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Infrared Yang–Mills theory as a spin system. A lattice approach

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Abstract

To verify the conjecture that Yang–Mills theory in the infrared limit is equivalent to a spin system whose excitations are knot solitons, a numerical algorithm based on the inverse Monte Carlo method is proposed. To investigate the stability of the effective spin field action, numerical studies of the renormalization group flow for the coupling constants are suggested. A universality of the effective spin field action is also discussed.

1. It was conjectured [1] that the $SU(2)$ Yang–Mills theory in the infrared region can be described as a spin system with the following action

$$S = \int dx \left\{ m^2 (\partial_\mu \boldsymbol{n})^2 + g^{-2} \left[\boldsymbol{n} \cdot (\partial_\mu \boldsymbol{n} \times \partial_\nu \boldsymbol{n}) \right]^2 \right\}, \quad (1)$$

where $\boldsymbol{n}^2 = 1$ (boldface letters stand for three-vectors). The path integral representation of the effective action (1) can be deduced from the Yang–Mills theory path integral via an implicit change of integration variables [2]. The analysis can be extended to the $SU(N)$ case [3,4]. Nonperturbative excitations of the effective theory are knot solitons [5]. Knot solitons look more like stringy excitations, which is believed to be a right physical picture of nonperturbative excitations of gauge fields. Yet, if the effective action (1) turns out to be a good approximation to the Yang–Mills theory in the infrared limit, the nonperturbative dynamics can be studied by quantum soliton theory methods. The mass gap in the spectrum of quantum Yang–Mills theory would therefore naturally be introduced as the lowest energy bound in the quantum soliton spectrum. A few important questions are to be addressed to validate or invalidate this attractive picture. First, what is the actual value of the mass scale m^2 which determines the low energy bound (in the classical soliton theory) [6]? Second, is the effective action stable from the point of view of the renormalization group flow of its coupling constants? Third, how big are the higher order corrections to (1)? The purpose of this Letter is to set up a numerical approach to answer these questions.

The existence of degrees of freedom whose dynamics dominates in the infrared region of Yang–Mills theory was established in numerical simulations [7–9] of lattice Yang–Mills theories some time ago. It was observed that dominant contributions to the string tension come from topological defects (monopoles) which occur in typical vacuum configurations of gauge fields when the latter are taken in a special gauge known as the maximal Abelian gauge. Topological defects unavoidably occur in any gauge that breaks the gauge group to its maximal Abelian subgroup [10]. In

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classical theory, it is evident from the fact that the homotopies of the gauge group and its maximal Abelian subgroup are different [11] (see more on the gauge fixing problem in quantum gauge theories in [12]). The importance of the above numerical discovery is that the defects alone are sufficient to reproduce essential nonperturbative features of Yang–Mills theory. The numerical procedure of singling out topological defects in Abelian projections of the lattice Yang–Mills theory can therefore be used in a theoretical analysis to parameterize the relevant degrees of freedom of the Yang–Mills connection and compute their effective action. It should be noted that the gauge fixing here does not serve its conventional purpose—removing nonphysical degrees of freedom—but rather it becomes an auxiliary tool to identify the degrees of freedom relevant for the infrared physics of the Yang–Mills theory. After a change of variables, that splits all the Yang–Mills degrees of freedom into the "infrared relevant" ones and the rest, is found, the effective action can be computed in any convenient gauge and its gauge invariance can be established by the standard BRST technique [2,3].

In our earlier works [2,3], a relation between the spin field $\boldsymbol{n}$ and topological defects of the connection in the maximal Abelian gauge has been found

$$\boldsymbol{A}_\mu = g^{-1}\partial_\mu \boldsymbol{n}\times\boldsymbol{n} + \boldsymbol{n}C_\mu + \boldsymbol{W}_\mu, \tag{2}$$

where $\boldsymbol{W}_\mu$ satisfies the following conditions. It is perpendicular to $\boldsymbol{n}$ and

$$\partial_\mu \boldsymbol{W}_\mu + (\boldsymbol{\alpha}_\mu + \boldsymbol{n}C_\mu)\times\boldsymbol{W}_\mu \equiv \nabla_\mu(\boldsymbol{\alpha}+\boldsymbol{n}C)\boldsymbol{W}_\mu = 0. \tag{3}$$

Here $\boldsymbol{\alpha}_\mu = g^{-1}\partial_\mu\boldsymbol{n}\times\boldsymbol{n}$ is the connection introduced in [13]. Relation (2) is a change of variables in the space of connections. Indeed, the original variables $\boldsymbol{A}_\mu$ have 12 independent scalar functions. There are two independent scalar functions in $\boldsymbol{n}$, four in C_μ, and, hence, there must be six independent scalar functions in $\boldsymbol{W}_\mu$. This is the case indeed because 12 components of $\boldsymbol{W}_\mu$ satisfy six independent conditions: four in $\boldsymbol{n}\cdot\boldsymbol{W}_\mu = 0$ and two in (3). Note that if $\boldsymbol{W}_\mu$ is perpendicular to $\boldsymbol{n}$, then covariant derivatives of $\boldsymbol{W}_\mu$ with respect the connection $\boldsymbol{\alpha}_\mu + \boldsymbol{n}C_\mu$ is always perpendicular to $\boldsymbol{n}$. The inverse transformation can be found by multiplying (2) by $\boldsymbol{n}$ using first the dot and then cross products. The obtained relations allow one to express C_μ and $\boldsymbol{W}_\mu$ as functions of $\boldsymbol{A}_\mu$ and $\boldsymbol{n}$. Substituting them into (3), an equation for $\boldsymbol{n}$ as a functional of $\boldsymbol{A}_\mu$ is derived. It has been shown that for a given $\boldsymbol{A}_\mu$, the corresponding spin field can be computed as [2]

$$\boldsymbol{n} = \frac{1}{2}\,\mathrm{tr}\big(\boldsymbol{\tau}\Omega_A^\dagger\tau_3\Omega_A\big), \quad \boldsymbol{\tau} = (\tau_1,\tau_2,\tau_3), \tag{4}$$

where τ_i are the Pauli matrices, $\mathrm{tr}(\tau_i\tau_j) = 2\delta_{ij}$ and Ω_A is a group element which depends on $\boldsymbol{A}_\mu$ so that the gauge transform of $\boldsymbol{A}_\mu$ with Ω_A satisfies the maximal Abelian gauge. Topological numbers of the defects have an integral representation via the spin field $\boldsymbol{n}$ [2].

The gauge transformation Ω_A is in general singular (it might not even be single valued in spacetime). In other words, for a typical vacuum configuration of gauge fields, the maximal Abelian gauge can *only* be achieved by a singular gauge transformation. According to numerical simulations [7], the third component of the gauge fixed configuration $\boldsymbol{A}_\mu^\Omega$, which is associated with the unbroken $U(1)$ subgroup, carries Dirac magnetic monopoles. The monopoles alone contribute about 90% to the energy of the flux tube (string) between static sources in the Yang–Mills theory. The monopole spacetime trajectories are determined by singularities of Ω_A. Hence, by taking Abelian connections of the monopoles and applying the inverse gauge transformation $\Omega_A^\dagger$ to them, one can parameterize the relevant (or "monopole producing") degrees of freedom of a generic Yang–Mills connection. The result of this procedure is given by Eqs. (2) and (3). By construction, the spin field $\boldsymbol{n}$ carries all the information about the spacetime distribution of the defects, and, hence, its effective theory should describe the infrared physics of the original Yang–Mills theory.

Using the path integral representation of the spin field effective action [2], one-loop calculations have been done in [14].[2] They show that the action (1) is to be modified by adding terms containing higher order time derivatives (which might be a source of instability of knot solitons). The one-loop renormalization group flow for the effective action parameters also indicates that the effective action might be stable in the infrared region [14]. The results of [14] seems encouraging

[2] See also [15] where different approaches, not related to the observations in lattice gauge theories, have been explored.

and deserve further studies, particularly, by some nonperturbative methods.

Since the topological defects in the gauge-fixed theory are not solutions to the classical equations motion with a finite energy or action, it is rather hard to give them an accurate mathematical meaning in continuum quantum theory. The nonlinear change of variables (2) makes sense for classical connections whose values are well-defined almost everywhere in spacetime. Typical quantum fields that form support of the path integral measure are distributions rather than classical smooth functions. The change of variables (2) is, in fact, ill-defined (because it involves products of distributions) unless some short-distance regularization is implemented. This can be achieved either by defining the path integral perturbatively with an ultraviolet cutoff, or by using the lattice (nonperturbatively defined) path integral for gauge theories. This latter approach is adopted in the Letter.

Here we develop the idea, first suggested in [2], of using the inverse Monte Carlo method [16] to find out whether (1) is indeed a good approximation to the infrared Yang–Mills theory. Within the framework of lattice Yang–Mills theory, an explicit numerical algorithm is proposed to compute and study the effective action for the spin field and the renormalization group flow of its parameters, which comprises the main goal of the Letter.

2. In lattice gauge theories, the dynamical variables are group elements $u_{x\mu} \equiv u_l \in SU(2)$ associated with each link, where x enumerates lattice sites, and μ indicates the direction of the link from the site x. Let $\{u_l\}$ be a Wilson ensemble of link variables distributed with the Boltzman probability $Z_W^{-1}e^{-S_W}$ with S_W being the Wilson action of link variables and Z_W the normalization factor (the partition function). The first step of the proposed numerical simulations is to generate an ensemble of the spin field from the Wilson ensemble. The spin field in the decomposition (2) is defined by Eq. (4). Thus, for every configuration u_l one has to find a configuration of gauge group elements $\Omega_x(u_l)$ such that the gauge transformed configuration

$$u_l^{\Omega} = \Omega_x u_l \Omega_{x+\mu}^{\dagger}, \tag{5}$$

satisfies the maximal Abelian gauge. Here $x+\mu$ denotes the lattice site next to x in the direction μ. The group elements $\Omega_x(u_l)$ can be found by maximizing the function [17]

$$\chi_u(\Omega) = \sum_l \operatorname{tr}\big[\tau_3\big(u_l^{\Omega}\big)^{\dagger}\tau_3 u_l^{\Omega}\big], \tag{6}$$

for each configuration u_l of the Wilson ensemble. The collection of group elements Ω_x at all lattice sites is regarded as variables, while u_l are just parameters. For every configuration u_l the function χ_u can have many local maxima. This is an evidence of the Gribov problem in lattice gauge theories (see for a review [12] and references therein). For every element of the Wilson ensemble u_l, one should take $\Omega_x(u)$ at which χ_u attains its absolute maximum. Finding an absolute maximum of χ_u is a difficult, if not impossible, task in the numerical gauge fixing. The state-of-the-art extrapolation toward the global maximum of χ_u can be found in [9]. The ensemble of the spin field is then computed as $\boldsymbol{n}_x(u_l) = \frac{1}{2}\operatorname{tr}(\boldsymbol{\tau}\Omega_x^{\dagger}\tau_3\Omega_x)$.

It is also possible to find the lattice version of the change of variables (2) and therefore to obtain a system of cubic equations whose solution defines the spin field components as functions of link variables (the lattice analog of the equations for the spin field suggested in [2]). Define an algebra element at each site $n_x = \Omega_x^{\dagger}\tau_3\Omega_x$ which satisfies the constraint $\operatorname{tr} n_x^2 = 2$. Combining (5) and (6) and introducing the Lagrange multiplier ξ_x to take into account the constraint on n_x, the extreme value problem for (6) is equivalent to the extreme value problem for the function

$$\tilde{\chi}_u(n) = \sum_x \Bigg\{ \sum_{\mu} \operatorname{tr}\big[n_{x+\mu}u_l^{\dagger}n_x u_l\big] + \xi_x\left(\frac{1}{2}\operatorname{tr} n_x^2 - 1\right)\Bigg\}. \tag{7}$$

Setting the variations of $\tilde{\chi}_u(n)$ with respect to n_x and ξ_x to zero, the following equation for n_x can be deduced

$$\varphi_x(n,u) \equiv \sum_{\mu}\big(u_{x-\mu,\mu}^{\dagger}n_{x-\mu}u_{x-\mu,\mu} + u_{x,\mu}n_{x+\mu}u_{x,\mu}^{\dagger}\big) + 2\xi_x n_x = 0. \tag{8}$$

Now assume that $\Omega_x(u)$ is a local maximum of (6) and, hence, n_x is a solution to (8), then u_l^{Ω} satisfies the maximal Abelian gauge and, by construction, $n_x = \boldsymbol{n}_x \cdot \boldsymbol{\tau}$. The Lagrange multiplier $\xi_x = \xi_x(u,n)$ is fixed

by multiplying (8) by n_x and taking the trace. After the substitution of $\xi_x = \xi_x(u, n)$ into (8), one gets the equation for the components of the spin field. Only two scalar equations in (8) are independent. They determine two independent components of the spin field as functions of the link variables.

As has been mentioned above, the group elements Ω_x and hence the spin field are not regular everywhere in space in the continuum theory. It is not difficult to give an example of Ω_x and the corresponding spin field $\boldsymbol{n}_x$ such that the connection $\boldsymbol{\alpha}_\mu$ coincides with the Wu–Yang monopole (e.g., take $\boldsymbol{n} = \boldsymbol{x}/r$, $r = |\boldsymbol{x}|$) which, in the maximal Abelian gauge, contains a Dirac monopole at the origin. The spin field is ill-defined at the position of the monopole. In the lattice gauge theory, the topological defects occur on the dual lattice sites [17]. For every configuration u_l, the configuration $\Omega_x(u_l)$ and, hence, $\boldsymbol{n}_x$ are well-defined and contain all the information about locations of the defects (magnetic monopoles) on the dual lattice and their topological numbers (magnetic charges). Consider an elementary cube of the spatial lattice and spins on its vertices. Frankly speaking, with an isolated defect present at the cube center, the spins are directed outward (or inward) the cube. In the confinement phase the monopole–antimonopole pairs (or monopole loops, when describing topological defects by their spacetime trajectories) are condensed (no isolated defects), therefore the above simple visual picture would not be valid. However, $\boldsymbol{n}_x$ is still well-defined at each lattice site and its dynamics can be studied.

3. The configurations of gauge fields u_l are distributed with the Boltzman probability $Z_W^{-1}e^{-S_W}$. The spin field configurations obtained from the Wilson ensemble must also be distributed with some probability $Z_s^{-1}e^{-S}$ where S is the unknown effective action of the spin field. The problem is therefore: given an ensemble of $\boldsymbol{n}_x$, find the corresponding probability or the effective action $S(\boldsymbol{n})$.

Any correlator of the spin field can be computed by the Monte Carlo method since the ensemble is known

$$\begin{aligned}\big\langle F(\boldsymbol{n})\big\rangle_n &\equiv Z_s^{-1}\int D\boldsymbol{n}\, e^{-S}F(\boldsymbol{n})\\ &= \frac{1}{M}\sum_{\{\boldsymbol{n}\}} F(\boldsymbol{n}) + \mathrm{O}\big(M^{-1/2}\big),\end{aligned} \tag{9}$$

where the sum is taken over the ensemble of the spin field, $Z_s = \int D\boldsymbol{n}\, e^{-S}$ is the partition function, $D\boldsymbol{n} = \prod_x d\boldsymbol{n}_x$ and the integration over a spin at each site implies the integration over a unit two-sphere. Parameterizing the spin vector by the spherical angles

$$\boldsymbol{n}_x = (\cos\phi_x \sin\theta_x, \sin\phi_x \sin\theta_x, \cos\phi_x), \tag{10}$$

we get $d\boldsymbol{n}_x = d\phi_x\, d\theta_x \sin\theta_x$ where $\phi_x \in [0, 2\pi)$ and $\theta_x \in [0, \pi]$. The expectation value (9) is also realized as an expectation value with respect to the original Wilson ensemble. Note that Eq. (8) defines the spin field as a function of link variables $\boldsymbol{n}_x = \boldsymbol{n}_x(u)$. Hence,

$$\big\langle F(\boldsymbol{n})\big\rangle_u = Z_W^{-1}\int Du\, e^{-S_W(u)}F\big(\boldsymbol{n}(u)\big), \tag{11}$$

where Z_W is the partition function for the Wilson action. In principle, this observation can be used to determine the effective action directly via the original Wilson ensemble. The idea is the same as in the continuum case [2]. Define the function $\Delta(u, \boldsymbol{n})$ by the condition

$$\int D\boldsymbol{n}\, \Delta(u, \boldsymbol{n}) \prod_x \delta\big(\boldsymbol{\varphi}_x(\boldsymbol{n}, u)\big) = 1, \tag{12}$$

where $\boldsymbol{\varphi}_x = \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau}\varphi_x)$ (see (8)), which leads to $\Delta(u, \boldsymbol{n}) = \det(\partial\varphi_x^a/\partial n_y^b)$. The parameterization (10) must be used to compute the derivatives and also the identity $\boldsymbol{\varphi}_x \cdot \boldsymbol{n}_x = 0$ is to be taken into account when computing the determinant. In spacetime the matrix $\partial\varphi_x^a/\partial n_y^b$ appears to be sparse because nonzero elements can only occur for $y = x$, $x \pm \mu$. Substituting the identity (12) into the integrand in (11), changing the order of integration and comparing it with (9), it is not difficult to deduce that

$$S(\boldsymbol{n}) = -\ln\left\langle \Delta(u, \boldsymbol{n}) \prod_x \delta\big(\boldsymbol{\varphi}_x(\boldsymbol{n}, u)\big)\right\rangle_u, \tag{13}$$

where $\boldsymbol{n}_x$ is now held fixed in the average over the Wilson ensemble.

Although (13) defines the spin field effective action as an expectation value of some function of the link variables, we are interested only in its behavior in the infrared region. In the continuum case this amounts to the so-called gradient expansion of the (nonlocal) effective action. In the numerical approach, the action

is sought in the form

$$S \approx \sum_i \lambda_i S_i, \tag{14}$$

where S_i are some specified functions of the spin field, whereas the coupling constants λ_i are to be determined. Since we are interested to compare the effective action with (1), we first find all possible local and independent terms that are of the forth order in derivatives and might contribute to the gradient expansion of S. In addition to the two terms in (1) it is necessary to include $\partial_\mu^2 \boldsymbol{n} \cdot \partial_\nu^2 \boldsymbol{n}$ and $[(\partial_\mu \boldsymbol{n})^2]^2$. The four terms are all independent Lorentz and isotopic invariant terms containing up to four derivative operators. In fact, there is one more invariant term which can be built by contracting the Lorentz tensor $\partial_\mu \boldsymbol{n} \times \partial_\nu \boldsymbol{n}$ with its dual (like a θ-term in the Yang–Mills action). Higher order terms can be classified accordingly by contracting invariant irreducible tensors of the isotopic and Lorentz groups with isotopic tensor products of the spin field and its derivatives. Then the renormalization group flow of the constants λ_i must be studied as high momentum components of the spin field are removed (integrated out). The renormalization group flow would show whether or not the effective action is stable (or, in other words, is a good approximation to (13)) in the infrared limit, and thereby validate or invalidate the conjecture.

We set

$$S_i = \frac{1}{m_i} \sum_x S_{ix}, \tag{15}$$

where m_i is the number of spins involved into a local interaction S_{ix}. Let $\boldsymbol{s}_{x,\mu} = \boldsymbol{n}_{x+\mu} - \gamma_{x\mu} \boldsymbol{n}_x$, where $\gamma_{x\mu} = \boldsymbol{n}_{x+\mu} \cdot \boldsymbol{n}_x$ is defined by the condition $\boldsymbol{s}_{x,\mu} \cdot \boldsymbol{n}_x = 0$ to make the correspondence with the continuum theory $\partial_\mu \boldsymbol{n} \cdot \boldsymbol{n} = 0$. Let a be a lattice spacing. The local spin interactions are written as

$$S_{1x} = a^2 \sum_\mu \big[\boldsymbol{s}_{x,\mu}^2 + \boldsymbol{s}_{x-\mu,\mu}^2\big], \tag{16}$$

$$S_{2x} = \left[\sum_\mu \boldsymbol{s}_{x,\mu}^2\right]^2 + \left[\sum_\mu \boldsymbol{s}_{x-\mu,\mu}^2\right]^2, \tag{17}$$

$$S_{3x} = \sum_{\mu,\nu} \big[(\boldsymbol{s}_{x,\mu} \times \boldsymbol{s}_{x,\nu})^2 + (\boldsymbol{s}_{x-\nu,\mu} \times \boldsymbol{s}_{x-\nu,\nu})^2 + (\boldsymbol{s}_{x-\mu,\mu} \times \boldsymbol{s}_{x-\mu,\nu})^2\big], \tag{18}$$

$$S_{4x} = \left[\sum_\mu (\boldsymbol{s}_{x,\mu} - \boldsymbol{s}_{x-\mu,\mu})\right]^2 + \big[x \to x' = x + \mu\big] + \big[x \to x' = x - \mu\big]. \tag{19}$$

In the continuum limit, the action (15) goes into (1) with the additional terms described above. The θ-term has the same form as S_{3x} where instead of the sum of squares, the sum of the dot products of each vector and its Lorentz dual has to be taken. Local spin interactions giving rise to terms with higher powers of ∂_μ in the continuum limit can be constructed similarly by using the correspondence rule: $\partial_\mu \boldsymbol{n} \to \boldsymbol{s}_{x,\mu}$, $2\partial_\mu \partial_\nu \boldsymbol{n} \to \boldsymbol{s}_{x+\mu,\nu} - \boldsymbol{s}_{x,\nu} + \boldsymbol{s}_{x+\nu,\mu} - \boldsymbol{s}_{x,\mu}$, etc.

Note that two terms in S_{1x} give the same contribution as the sum over x is taken in (15) and therefore $m_1 = 2$. Similarly, each of three terms in S_{ix} $(i = 2, 3, 4)$ gives the same contribution to (15) and $m_i = 3$. The reason the equivalent term are given in S_{ix} is that S_{ix} is the part of the action S_i that contains all terms involving the spin $\boldsymbol{n}_x$ at a fixed site x. This representation will be useful in what follows.

4. Here we formulate the inverse Monte Carlo algorithm for computing λ_i. The inverse Monte Carlo method is well-known in studies of the real space renormalization group of spin systems [16]. It has also been applied to compute an effective action for monopole currents in the maximal Abelian projection [18]. The use of the spin field order parameter $\boldsymbol{n}_x$ rather than the monopole current is more appealing because of several reasons (relations to the quantum soliton theory, similarities between strings and knot solitons) pointed out after (1).

Let S_x denote all terms in S that contain the spin $\boldsymbol{n}_x$ at a fixed site x, $S_x = \sum_i \lambda_i S_{ix}$. For every $S_i(\boldsymbol{n})$ we construct a new function

$$\overline{S_i}(\boldsymbol{n}, \lambda) = \frac{1}{m_i} \sum_x Z_x^{-1} \int d\boldsymbol{n}_x \, e^{-S_x} S_{ix} \equiv \sum_x \overline{S}_{ix}, \tag{20}$$

where $Z_x = Z_x(\boldsymbol{n}) = \int d\boldsymbol{n}_x \, e^{-S_x}$. The bar in $\overline{S}_{ix}$ denotes an expectation value carried out with respect to the effective action $\langle \cdots \rangle_n$ but calculated for only one spin, $\boldsymbol{n}_x$. The environment (i.e., neighboring spins) is held fixed. So, $\overline{S}_{ix}$ depends only on the spins at the sites neighboring with x, i.e., on $\boldsymbol{n}_{x\pm\mu}$ and

$\boldsymbol{n}_{x\pm 2\mu}$. Taking the expectation value of $\overline{S}_i$, we find the identity

$$\langle S_i(\boldsymbol{n})\rangle_u = \langle \overline{S}_i(\boldsymbol{n},\lambda)\rangle_u. \tag{21}$$

Using the Monte Carlo method (9), the l.h.s. of (21) can be computed, while the r.h.s. cannot. The integral over $\boldsymbol{n}_x$ in (20) cannot be computed for given configurations of neighboring spins because the true values of λ_i are not known. Had the coupling constants been known, the ordinary integral in (20) could have been computed, for instance, numerically for any given $\boldsymbol{n}_{x\pm\mu}$ and $\boldsymbol{n}_{x\pm 2\mu}$.

Suppose some trial values $\tilde{\lambda}_i$ of the coupling constants are taken to compute the r.h.s. of (20). The equality

$$\langle \overline{S}_i(\boldsymbol{n},\lambda)\rangle_u = \langle \overline{S}_i(\boldsymbol{n},\tilde{\lambda})\rangle_u, \tag{22}$$

holds if and only if $\lambda_i = \tilde{\lambda}_i$. This is used to set up an iterative algorithm to find the true coupling constants. Eq. (22) is regarded as a system of nonlinear equations where the l.h.s. is known (cf. (21)). It can be solved numerically by Newton's method or some of its alterations. For $\tilde{\lambda}_i \approx \lambda_i$ we have

$$\begin{aligned}&\langle S_i\rangle_u - \langle \overline{S}_i(\boldsymbol{n},\tilde{\lambda})\rangle_u \\ &\quad\approx \sum_j \left\langle \frac{\partial}{\partial\lambda_j}\overline{S}_i(\boldsymbol{n},\lambda)\right\rangle_u\Big|_{\lambda=\tilde{\lambda}} (\lambda_j - \tilde{\lambda}_j).\end{aligned} \tag{23}$$

Using the definition (20), it is not difficult to show that

$$\frac{\partial}{\partial\lambda_j}\overline{S}_i = \overline{S}_j\overline{S}_i - \overline{S_jS_i}. \tag{24}$$

The function $\overline{S_jS_i}$ is defined by (20) where S_{ix} is replaced by $S_{jx}S_{ix}$. The true values of the coupling constants are computed by the iterating procedure

$$b_i(\lambda^{(n)}) = \sum_j A_{ij}(\lambda^{(n)})(\lambda_j^{(n+1)} - \lambda_j^{(n)}), \tag{25}$$

$$b_i(\lambda^{(n)}) = \langle S_i(\boldsymbol{n})\rangle_u - \langle \overline{S}_i(\boldsymbol{n},\lambda^{(n)})\rangle_u, \tag{26}$$

$$A_{ij}(\lambda^{(n)}) = \langle \overline{S}_i\overline{S}_j - \overline{S_iS_j}\rangle_u\big|_{\lambda=\lambda^{(n)}}, \tag{27}$$

where $\lambda_i^{(0)} = \tilde{\lambda}_i$ and $\lambda_i^{(n)} \to \lambda_i$ as $n\to\infty$.

The convergence depends on the choice of the trial constants $\tilde{\lambda}_i$. If iterations take many cycles, statistical errors are likely to introduce instabilities in the solution. A similar problem was encountered in [19]. The solution there was to find the iteration limit through a linear mapping of the space of λ_i. Another method to compute the coupling constants is to use the Schwinger–Dyson equations [20]. In principle, the coupling constants can be compared with their "exact" values defined through (13). The expectation value in the r.h.s. of (13) can be expanded into a series over the spin field around some specific spin field configuration. The expansion coefficients can be computed my the Monte Carlo method in the Wilson ensemble. For instance, the mass scale λ_1 can be obtained by taking the second derivative of (13) with respect to the spin field at the particular configuration $n_x^a = \delta^{3a}$ (as was suggested in the continuum case [2]). This procedure involves, however, computations of the determinant, which is very costly. Eq. (11) can be used to measure the goodness of the approximation (15)–(19).

5. Having found the coupling constants, the renormalization group flow for them has to be investigated to prove the stability of the effective action in the limit of large wave lengths. With this purpose, we use the representation (10) to take into account the constraint $\boldsymbol{n}_x^2 = 1$. Let the matrix f_{xk} be a discrete Fourier transform associated with the lattice, $\sum_x f^*_{k'x}f_{xk} = \delta_{k'k}$ and $\sum_k f^*_{x'k}f_{kx} = \delta_{x'x}$. The sum over k implies the sum over all momentum vectors allowed by the lattice. Given the ensemble of θ_x and ϕ_x, the Fourier components θ_k and ϕ_k can be computed.

Next the spin field ensemble can be generated for all momenta bounded from above by some scale Λ_1

$$\theta_x(\Lambda_1) = \sum_{k\in K_1} f_{xk}\theta_k. \tag{28}$$

Similarly for $\phi_x(\Lambda_1)$. The sum in (28) is extended over those vectors k whose norm does not exceed the scale Λ_1. This subset in the momentum space is denoted K_1. The ensemble $\boldsymbol{n}_x(\Lambda_1)$ can be used as the input for the inverse Monte Carlo procedure described in the previous section to compute new coupling constants $\lambda_i(\Lambda_1)$.

Repeating this procedure for successively smaller scales $\Lambda_{k+1} < \Lambda_k$ we can obtain the sequence of the coupling constants $\lambda_i(\Lambda_k)$, $k = 0, 1, 2, \ldots$, where $k = 0$ corresponds to the coupling constants computed with the original spin field ensembles. By truncating the sum over momenta in (28) we generate a spin field ensemble in the infrared region (large wave lengths). Hence the sequence $\lambda_i(\Lambda_k)$ describes the behavior

of the coupling constants as functions of the scale Λ that restricts allowed momenta from above in the effective theory, i.e., as Λ decreases, the infrared limit is approached.

The explicit removal of Fourier modes can be strongly affected by the breaking of rotational symmetry on coarse lattices or for larger Λ's. So, the block spin decomposition [16,21] might be a more attractive procedure to study the renormalization group flow. The idea is to average spins over elementary cells (blocks) of the original lattice. For instance, the angular variables θ_x and ϕ_x are specified at elementary cubic cell vertices. Consider a new lattice with mesh $2a$ which is constructed as follows. Let the point y be the center of the elementary cell. The neighboring sites are then $y \pm 2\mu$. So, each site y of the new lattice is inside an elementary cube C_y of the original lattice, and the cubes C_y and $C_{y'}$ do not have common vertices if $y \neq y'$, while C_y and $C_{y'}$ coincide if $y = y'$. Define

$$\theta_y = 2^{-D-1} \sum_{x \in C_y} \theta_x, \tag{29}$$

and similarly for ϕ_y, where D is the lattice dimension. That is, θ_y is an average value of θ over all vertices of one elementary cube. The spin field $\boldsymbol{n}_y$ is defined by (10) where $x \to y$. The averaging (29) is also equivalent to removing short wave length components of the spin field. Doing this procedure for a successively larger lattice spacing ($2a$, $4a$, etc.) and computing the coupling constants on each step, we can again generate the renormalization group flow $\lambda_i(\Lambda_k)$ (where $\Lambda_k \sim 2^{-k}/a$).

The behavior of $\lambda_i(\Lambda)$ allows one to verify whether the effective action (1) (possibly with extra terms) is stable in the infrared region as was observed in [14] in the one-loop approximation. For instance, it is critical to observe the right signs of $\lambda_{1,3}$ (cf. (1)) because λ_1 sets the mass scale for knot solitons, while λ_3 should reproduce the running gauge coupling constant g in the continuum limit. Relatively large (and growing) values of $\lambda_{2,4}$ would mean instability of knot solitons. It would also indicate that the approximation (1) is not justified and the higher order spin interactions are relevant for the spin field dynamics. In this case more higher order terms have to be included into (15) and the renormalization group flow of the corresponding coupling constants is to be computed.

It should also be noted that the lattice size becomes important in studies of $\lambda_i(\Lambda)$. It is clear that $\Lambda_{\max} = \Lambda_0 \sim 1/a$ and $\Lambda_{\min} \sim 1/L$ where a is a lattice mesh and $L = Na$ with N being the number of lattice sites in one direction. Values of Λ cannot be taken too close to $\Lambda_{\min}$ because the characteristic scaling behavior of the dimensional λ_i in the continuum limit [22] must still be observable.

Let us summarize the essential steps of the algorithm. First, a Wilson ensemble of the link variables is generated. Then the spin field ensemble is computed. For every element of the ensemble the functions $\overline{S_i}$ are computed for some trial coupling constants. The integral over a spin field at a fixed site x involved in the definition of $\overline{S_i}$ has to be done numerically. Then the iterating algorithm is applied to compute the coupling constants. Finally, the procedure is repeated for several ensembles of the spin field which are obtained from the original ensemble by truncating (integrating out) short wave length components. This gives the renormalization group flow of the coupling constants which can tell us about the stability of the effective action in the infrared limit. Using an appropriate scaling of λ_1 in the continuum limit, one can compute the mass scale of knot solitons supported by the effective action (1), while the coefficients at the additional terms would determine their stability.

As a final remark, it should be noted that the choice of the spin field ensemble suggested above is associated with the maximal Abelian gauge. It might be interesting to investigate the effective action if different Abelian gauges are used to obtain the spin field ensemble. The purpose of such a study is to seek a numerical evidence that the effective dynamics of Yang–Mills fields in the infrared region is indeed governed by configurations that exhibit topological defects when taken in any Abelian gauge [23]. A generalization of (8) for the spin field ensemble can be defined as follows. Let $\Omega_x(u)$ be a gauge transformation such that u_l^{Ω} satisfies some Abelian gauge. The latter implies in particular that there exists a field defined at each x, $\overline{\Phi}_x = \overline{\Phi}_x(u^{\Omega})$ which is diagonal $[\tau_3, \overline{\Phi}_x] = 0$ and also $\overline{\Phi}_x(u^{\Omega}) = \Omega_x^{\dagger} \Phi_x(n,u) \Omega_x$. Therefore, a general equation for the spin field reads

$$\big[\Phi_x(n,u), n_x\big] = 0, \tag{30}$$

where Φ_x is any function of n_x and u_l which transforms according to the adjoint representation of the

gauge group. An example of the field $\Phi_x(n,u)$ is given by the first two terms in Eq. (8), and Eq. (8) itself is clearly equivalent to (30). Link variables in two different Abelian gauges are related to each other by a singular gauge transformation. Eq. (30) is covariant under such transformations and, hence, represent the most general equation for the spin field components.

The simplest class of Abelian gauges is obtained by taking Φ_x to be independent of the spin field. In this case, the spin field is identified as a solution of (30): $n_x = \Phi_x(u)/|\Phi_x(u)|$ where the vertical bars mean the norm with respect the trace scalar product in the Lie algebra. This is the case of the Polyakov Abelian gauge [24] and the so-called Laplacian Abelian gauge [25]. The spin field ensemble in the case of the Polyakov Abelian gauge is probably the simplest one to generate. In fact it is possible to average over all such choices of Φ in the Yang–Mills path integral (dynamical Abelian gauge [26]) by using a supersymmetric extension of the theory.

The $SU(N)$ generalization of (30) is also straightforward. Take $N-1$ orthonormal and commutative spin fields, $(n_x^a, n_x^b) = \delta^{ab}$ and $[n_x^a, n_x^b] = 0$ where $a, b = 1, 2, \ldots, N-1$. They should satisfy the equation $[\Phi_x(n,u), n_x^a] = 0$ where Φ_x is some operator that transforms under the adjoint representation. In the simplest case when Φ_x depends only on the link variables, the solution for the spin fields is obtained by orthogonalizing the commutative fields $\Phi_x^a \sim d^{(a)}_{i_1 i_2 \cdots i_{a+1}} e^{i_1} \Phi_x^{i_2} \cdots \Phi_x^{i_{a+1}}$ where e^i is an orthogonal basis in the Lie algebra $su(N)$, $\Phi_x = e^i \Phi_x^i$ and $d^{(a)}_{i_1 i_2 \cdots i_{a+1}}$ are the $N-1$ symmetric irreducible invariant tensors of $su(N)$. Since for any Lie algebra element Φ_x it always possible to construct $N-1$ linearly independent elements Φ_x^a (using the tensors $d^{(a)}$) that commute with Φ_x and amongst each other, the spin field can also be specified by $[\Phi_x^b(n,u), n_x^a] = 0$ for some choice of $\Phi_x^b(n,u)$. A generalization of Eq. (8) to the $SU(N)$ case is simple. One has to replace n_x by n_x^a in the first two terms (this defines $\Phi_x^a(n,u)$) and $\xi_x n_x$ by $\sum_b \xi_x^{ab} n_x^b$ in the third one where $\xi_x^{ab} = (\Phi_x^a, n_x^b)$.

Note added

After submission of the paper for publication, I have learned that the group of A. Wipf have done numerical simulations on a 16^4 lattice [27]. The mass gap in the spin field spectrum and possible global $SO(3)$ symmetry breaking (a preferable direction of the spin field) have been reported so far. The existence of the mass gap around the lowest glueball mass should certainly be expected because of the Abelian (and monopole) dominance. The interesting question of the renormalization group flow of the coupling constants is still to be studied.

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References

[1] L.D. Faddeev, A.J. Niemi, Phys. Rev. Lett. 82 (1999) 1624; L.D. Faddeev, A.J. Niemi, Phys. Lett. B 449 (1999) 214.
[2] S.V. Shabanov, Phys. Lett. B 458 (1999) 322; and also, S.V. Shabanov, hep-th/0004135.
[3] S.V. Shabanov, Phys. Lett. B 463 (1999) 263.
[4] L.D. Faddeev, A.J. Niemi, Phys. Lett. B 464 (1999) 90.
[5] L.D. Faddeev, Quantization of solitons, IAS preprint, IAS-75-QS70, Princeton, 1975; L.D. Faddeev, A.J. Niemi, Nature 378 (1997) 58; J. Gladikowski, M. Hellmund, Phys. Rev. D 56 (1997) 5194; R.A. Battye, P.M. Sutcliffe, Phys. Rev. Lett. 81 (1998) 4798; J. Hietarinta, P. Salo, Phys. Lett. B 451 (1999) 60.
[6] L. Kapitansky, A. Vakulenko, Sov. Phys. Dokl. 24 (1979) 433; R.S. Ward, Nonlinearity 12 (1999) 241.
[7] T. Suzuki, I. Yotsuyanagi, Phys. Rev. D 42 (1990) 4257; J.D. Stack, S.D. Neiman, R.J. Wensley, Phys. Rev. D 50 (1994) 3399.
[8] Various aspects of Abelian projections in lattice gauge theories are discussed by M.N. Chernodub, M.I. Polikarpov, in: P. van Baal (Ed.), Confinement, Duality, and Nonperturbative Aspects of QCD, NATO Adv. Sci. Inst. Ser. B Phys., Vol. 368, Plenum Press, London, 1998, p. 378; R.W. Haymaker, Phys. Rep. 315 (1999) 153.
[9] G. Bali, V. Bornyakov, M. Mueller-Preussker, K. Schilling, Phys. Rev. D 54 (1996) 2863.
[10] G. 't Hooft, Nucl. Phys. B 190 (1981) 455.

[11] I.M. Singer, Commun. Math. Phys. 60 (1978) 7.
[12] S.V. Shabanov, Phys. Rep. 326 (2000) 1.
[13] Y.M. Cho, Phys. Rev. D 21 (1980) 1080;
Y.M. Cho, Phys. Rev. D 23 (1981) 2415.
[14] H. Gies, Phys. Rev. D 63 (2001) 125023, hep-th/0102026.
[15] E. Langmann, A.J. Niemi, Phys. Lett. B 463 (1999) 252;
Y.M. Cho, H. Lee, D.G. Pak, hep-th/9905215.
[16] R.H. Swendsen, Phys. Rev. Lett. 52 (1984) 1165;
R.H. Swendsen, Phys. Rev. B 30 (1984) 3866;
M. Cruetz, A. Gocksch, M. Ogilvie, Phys. Rev. Lett. 53 (1984) 875.
[17] A.S. Kronfeld, G. Schierholtz, U.J. Wiese, Nucl. Phys. B 293 (1987) 461.
[18] H. Shiba, T. Suzuki, Phys. Lett. B 343 (1995) 315;
H. Shiba, T. Suzuki, Phys. Lett. B 351 (1995) 519.
[19] M. Alford, I.T. Drummond, R.R. Horgan, M. Peardon, H. Shanahan, Phys. Rev. D 63 (2001) 074501.
[20] A. Gonzalez-Arroyo, M. Okawa, Phys. Rev. D 35 (1987) 672.
[21] S.M. Catterall, M. Hasenbusch, R.R. Horgan, R.L. Renken, Phys. Rev. D 58 (1998) 074510.
[22] M. Teper, hep-th/9812187.
[23] A. Di Giacomo, B. Lucini, L. Montesi, G. Pafutti, Nucl. Phys. Proc. Suppl. 65 (1998) 540;
A. Di Giacomo, B. Lucini, L. Montesi, G. Pafutti, Phys. Rev. D 61 (2000) 034503.
[24] N. Weiss, Phys. Rev. D 24 (1981) 457.
[25] A.J. van der Sijs, Nucl. Phys. Proc. Suppl. 53 (1997) 535;
L. Del Debbio et al., Phys. Rev. D 58 (1998) 094501.
[26] S.V. Shabanov, Mod. Phys. Lett. A 11 (1996) 1081, hep-th/9611228;
S.V. Shabanov, Nucl. Phys. Proc. Suppl. 54A (1997) 213.
[27] L. Dittmann, T. Heinzl, A. Wipf, hep-lat/0110026.