denotes certain signed Schur functions indexed by a partition with empty $k$-core whose construction is best explained through an example.

For instance, in the expansion of $s_{113}(x^3)$, since $n = 5$ and $k = 3$, one of the terms in (a) is that which corresponds to the triple of partitions $l_0 = (1), l_1 = (1^2), l_2 = (2)$. By using the Remmel–Whitney rule for multiplying Schur functions, we obtain

$$c_{(1,1^2,2)}^{(113)} = 2.$$
To construct $SS_{(1), (1^2), (2)}(x)$, we proceed as follows. First of all, we represent $(1), (1^2), \text{ and } (2)$ as partitions with an equal number of parts, that is, we write $(0,1)$ instead of $(1), (1^2)$ since it already has two parts, and $(0,2)$ instead of $(2)$. This given, we construct the circle diagram given in Fig. 5.

The precise rule to construct the circle diagram is that for each column $k$, if $I_k = (i_1, i_2, \ldots, i_m)$, then in the $k$th column we circle the dots in rows $i_1, i_2 + 1, i_3 + 2, \ldots, i_m + m - 1$. Next, we label the dots in the circle diagram with the numbers $0, 1, 2, 3, \ldots$ successively from left to right and from top to bottom and record the labels that correspond to circled dots. This gives the labeled diagram in Fig. 6.

In the case of a general $k$-tuple $I_0, I_1, \ldots, I_{k-1}$, we obtain a circle diagram with $m$ circles in each column, where $m$ is the maximum number of non-zero parts appearing in any of the partitions $I_0, I_1, \ldots, I_{k-1}$.

Let $b_1 < b_2 < b_3 < \cdots < b_{m,k}$ be the labels placed on the circles, and $q_{s,1} < q_{s,2} < \cdots < q_{s,m}$ be the labels appearing in the column corresponding to the partition $I_s$. Finally, let $\text{inv}(I_0, \ldots, I_{k-1})$ denote the number of inversions of the permutation $q_{0,1}q_{0,2}\cdots q_{0,m}q_{1,1}q_{1,2}\cdots q_{1,m}\cdots q_{k-1,1}q_{k-1,2}\cdots q_{k-1,m}$ (12)
and let \( \text{sh}(I_0, \ldots, I_{k-1}) = (b_1, b_2 - 1, b_3 - 2, \ldots, b_{m,k} - m \cdot k + 1) \). Here, given any sequence \( s = s_1 \cdots s_n \) of integers,

\[
\text{inv}(s) = \sum_{1 \leq i < j \leq n} \chi(s_i > s_j),
\]

where for any statement \( A \), \( \chi(A) = 1 \) if \( A \) is true and \( \chi(A) = 0 \) if \( A \) is false. This given, we set

\[
\text{SS}_{I_0, \ldots, I_{k-1}} = (-1)^{\binom{n}{2}} + \text{inv}(I_0, \ldots, I_{k-1}) \text{sh}(I_0, \ldots, I_{k-1})
\]

and write

\[
\text{sgn}_k(\text{sh}(I_0, \ldots, I_{k-1})) = (-1)^{\binom{n}{2}} + \text{inv}(I_0, \ldots, I_{k-1}).
\]

We note that \( I_0, \ldots, I_{k-1} \) is also called the \( k \)-quotient of \( \text{sh}(I_0, \ldots, I_{k-1}) \), (see [6]). Going back to our particular example, we can easily see that to calculate the number of inversions of the permutation \( 0, 6, 4, 7, 2, 11 \), we need only count, for each circle in the diagram of Fig. 6, the number of circles that are northeast of it, and add all these counts. This gives

\[
\text{inv}((1),(12),(2)) = 0 + 0 + 1 + 2 + 1 + 0 = 4
\]

at the same time, we have \( \binom{3}{2} \binom{3}{1} = \binom{3}{2} \binom{3}{2} = 3 \) and

\[
\text{sh}(I_0, \ldots, I_{k-1}) = (0 - 0, 2 - 1, 4 - 2, 6 - 3, 7 - 4, 11 - 5) = (0, 1, 2, 3, 3, 6).
\]

So we, finally, obtain in this case \( \text{SS}_{(1),(12),(2)} = -s_{12336}(x) \).

There is an alternative method for computing the signed Schur functions \( \text{SS}_{I_0, \ldots, I_{k-1}} \), which occurs in the SXP-algorithm due to Carbonara, Remmel and Yang [1,2]. To describe their method, we need to define special rim hook tabloids. Given a Ferrers diagram \( \lambda \), a rim hook \( h \) of \( \lambda \) is a consecutive sequence of cells along the northeast boundary of \( \lambda \) such that any two consecutive cells of \( h \) share an edge, and the removal of the cells of \( h \) from \( \lambda \) results in a Ferrers diagram corresponding to another partition. We let \( r(h) \) denote the number of rows of \( h \) and \( c(h) \) denote the number of columns of \( h \). We say that \( h \) is special if \( h \) has at least one cell in the first column of \( \lambda \). For example, Fig. 7 pictures all special rim hooks of \( \lambda = (2,2,4) \).

This given, a special rim hook tabloid \( T \) of shape \( \lambda \) and type \( \mu = (\mu_1, \ldots, \mu_k) \) is a filling of the Ferrers diagram of \( \lambda \) with special rim hooks \( (h_1, \ldots, h_k) \) such that \( (|h_1|, \ldots, |h_k|) \) is a rearrangement of \( (\mu_1, \ldots, \mu_k) \) where \( |h_i| \) denotes the number of cells...
of \( h_i \). To be more precise, one can think of a special rim hook tabloid \( T \) as a sequence of shapes \( \{ \phi = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(k)} = \lambda \} \) such that for all \( i \geq 1 \), \( \lambda^{(i)}/\lambda^{(i-1)} \) is a special rim hook of \( \lambda^{(i)} \) and \( (|\lambda^{(1)}/\lambda^{(0)}|,|\lambda^{(2)}/\lambda^{(1)}|,\ldots,|\lambda^{(k)}/\lambda^{(k-1)}|) \) is a rearrangement of \( \mu \).

The sign of \( T \), \( \text{sgn}(T) \), and is defined by

\[
\text{sgn}(T) = \Pi \text{sgn}(\lambda^{(i)}/\lambda^{(i-1)}),
\]

where if \( h \) is a rim hook

\[
\text{sgn}(T) = (-1)^{r(h)-1}.
\]

For example, Fig. 8 shows a special rim hook tabloid \( T \) of shape \((2,3,4,4,5,5)\), type \((2,4,5,6,6)\) with \( \text{sgn}(T) = 1 \).

Given a special rim hook tabloid \( T \), we let \( h_i(T) \) denote the rim hook which starts in the \( i \)th row from the top and we write \( |h_i(T)| \) for the length of \( h_i(T) \). In the case where there is no rim hook that starts in row \( i \) of \( T \) such as is the case of row 3 for the special rim hook pictured in Fig. 8, we think of \( h_i(T) \) as the empty rim hook and let \( |h_i(T)| = 0 \). It is easy to see that a special rim hook tabloid is completely determined by the lengths of its special rim hooks \( h_1(T), h_2(T), \ldots \).

This given, we are now in a position to describe the alternative method for computing the signed Schur function \( SS_{I_0,\ldots,I_{k-1}} \) that appears in the SXP-algorithm. Namely, suppose \( \lambda_j = (i_{j1}^1, i_{j2}^1, \ldots, i_{jm}^j) \) for \( j = 0, 1, \ldots, k-1 \). Then let \( T = T_{I_0,\ldots,I_{k-1}} \) be the special rim hook tabloid such that \( h_{1+j+r, k}(T) = k \cdot i_{j+r+1}^j \) for \( j = 0, \ldots, k-1 \) and \( r = 0, \ldots, m-1 \). Then

\[
SS_{I_0,\ldots,I_{k-1}} = \text{sgn}(T) s_{\text{sh}(T)},
\]

where \( \text{sh}(T) \) denotes the shape of \( T \). Eq. (17) is best explained by an example. Suppose \( k = 3 \), \( I_0 = (1, 2, 2) \), \( I_1 = (0, 1, 2) \), and \( I_2 = (1, 1, 3) \). Thus, we have the triple of diagrams shown in Fig. 9.

To get the sequence \( |h_1(T_{I_0,I_1,I_2})|, \ldots, |h_6(T_{I_0,I_1,I_2})| \), just read the lengths of the top rows of the diagrams from left to right, then read the lengths of the second row of

\[
\begin{align*}
|h_1(T)| &= 4 \\
|h_2(T)| &= 6 \\
|h_3(T)| &= 0 \\
|h_4(T)| &= 6 \\
|h_5(T)| &= 2 \\
|h_6(T)| &= 5
\end{align*}
\]
the diagrams from left to right, and so on. In our example, we would produce the sequence

\[ 1, 0, 1, 2, 1, 1, 2, 2, 3. \] (18a)

Then the sequence \( |h_1(T_{l_0, l_1, l_2})|, \ldots, |h_q(T_{l_0, l_1, l_2})| \) is the result of multiplying each element of (18a) by \( k \). Since \( k = 3 \), in our case, we obtain the sequence

\[ 3, 0, 3, 6, 3, 6, 6, 9. \] (18b)

Given the sequence in (18b), the easiest way to construct \( T \) is to start from the left and construct \( T \) from the bottom up. For example, the bottom rim hook is of length 9, so \( T \) starts out as follows:

Here to give a more compact representation of \( T \), we draw a dot, \( \bullet \), in place of a cell. Next we add a rim hook of size 6, to get

Continuing on in this way, it is easy to construct \( T = T_{l_0, l_1, l_2} \) pictured in Fig. 10. Thus,

\[ SS_{l_0, l_1, l_2} = \text{sgn}(T)s_{\text{sh}(T)} = -s_{(1,2,3,4,4,4,6,6,9)}. \]

2. The computation of \( s_2[s_{(n^k)}] \) and \( s_1[s_{(n^k)}] \)

In this section, we will show that if \( \lambda = (n^k) \) is of rectangular shape, then there are surprisingly simple formulas for the Schur function expansions of \( s_2[s_\lambda] \) and \( s_1[s_\lambda] \). To state our formulas, we need to define \((k,n)\)-balanced shapes.

**Definition 2.1.** A partition \( \lambda = (\lambda_1, \ldots, \lambda_{2k}) \) of \( 2nk \) where \( 0 \leq \lambda_1 \leq \cdots \leq \lambda_{2k} \) is \((k,n)\)-balanced if for all \( 1 \leq i \leq k \), \( \lambda_i + \lambda_{2k+1-i} = 2n \).
A geometric picture of \((k,n)\)-balanced shapes is obtained by considering those pairs of partitions \(\alpha\) and \(\beta\) that fit together to fill the rectangle \((n^k)\) as pictured in Fig. 11. In such a situation, we will say \(\alpha\) and \(\beta\) are complementary relative to the rectangle \((n^k)\). Thus, \(\alpha = (\alpha_1, \ldots, \alpha_k)\) and \(\beta = (\beta_1, \ldots, \beta_k)\) are complementary relative to \((n^k)\) if and only if for all \(i \leq k\), \(\alpha_i + \beta_{k+1-i} = n\). Given a pair of complementary shapes \(\alpha\) and \(\beta\) relative to \((n^k)\), let \(\lambda(n^k, \alpha, \beta)\) denote the partition that is obtained by placing the partition \(\alpha\) to the right of \((n^k)\), and \(\beta\) on top of \(n^k\), see Fig. 12.
It is then easy to see that \( \lambda \) is \((k,n)\)-balanced iff \( \lambda = \lambda(n^k, \alpha, \beta) \) for some pair of complementary shapes \( \alpha \) and \( \beta \) relative to \((n^k)\). We let \( B(k,n) \) devote the set of all \((k,n)\)-balanced shapes. For \( \lambda = \lambda(n^k, \alpha, \beta) \in B(n,k) \), we defined the sign of \( \lambda \) by

\[
\text{sgn}(\lambda) = (-1)^{|\beta|} = (-1)^{\sum_{i=1}^{k} \lambda_i}.
\]

This given, we will prove the following.

**Theorem 1.**

(a) \( s_2[s_{(n^k)}] = \sum_{\lambda \in B(n,k)} s_{\lambda} \),

(b) \( s_1[s_{(n^k)}] = \sum_{\lambda \in B(n,k)} s_{\lambda} \).

**Proof.** In light of (7), to compute \( s_2[s_{(n^k)}] \) and \( s_1[s_{(n^k)}] \), we need only compute \( s_{(n^k)} \cdot s_{(n^k)} \) and \( p_2[s_{(n^k)}] \). Indeed, it is easy to see that Theorem 1 immediately follows from the next two lemmas.

**Lemma 1.**

\[
s_{(n^k)} \cdot s_{(n^k)} = \sum_{\lambda \in B(n,k)} s_{\lambda}.
\]

**Lemma 2.**

\[
p_2[s_{(n^k)}] = \sum_{\lambda \in B(n,k)} \text{sgn}(\lambda) s_{\lambda}.
\]

Lemma 1 is an easy consequence of the Remmel–Whitney rule. That is, consider the reverse lexicographic filling of \((n^k) \ast (n^k)\), see Fig. 13.

It is easy to see that in any \((n^k) \ast (n^k)\)-compatible tableaux, the numbers 1, \ldots, \(nk\) must be in the positions pictured in Fig. 14.
It is also easy to see that no number $x$ with $nk + 1 \leq x \leq 2nk$ can lie in the region northeast of the rectangle $(n^k)$, i.e., the region indicated by a (*) in Fig. 14, since such an $x$ would require that there be $\geq n + 1$ numbers to its right in $rl((n^k) \ast (n^k))$ and $\geq k + 1$ numbers below it in $rl((n^k) \ast (n^k))$. Thus, if $D$ is $(n^k) \ast (n^k)$-compatible, $D$ must be of shape pictured in Fig. 12 for some $\alpha$ and $\beta$ that are not necessarily complementary with respect to $(n^k)$. However, the numbers in the shape $\alpha$ are completely forced by conditions 2a and 2b of the definition of compatible tableaux. That is, condition 2b forces that elements in the $l$th row of the upper rectangle of $rl((n^k) \ast (n^k))$ must lie in row $l$ or higher in $D$. Thus, the elements in the first row of the $\alpha$ part of $D$ come from the first row of the upper rectangle of $rl((n^k) \ast (n^k))$, the elements in the second row of the $\alpha$ part of $D$ come from the second row of the upper rectangle of $rl((n^k) \ast (n^k))$, etc. But then, condition 2a forces that the elements in the $l$th row of the $\alpha$ part of $D$ must be the largest elements in the $l$th row of the upper rectangle of $rl((n^k) \ast (n^k))$ for $l = 1, \ldots, k$. Thus, if $\alpha = (\alpha_1, \ldots, \alpha_k)$, then row 1 of the $\alpha$ part of $D$ is filled with the numbers $(n + 1)k, (n + 1)k - 1, \ldots, (n + 1)k - \alpha_1 + 1$, row 2 of the $\alpha$ part of $D$ is filled with $(n + 2)k, (n + 2)k - 1, \ldots, (n + 2)k - \alpha_2 + 1$, etc.

We are now left with the following situation. Suppose $\gamma$ is the complementary shape of $\alpha$ relative to $(n^k)$. The numbers not accounted for in $rl((n^k) \ast (n^k))$ are pictured in Fig. 15.

The relative positions of $\alpha$ and $\beta$ ensure that all compatibility conditions between elements in the $\alpha$ part of the upper rectangle of $rl((n^k) \ast (n^k))$ and the $\gamma$ part of the upper rectangle of $rl((n^k) \ast (n^k))$ are automatically met. Thus, the only thing that we have to do is to ensure $D$ is $(n^k) \ast (n^k)$-compatible is to place the numbers in Fig. 15.
into the shape $\beta$ according to conditions 2a and 2b. But it is easy to see that conditions 2a and 2b force the numbers $1 + nk, \ldots, \gamma_1 + nk$ to lie in the first row of $\beta$. Next, the numbers $1 + (n+1)k, \ldots, \gamma_2 + (n+1)k$ must be placed on top of $1 + nk, \ldots, \gamma + nk$ and be placed as far left as possible which means the numbers from the first 2 rows of Fig. 15 will fill a shape of the form $(\gamma_1, \gamma_2)$. Next, the numbers $1 + (n + 2)k, \ldots, \gamma_3 + (n + 2)k$ must be placed on the outside of the shape $(\gamma_1, \gamma_2)$ and be placed as far left as possible which means the numbers from the first 3 rows of Fig. 15 will fill a shape of the form $(\gamma_1, \gamma_2, \gamma_3)$. Continuing on in this manner, we see that the positions of the numbers in the first $l$ row of Fig. 15 are completely forced and they fill of shape $(\gamma_1, \ldots, \gamma_l)$.

But then, $\beta = \gamma$, and hence $D$ must be of shape $\lambda(n^k, \alpha, \gamma)$. Thus, we have shown that every $(n^k) \ast (n^k)$-compatible tableau has a $(k, n)$-balanced shape of shape $\lambda(n^k, \alpha, \gamma)$. Moreover, our argument shows that given $\alpha$ and $\gamma$ which are complementary with respect to $(n^k)$, there is a unique $(n^k) \ast (n^k)$-compatible tableaux of shape $\lambda(n^k, \alpha, \gamma)$ which proves Lemma 1.

Lemma 1.2 was first observed by Proctor and is proved by Sundquist [12]. Both Proctor and Sundquist showed that Lemma 1.2 is a consequence of certain identities involving Pfaffians. We will show that Lemma 1.2 can be given a much more direct proof from the SXP-algorithm.

Suppose $\lambda = (\lambda_1, \ldots, \lambda_{2k})$ is $(k, n)$-balanced so that

$$\lambda_i + \lambda_{2k+1-i} = 2n$$

for $1 \leq i \leq k$. (20)

Let $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_{2k})$ where $\hat{\lambda}_i = \lambda_i + i - 1$ for $i = 1, \ldots, 2k$. Then clearly

$$\hat{\lambda}_i + \hat{\lambda}_{2k+1-i} = 2n + 2k - 1$$

for $1 \leq i \leq k$. (21)

It follows from (21) that for each $i$, the pair $(\hat{\lambda}_i, \hat{\lambda}_{2k+1-i})$ contains one even element and one odd element. Next, let $\sigma$ be the permutation that rearranges $(\hat{\lambda}_1, \ldots, \hat{\lambda}_{2k})$ so that all the even elements of the sequence come first in increasing order and are followed by the odd elements in increasing order. It is easy to prove by induction that

(A) $\text{sgn}(\sigma) = (-1)^{\text{inv}(\sigma)} = (-1)^{\text{Odd}(\hat{\lambda}_1, \ldots, \hat{\lambda}_{2k})} = (-1)^{\sum_{i=1}^{k} \hat{\lambda}_i}$

(22)

where Odd$(\hat{\lambda}_1, \ldots, \hat{\lambda}_{2k})$ equals the number of odd elements in $\hat{\lambda}_1, \ldots, \hat{\lambda}_k$ and

(B) $\hat{\lambda}_{\sigma_i} + \hat{\lambda}_{\sigma_{k+1-i}} = 2n + 2k - 1$ for $i = 1, \ldots, k$. (23)

Now, let $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_k)$ be defined by

$$\alpha_i = \left(\frac{\hat{\lambda}_{\sigma_i}}{2}\right) - i + 1$$

for $i = 1, \ldots, k$. (24)

and

$$\beta_i = \left(\frac{\hat{\lambda}_{\sigma_{k+1-i}} - 1}{2}\right) - i + 1$$

for $i = 1, \ldots, k$. (25)
It follows from (23)–(25) that
\[ \alpha_i + \beta_{k+1-i} = n \quad \text{for } i = 1, \ldots, k. \] (26)

Thus, \( \alpha \) and \( \beta \) are complementary shapes relative to \( (n^k) \). Moreover, if we consider
the circle diagram for \( k = 2 \), with \( I_0 = \alpha \) and \( I_1 = \beta \), then \( \hat{\lambda}_{\sigma_1}, \ldots, \hat{\lambda}_{\sigma_4} \) are the labels in
the first column and \( \hat{\lambda}_{\sigma_{5}}, \ldots, \hat{\lambda}_{\sigma_{8}} \) are the labels in
the second column. It follows that
\[ \text{sh}(I_0, I_1) = \lambda \] (27)

and
\[ (-1)^{\sigma}(\hat{\lambda})^{+ \text{inv}(I_0, I_1)} = (-1)^{\sigma}(-1)^{\sum_{i=1}^{k} \hat{\gamma}_i} \]
\[ = (-1)^{\sigma} \sum_{i=1}^{k} (-1)^{\hat{\gamma}_i}. \] (28)

Since all our steps starting at (20) are reversible, we have shown that there is a
1:1 correspondence \( \theta \) between \((k,n)\)-balanced shapes \( \lambda \) and pairs of complementary
partitions \( \alpha \) and \( \beta \) relative to \( (n^k) \). Moreover, our arguments show that if \( \theta(\lambda) = (\alpha, \beta) \), then
\[ \text{SS}(\alpha, \beta) = \text{sgn}(\lambda) \gamma. \] (29)

(We note that it is not the case that if \( \theta(\lambda) = (\alpha, \beta) \), then \( \lambda = \gamma(n^k, \alpha, \beta) \). For example,
if \( n = 4 \) and \( k = 3 \), then \( \lambda = \lambda(4^3, (1, 2), (2, 3, 4)) = (2, 3, 4, 4, 5, 6) \). Thus \( \lambda = (2, 4, 6, 7, 9, 11) \) so that \( \theta(\lambda) = ((1^3), (3^3)) \).)

Now by (10),
\[ p_{2}[s_{(n^k)}] = \sum_{\alpha, \beta} \langle s_{s\beta}, s_{(n^k)} \rangle \text{SS}(\alpha, \beta)(x). \] (30)

But \( \langle s_{s\beta}, s_{(n^k)} \rangle = 0 \) unless \( \alpha, \beta \subseteq (n^k) \), and if \( \alpha, \beta \subseteq (n^k) \), then
\[ \langle s_{s\beta}, s_{(n^k)} \rangle = \langle s_{\beta}, s_{(n^k)} \rangle. \] (31)

However, by the exact same argument that we used to conclude that there is a unique
compatible tableau relative to the reverse lexicographic filling of Fig. 15, there is a
unique \((n^k)/\alpha)\)-compatible tableau of shape \( \gamma \) where \( \gamma \) is the complementary shape for
\( \alpha \) relative to \( (n^k) \). Thus \( \langle s_{s\beta}, s_{(n^k)} \rangle = 0 \) if \( \beta \neq \gamma \) and equals 1 if \( \beta = \gamma \). Let \( \text{CP}(n^k) \) be
the set of all complementary pairs \((\alpha, \beta)\) relative to \((n^k)\). Then we have shown that
(30) reduces to
\[ p_{2}[s_{(n^k)}] = \sum_{(\alpha, \beta) \in \text{CP}(n^k)} \text{SS}(\alpha, \beta). \] (32)
But by (29),
\[ \sum_{(\alpha, \beta) \in \mathcal{CP}(\pi^\lambda)} SS_{(\alpha, \beta)} = \sum_{\lambda \in \mathcal{B}(k, n)} \text{sgn}(\lambda)s_{\lambda} \]

which proves Lemma 2.

3. The computation of \( s_{(a,b)}s_{(a,b)} \)

Let \( a + b = n \), \( \lambda \vdash 2n \) and let \( D \) denote the skew diagram \( (a, b) \times (a, b) \) in Fig. 16:

Then
\[ s_{(a,b)}^2 = \sum_{\lambda \vdash [(a,b),(a,b)]} t_{\lambda} s_{\lambda}, \tag{33} \]

where \( t_{\lambda} \) is the number of \( D \)-compatible standard tableaux of shape \( \lambda \). Next, we shall apply the Remmel-Whitney algorithm to expand a typical term in \( s_{(a,b)}^2 \). The reverse lexicographic filling for the skew shape \( (a, b) \times (a, b) \) is shown in Fig. 17.

It is then easy to see that in our skew Schur function expansion rule that the \( D \)-compatible tableaux will have the following properties.

I. The numbers 1 through \( b \) will be in the first row, and the numbers \( b + 1, \ldots, n \) will be in the second row so that we can recover the shape \( (a, b) \) by looking at the shape occupied by the numbers \( 1, \ldots, n \) in Fig. 18.

II. The numbers \( n + 1, \ldots, n + b \), are placed on the outside of the shape corresponding to the numbers \( 1, \ldots, n \) in such a way that no two cells containing numbers between \( n + 1 \) and \( n + b \) lie in the same column, and the numbers in such cells increase from left to right.

![Fig. 16.](image1)

![Fig. 17.](image2)

![Fig. 18.](image3)
III. The numbers $n + b + 1, \ldots, 2n$ will be placed on the outside of the shape corresponding to the numbers $1, \ldots, n, n + 1, \ldots, n + b$ in such a way that no two cells containing numbers greater than $n + b$ lie in the same column, and the numbers in such cells increase from left to right.

We can replace any $D$-compatible tableau by a three-colored diagram where we color the cells filled with numbers $1, \ldots, n$ white, the cells filled with numbers $n + 1, \ldots, n + b$ red, and the cells filled with numbers $n + b + 1, \ldots, 2n$ blue. Of course, we can recover the original $D$-compatible tableaux by filling the white cells as pictured in Fig. 18, filling the red cells from left to right with the numbers $n + 1, \ldots, n + b$, and filling the blue cells from left to right with the numbers $n + b + 1, \ldots, 2n$.

Thus, the $D$-compatible tableaux corresponding to terms on RHS of (33) correspond exactly to three-colored diagrams that consist of Ferrers diagrams with at most four rows whose cells are colored white, blue, and red in such a way that

- the shape of white cells in $(a, b)$;
- the red and white cells form a shape $\gamma$ such that $\gamma/(a, b)$ is a skew row (i.e., no two red cells lie in the same column) and $|\gamma/(a, b)| = b$;
- blue cells occupy $\lambda/\gamma$ and $\lambda/\gamma$ is a skew row and $|\lambda/\gamma| = a$;
- note that condition (b) of the Remmel-Whitney rule forces that in a $D$-compatible tableau $T, 2n$ must be to the northeast of $n + b$, $2n - 1$ must be to the northeast of $n + b - 1, \ldots$, and $n + b + 1$ must be to the northeast of $n + b - a + 1$.

It follows that for any given blue cell $C$, the number of red cells that are strictly below and weakly to the right of $C$ is greater than or equal to the number of blue cells that are strictly to the right and weakly below $C$. Notice that there are two basic facts which follow from the rules we have to keep in mind.

Any column of height 2 in $\gamma/(a, b)$ must be filled with a blue cell on top of a red cell and by (d) there are no blue cells in the first row.

Fig. 20 shows an example of a typical $D$-compatible tableau and its corresponding three-color diagram. Note that any small letter that is either inside a cell or inside a row in the subsequent figures denotes the first letter of the color either of a single cell or of all the cells in a row. So, for example, $b =$ blue cell, $r =$ red cell, and so on. A cell or row with no letter inside means that the corresponding cell or row is white.

The examples shown in Figs. 19 and 20 show how, we can recover the actual $D$-compatible tableau from a three-colored diagram satisfying (a)–(d). So from now on, for convenience, we will replace numbers with colors.

Now, we are ready to compute all the $D$-compatible tableaux of shape $\lambda$ corresponding to terms on RHS of (33). Note that for all cases, we have $\lambda_1 \leq a \leq \lambda_3$, $\lambda_2 \leq b \leq \lambda_4$. In fact, $a \geq \lambda_1$, $b \geq \lambda_2$ by II and III since otherwise we will have two red or two blue cells in the same column and the conditions $a \leq \lambda_3$, $b \leq \lambda_4$ are automatic.

Given that $\lambda_1 \leq a \leq \lambda_3$ and $\lambda_2 \leq b \leq \lambda_4$, there are four cases that come up and these are pictured in Figs. 21–24.

**Case 1:** $a < \lambda_2$, $b < \lambda_3$

**Case 2:** $a < \lambda_2$, $\lambda_3 \leq b$
Case 3: $\lambda_2 \leq a$, $b < \lambda_3$

Case 4: $\lambda_2 \leq a$, $\lambda_3 \leq b$

In the above figures, the cells that are colored either red or blue are forced by the rules. The regions in the diagrams that are labeled by $A_2$ and $A_3$ represent those cells in the second and third rows, respectively, whose colors are not forced and hence may be colored either red or blue.

Now let us fix some notation. Given a three-colored diagram $D$ satisfying conditions (a)–(d), let

- $b_2 =$ the number of blue cells in $A_2$,
- $b_3 =$ the number of blue cells in $A_3$, 
\( r_2 \) = the number of red cells in \( A_2 \),

\( r_3 \) = the number of red cells in \( A_3 \),

\( F_{b_i} \) = the number of forced blue cells in row \( i \) for \( i = 1, 2, 3 \),

\( F_{r_i} \) = the number of forced red cells in row \( i \) for \( i = 2, 3, 4 \).

We now want to translate conditions (a)-(d) into equalities and inequalities on these quantities. In each case, our conditions on the diagrams ensure that condition (a) holds and that the regions consisting of all red squares or all blue squares are skew rows. Thus, to complete our translation of conditions (a)-(c), we need only write equations to express that \( |\lambda_1| = a \) and \( |\gamma(a, b)| = b \). These can be expressed as follows.

\[
\begin{align*}
(E_1) & \quad a = F_{b_1} + F_{b_2} + F_{b_3} + b_2 + b_3,
(E_2) & \quad b = Fr_2 + Fr_3 + Fr_4 + r_2 + r_3.
\end{align*}
\]

Of course, our definitions of \( b_2, r_2, b_3, r_3 \) force

\[
(E_3) \quad r_2 + b_2 = |A_2|
\]

and

\[
(E_4) \quad r_3 + b_3 = |A_3|.
\]

Next, condition (d) applied to the blue cells in row 3 is equivalent to

\[
(E_5) \quad b_3 + F_{b_3} \leq F_{r_4},
\]

and condition (d) applied to the blue cells in row 2 is equivalent to

\[
(E_6) \quad b_2 + F_{b_2} \leq r_3 + F_{r_3} + (F_{r_4} - F_{b_3} - b_3).\]
Since for any given $\lambda$, $F_r$ and $F_b$ are fixed, the only other conditions that we have are that

\[(E_7) \quad b_2, r_2, b_3, r_3 \geq 0.\]

Next, we want to write these conditions in a more concise form. Clearly $E_1, E_2,$ and $E_7$ imply the following inequalities:

\[(C_1) \quad a \geq F_b_1 + F_b_2 + F_b_3.\]

\[(C_2) \quad b \geq F_r_2 + F_r_3 + F_r_4.\]

By $E_3, E_4,$ and $E_7$, we can replace $r_2$ and $r_3$ by $|A_2| - b_2$ and $|A_3| - b_3$, respectively. Then using the fact that $n = a + b = F_r_2 + F_r_3 + F_r_4 + r_2 + r_3 + F_b_1 + F_b_2 + F_b_3 + b_2 + b_3$, it is easy to see that we can replace $E_1-E_4$, and $E_7$ by

\[(C_3) \quad b_2 + b_3 = a - (F_b_1 + F_b_2 + F_b_3)\]

and

\[(C_4) \quad 0 \leq b_3 \leq |A_3|, \quad 0 \leq b_2 \leq |A_2|.\]

$E_5$ is equivalent to

\[(C_5) \quad b_3 \leq F_r_4 - F_b_3.\]

Next, using the fact that $b_2 = a - F_b_1 - F_b_2 - F_b_3 - b_3$, it is easy to check that $E_6$ can be rewritten in the form

\[(C_6) \quad b_3 \leq F_b_1 + F_r_3 + F_r_4 + |A_3| - a.\]

Let

\[u_3 = \min(a - (F_b_1 + F_b_2 + F_b_3), |A_3|, F_r_4 - F_b_3, F_b_1 + F_r_3 + F_r_4 + |A_3| - a),\]

and

\[\ell_3 = \max(0, a - (F_b_1 + F_b_2 + F_b_3) - |A_2|).\]

Note that $C_3$ and $C_4$ force the following lower bound for $b_3$.

\[(C_7) \quad \ell_3 \leq b_3.\]

Finally, $C_3-C_6$ force the following upper bound on $b_3$.

\[(C_8) \quad b_3 \leq u_3.\]

In fact, it is not difficult to check that $(C_7 \& C_8)$ are equivalent to $(C_3 \& C_4 \& C_5 \& C_6)$. 

Thus, in the presence of $C_1$ and $C_2$, the number of three-colored diagrams meeting conditions (a)-(d) is just the number of $b_3$'s satisfying $\ell_3 \leq b_3 \leq \mu_3$. Hence, we have proved the following:

**Theorem 2.** Let $\langle , \rangle$ denote the Hall inner product on symmetric functions. Then

$$\langle s_{2,a,b}^2, s_{3} \rangle = (1 + (u_3 - \ell_3))\chi(C_1)\chi(C_2)\chi(u_3 \geq \ell_3),$$

where for any statement $P$, $\chi(P) = 0$, if $P$ is false, and $\chi(P) = 1$, if $P$ is true.

Now, to complete our argument, we must examine how the general conditions specialize in each of our four cases.

**Case 1:** In this case,

\begin{align*}
|A_2| &= a - \lambda_1, \\
|A_3| &= b - \lambda_2, \\
Fb_1 &= \lambda_1, \\
Fb_2 &= \lambda_2 - a, \\
Fb_3 &= \lambda_3 - b, \\
Fr_2 &= \lambda_1, \\
Fr_3 &= \lambda_2 - a, \\
Fr_4 &= \lambda_4 - b.
\end{align*}

Then, by substituting, we get

\begin{align*}
(C_1) \quad &a \geq \lambda_1 + \lambda_2 - a + \lambda_3 - b \iff 2a + b \geq \lambda_1 + \lambda_2 + \lambda_3 \\
&\iff 2a + b - 2n \geq \lambda_1 + \lambda_2 + \lambda_3 - 2n \iff -b \geq -\lambda_4 \\
&\iff \lambda_4 \geq b \text{ (no condition)}
\end{align*}

\begin{align*}
(C_2) \quad &b \geq \lambda_1 + \lambda_2 - a + \lambda_4 - b \iff a + 2b \geq \lambda_1 + \lambda_2 + \lambda_4 \\
&\iff a + 2b - 2n \geq \lambda_1 + \lambda_2 + \lambda_4 - 2n \iff -a \geq -\lambda_3 \\
&\iff \lambda_3 \geq a \text{ (no condition)}.
\end{align*}

Thus, $\chi(C_1)\chi(C_2) = 1$ in this case. Next, we calculate $u_3$ and $\ell_3$. Substituting in (34) we get

\begin{align*}
u_3 \leq \min(a - (\lambda_1 + \lambda_2 - a + \lambda_3 - b), b - \lambda_2, \lambda_4 - b - (\lambda_3 - b), \lambda_1 + \lambda_2 - a + \lambda_4 \\
&-b + b - \lambda_2 - a)
= \min(2a + b - (\lambda_1 + \lambda_2 + \lambda_3), b - \lambda_2, \lambda_4 - \lambda_3, \lambda_1 + \lambda_4 - 2a).
\end{align*}

Note that $2(a + b) = 2n = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ so that

\begin{align*}
2a + b - (\lambda_1 + \lambda_2 + \lambda_3) &= 2n - b - (\lambda_1 + \lambda_2 + \lambda_3) \\
&= (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) - b - (\lambda_1 + \lambda_2 + \lambda_3).
\end{align*}

\begin{align*}
&= \lambda_4 - b.
\end{align*}
Table 1

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>A_{2}</td>
</tr>
<tr>
<td>$</td>
<td>A_{3}</td>
</tr>
<tr>
<td>$F_{b_1}$</td>
<td>$\lambda_1$</td>
</tr>
<tr>
<td>$F_{b_2}$</td>
<td>$\lambda_2 - a$</td>
</tr>
<tr>
<td>$F_{b_3}$</td>
<td>$\lambda_3 - b$</td>
</tr>
<tr>
<td>$F_{b_4}$</td>
<td>$\lambda_4 - b$</td>
</tr>
<tr>
<td>$\chi(C_1)$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi(C_2)$</td>
<td>1</td>
</tr>
<tr>
<td>$u_3$</td>
<td>min$(\lambda_4 - \lambda_3, 2b - \lambda_2 - \lambda_3)$</td>
</tr>
<tr>
<td>$\ell_3$</td>
<td>0</td>
</tr>
<tr>
<td>$(s_{(a,b)}, s_l)$</td>
<td>$[1 + u_3] \chi(2a + 2b)$</td>
</tr>
</tbody>
</table>

Thus, since $b < \lambda_3$, $\lambda_4 - b > \lambda_4 - \lambda_3$. Similarly,

\[
\lambda_1 + \lambda_4 - 2a = \lambda_1 + \lambda_4 - 2a - 2b + 2b
\]
\[
= \lambda_1 + \lambda_4 - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) + 2b
\]
\[
= 2b - \lambda_2 - \lambda_3 = (b - \lambda_2) + (b - \lambda_3).
\]

Thus, since $b < \lambda_3$, $(b - \lambda_2) + (b - \lambda_3) < b - \lambda_2$. Hence,

\[
u_3 = \min(\lambda_4 - \lambda_3, 2b - \lambda_2 - \lambda_3)\]

Substituting in (35), we get

\[
\ell_3 = \max(0, a - (\lambda_1 + \lambda_2 - a + \lambda_3 - b) - a + \lambda_1)
\]
\[
= \max(0, a + b - \lambda_2 - \lambda_3) = 0
\]

since $\lambda_2 > a$ and $\lambda_3 > b$.

Thus, in case 1,

\[
(s_{(a,b)}, s_l) = [1 + \min(\lambda_4 - \lambda_3, 2b - \lambda_2 - \lambda_3)] \chi(2b - \lambda_2 - \lambda_3 \geq 0).
\]

The computations in the other cases are similar. Rather than give all the details, we will simply summarize the results of the computations in each of the four cases in Tables 1 and 2. That is, we will list the values of $u_3, \ell_3, b_3, b_2, r_3, r_2, F_{b_1}$ for $i = 1, \ldots, 3$, and $F_{r_i}$ for $i = 2, 3, 4$ in each case; give the result of conditions $C_1$ and $C_2$ by computing $\chi(C_1)$ and $\chi(C_2)$; and finally give an expression for $(s_{(a,b)}, s_l)$.

We then have proved the following theorem.

**Theorem 2.** Let $d_i = (s_{(a,b)}, s_l)$. Then

(i) $d_i = 0$ if $\lambda$ has more than four parts,
Table 2

<table>
<thead>
<tr>
<th>Case 3</th>
<th>Case 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\lambda_2 - \lambda_1$</td>
</tr>
<tr>
<td>$</td>
<td>\lambda_3 - a$</td>
</tr>
<tr>
<td>$b - a$</td>
<td>$\lambda_1$</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\lambda_3 - b$</td>
<td>$\lambda_1$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\lambda_1$</td>
</tr>
<tr>
<td>$\lambda_4 - b$</td>
<td>$\lambda_4 - b$</td>
</tr>
<tr>
<td>$\lambda_4 - b$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\chi(C_1)$</td>
<td>$\chi(2b \geq \lambda_1 + \lambda_4)$</td>
</tr>
<tr>
<td>$u_3 \min(a + b - \lambda_1 - \lambda_3, b - a, \lambda_4 - \lambda_3, 2b - \lambda_2 - \lambda_3)$</td>
<td>$\min(a - \lambda_1, \lambda_3 - a, \lambda_4 - b)$</td>
</tr>
<tr>
<td>$\lambda_4 - \lambda_3, 2b - \lambda_2 - \lambda_3)$</td>
<td></td>
</tr>
<tr>
<td>$\max(0, a + b - \lambda_2 - \lambda_3)$</td>
<td>$a - \lambda_2$</td>
</tr>
<tr>
<td>$\langle s_{(a,b)}^2, s_{(a,b)} \rangle_x$</td>
<td>$[1 + u_3 - \delta_x] \chi(u_3 \geq \delta_x)$</td>
</tr>
<tr>
<td>$[1 + u_3 - \delta_x] \chi(u_3 \geq \delta_x)$</td>
<td>$\chi(2b \geq \lambda_1 + \lambda_4)$</td>
</tr>
</tbody>
</table>

(ii) $d_x = 0$ if $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and it is not the case that $\lambda_1 \leq a \leq \lambda_3$ and $\lambda_2 \leq b \leq \lambda_4$.

(iii) if $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4), \lambda_1 \leq a \leq \lambda_3$, and $\lambda_2 \leq b \leq \lambda_4$, then

(a) $d_x = (1 + x_1) \chi(2b \geq \lambda_2 + \lambda_3)$ where $x_1 = \min(\lambda_4 - \lambda_3, 2b - \lambda_2 - \lambda_3)$ if $a < \lambda_2$ and $b < \lambda_3$.

(b) $d_x = (1 + x_2) \chi(\lambda_3 + \lambda_4 \geq 2b)$ where $x_2 = \min(\lambda_3 + \lambda_4 - 2b, \lambda_3 - \lambda_2)$ if $a < \lambda_2$ and $b \geq \lambda_3$.

(c) $d_x = (1 + x_3 - y_3) \chi(x_3 \geq y_3) \chi(2b \geq \lambda_1 + \lambda_4)$, where $y_3 = \max(0, a + b - \lambda_2 - \lambda_3)$ and $x_3 = \min(a + b - \lambda_1 - \lambda_3, b - a, \lambda_4 - \lambda_3, 2b - \lambda_2 - \lambda_3)$ if $a \geq \lambda_2$ and $b < \lambda_3$.

(d) $d_x = (1 + x_4 - y_4) \chi(x_4 \geq y_4) \chi(2b \geq \lambda_1 + \lambda_4)$, where $x_4 = \min(a - \lambda_1, \lambda_3 - \lambda_2, \lambda_4 - b)$ and $y_4 = a - \lambda_2$ if $a \geq \lambda_2$ and $b \geq \lambda_3$.

We end the section by computing $\langle s_{(r,s)} s_{(h,k)}, s_{(a,b)} \rangle = c_{(r,s),(h,k)}$ since we will need it for the computation of $p_2[s_{(a,b)}]$ in Section 3.

The following lemma reduces the computation of the coefficients $c_{(r,s),(h,k)}$ to Pieri’s rule.

**Lemma 3.** Let $a + b = n$ and $(r,s), (h,k) \subseteq (a, b)$ such that $|(r,s) + (h,k)| = n$, then

$$\langle s_{(r,s)} s_{(h,k)}, s_{(a,b)} \rangle = \langle s_{(r+h,r+k)} s_{(-r,-k)}, s_{(a,b)} \rangle.$$ 

**Proof.** Let $F_{(r,s)}$ and $F_{(h,k)}$ be the Ferrers diagrams associated with the partitions $(r,s)$ and $(h,k)$.

By the same sort of reasoning that we used to compute $s_{(a,b)} s_{(a,b)}$, the $(r,s)\ast(h,k)$-compatible tableaux of shape $(a, b)$ are in one correspondence with the colored diagrams pictured in Fig. 26 where the distribution of white, red, blue, and green cells match the distribution pictured in Fig. 25.
Since the blue and red cells are fixed by the rules, these configurations depend on the green cells of total length \( s - r \) and it makes no difference if we put them at the end of the two rows, as Fig. 27 shows.

But these configurations are those that correspond to the \((r + h, r + k) \ast (s - r)\)-compatible tableaux of shape \((a, b)\), and this proves the lemma.

The following lemma provides some restrictions on \( a \) and \( b \) that will be useful for the sequel.

**Lemma 4.** Suppose \( a + b = n \), \((r, s), (k) \subseteq (a, b)\), and \(|(r, s) + (k)| = n\).

Then

\[
\langle s_{(r,s)} s_k, s_{(a,b)} \rangle = \langle s_{(s-r)} s_k, s_{(a-r,b-r)} \rangle.
\]

**Proof.** Consider the skew diagram (Fig. 28)

According to Pieri’s rule, the generic two rows diagram that comes up from the product \( s_{(r,s)} s_k \) is shown in Fig. 29.
According to the fact that \( (s_\lambda, s_\mu) = \delta_{\lambda\mu} \), the Kronecker delta, then the coefficients

\[
(s(r,s)S_k, S(a,b)) = \delta_{r\lambda} = \delta_{s\mu}
\]

are equal to 1 when the first row in Fig. 29 is equal to \( a \) and the second row is equal to \( b \) and are equal to 0 otherwise.

But if we cut the diagram in Fig. 29 where the green part starts, we get the diagram shown in Fig. 30.

The diagram of Fig. 30 corresponds to the \((s-r)\ast k\)-compatible tableaux that count the coefficient

\[
(s(s-r)S_k, S(a-r,b-r)).
\]

From Figs. 29 and 30, it is easy to see that the corresponding coefficients (36) and (37) are equal.

Finally, we have:

**Theorem 4.** \( (s(r,s)S_k, S(a,b)) = \chi(r+h \leq a \leq \min(k + r, s + h)) \).

**Proof.** By first applying Lemma 3 and then Lemma 4, we get

\[
(s(r,s)S_k, S(a,b)) = (s(r+h,r+k)S(s-r), S(a,b))
\]

\[
= (s(k-h)S(s-r), S(a-(r+h), b-(r+h))) \chi(a \geq r + h)
\]

\[
= \chi(a \geq r + h) \chi(a - (r + h) \leq \min(k - h, s - r))
\]

\[
= \chi(r + h \leq a \leq \min(k + r, s + h)).
\]
4. The computation of \( s(a,b) \) \([P2]\)

By the SXP-algorithm of Section 1,

\[
S(a,b)[P2] = \sum_{\lambda^{(1)}, \lambda^{(2)} \subseteq (a,b)} \langle \delta_{\lambda^{(1)}}, \delta_{\lambda^{(2)}}, s(a,b) \rangle SS(\lambda^{(1)}, \lambda^{(2)}).
\]

(38)

Note that by Theorem 4, the coefficients \( \langle \delta_{\lambda^{(1)}}, \delta_{\lambda^{(2)}}, s(a,b) \rangle \) \( \in \{0, 1\} \).

As it has been stated in Section 1, the correspondence between circle diagrams and special rim hook tabloids provides an alternative way to compute \( SS(\lambda^{(1)}, \lambda^{(2)}) \).

Thus, we can rewrite \( s(a,b)[P2] \) as follows:

\[
S(a,b)[P2] = \sum_{\langle \lambda^{(2)}, \lambda^{(2)} \subseteq (a,b) \rangle \atop {\sum a_i = a+b \atop {a_1, a_2 < a \atop a_3, a_4 < b}}} \langle \delta_{\lambda^{(1)}}, \delta_{\lambda^{(2)}}, s(a,b) \rangle \text{sgn}(T_{(a_1, a_2)}(a_3, a_4)) s_{sh}(T_{(a_1, a_2)}(a_3, a_4)).
\]

(39)

Thus, we have to analyze all special rim hook tabloids \( T \) of the form \( T = T_{(a_1, a_2)}(a_3, a_4) \). By (17), it follows that \( T \) must have four rim hooks, \( h_i(T) = h_i \) for \( i = 1, 2, 3, 4 \), that satisfy the following conditions:

1. \( |h_i| = 2a_i \) for \( i = 1, 2, 3, 4 \),
2. \( |h_1| \leq |h_3| \), and
3. \( |h_2| \leq |h_4| \).

Note that if \( T \) is a special rim hook tabloid satisfying (1)–(3), then \( T = T_{(a_1, a_2)}(a_3, a_4) \) and, hence,

\[
SS_{(a_1, a_2)}(a_3, a_4) = \text{sgn}(T) s_{sh}(T).
\]

In addition, it is easy to see from the circle diagram construction of \( SS_{\lambda, \lambda} \) that if \( SS_{\lambda, \lambda} = \pm SS_{\lambda, \lambda} \), then \( I_0 = I_1 \) and \( J_0 = J_1 \). This means that there can be at most one special rim hook tabloid \( T \) satisfying conditions (1)–(3) for any given shape \( \lambda \).

Next, we will show that the special rim hook tabloids \( T \) of shape \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \) satisfying (1)–(3) can be classified according to one of six cases depending on the parity of \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \). These six cases will be summarized in Table 3.

First, observe that neither rim hook \( h_1 \) nor \( h_2 \) can end in row 4. That is, if \( h_1 \) ends in row 4, then the \( T \) must be of the form shown in Fig. 31.

But it is easy to see from Fig. 31 that no matter how we place \( h_2, h_3, \) and \( h_4 \), \( |h_3| \) will be strictly less than \( |h_1| \) which violates condition (2). A similar argument will show that if \( h_2 \) ends in row 4, then \( |h_2| > |h_4| \) which violates condition (3). Thus \( h_1 \) must end in one of the rows 1, 2, or 3, and \( h_2 \) must end in either row 2 or 3. Hence, we have six cases to analyze, depending on the rows in which \( h_1 \) and \( h_2 \) end. We will analyze two of these cases explicitly.
Table 3

<table>
<thead>
<tr>
<th>Case</th>
<th>Conditions</th>
<th>$T$</th>
<th>$T([k],[d],[k],[d])$</th>
<th>$(p_2(s_{a,b}^t), s_d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>all $\lambda_i$ even</td>
<td>$T\left(\frac{\lambda_1 \lambda_2}{2}, \frac{\lambda_3 \lambda_4}{2}\right)$</td>
<td>$X\left(\frac{\lambda_1 + \lambda_2}{2}\right) \leq a \leq \min\left{\frac{\lambda_2 + \lambda_3}{2}, \frac{\lambda_4 + \lambda_5}{2}\right}$</td>
<td></td>
</tr>
<tr>
<td>Case 2</td>
<td>$\lambda_1, \lambda_2$ even, $\lambda_3, \lambda_4$ odd</td>
<td>$T\left(\frac{\lambda_1 \lambda_2}{2}, \frac{\lambda_3 \lambda_4}{2}\right)$</td>
<td>$X\left(\frac{\lambda_1 + \lambda_2}{2}\right) \leq a \leq \min\left{\frac{\lambda_2 + \lambda_3}{2}, \frac{\lambda_4 + \lambda_5}{2}\right}$</td>
<td></td>
</tr>
<tr>
<td>Case 3</td>
<td>$\lambda_1, \lambda_4$ even, $\lambda_2, \lambda_3$ odd</td>
<td>$T\left(\frac{\lambda_1 \lambda_2}{2}, \frac{\lambda_3 \lambda_4}{2}\right)$</td>
<td>$X\left(\frac{\lambda_1 + \lambda_2}{2}\right) \leq a \leq \min\left{\frac{\lambda_2 + \lambda_3}{2}, \frac{\lambda_4 + \lambda_5}{2}\right}$</td>
<td></td>
</tr>
<tr>
<td>Case 4</td>
<td>$\lambda_1, \lambda_2$ odd, $\lambda_3, \lambda_4$ even</td>
<td>$T\left(\frac{\lambda_1 \lambda_2}{2}, \frac{\lambda_3 \lambda_4}{2}\right)$</td>
<td>$X\left(\frac{\lambda_1 + \lambda_2}{2}\right) \leq a \leq \min\left{\frac{\lambda_2 + \lambda_3}{2}, \frac{\lambda_4 + \lambda_5}{2}\right}$</td>
<td></td>
</tr>
<tr>
<td>Case 5</td>
<td>all $\lambda_i$ odd</td>
<td>$T\left(\frac{\lambda_1 \lambda_2}{2}, \frac{\lambda_3 \lambda_4}{2}\right)$</td>
<td>$X\left(\frac{\lambda_1 + \lambda_2}{2}\right) \leq a \leq \min\left{\frac{\lambda_2 + \lambda_3}{2}, \frac{\lambda_4 + \lambda_5}{2}\right}$</td>
<td></td>
</tr>
<tr>
<td>Case 6</td>
<td>$\lambda_1, \lambda_4$ odd, $\lambda_2, \lambda_3$ even</td>
<td>$T\left(\frac{\lambda_1 \lambda_2}{2}, \frac{\lambda_3 \lambda_4}{2}\right)$</td>
<td>$X\left(\frac{\lambda_1 + \lambda_2}{2}\right) \leq a \leq \min\left{\frac{\lambda_2 + \lambda_3}{2}, \frac{\lambda_4 + \lambda_5}{2}\right}$</td>
<td></td>
</tr>
</tbody>
</table>
Case 1: $h_1$ ends in row 1, $h_2$ ends in row 2.

There are two subcases:

subcase 1.1: $h_3$ ends in row 3

subcase 1.2: $h_3$ ends in row 4

In subcase 1.1, it follows from the picture that $|h_1| = \lambda_1$, $|h_2| = \lambda_2$, $|h_3| = \lambda_3$, and $|h_4| = \lambda_4$ which implies that $\lambda_i$ are all even. Conversely, suppose we have a shape $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ where $\lambda_i$ are all even, then $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$. This means that if we take the special rim hook tabloid $T$ pictured in subcase 1.1, then $|h_i(T)| = \lambda_i$ for all $i$ so $|h_i(T)|$ is even for all $i$ and automatically $|h_1(T)| \leq |h_3(T)|$ and $|h_2(T)| \leq |h_4(T)|$. Thus,
$T$ will be of the form

$$T = T_{\left( \frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \frac{\lambda_3}{2} \right)} \left( \frac{\lambda_4}{2} \right).$$

In subcase 1.2, it follows from the picture that $|h_1| = \lambda_1$, $|h_2| = \lambda_2$, $|h_3| = \lambda_3 - 1$, and $|h_4| = \lambda_4 - 1$ which implies that $\lambda_1, \lambda_2$ are even, $\lambda_3, \lambda_4$ are odd. Conversely, suppose we have a shape $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ where $\lambda_1, \lambda_2$ are even, and $\lambda_3, \lambda_4$ are odd, then $\lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4$. This means that if we take the special rim hook tabloid $T$ pictured in subcase 1.2, then $|h_1(T)| = \lambda_1$, $|h_2(T)| = \lambda_2$, $|h_3(T)| = \lambda_3 - 1$, and $|h_4(T)| = \lambda_4 - 1$. Consequently, $|h_i(T)|$ is even for all $i$ and $|h_1(T)| = \lambda_1 < \lambda_4 + 1 = |h_3(T)|$, $|h_2(T)| = \lambda_2 \leq \lambda_3 - 1 = |h_4(T)|$. Thus $T$ will be of the form

$$T = T_{\left( \frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \frac{\lambda_3}{2} \right)} \left( \frac{\lambda_4 - 1}{2} \right).$$

Case 2: $h_1$ ends in row 1, $h_2$ ends in row 3.

There are two subcases:

subcase 2.1: $h_3$ ends in row 3

subcase 2.2: $h_3$ ends in row 4

In subcase 2.1, it follows from the picture that $|h_1| = \lambda_1$, $|h_2| = \lambda_3 + 1$, $|h_3| = \lambda_2 - 1$, and $|h_4| = \lambda_4$, which implies that $\lambda_1, \lambda_4$ are even, and $\lambda_2, \lambda_3$ are odd. Conversely, suppose we have $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ where $\lambda_1, \lambda_4$ are even and $\lambda_2, \lambda_3$ are odd, then $\lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4$. 
This means that if we take the special rim hook tabloid \( T \) pictured in subcase 2.1, then 
\[ |h_1(T)| = \lambda_1, \quad |h_2(T)| = \lambda_3 + 1, \quad |h_3(T)| = \lambda_2 - 1, \quad \text{and} \quad |h_4(T)| = \lambda_4. \]
Consequently, \( |h_i(T)| \) is even for all \( i \), \( |h_1(T)| = \lambda_1 < \lambda_2 - 1 = |h_3(T)| \), and \( |h_2(T)| = \lambda_3 + 1 \leq \lambda_4 = |h_4(T)| \).
Thus, \( T \) will be of the form
\[ T = T_{\left( \frac{\lambda_2 - 1}{2}, \frac{\lambda_2 + 1}{2}, \frac{\lambda_3}{2}, \frac{\lambda_4}{2} \right)} \]

In subcase 2.2, it follows from the picture that 
\[ |h_1| = \lambda_1, \quad |h_2| = \lambda_3 + 1, \quad |h_3| = \lambda_4 + 1, \quad \text{and} \quad |h_4| = \lambda_2 - 2, \]
which implies that \( |h_2| = \lambda_3 + 1 > |h_4| = \lambda_2 - 2 \) violating condition (3).

The remaining four cases can be analyzed in a similar manner. The net result is that there are six types of special rim hook tableaux \( T_{(\lambda_1, \lambda_3), (\lambda_2, \lambda_4)} \) meeting conditions (1)–(3), and in each case the special rim hook tableau determines the parity of \( \lambda_1, \lambda_2, \lambda_3, \) and \( \lambda_4 \). Moreover, the value of \( \text{sgn}(T_{(\lambda_1, \lambda_3), (\lambda_2, \lambda_4)}) \) and \( \langle s_{(\lambda_1, \lambda_3), (\lambda_2, \lambda_4)} : s_{(\lambda_1, \lambda_3), (\lambda_2, \lambda_4)} \rangle \) can be easily calculated, thus giving the coefficient \( \langle p_2[s_{(a,b)}], s_{(a,b)} \rangle \) in each case. This information is summarized in Table 3.

Note that several entries in the last column of Table 3 are identical and observing that \( \lambda_1 + \lambda_3 - 1 \leq \lambda_2 + \lambda_4 + 1 \), we can summarize Table 3 in the following theorem.

**Theorem 5.** Let \( c_\lambda = \langle p_2[s_{(a,b)}], s_{(a,b)} \rangle \). Then

(i) \( c_\lambda = 2((\lambda_1 + \lambda_3)/2 \leq \lambda \leq \min((\lambda_2 + \lambda_3)/2, (\lambda_1 + \lambda_4)/2)) \), if \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \) and either \( \lambda_i \) are even or all \( \lambda_i \) are odd,

(ii) \( c_\lambda = -2((\lambda_1 + \lambda_3)/2 \leq \lambda \leq \min((\lambda_2 + \lambda_3)/2, (\lambda_1 + \lambda_4)/2)) \), if \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \) and either \( \lambda_1, \lambda_2 \) are even and \( \lambda_3, \lambda_4 \) are odd, or \( \lambda_1, \lambda_2 \) are odd and \( \lambda_3, \lambda_4 \) are even,

(iii) \( c_\lambda = 2((\lambda_1 + \lambda_3 + 1)/2 \leq \lambda \leq \min((\lambda_2 + \lambda_3)/2, (\lambda_1 + \lambda_4)/2)) \), if \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \) and either \( \lambda_1, \lambda_4 \) are even and \( \lambda_2, \lambda_3 \) are odd or \( \lambda_1, \lambda_4 \) are odd and \( \lambda_2, \lambda_3 \) are even, and

(iv) \( c_\lambda = 0 \) is otherwise.

Note that by Theorem 5, \( \langle p_2[s_{(a,b)}], s_{(a,b)} \rangle \in \{0, \pm 1\} \) for all \( \lambda \). It follows from (7) that for \( a \leq 2 \), if \( t_\lambda = \langle s_{(a,b)} s_{(a,b)} \rangle \) is even, then \( t_\lambda/2 = \langle s_{(a,b)} s_{(a,b)} \rangle \). That is, we know \( \langle s_{(a,b)} s_{(a,b)} \rangle \), \( s_{(a,b)} \rangle = \frac{1}{2}((s_{(a,b)} s_{(a,b)}) \pm \langle p_2[s_{(a,b)}], s_{(a,b)} \rangle)^2 \) must be an integer so that if \( t_\lambda \) is even, then \( \langle p_2[s_{(a,b)}], s_{(a,b)} \rangle = 0 \). If \( t_\lambda \) is odd, then \( \langle s_{(a,b)} s_{(a,b)} \rangle \) determines the sign of \( \langle p_2[s_{(a,b)}], s_{(a,b)} \rangle \) to determine whether we want \( t_\lambda + 1/2 \) or \( t_\lambda - 1/2 \).

This observation allows us to use Theorems 3 and 5 to easily evaluate (7) when \( \mu = (a, b) \). This leads to the following theorem.

**Theorem 6.** Suppose \( a + b = n \) and \( \lambda \) is a partition of \( 2n \).

Let \( t_\lambda = \langle s_{(a,b)} s_{(a,b)} \rangle \), \( \mu_\lambda = \langle p_2[s_{(a,b)}], s_{(a,b)} \rangle \) and \( v_\lambda = \langle s_{(a,b)} s_{(a,b)} \rangle \). Then

(i) \( \mu_\lambda = v_\lambda = 0 \), if \( \lambda \) has more than four parts,

(ii) If \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \), then

(a) \( \mu_\lambda = v_\lambda = t_\lambda/2 \) if \( t_\lambda \) is even

(b) \( \mu_\lambda = (t_\lambda + 1)/2 \) and \( v_\lambda = (t_\lambda - 1)/2 \) if \( t_\lambda \) is odd and either all \( \lambda_i \)'s are even or all \( \lambda_i \)'s are odd

(c) \( \mu_\lambda = t_\lambda - 1/2 \) and \( v_\lambda = t_\lambda + 1/2 \) if otherwise.
Remark. Considering that

\[(s_\lambda[s_\mu])' = \begin{cases} 
    s_\lambda[s_\mu'] & \text{if } |\mu| \text{ is even}, \\
    s_\lambda'[s_\mu'] & \text{if } |\mu| \text{ is odd},
\end{cases}\]

where for any sum \[\sum c_v s_v, (\sum c_v s_v)'\] denotes the sum \[\sum c_v s_v',\] and \[v'\] denotes the conjugate of \[v,\] by the previous theorem we can also derive explicit formulas for the Schur function expansion of the plethysms \[s_2[s_\mu]\] and \[s_1[s_\mu],\] where \[\mu\] has two columns.

References