Local spectral gaps on loop spaces

Andreas Eberle *

Mathematical Institute, 24–29 St. Giles, Oxford OX1 3LB, UK

Received 23 March 2002

Abstract

We prove Poincaré inequalities w.r.t. the distributions of Brownian bridges on sets of loops with jumps of limited size over compact Riemannian manifolds. Moreover, we study the asymptotic behavior of the second Dirichlet eigenvalues as the time parameter $T$ of the underlying Brownian bridge tends to 0. This behavior depends crucially on the geodesics contained in the set of loops considered. In particular, for different choices of a Riemannian metric on the base manifold, qualitatively different asymptotic behaviors can occur. The proof of the basic Poincaré inequality is based on the construction of the Brownian bridge by consecutive bisection of the parametrization interval.

© 2003 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

Keywords: Loop space; Brownian bridge; Poincaré inequality; Geodesic
1. Introduction and main results

1.1. Introduction

Let $M$ be a compact connected and simply connected Riemannian manifold, and let $x \in M$. In [23], S. Fang proved a Poincaré inequality of type

$$\text{Var}(F; P_x) \leq C \cdot \int_{\Omega_x} (DF, DF)_ω P_x(dω)$$

(1.1)

on the based path space $\Omega_x = \{ω \in C([0, 1], M); \ ω(0) = x\}$. Here $P_x$ is the distribution on $\Omega_x$ of an $M$-valued Brownian motion starting at $x$, $\text{Var}(F; P_x) := \int (F - \int F dP_x)^2 dP_x$ denotes the variance of a function $F$ w.r.t. $P_x$, $(\cdot, \cdot)$ is an $H^1$ type metric on $\Omega_x$, and $D$ is the corresponding gradient operator, cf., e.g., the lecture notes [27] by E. Hsu, or cf. below for the precise definitions of corresponding objects on pinned path spaces.

The inequality (1.1) holds for all smooth cylinder functions $F$ on $\Omega_x$ with a joint finite constant $C$ that depends only on bounds of the Ricci curvature on the base manifold $M$.

It is equivalent to a spectral gap of size $\lambda = 1/C$ between the first (zero) eigenvalue and the rest of the spectrum of the corresponding self-adjoint operator $D^*D$ on $L^2(\Omega_x; P_x)$. Fang’s original proof is based on a manifold version of the Clark–Ocone formula. Later, a logarithmic Sobolev inequality on $\Omega_x$ (which is stronger than (1.1)) has been proven by different methods in [3,5,26]. The proof given by E. Hsu in [26] relies crucially on the Markov property of Brownian motion, the proof in [3] uses an embedding of $M$ into an Euclidean space, and the arguments used in [5] extend those in [23].

From the geometrical point of view, analysis on loop spaces over Riemannian manifolds seems more interesting than on path spaces, cf., e.g., [39,42]. On the loop space $Λ_x = \{ω \in Ω_x; \ ω(1) = x\}$, the measure $P_x$ should be replaced by the distribution of a Brownian bridge, which is obtained from $P_x$ by restriction to the submanifold $Λ_x \subset Ω_x$.

Attempts to prove an inequality similar to (1.1) on $Λ_x$ did only yield considerably modified estimates, cf., e.g., [11,24]. In fact, it has been shown in [15] that the natural counterpart of the inequality (1.1) on $Λ_x$ does not hold if $M$ contains a closed geodesic $γ : S^1 \to M$ such that the curvature is constant and strictly negative on a neighborhood of $γ(S^1)$, and $x$ is close to $γ(S^1)$. In particular, for every compact differentiable manifold $M$ and $x \in M$, we can choose a Riemannian metric on $M$ such that a global Poincaré inequality on $Λ_x$ does not hold.

If we do not want to modify the framework or restrict to a special class of Riemannian manifolds then we will be able to prove Poincaré inequalities, spectral gaps respectively, only on appropriate subsets of loop spaces. Estimates for spectral gaps on a sequence of subsets that exhaust the full space could then give information on the global geometry of a loop space and the underlying Riemannian manifold respectively. The aim of this work is to prove such local Poincaré inequalities, and to derive first estimates on the local spectral gaps. These estimates are asymptotic statements as the time parameter of the underlying Brownian bridge goes to 0. They enable us to work out more clearly relations between the geometry of the base manifold and Poincaré type inequalities on loop spaces, which were
indicated by the results in [15]. In particular, we will distinguish between three different classes of loop spaces with qualitatively different behavior of the local spectral gaps, cf. Corollary 1.5 and the comments and examples below.

Our method of proof is partially motivated by E. Hsu’s original proof of the logarithmic Sobolev inequality on a based path space. However, instead of viewing the Brownian bridge as a Markov process in time in the usual way, our proofs rely on the construction of the bridge by consecutive bisection of the parametrization interval $[0, 1]$. The Brownian bridge can thus be viewed as a kind of Markov chain with growing state space $M^{2^n}, n \in \mathbb{N}$. This allows us to apply techniques for estimating spectral gaps w.r.t. the laws of Markov chains. To obtain a strictly positive lower bound for the spectral gap in the limit $n \to \infty$, some highly nontrivial variance estimates for functionals on Wiener space that have been proven by Malliavin and Stroock [33] play a crucial rôle.

1.2. Framework

We now describe the framework needed to state our results in detail. For more background information on the problems considered in this article, we refer to [14] and the references therein. For more background on the framework see, e.g., [3, 8–10, 26, 27, 32].

We fix a compact connected Riemannian manifold $M$ and $x, y \in M$. Simply connectedness is not required in the moment. We consider the pinned path space

$$\Omega_{x,y} = \{\omega \in C([0, 1], M); \omega(0) = x, \omega(1) = y\}$$

endowed with the supremum distance,

$$d_\infty(\omega, \sigma) = \sup_{s \in [0, 1]} d(\omega(s), \sigma(s)).$$

In particular, $\Lambda_x = \Omega_{e,x}$. The global injectivity radius $\text{inj}(M)$ is the infimum of all $r \geq 0$ such that the exponential map $\exp_x$ is a diffeomorphism from the ball $B(0, r)$ in the tangent space $T_xM$ onto the ball $B(x, r)$ in $M$ for all $x \in M$. Since $M$ is compact, $\text{inj}(M) > 0$. We fix $R \in (0, \text{inj}(M))$. We want to derive estimates for second Dirichlet eigenvalues on the sets

$$\Omega_{x,y}^{R,N} = \{\omega \in \Omega_{x,y}; \max_{i=0,1,\ldots,N-1} \sup_{s,t \in [(i/N,(i+1)/N)]} d(\omega(s), \omega(t)) < R\}, \quad (1.2)$$

$N \in \mathbb{N}$. Clearly, $\Omega_{x,y} = \bigcup_{N \in \mathbb{N}} \Omega_{x,y}^{R,N}$.

For $T > 0$, let $P^T_{x,y}$ denote the distribution on $\Omega_{x,y}$ of the $M$-valued Brownian bridge from $x$ to $y$ in time $T$, reparametrized to the interval $[0, 1]$. Hence $P^T_{x,y}$ is the unique probability measure on the Borel $\sigma$-algebra of $\Omega_{x,y}$ such that

$$\int f(\omega(s_1), \omega(s_2), \ldots, \omega(s_k)) P^T_{x,y}(d\omega)$$
\[
\frac{1}{p_T(x, y)} \int_{M^k} f(x_1, x_2, \ldots, x_k) p_{x_1}^T(x_1, x_1) p_{x_2-x_1}^T(x_1, x_2) \times \cdots \times p_{x_k-x_{k-1}}^T(x_{k-1}, x_k) p_{x_k}^T(x_k, y) \prod_{i=1}^{k} V(dx_i)
\]

holds for all \(k \in \mathbb{N}, f \in C^\infty(M^k),\) and \(0 < s_1 < s_2 < \cdots < s_k < 1.\) Here \(p_t(x, y)\) denotes the heat kernel of \(\Delta\) on \(M.\)

For \(\omega \in \Omega_{x,y},\) the tangent space \(T_{\omega} \Omega_{x,y}\) to the Banach manifold \(\Omega_{x,y}\) consists of all continuous vector fields \(X : [0, 1] \to TM\) along \(\omega\) such that \(X_0 = 0\) and \(X_1 = 0.\) To introduce an \(H^1\) metric on \(\Omega_{x,y},\) the covariant derivative along a path \(\omega\) has to be defined first. The usual parallel transport, and hence also the usual covariant derivative, are only defined along absolutely continuous paths on \(M,\) but these paths form a set of measure 0 w.r.t. each of the measures \(P_{T_{x,y}}^T, T > 0.\) On the other hand, every Brownian bridge is a semimartingale (w.r.t. the augmentation of the filtration generated by the process), and thus there is a well-defined notion of stochastic parallel transport along its paths, cf., e.g., [22]. In fact, a joint version of the stochastic parallel transports w.r.t. all the bridges can be constructed in the following way: all paths in \(C(0, 1, M)\) are approximated by piecewise minimal geodesic interpolations adapted to a subsequence of the sequence of dyadic partitions of the interval \([0, 1],\) and the limit of the corresponding parallel transports is taken. The set \(\Omega_{x,y}^{reg}\) of all \(\omega \in C([0, 1], M)\) such that this limit exists is a measurable subset of \(C([0, 1], M).\) For every \(\omega \in \Omega_{x,y}^{reg,}\) the limit of the parallel transports along minimal geodesics yields a continuous family of isometries

\[
\tau_s(\omega) : T_{\omega(0)}M \to T_{\omega(s)}M, \quad 0 \leq s \leq 1.
\]

Moreover, the sequence of partitions can be chosen in such a way that for all \(a, b \in M,\)

\[
P_{a,b}^{T_{x,y}}[\Omega_{x,y} \cap \Omega_{x,y}^{reg}] = 1,
\]

and the process \((\tau_s)_{0 \leq s \leq 1}\) under \(P_{a,b}^T\) is a version of the stochastic parallel transport w.r.t. the corresponding Brownian bridge.\(^1\)

From now on we fix \(\Omega_{x,y}^{reg,}\) and \((\tau_s)_{0 \leq s \leq 1}\) as above. We set \(\Omega_{x,y}^{reg} := \Omega_{x,y} \cap \Omega_{x,y}^{reg}.\) For \(s, t \in [0, 1]\) and \(\omega \in \Omega_{x,y}^{reg}\) let

\[
\tau_{s,t}(\omega) := \tau_t(\omega)\tau_s(\omega)^{-1},
\]

\(^1\) In fact, let \(P_{a,b}^T\) denote the distribution on \(C([0, 1], M)\) of Brownian motion on \(M\) starting at \(a,\) speeded up with factor \(T^{-1}\). Then we can find a subsequence of the sequence of dyadic partitions such that the parallel transports along the corresponding piecewise minimal geodesic interpolations of \(\omega\) converge uniformly to the stochastic parallel transport for \(P_{a,b}^T\) a.e. path \(\omega \in C([0, 1], M)\) for every \(a \in M,\) cf. Chapter VI, Theorem 7.3 in [29] and (8.15) in [22]. Since the measures \(P_{a,b}^T\) and \(p_{T_{a,b}}^T\) are equivalent on the \(\sigma\)-algebra generated by \(\omega \mapsto \omega(s), 0 \leq s \leq 1/2,\) the same approximations of the stochastic parallel transport w.r.t. \(P_{a,b}^T\) converge uniformly on \([0, 1/2]\) \(P_{a,b}^T\)-a.s. for all \(a, b \in M.\) Since the time reversal of \(P_{a,b}^T\) is \(P_{b,a}^T\), we can use a similar procedure to find a subsequence of the already chosen subsequence of partitions such that the approximations to the stochastic parallel transport converge also uniformly on \([1/2, 1]\) \(P_{a,b}^T\)-a.s. for all \(a, b \in M.\)
which is an isometry from $T_{\omega(s)}M$ to $T_{\omega(t)}M$. For $\omega \in \Omega_{x,y}^\reg$, a continuous vector field $X \in T_{\omega}\Omega_{x,y}$ is called absolutely continuous if $s \mapsto \tau_{s,t}(\omega)X_s$ is an absolutely continuous function with values in the vector space $T_{\omega(t)}M$ for some, or, equivalently, all $t \in [0, 1]$. If $X$ is absolutely continuous, an integrable vector field $\nabla X/\text{d}s$ along $\omega$ is defined by:

$$\nabla X/\text{d}s(s) = \frac{d}{de} (\tau_{s+e,s}(\omega)X_s)\bigg|_{e=0} \text{ for a.e. } s \in [0, 1].$$

The $H^1$ tangent space to $\Omega_{x,y}$ at $\omega$ is the Hilbert space

$$T^1_{\omega} \Omega_{x,y} = \{ X \in T_{\omega} \Omega_{x,y}; \ (s \mapsto \tau_{s,0}(\omega)X_s) \in H^{1,2}(0, 1; T_0M) \}$$

with inner product

$$(X, Y)_\omega = \int_0^1 \left( \frac{\nabla X}{\text{d}s}(s), \frac{\nabla Y}{\text{d}s}(s) \right)_{\omega(s)} \text{d}s.$$

Notice that if $\omega$ is smooth, then $\tau$ is the usual parallel transport, $\nabla/\text{d}s$ is the standard covariant derivative, and $T^1_{\omega} \Omega_{x,y}$ is the tangent space to the Hilbert manifold of $H^1$ paths from $x$ to $y$, cf. [30].

Let $\mathcal{F}C^\infty$ denote the space of all smooth cylinder functions $F : C([0, 1], M) \rightarrow \mathbb{R}$ of type

$$F(\omega) = f(\omega(s_1), \omega(s_2), \ldots, \omega(s_k))$$

for some $k \in \mathbb{N}$, $f \in C^\infty(M^k)$, and $s_1, \ldots, s_k \in [0, 1]$. For $\omega \in C([0, 1], M)$, a continuous vector field $X$ along $\omega$, and $F \in \mathcal{F}C^\infty$, the directional derivative $XF$ is given by:

$$XF = \sum_{i=1}^k (X(s_i) f)(\omega(s_1), \ldots, \omega(s_k)).$$

where $X(s_i)$ denotes the application of the derivative in direction $X_s$ to the $i$th component on $M^k$. For $F \in \mathcal{F}C^\infty$ and $\omega \in \Omega_{x,y}$, the $H^1$ gradient $D^0 F(\omega) \in T^1_{\omega} \Omega_{x,y}$ is defined by:

$$(D^0 F(\omega), X)_\omega = X F \text{ for all } X \in T^1_{\omega} \Omega_{x,y}.$$
for $0 \leq t \leq 1$ and $\omega \in \Omega_{x,y}^{\text{reg}}$. Here $\text{grad}^{(i)}$ denotes the application of the gradient to the $i$th component on $M^k$, and $G_0^0(s,t) = s \wedge t - s \cdot t$, $s,t \in [0,1]$, is the Green’s function of the operator $-d^2/ds^2$ with Dirichlet boundary conditions on $(0,1)$. In particular,

$$\left(D^0 F(\omega), D^0 G(\omega)\right)_\omega = \sum_{i,j=1}^k G_0^0(s_i,s_j) \cdot \langle \tau_{s_i,s_j}(\omega) \text{grad}^{(i)} f(\omega(s_1), \ldots, \omega(s_k)), \text{grad}^{(j)} f(\omega(s_1), \ldots, \omega(s_k))\rangle_{\omega(s_j)},$$

(1.4)

which is a bounded function on $\Omega_{x,y}^{\text{reg}}$.

In the sequel, we will usually extend the parallel transport $\tau$ from $\Omega_{x,y}^{\text{reg}}$ to $\Omega_{x,y}$ in an arbitrary measurable way. The bundle $T_1 \Omega_{x,y}$ then extends to a measurable field of Hilbert spaces over $\Omega_{x,y}$, and $D^0 F$ extends to a bounded measurable section of this measurable field of Hilbert spaces, cf. [13] or Appendix D in [18]. Since we will always be dealing with classes w.r.t. one of the measures $P_{T,x,y}$, $T > 0$, this extension will have no effect on our considerations, and is used only for notational convenience.

Now fix $T > 0$. Since $P_{T,x,y}^{\Omega_{x,y}^{\text{reg}}} = 1$, the functions $\omega \mapsto (D^0 F(\omega), D^0 G(\omega))_\omega$, $F,G \in FC^\infty$, represent unique classes in $L^\infty(\Omega_{x,y}; P_{T,x,y})$. We define a quadratic form $\mathcal{E}_{x,y}^T$ with domain $FC^\infty$ on $L^2(\Omega_{x,y}; P_{T,x,y})$ by:

$$\mathcal{E}_{x,y}^T(F,G) = \int_{\Omega_{x,y}} \left(D^0 F(\omega), D^0 G(\omega)\right)_\omega P_{x,y}(d\omega).$$

(1.5)

It can be shown that $\mathcal{E}_{x,y}^T$ is closable on $L^2(\Omega_{x,y}; P_{x,y}^T)$, i.e., the completion of $FC^\infty$ w.r.t. the inner product

$$(F,G)_{1,2} = \mathcal{E}_{x,y}^T(F,G) + \int FG dP_{x,y}$$

is embedded (one-to-one) into $L^2(\Omega_{x,y}; P_{x,y}^T)$, cf. [12]. We denote the closure of the quadratic form again by $\mathcal{E}_{x,y}^T$, and its domain (i.e., the subspace of $L^2(\Omega_{x,y}; P_{x,y}^T)$ corresponding to the completion of $FC^\infty$ w.r.t. $(\cdot, \cdot)_{1,2}$) by $H^{1,2}(\Omega_{x,y}; P_{x,y}^T)$. The spaces $FC^\infty$ and (thus) $H^{1,2}(\Omega_{x,y}; P_{x,y}^T)$ are dense subspaces of $L^2(\Omega_{x,y}; P_{x,y}^T)$. The gradient $D^0$ extends to a closed linear operator $(D^0, H^{1,2}(\Omega_{x,y}; P_{x,y}^T))$ from $L^2(\Omega_{x,y}; P_{x,y}^T)$ to the direct integral $L^2(T_1 \Omega_{x,y}; P_{x,y}^T)$, i.e., the space of all square integrable sections of the measurable field of Hilbert spaces $T_1 \Omega_{x,y}$. The cylinder functions in $FC^\infty$ are contained in the domain of the nonnegative definite self-adjoint operator

$$\mathcal{L}_{x,y}^T = (D^0)^* D^0$$

which is associated to the form $(\mathcal{E}_{x,y}^T, H^{1,2}(\Omega_{x,y}; P_{x,y}^T))$ on $L^2(\Omega_{x,y}; P_{x,y}^T)$, cf. Enchev and Stroock [21]. Here $(D^0)^*$ denotes the adjoint w.r.t. $P_{x,y}^T$. An explicit expression for $\mathcal{L}_{x,y}^T F$, $F \in FC^\infty$, is given in [21].
Remark. For $M$ replaced by $\mathbb{R}^d$, $P^T_{x,y}$ is a Gaussian measure, $D^0$ is the Malliavin gradient on the corresponding abstract Wiener space, and $L^T_{x,y}$ is the corresponding Ornstein–Uhlenbeck operator, cf. [32].

For an open subset $\mathcal{U} \subseteq \Omega_{x,y}$ we set:

$$H^1_{0,1}(\mathcal{U}; P^T_{x,y}) := \left\{ F \in H^{1,2}(\Omega_{x,y}; P^T_{x,y}); \ F = 0 \text{ outside } \mathcal{U} \right\}.$$  

(1.6)

This is a closed subspace of $H^{1,2}(\Omega_{x,y}; P^T_{x,y})$. If $\mathcal{U}$ has nonempty interior and a sufficiently regular boundary then $H^1_{0,1}(\mathcal{U}; P^T_{x,y})$ is dense in $L^2(\mathcal{U}; P^T_{x,y})$ (we identify functions in $H^1_{0,1}(\mathcal{U}; P^T_{x,y})$ with their restrictions to $\mathcal{U}$). The restriction of $\mathcal{E}^T_{x,y}$ to $H^1_{0,1}(\mathcal{U}; P^T_{x,y})$ is a closed quadratic form on $L^2(\mathcal{U}; P^T_{x,y})$. Its generator can be viewed as the realization of the operator $L^T_{x,y}$ with Dirichlet boundary conditions on $\mathcal{U}$.

1.3. Local spectral gaps

For $T > 0$ let $\mathbb{E}^T_{x,y}[\cdot] \mid \cdot$ denote the conditional expectation w.r.t. $P^T_{x,y}$. We set $\inf \emptyset := \infty$. For an open subset $\mathcal{U} \subseteq \Omega_{x,y}$ let

$$\lambda(\mathcal{U}; P^T_{x,y}) = \inf_{F \in \mathcal{D}} \frac{\int_{\mathcal{U}} (D^0 F, D^0 F) \, dP^T_{x,y}}{\int_{\mathcal{U}} (F - \mathbb{E}^T_{x,y}[F|\mathcal{U}])^2 \, dP^T_{x,y}},$$

where $\mathcal{D}$ consists of all nonconstant functions $F \in H^{1,2}_{0,1}(\mathcal{U}; P^T_{x,y})$. Below we will also consider $\lambda^+ (\mathcal{U}; P^T_{x,y})$ which is defined similarly but with the infimum taken only over $F \in \mathcal{D}$ with $\int_{\mathcal{U}} F \, dP^T_{x,y} = 0$. Both are lower bounds for the generalized second lowest eigenvalue $\lambda^\text{Dhr}_{2,1}(\mathcal{U}; P^T_{x,y})$ of the self-adjoint realization of the operator $L^T_{x,y}$ with Dirichlet boundary conditions on $L^2(\mathcal{U}; P^T_{x,y})$, which is given by

$$\lambda^\text{Dhr}_{2,1}(\mathcal{U}; P^T_{x,y}) := \sup_{G} \left\{ \frac{\int_{\mathcal{U}} (D^0 F, D^0 F) \, dP^T_{x,y}}{\int_{\mathcal{U}} F^2 \, dP^T_{x,y}}; \ F \perp G \text{ in } L^2(\mathcal{U}; P^T_{x,y}) \right\}$$

with the supremum taken over all nonzero $G \in L^2(\mathcal{U}; P^T_{x,y})$, and the infimum taken over all $F \in H^{1,2}_{0,1}(\mathcal{U}; P^T_{x,y})$ orthogonal to $G$ in $L^2(\mathcal{U}; P^T_{x,y})$. Notice that

$$\lambda(\Omega_{x,y}; P^T_{x,y}) = \lambda^+ (\Omega_{x,y}; P^T_{x,y}) = \lambda^\text{Dhr}_{2,1}(\Omega_{x,y}; P^T_{x,y})$$

(1.7)

is the generalized second lowest eigenvalue of the operator $L^T_{x,y}$ on $L^2(\Omega_{x,y}; P^T_{x,y})$. The zero eigenspace of this operator is one-dimensional if $M$ is simply connected, cf. [1]. Hence in this case, $\lambda(\Omega_{x,y}; P^T_{x,y})$ is the gap between the zero eigenvalue and the rest of the spectrum of $L^T_{x,y}$. It has been shown in [15,16] that this gap vanishes if there exists a closed geodesic $\gamma : S^1 \rightarrow M$ such that the sectional curvature is constant and strictly negative in
a neighborhood of $\gamma(S^1)$, and $x$ and $y$ are sufficiently close to $\gamma$. It is not known if there exists a compact simply connected Riemannian manifold $M$ such that the gap is strictly positive, cf. Aida [2] for a corresponding result on certain noncompact manifolds.

Our first main result implies that at least on sufficiently small balls $U$ around constant loops, $\lambda(U; P_{x,y}^T)$ does not vanish. For $R > 0$ and $a, b \in M$ let:

$$\Omega^R_{a,b} \coloneqq \Omega^R_{a,b} \coloneqq \left\{ \omega \in \Omega_{a,b}; \sup_{s,t \in [0,1]} d(\omega(s), \omega(t)) < R \right\}.$$ 

This set is nonempty if and only if $d(a, b) < R$. For $T > 0$ we set:

$$\tilde{\lambda}(T, R) = \inf \{ \lambda(\Omega^R_{a,b}; P_{a,b}^T); a, b \in M \}.$$ 

(1.8)

**Theorem 1.1.** There exist $R_0, C \in (0, \infty)$ such that for every $R \in (0, R_0)$, $\tilde{\lambda}(T, R) > 0$ for all $T > 0$, and

$$\liminf_{T \downarrow 0} T \cdot \tilde{\lambda}(T, R) \geq 1 - CR^2.$$ 

(1.9)

**Remarks.** (1) For $M$ replaced by $\mathbb{R}^d$,

$$\lim_{T \downarrow 0} T \cdot \lambda(\Omega^R_{a,b}; P_{a,b}^T) = 1 \quad \text{for all } a, b \text{ with } |a - b| < R.$$ 

Hence as $R \downarrow 0$, the lower bound (1.9) for the asymptotics of $\lambda(\Omega^R_{a,b}; P_{a,b}^T)$ as $T \downarrow 0$ approaches the sharp lower bound from the flat case.

(2) As $T \downarrow 0$, the bottom of the spectrum of the self-adjoint realization of $L_{a,b}^T$ with Dirichlet boundary conditions on $\Omega^R_{a,b}$, $d(a, b) < R$, approaches 0. Hence the estimate (1.9) for the second generalized Dirichlet eigenvalue implies a corresponding estimate for the asymptotics of the gap between the first and second generalized Dirichlet eigenvalue of $L_{a,b}^T$.

The proof of Theorem 1.1 is given in Section 5 below.

Next, we consider the larger sets $\Omega^R_{a,b} \coloneqq \Omega^R_{a,b}$ defined by (1.2). We first point out that these sets exhaust the full loop space $\Omega_{x,y}$ in an appropriate way:

**Lemma 1.2.** For every $R, T > 0$, the union $\bigcup_{N \in \mathbb{N}} H^{1,2}_{0}((\Omega^R_{x,y} N; P_{x,y}^T)$ is a dense subspace of $H^{1,2}_{0}(\Omega_{x,y}; P_{x,y}^T)$. In particular,

$$\lim_{N \to \infty} \lambda(\Omega^R_{x,y} N; P_{x,y}^T) = \lim_{N \to \infty} \lambda^*(\Omega^R_{x,y} N; P_{x,y}^T) = \lambda(\Omega_{x,y}; P_{x,y}^T).$$ 

(1.10)

From now on we fix the constant $R_0 > 0$ for which the assertion of Theorem 1.1 is proven below. By the assertion of the theorem, and by arguments similar to those used in its proof, we obtain the
Corollary 1.3. For all $T > 0$, $R \in (0, R_0)$, $N \in \mathbb{N}$, and $x, y \in M$, 

$$\lambda(\Omega_{x,y}^{R,N}; P_{x,y}) > 0.$$ 

In the form stated, the assertion does not depend on the geometric and topological properties of $M$. In fact, it even holds if $M$ is not simply connected. The reason is that the indicator functions of the homotopy classes do not satisfy Dirichlet boundary conditions on $\Omega_{x,y}^{R,N}$. The geometry becomes crucial, however, if we make statements about the size of $\lambda(\Omega_{x,y}^{R,N}; P_{x,y})$. For fixed $T > 0$, good estimates for this quantity are difficult to obtain. As a first step in this direction, we will now show how the asymptotic behavior of $\lambda(\Omega_{x,y}^{R,N}; P_{x,y})$ as $T \downarrow 0$ depends on the Riemannian structure on the base manifold $M$. Let 

$$L(\omega) = \sup_{i=0}^{k-1} d(\omega(s_i), \omega(s_{i+1}))$$

and 

$$E(\omega) = \frac{1}{2} \sup_{i=0}^{k-1} \frac{d(\omega(s_i), \omega(s_{i+1}))^2}{s_{i+1} - s_i}$$

denote the (possibly infinite) length and energy of a path $\omega \in C([0,1], M)$. The suprema are taken over all partitions $0 = s_0 < s_1 < \cdots < s_k = 1$. The set of all geodesics in $\Omega_{x,y}$ that are local minima of the energy functional $E$ is denoted by $\Gamma_{x,y}^{\text{min}}$. We fix a global energy minimum $\gamma_{x,y} \in \Gamma_{x,y}^{\text{min}}$. For $\gamma \in \Gamma_{x,y}^{\text{min}}$ we set: 

$$M_{x,y}^{R,N}(\gamma) = \inf \{ E(\omega); \omega \in \tilde{\Omega}_{x,y}^{R,N}, \omega(0) = \gamma, \omega(1) = \gamma_{x,y} \}.$$ 

Here 

$$\tilde{\Omega}_{x,y}^{R,N} = \Omega_{x,y}^{R,N} \cup \left\{ \omega \in \partial \Omega_{x,y}^{R,N}; \max_{0 \leq i < N-1} d(\omega(i), \omega(i+1)) = R \right\},$$

and $\tilde{\Omega}_{x,y}^{R,N}$ denotes the set of all paths in $\hat{\Omega}_{x,y}^{R,N}$ that project to a continuous path on the space $\hat{\Omega}_{x,y}^{R,N} / \sim$ obtained by identifying $\hat{\Omega}_{x,y}^{R,N} \setminus \Omega_{x,y}^{R,N}$ to a single point. Let 

$$m_{x,y}^{R,N} = \max \{ M_{x,y}^{R,N}(\gamma) - E(\gamma); \gamma \in \Gamma_{x,y}^{\text{min}}, L(\gamma) \leq NR \},$$

and 

$$\tilde{m}_{x,y}^{R,N} = \max \{ \left( M_{x,y}^{R,N}(\gamma) - \frac{NR^2}{2} \right) - E(\gamma); \gamma \in \Gamma_{x,y}^{\text{min}}, L(\gamma) \leq NR \}.$$ 

By combining Theorem 1.1 and the techniques used in its proof with the results on discretized path spaces proven in [17], we obtain:
Theorem 1.4. Let $N \in \mathbb{N}$ and $R \in (0, R_0)$ with $NR > d(x, y)$. Then,

(i) $\lim \inf_{T \downarrow 0} T \log \lambda(\Omega_{x,y}^{R,N} ; p_{x,y}^T) \geq -m_{x,y}^{R,N}$.

(ii) $\lim \sup_{T \downarrow 0} T \log \lambda(\Omega_{x,y}^{R,N} ; p_{x,y}^T) \leq -\tilde{m}_{x,y}^{R,N}$.

Suppose that $y$ is not conjugate to $x$. If there exists only one geodesic in $\Omega_{x,y}$ of length $\leq N \cdot R$ which is a local minimum of the energy functional, then:

$$\lim \inf_{T \downarrow 0} T \lambda^*(\Omega_{x,y}^{R,N} ; p_{x,y}^T) > 0.$$

In particular, $\lambda(\Omega_{x,y}^{R,N} ; p_{x,y}^T)$ goes to 0 exponentially fast as $T \downarrow 0$ if $E : \Omega_{x,y} \to \mathbb{R}$ has several strict local minima of length $< NR$ (i.e., energy $< NR^2/2$), whereas $\lambda^*(\Omega_{x,y}^{R,N} ; p_{x,y}^T)$ grows asymptotically of order $1/T$ if there is only one local minimum of length $\leq NR$. The proof of the theorem is given in Section 6 below.

Remarks. (1) The result of Theorem 1.4(ii) is motivated by Witten’s proof of the Morse inequalities, cf. [41]. For more background on critical points of the energy functional and the classical approach to Morse theory on loop spaces cf. Milnor [34], and Cheeger and Ebin [7].

(2) The reason why we do not obtain the exponential decay rate $m_{x,y}^{R,N}$ in the upper bound in Theorem 1.4(i) in general is that the first Dirichlet eigenvalues on $\Omega_{a,b}^R$ w.r.t. $p_{a,b}^{T/N}$ are strictly positive for all $a, b \in M$ with $d(a, b) < R$, and, in general, can only be expected to go to zero with exponential decay rate $(NR^2 - d(a, b)^2)/2$ as $T \downarrow 0$. An explicit lower bound for the linear growth rate in the second case can be derived from the proof of Theorem 1.4. We point out, however, that this lower bound is far from optimal, and, in particular, depends heavily on $N$. The reason is an unprecise estimation of the $H^1$ metric on $\Omega_{x,y}^{R,N}$ by the $L^2$ metric and vice versa, which is used in the proof of the theorem.

On discretized loop spaces there is a more precise estimate, cf. [17, Corollary 1.3(ii)]. It is not clear to me how to obtain a similar estimate in the nondiscrete case.

We finally look at the behavior of the exponential decay rate of $\lambda(\Omega_{x,y}^{R,N} ; p_{x,y}^T)$ as $T \downarrow 0$ for large $N$. Let

$$m_{x,y} = \sup \{ M_{x,y}(\gamma) - E(\gamma) : \gamma \in \Gamma_{x,y}^{\min} \} \in [0, \infty],$$

where

$$M_{x,y}(\gamma) = \inf \{ \sup(E \circ H) : H \in C([0, 1], \Omega_{x,y}), H(0) = \gamma, H(1) = \gamma_{x,y} \}$$

is the elevation of the lowest mountain pass in the energy landscape needed to go from $\gamma$ to $\gamma_{x,y}$ via a homotopy with fixed end-points. It has been shown in the proof of Corollary 1.5
in [17] that for every $R > 0$, $m_{R,N}^{x,y} \to m_{x,y}$ as $N \to \infty$. Similarly, one verifies that $\tilde{m}_{R,N}^{x,y} \to m_{x,y}$ as well. Hence by Theorem 1.4:

**Corollary 1.5.** For every $R \in (0, R_0)$,

$$\lim_{N \to \infty} \liminf_{T \downarrow 0} T \log \lambda(\Omega_{R,N}^{x,y} ; P_{x,y}^T) = \lim_{N \to \infty} \limsup_{T \downarrow 0} T \log \lambda(\Omega_{R,N}^{x,y} ; P_{x,y}^T) = -m_{x,y}.$$

To understand what this means, we notice that there are three cases with qualitatively different behavior: $m_{x,y} = \infty$, $m_{x,y} \in (0, \infty)$, and $m_{x,y} = 0$. If $M$ is not simply connected then $m_{x,y} = \infty$. The same happens if there exists a nonconstant closed geodesic $\gamma : S^1 \to M$ such that the sectional curvature is strictly negative on $\gamma(S^1)$, and $x$ and $y$ are close to $\gamma(S^1)$, cf. Theorem 1.4 in [17]. Recall that the loop spaces for which the absence of a global spectral gap on $\Omega_{x,y}$ has been proven in [15] are of this type. On the other hand, if the Ricci curvature is strictly positive on $M$, then $\Gamma_{x,y}^{\min}$ is finite, whence $m_{x,y} < \infty$ if $M$ is moreover simply connected, cf. again Theorem 1.4 in [17]. If there are at least two strict local minima of the energy functional on $\Omega_{x,y}$ then $m_{x,y} > 0$, and if $\gamma_{x,y}$ is the only local minimum then $m_{x,y} = 0$.

**Example.** Suppose that $M$ is a two-dimensional ellipsoid with no two axes of equal length, and let $x, y \in M$ be nonconjugate. Since the Ricci curvature is strictly positive, $m_{x,y}$ is finite. Moreover, we can both find $x, y \in M$ such that $m_{x,y} = 0$, and such that $m_{x,y} > 0$. For example, if $x$ and $y$ are both on the longest of the three principal ellipses then $m_{x,y} = 0$. If, on the other hand, $x$ and $y$ are both on the shortest principal ellipse, and $y$ is sufficiently close to the cut point of $x$ along the ellipse, then $m_{x,y} > 0$. In the loop space case, i.e., for $x = y$, $m_{x,y} = 0$ provided the ellipsoid is sufficiently round (i.e., the lengths of the axes do not differ too much). If, however, one of the axes is much longer than the others, and $x$ is on (or close to) the shortest principal ellipse, then $m_{x,y} > 0$.

1.4. Notation and overview

Throughout the rest of this article, $M$ is a compact connected Riemannian manifold with metric $\langle \cdot, \cdot \rangle$, and $d$ denotes the dimension of $M$. The Riemannian manifold is always endowed with its Levi-Civita connection. The global injectivity radius on $M$ is denoted by $\text{inj}(M)$. We set $\kappa = 0$ if the sectional curvature on $M$ is nonnegative, and we let $\kappa$ denote the infimum of the sectional curvature else. Similarly, $\kappa$ denotes the positive part of the supremum of the sectional curvature on $M$, and $\kappa := \max(-\kappa, \kappa)$, i.e., $\kappa$ is the supremum of the absolute value of the curvature. The Riemannian volume on $M$ is denoted by $\text{Var}$, and the heat kernel of $\frac{1}{2} \Delta$ by $p_t(a,b)$. Moreover,

$$\text{Var}(f ; \mu) := \int \left( f - \int f \, d\mu \right)^2 \, d\mu$$

denotes the variance of a square-integrable function $f$ w.r.t. a probability measure $\mu$. 

Basic ingredients in the proof of Theorem 1.1 are asymptotic estimates for spectral gaps on the base manifold w.r.t. the distribution of the mid-point of a Brownian bridge, a commutation formula for derivatives and expectation values w.r.t. Brownian bridges, and some highly nontrivial variance estimates by Malliavin and Stroock [33]. These tools are provided in Sections 2 and 3 below. In Section 4, they are used to derive the key estimate in the proof of Theorem 1.1. The theorem itself is obtained by iterating this estimate, cf. Section 5. All other results stated above are proven in Section 6. The main results of this article have been announced in [19].

2. Poincaré inequalities w.r.t. distributions of mid-points of Brownian bridges

For \( r \in (0, \text{inj}(M)) \), \( a, b \in M \), and \( T > 0 \) let

\[
U_{a,b}^r = \{ z \in M; \ d(a, z) < r \text{ and } d(z, b) < r \},
\]

and

\[
\mu_{a,b}^T(dz) = \frac{p_{T/2}(a, z)p_{T/2}(z, b)}{p_T(a, b)V(dz)}.
\]

Moreover let

\[
c_T(r) = \sup_{a, b \in M} \left\{ \text{Var} \left( f; \mu_{a,b}^T \right); \ f \in C^\infty(M) \setminus \{0\}, \ f = 0 \text{ on } M \setminus U_{a,b}^r \right\},
\]

where we use the convention \( \sup \emptyset = -\infty \). Thus \( c_T(r) \) is the smallest constant such that the Poincaré type inequality,

\[
\text{Var}(f; \mu_{a,b}^T) \leq c_T(r) \cdot \int |\text{grad} f|^2 d\mu_{a,b}^T, \quad f \in C^\infty(M), \ f = 0 \text{ on } M \setminus U_{a,b}^r,
\]

holds for all \( a, b \in M \). Since \( \mu_{a,b}^T \) is equivalent to the Riemannian volume with density bounded from above and below, \( c_T(r) < \infty \) for all \( r, T > 0 \). The aim of this section is to prove the following result:

**Theorem 2.1.** There exist \( r_0 > 0 \) and a finite constant \( C \) depending only on \( M \) such that

\[
\limsup_{T \downarrow 0} T^{-1} c_T(r) \leq \frac{1 + Cr^2}{4} \quad \forall r \in (0, r_0].
\]

Fix \( \overline{R} \in (0, \text{inj}(M)) \) with \( \overline{R} < \pi / \sqrt{\kappa} \). Let \( I \) be an index set, and let \( E_i, \ i \in I \), be functions defined on neighborhoods of open balls \( B(y_i, \overline{R}) \), \( y_i \in M \). We assume that \( y_i \) is a global minimum of the functional \( E_i \) on \( B(y_i, \overline{R}) \) for every \( i \in I \). For a subset \( U \subseteq M \)
and a probability measure \( \nu \) on \( U \) that is equivalent to the volume measure with a smooth density \( \rho \) we set:

\[
\lambda^\text{Neu}_2(U; \nu) = \inf \frac{\int |\text{grad } f|^2 \, d\nu}{\text{Var}(f; \nu)},
\]

where the infimum is taken over all nonconstant restrictions to \( U \) of functions in \( C^\infty(M) \).

If the boundary of \( U \) is sufficiently regular then \( \lambda^\text{Neu}_2 \) is the second lowest Neumann eigenvalue of the operator \( \Delta + \langle \text{grad } \log \rho, \text{grad } \cdot \rangle \) on \( L^2(U; \nu) \). Theorem 2.1 is essentially a consequence of the following result:

**Theorem 2.2.** Suppose that:

1. The quadratic forms \( \text{Hess}_{y_i} E_i, \ i \in I \), are uniformly bounded from below by a constant \( \xi > 0 \), i.e.,

\[
\text{Hess}_{y_i} E_i(v, v) \geq \xi \cdot \langle v, v \rangle_{y_i}
\]

for all \( i \in I \) and \( v \in T_{y_i} M \).

2. The third derivatives \( \nabla^2 dE_i, \ i \in I \), are uniformly bounded on \( B(y_i, R) \) by a joint constant \( A \in (0, \infty) \), i.e.,

\[
\left| E_{i} \circ \gamma'''(0) \right| \leq A
\]

for every \( i \in I \) and every unit speed geodesic \( \gamma : (-1, 1) \to M \) with \( \gamma(0) \in B(y_i, R) \).

Let \( \varepsilon \leq \min(R, \xi/A) \), and let \( \nu_{T, \varepsilon}^i, \ T > 0, \ i \in I \), be the probability measures on \( B(y_i, \varepsilon) \) given by

\[
\nu_{T, \varepsilon}^i(dx) = e^{-E_i(x)/T} V(dx) \int_{B(y_i, \varepsilon)} e^{-E_i/T} \, dV.
\]

Then:

\[
\liminf_{T \downarrow 0} \left(T \cdot \inf_{i \in I} \lambda^\text{Neu}_2(B(y_i, \varepsilon); \nu_{T, \varepsilon}^i) \right) \geq \xi.
\]

Apart from the uniformity in \( i \), Theorem 2.2 is a consequence of Theorem 2.3 in [17]. The idea of the proof in [17] was to use an inverse ground state transform to map the operators whose spectrum we are interested in unitarily to Schrödinger operators on an \( L^2 \) space w.r.t. the volume measure. The Schrödinger operators thus appearing are precisely those studied in connection with Witten’s approach to the Morse inequalities [41], and techniques from semiclassical analysis developed in this context (cf. in particular [35]) can be applied to obtain asymptotic estimates on the low-lying eigenvalues as \( T \downarrow 0 \). We now verify that the assumptions from Theorem 2.3 in [17] are in fact satisfied in our case, and...
we show that the arguments used in the proof of this theorem do even yield the uniform estimates claimed above.

**Proof of Theorem 2.2.** Fix \( i \in I \). Since \( \varepsilon \leq R < \text{inj}(M) \), the boundary \( \partial B(y_i, \varepsilon) \) is a smooth \( (d-1) \)-dimensional submanifold that splits \( M \) into two components. Let \( \gamma : [0, \varepsilon] \to M \) be a unit speed geodesic with \( \gamma(0) = y_i \). By the assumptions, \( (E_i \circ \gamma)'(0) = 0 \), \( (E_i \circ \gamma)'(0) \geq \xi \), and \( |(E_i \circ \gamma)''(s)| \leq A \) for all \( s \in (0, \varepsilon) \). Hence \( (E_i \circ \gamma)''(s) \geq \xi - s A > 0 \), and

\[
(E_i \circ \gamma)'(s) \geq (\xi - s A) \cdot s / 2 \quad \text{for } s \in [0, \varepsilon].
\]

In particular, \( y_i \) is the only critical point of \( E_i \) in \( \overline{B(y_i, \varepsilon)} \), and

\[
\langle n, \text{grad} E_i \rangle \geq 0 \quad \text{on } \partial B(y_i, \varepsilon),
\]

where \( n \) denotes the outer normal vector field. Moreover, on \( \overline{B(y_i, \varepsilon)} \),

\[
|\text{grad} E_i|^2 \geq (\xi^2 / 4) \cdot d(\cdot, y_i)^2,
\]

and

\[
|\Delta E_i| \leq \left| \Delta E_i(y_i) \right| + d \cdot \varepsilon \cdot A \leq 2d \varepsilon. \quad (2.7)
\]

Theorem 2.3 in [17] now implies \( \lim_{T \to 0} T \cdot \lambda_{\text{Neu}}(B(y_i, \varepsilon); \nu^T, \varepsilon_i) = \alpha \), where \( \alpha \) is the lowest eigenvalue of \( \text{Hess}_{y_i} E_i \). In particular, \( \alpha \geq \xi \). It only remains to verify that the constants in the lower bound part of the proof of Theorem 2.3 in [17] can be chosen uniformly for the functions \( E_i \) with domain \( U_i := B(y_i, \varepsilon) \). By (2.6), (2.7) and the assumptions, this is the case:

In fact, by (2.6), we can choose \( \delta := \xi^2 / 4 \) in Step 3 of the proof of Theorem 2.3 in [17]. By (2.7), we then obtain:

\[
V_T \geq \frac{\delta}{4} T^{-6/5} - \frac{1}{2} T^{-1} 2d \varepsilon \geq \frac{\delta}{8} T^{-6/5}
\]

(and hence (2.24) in [17]) for all \( T \in (0, T_0) \) with \( T_0 := (\delta d \varepsilon / 8)^5 \). The constant \( T_1 \) in the proof in [17] can be chosen arbitrarily with \( 6T_1^{2/5} \leq \varepsilon \). The constants \( C_1 \) and \( C_2 \) appearing in (2.28) in [17] can be chosen depending only on the Riemannian structure of \( M \) and the bound \( A \) on the third derivatives of the functions \( E_i, i \in I \). Also the constants \( K_T \) have been chosen depending only on \( M \). With this choice of constants, we obtain the estimate (2.30) in [17] for each of the functions \( E_i, i \in I \), and always \( \kappa_1(y_1) \geq \xi \). Since \( y_i \) is the only critical point of \( E_i \) in \( \overline{U_i} \), we can now show as in (2.33) in [17] that with \( E = E_i \) for any \( i \in I \),

\[
\mathcal{E}_T(T) \geq K_T^{-2} \cdot (1 - \overline{T}^{1/5}) \cdot T^{-1} \cdot \xi \cdot \int_{U_i} g^2 \, dV
\]
for \( T \in (0, T_2) \) and \( g \in C^\infty(M) \) satisfying (2.34) in [17], where \( T_2 > 0 \) is chosen in such a way that \( \delta/8 > K_T^{-2} \cdot T^{1/5} \) for \( T \in (0, T_2) \), and
\[
\overline{C} := (18 + C_2) \cdot \xi^{-1} \cdot \sup \left\{ K_T^2 : 0 < T < T_2 \right\}.
\]

Hence as in (2.35) in [17], we obtain:
\[
\int_{U_i} |\nabla f|^2 \, d\nu_{T, \varepsilon} \geq K_T^{-2} (1 - \overline{C} T^{1/5}) T^{-1} \xi \cdot \int_{U_i} f^2 \, d\nu_{T, \varepsilon}
\]
for every \( i \in I \), \( T \in (0, T_2) \), and \( f \in C^\infty(M) \) such that the restriction of \( f \) to \( U_i \) is orthogonal in \( L^2(U_i; \nu_{T, \varepsilon}^i) \) to the function \( \hat{e}_T^i \) (depending on \( i \)) defined by (2.37). This means that
\[
\lambda_2^{\text{Neu}}(U_i; \nu_{T, \varepsilon}^i) \geq K_T^{-2} (1 - \overline{C} T^{1/5}) T^{-1} \xi
\]
for all \( T \in (0, T_2) \) and \( i \in I \). Since \( \overline{C} \), \( T_2 \), and \( K_T \) do not depend on \( i \), and \( \lim_{T \downarrow 0} K_T = 1 \), we obtain (2.4) above.

**Proof of Theorem 2.1.** For \( a, b \in M \) and \( z \in U_{a, b} \overline{R} \) let
\[
E_{a, b}(z) = d(a, z)^2 + d(z, b)^2.
\]
We want to apply Theorem 2.2 with these functions. By compactness, the third derivatives of \( (x, y) \mapsto d(x, y)^2 \) are uniformly bounded on \( \{(x, y) \in M \times M ; d(x, y) \leq \overline{R} \} \). Hence the third derivatives of the functions \( E_{a, b} \), \( a, b \in M \), are uniformly bounded on \( U_{a, b} \overline{R} \) by a joint constant \( A \). Moreover, a comparison argument with the space form of constant curvature \( \kappa \) shows that for \( a \in M \) and \( v \in T_{a} M \) for some \( z \in B(a, \overline{R}) \),
\[
\text{Hess}_z d(a, \cdot)^2(v, v) \geq 2 f_\kappa(d(a, z)) |v|^2,
\]
where
\[
f_\kappa(r) = \begin{cases} \sqrt{\kappa} r \cdot \cot(\sqrt{\kappa} r) & \text{if } \kappa > 0, \\ 1 & \text{if } \kappa = 0, \end{cases}
\]
(cf., e.g., [37, Chapter I, Theorem 1.1]). Note that since \( \overline{R} < \pi / \sqrt{\kappa} \), \( f_\kappa \) is decreasing and strictly positive on \( [0, \overline{R}] \). For \( a, b \in M \) with \( d(a, b) \leq \overline{R} \) let \( m_{a, b} \) denote the mid-point of the unique minimal geodesic \( \gamma_{a, b} \) connecting \( a \) and \( b \). Note that \( m_{a, b} \) is the global minimum of \( E_{a, b} \) on \( U_{a, b} \overline{R} \), and \( d(a, m_{a, b}) = d(m_{a, b}, b) = d(a, b) / 2 \). For \( r \in (0, \overline{R}] \) and \( a, b \in M \) with \( d(a, b) \leq r \),
\[
\text{Hess}_{m_{a, b}} E_{a, b} \geq 4 f_\kappa(r/2) \, ds^2
\]
by (2.8). Let  \( \bar{r} := \min(\bar{R}/2, 4 f_\tau(\bar{R}/4)) \). To simplify the notation let \( B_{a,b}(r) = B(m_{a,b}, r) \), \( r > 0 \). Since \( B_{a,b}(\bar{R}/2) \subset U_{a,b}^R \), Theorem 2.2 shows that for \( r \in (0, \bar{r}) \),
\[
\liminf_{T \downarrow 0} \left( T \cdot \inf \left\{ \lambda_{Neu}^2 (B_{a,b}(r); \nu_{a,b}^T) : a, b \in M \text{ with } d(a, b) \leq r \right\} \right) \geq 4 f_\tau(r/2),
\]
where \( \nu_{a,b}^T \) is the probability measure on \( B_{a,b}(r) \) given by
\[
\nu_{a,b}^T = e^{\frac{-E_{a,b}}{T}} \frac{dV}{\int_{B_{a,b}(r)} e^{\frac{-E_{a,b}}{T}} dV}.
\]
For two equivalent measures \( \mu \) and \( \nu \) on a set \( U \), we set:
\[
\text{osc}(\mu, \nu; U) = \sup_U \frac{d\mu}{d\nu} / \inf_U \frac{d\mu}{d\nu}.
\]
We can use results on the short time asymptotics of the heat kernel on a Riemannian manifold to derive estimates for the density of \( \mu_{T_{a,b}} \) w.r.t. \( \nu_{a,b}^T \): for \( a, b \in M \) with \( d(a, b) < \text{inj}(M) \) let \( \psi(a, b) \) be the unique vector in \( \exp^{-1}_a(\{ b \}) \) with length \( < \text{inj}(M) \), and let
\[
\psi(a, b) = \text{det}(d\psi(a, b) \exp_a)
\]
denote the corresponding Jacobian determinant of the exponential map. Since \( B_{a,b}(r) \subset U_{a,b}^r \), we obtain:
\[
\limsup_{T \downarrow 0} \sup_{a,b \in M \atop d(a, b) \leq r} \text{osc}(\mu_{a,b}^T, \nu_{a,b}^T; B_{a,b}(r)) \leq \frac{\Psi(2r)}{\psi(2r)}
\]
for \( r \in [0, \bar{r}] \), cf. Lemma 3.2 in [17]. Let \( \bar{\mu}_{a,b}^T \) be the probability measure on \( B_{a,b}(r) \) with density
\[
d\bar{\mu}_{a,b}^T / d\mu_{a,b} = 1 + \epsilon_{a,b}^T \cdot \chi_{B_{a,b}(r) \setminus B_{a,b}(r/2)},
\]
where
\[
\epsilon_{a,b}^T = \mu_{a,b}^T (M \setminus B_{a,b}(r)) / \mu_{a,b}^T (B_{a,b}(r) \setminus B_{a,b}(r/2)).
\]
Below, we will prove that there exists \( r_1 \in (0, \bar{r}) \) such that for \( r \in [0, r_1] \), \( \epsilon_{a,b}^T \) converges to 0 uniformly in \( a, b \in M \) with \( d(a, b) \leq \bar{r} \). Taking this for granted for the moment, we obtain by (2.11) and (2.12),
\[
\liminf_{T \downarrow 0} (T \cdot \inf \left\{ \lambda_2^N \left( B_{a,b}(r); \tilde{\mu}^T_{a,b} \right); \ a, b \in M \text{ with } d(a, b) \leq r \right\}) \\
\geq 4 f \kappa (r/2) \psi(2r) / \overline{\psi}(2r) \quad \forall r \in (0, r_1].
\]  
(2.13)

Now let \( r_0 := r_1/4 \). Note that \( U^{T}_{a,b} \) is empty if \( d(a, b) \geq 2r \). Since \( U^{T}_{a,b} \subset B_{a,b}(2r) \), and the restrictions of \( \mu^T_{a,b} \) and \( \tilde{\mu}^{T,4r}_{a,b} \) to the Borel \( \sigma \)-algebra of \( B_{a,b}(2r) \) coincide, (2.13) implies

\[
\limsup_{T \downarrow 0} T^{-1} c_T (r) \leq \frac{1}{4} f \kappa (2r)^{-1} \cdot \overline{\psi}(8r) / \psi(8r) \quad \forall r \in (0, r_0].
\]  
(2.14)

Since \( \overline{R} < \inf(M) \) and \( \overline{R} < \pi / \sqrt{R} \), Bishop’s comparison theorems imply:

\[
\overline{\psi}(r) \leq 1 + C_1 \cdot R^2 \quad \text{and} \quad \overline{\psi}(r) \geq 1 - C_2 \cdot R^2 \quad \forall r \in (0, \overline{R})
\]

with finite constants \( C_1, C_2 \) that do only depend on the dimension of \( M \), cf. (3.8) and (3.9) in [17]. Also,

\[
f \kappa (r) \leq 1 + C_3 \cdot R^2 \cdot r^2 \quad \forall r \in (0, \overline{R})
\]

with a universal constant \( C_3 \). The assertion of Theorem 2.1 is hence a consequence of (2.14). \( \square \)

It remains to show that for small \( r, \varepsilon^{T,r}_{a,b} \to 0 \) as \( T \downarrow 0 \) uniformly in \( a \) and \( b \):

**Claim.** There exists \( r_1 > 0 \) such that for every \( r \in [0, r_1] \),

\[
\sup_{a,b \in M, \ d(a,b) \leq r} \frac{\mu^T_{a,b}(M \setminus B_{a,b}(r))}{\mu^T_{a,b}(B_{a,b}(r) \setminus B_{a,b}(r/2))} \leq A_r \cdot e^{-\beta_r T} \quad \forall T \in (0, 1]
\]

holds with constants \( A_r, \beta_r \in (0, \infty) \).

**Proof of the claim.** For \( r \in [0, \tilde{r}] \) let

\[
E_1(r) := \sup \left\{ E_{a,b}(z); \ a, b \in M \text{ with } d(a, b) \leq \tilde{r}, \ z \in B_{a,b}(r) \right\}.
\]

Moreover, let

\[
E_2 := \inf \left\{ E_{a,b}(z); \ a, b \in M \text{ with } d(a, b) \leq \tilde{r}, \ z \in M \setminus B_{a,b}(\tilde{r}) \right\}.
\]

By continuity of the map \( (a, b, z) \mapsto E_{a,b}(z) \) and compactness of \( M \), \( E_2 > 0 \) and \( \lim_{r \downarrow 0} E_1(r) = 0 \). We fix \( r_1 > 0 \) such that \( E_1(r) \leq E_2/2 \) for all \( r \in [0, r_1] \). By Varadhan’s result [40],

\[
\lim_{T \downarrow 0} T \log p_T(a, b) = -d(a, b)^2 / 2 \quad \text{uniformly on } M \times M.
\]
In particular, for every $\delta > 0$ there exists a constant $K_5 \in (0, \infty)$ with

$$K_5^{-1} e^{-(1+\delta)E_{a,b}(\bar{z})/T} \leq pT/2(a, z) pT/2(z, b) \leq K_5 e^{-(1-\delta)E_{a,b}(\bar{z})/T}$$

for all $a, b, z \in M$ and $T \in (0, 1)$. Let $r \in [0, \bar{r}]$. For $a, b \in M$ with $d(a, b) \leq \bar{r}$ we obtain:

$$\int_{M \setminus B_{a,b}(\bar{r})} pT/2(a, z) pT/2(z, b) V(dz) \leq K_1/4 \cdot V(M) \cdot e^{-3E_2/4T},$$

and

$$\int_{B_{a,b}(r) \setminus B_{a,b}(r/2)} pT/2(a, z) pT/2(z, b) V(dz) \geq K_5^{-1} \cdot e^{-5E_1(r)/4T} \geq C_4 \cdot e^{-5E_2/8T}.$$

where $C_4$ is a strictly positive constant that can be chosen independently of $a$ and $b$. Hence

$$\frac{\mu_T^{M \setminus B_{a,b}(\bar{r})}}{\mu_T^{B_{a,b}(r) \setminus B_{a,b}(r/2)}} \leq K_1/4 \cdot V(M) \cdot C_4^{-1} \cdot e^{-E_2/8T}. \quad (2.15)$$

Moreover, by (2.12),

$$\frac{\mu_T^{B_{a,b}(\bar{r}) \setminus B_{a,b}(r)}}{\mu_T^{B_{a,b}(r) \setminus B_{a,b}(r/2)}} \leq \frac{\psi(\bar{r})}{\psi(r)} \cdot \int_{B_{a,b}(\bar{r}) \setminus B_{a,b}(r)} e^{-E_{a,b}/T} V(dV) \int_{B_{a,b}(r) \setminus B_{a,b}(r/2)} e^{-E_{a,b}/T} V(dV). \quad (2.16)$$

The first part of the proof of Theorem 2.2 shows that for $a, b \in M$ with $d(a, b) \leq \bar{r}$, $s \in (0, \bar{r})$, and $\theta \in T_{ma,b} M$ with $|\theta| = 1$,

$$\frac{\partial}{\partial s} E_{a,b}(\exp_{ma,b}(s \cdot \theta)) \geq 2s \cdot f_2(\bar{r}/2). \quad (2.17)$$

The interval $[r, \bar{r}]$ is the image of $[r/2, r]$ under the map $s \mapsto r + (\bar{r} - r)/(r/2) \cdot (s - r/2)$. By (2.17),

$$E_{a,b}(\exp_{ma,b}\left(s + \frac{r - r}{r/2} \cdot (s - r/2) \cdot \theta\right)) - E_{a,b}(\exp_{ma,b}(s \cdot \theta)) \geq r \cdot f_2(\bar{r}/2)$$

for all $s \in [r/2, r]$. An estimation in polar coordinates thus shows that the right-hand side of (2.16) is dominated from above by $C_5 \cdot \exp(-r \cdot f_2(\bar{r}/2)/T)$ for some finite constant $C_5$ that can be chosen independently of $a, b \in M$ with $d(a, b) \leq \bar{r}$. The claim is hence a consequence of (2.15) and (2.16). \qed
3. Differentiation of the law of a Brownian bridge w.r.t. the starting point

In this section, we prove a formula for the derivatives of expectations w.r.t. pinned Wiener measure when the starting point of the corresponding Brownian bridge is varied. We then apply a result of Malliavin and Stroock [33] to obtain estimates for the derivatives. Fix $T > 0$ and $a, b \in M$. The expectation w.r.t. $P_{T, a, b}$ will be denoted by $E_{T, a, b}$.

Recall that the $M$ valued stochastic process $\Pi_s : C([0, 1], M) \to M$, $\Pi_s(\omega) = \omega(s)$, is a semimartingale (Brownian bridge) w.r.t. the probability measure $P_{T, a, b}$ and the augmentation $(F^T_{s, a, b})_{0 \leq s \leq 1}$ of the filtration $F_s = \sigma(\Pi_u; 0 \leq u \leq s)$. The $T_a M$ valued lifting (anti-development) of $(\Pi_s)_{0 \leq s \leq 1}$ is denoted by $(B_s)_{0 \leq s \leq 1}$. Suppose that $X_s(\omega) = \tau_s(\omega)H_s(\omega)$, $0 \leq s \leq 1$, $\omega \in \Omega_{a, b}$, for some $T_a M$ valued, $(F^T_{s, a, b})$-adapted stochastic process $(H_s)_{0 \leq s \leq 1}$ on $\Omega_{a, b}$ such that for $P_{T, a, b}$-a.e. $\omega$, $s \mapsto H_s(\omega)$ is absolutely continuous and

$$E_{T, a, b}\left[\int_0^1 |H'_s|^2 \, ds\right] < \infty.$$ 

Thus the stochastic parallel transport $\tau_s$ and the anti-development $B_s$, $0 \leq s \leq u$, w.r.t. the Brownian bridge $((\Pi_s)_{0 \leq s \leq u}, P_{T, a, b})$ up to time $u$ are also versions of the stochastic parallel transport and the anti-development respectively w.r.t. the Brownian motion $((\Pi_s)_{0 \leq s \leq u}, P^T_a)$. In particular, $(B_s)_{0 \leq s \leq u}$ is a $T_u M$ valued Brownian motion with time parameter $T$ w.r.t. $P^T_a$. If $U$ is an isometry from $T_u M$ to $T_z M$ for some $z$, we set $\text{Ric}_U := U^{-1}\text{Ric}_z U \in \text{End}(T_u M)$. Then,

$$\delta^T_u X = \int_0^1 \left(T^{-1}H'_s + \frac{1}{2}\text{Ric}_z(H_s)\right) \cdot dB_s$$

is well-defined both in $L^2$ w.r.t. $P^T_a$ and w.r.t. $P^T_{a, b}$. The existence of the corresponding integral from 0 to 1 w.r.t. $P^T_{a, b}$ is nontrivial, and can only be expected if $X_1$ vanishes. We state the following slight extension of [27], Proposition 3.4:
Lemma 3.1. Suppose that $H_s(\omega) = A_s(\omega) \cdot h_s$ for some $h \in H^{1,2}(0, 1; T_aM)$ with $h_1 = 0$, and an $(\mathcal{F}_T^{a,b})$ adapted, $\text{End}(T_aM)$ valued process $(A_s)_{0 \leq s \leq 1}$ such that $s \mapsto A_s(\omega)$ is $C^1$ for every $\omega \in \Omega_{a,b}$, and

$$\sup_{\omega \in \Omega_{a,b}} \sup_{s \in [0,1]} |A'_s(\omega)| < \infty. \quad (3.3)$$

Then the $L^1(\Omega_{a,b}; P_a)\text{-}\lim$ of $\delta^T u X$ as $u \uparrow 1$ exists, and the limit is in $L^2(\Omega_{a,b}; P_a)$. We denote this limit by:

$$\delta^T X = \int_0^1 \left( T^{-1} H'_s + \frac{1}{2} \text{Ric}_{\tau_s}(H_s) \right) \cdot dB_s. \quad (3.4)$$

The proof of Lemma 3.1 is almost exactly the same as in [27], where the result has been shown under the additional assumptions $T = 1, h_0 = 0$, and $A_s(\omega) = 1$ for all $s$ and $\omega$. Just note that by (3.3),

$$|H_s(\omega)| \leq C_1 \cdot |h_s| \quad \text{and} \quad |H'_s(\omega)| \leq C_1 \cdot (|h'_s| + |h_s|) \quad \forall s, \omega \quad (3.5)$$

with some constant $C_1$ that does not depend on $s$ and $\omega$. These estimates are all that is needed to carry over the proof to the case $A \neq 1$. The case $h_0 \neq 0$ can be included without modification, and the case $T \neq 1$ can be treated similarly to the case $T = 1$. From Hsu’s proof we just recall that the key ingredients are the estimates:

$$|\text{grad}_a \log p_t(\cdot, b)| \leq C \cdot \left( \frac{d(a, b)}{t} + \frac{1}{\sqrt{t}} \right), \quad (3.6)$$

and

$$|\text{Hess}_a \log p_t(\cdot, b)| \leq C \cdot \left( \frac{d(a, b)^2}{t^2} + \frac{1}{t} \right) \quad (3.7)$$

for all $a, b \in M$ and $t \in (0, 1]$ with a constant $C$ depending only on $M$. These estimates have been proven in [28,36,38].

Theorem 3.2. Let $v \in T_aM$. Suppose that $X_s = \tau_s H_s$ with $H$ as in Lemma 3.1, and that $X_0 = v P_a a$-a.s. Then,

$$d_a(E^T_{a,b}[F])[v] = E^T_{a,b}[XF] - \text{Cov}(\delta^T X, F; P^T_{a,b}) \quad (3.8)$$

for all smooth cylinder functions $F: C([0, 1], M) \to \mathbf{R}$ of type $F(\omega) = f(\omega(s_1), \ldots, \omega(s_n))$, $n \in \mathbb{N}$, $0 < s_1 < s_2 < \cdots < s_n < 1$, $f \in C^\infty(M^n)$.
Here $X F$ is defined as in the introduction, and $\text{Cov}(\cdot, \cdot; P_{a,b}^T)$ denotes the covariance w.r.t. the probability measure $P_{a,b}^T$.

**Remark.** In the case $v = 0$ the proof given below also shows that $E_{a,b}^T[\delta^T X] = 0$, cf. (3.13). Hence in this case, (3.8) yields

$$E_{a,b}^T [XF] = E_{a,b}^T [F \delta^T X].$$

Theorem 3.2 thus is an extension of the integration by parts identity w.r.t. $P_{a,b}^T$. In fact, the proof can be carried out similarly to the one of the i.b.p. identity given in [27]:

**Proof.** Let $F$ be a smooth cylinder function as above, and let $u < 1$ such that $F$ is $\mathcal{F}_u$ measurable. By [9, Theorem 4.11],

$$d_a \left( E_T^T [F] [v] \right) = E_{a}^T [XF] - E_{a}^T [F \delta^T X].$$

Note that in [9], the Itô integral defining $\delta^T X$ is taken from 0 to 1, but for $\mathcal{F}_u$ measurable $F$ this does not change formula (3.2). Also, Driver’s formula is stated only for $T = 1$, but since $P_{a,b}^T$ is the image of the distribution on $C([0, 1], M)$ of a standard Brownian motion starting at $a$ w.r.t. the transformation $\Phi_T: \omega \mapsto \omega(\cdot / T)$, (3.9) follows from this result applied on $C([0, 1], M)$ with $\tilde{F}(\omega) = F(\Phi_T(\omega|_{[0, T]}))$ and $\tilde{H}_s(\omega) = H_s/T(\Phi_T(\omega|_{[0, T]}))$ for $s \leq T$, 0 else.

From (3.9), (3.8) can be obtained similarly as in the proof of Theorem 3.5 (integration by parts identity) in [27]. As usual, we use the notation $vf = da[f[v]]$ for a function $f: M \to \mathbb{R}$. By (3.1),

$$v E_{a}^T [F] = p_T(a, b)^{-1} E_{a}^T [F p_{(1-u)T}(\Pi_u, b)] + v \log p_T(\cdot, b) E_{a}^T [F].$$

Applying (3.9) with $F$ replaced by $F \cdot p_{(1-u)T}(\Pi_u, b)$ yields

$$E_{a,b}^T [XF] + E_{a,b}^T [F X \log p_{(1-u)T}(\Pi_u, b)] - E_{a,b}^T [F \delta^T X].$$

As $u \uparrow 1$, $\delta_u^T X \to \delta^T X$ in $L^1(\Omega_{a,b}; P_{a,b}^T)$ by Lemma 3.1. Moreover, by (3.6) and (3.5),

$$|X \log p_{(1-u)T}(\Pi_u, b)| \leq |X_u| \cdot |\text{grad}_{\Pi_u} \log p_{(1-u)T}(\cdot, b)| \leq C \cdot C_1 \cdot |h_u| \cdot \frac{d(\Pi_u, b)}{(1-u)T} + \frac{1}{\sqrt{(1-u)T}}.$$

Since $h_1 = 0$,

$$|h_u| = \left| \int_u^1 h'_r \, dr \right| \leq \sqrt{1-u} \cdot \int_u^1 |h'_r|^2 \, dr.$$
Hence
\[
E_{a,b}^T[|X \log p_{(1-u)T}(\Pi_{a,b})|] \leq C \cdot C_1 \cdot T^{-1/2} \cdot \int_0^1 |h'_t|^2 \, dt \cdot \frac{1 + E_{a,b}^T[d(\Pi_{a,b})]}{\sqrt{(1-u)T}}
\]
which converges to 0 as \( u \uparrow 1 \). By combining (3.10) and (3.11), and letting \( u \) tend to 1, we obtain:
\[
vE_{a,b}^T[F] = E_{a,b}^T[XF] - v \log p_T(\cdot, b)E_{a,b}[F].
\]
(3.12)
In particular, for \( F = 1 \) we have:
\[
E_{a,b}^T[\delta^T X] = v \log p_T(\cdot, b).
\]
(3.13)
By inserting (3.13) in (3.12), we obtain (3.8).

By Theorem 3.2 we see that to control the derivative \( vE_{a,b}^T[F] \), we need an estimate for the variance of \( \delta^T X \) for some appropriate vector field \( X \) with \( X_0 = v \). It has been realized by Malliavin and Stroock [33] that there is a particular choice for \( X \), for which \( \text{Var}(\delta^T X; P^T_{a,b}) \) is not as singular for small \( T \) as for most other choices. Let \( A^{T,a}: \Omega \to C^1([0, 1], \text{End}(T_aM)) \) be \( P^T_{a,b} \)-a.s. and \( P^T_{a,b} \)-a.s. defined by:
\[
\frac{d}{ds}A_{s}^{T,a}(\omega) + \frac{T}{2} \text{Ric}_{\tau_s(\omega)}A_{s}^{T,a}(\omega) = 0, \quad A_0^{T,a}(\omega) = \text{id}_{T_aM}.
\]
(3.14)
For \( a, b \in M \) with \( b \notin \text{Cut}(a) \) let \( \gamma_{a,b} \) denote the unique minimal geodesic in \( \Omega_{a,b} \), and let \( Y_{a,b}^v, v \in T_aM \), denote the unique Jacobi field along \( \gamma_{a,b} \) with boundary conditions \( Y_{a,b}^v(0) = v \) and \( Y_{a,b}^v(1) = 0 \). Recall that we assume that \( \tau_s(\omega) \) is the usual parallel transport along \( \omega \) if \( \omega \) is smooth. Let \( \xi_{a,b}^v \in C^\infty([0, 1], T_aM) \) be given by:
\[
\xi_{a,b}^v(s) = \tau_s(\gamma_{a,b})^{-1}Y_{a,b}^v(s), \quad 0 \leq s \leq 1,
\]
i.e., \( \xi_{a,b}^v(0) = v, \xi_{a,b}^v(1) = 0 \), and
\[
\frac{d^2}{ds^2}\xi_{a,b}^v(s) + R_{\tau_s(\gamma_{a,b})}(\xi_{a,b}^v(s), \gamma_{a,b}'(0))\gamma_{a,b}'(0) = 0 \quad \text{for } s \in (0, 1),
\]
where \( R_U(v_1, v_2)v_3 := U^{-1}R(Uv_1, Uv_2)Uv_3 \) for \( v_1, v_2, v_3 \in T_aM \) and an isometry \( U: T_aM \to T_zM, z \in M \). Let \( S_aM = \{ v \in T_aM; \ |v| = 1 \} \).

**Theorem 3.3** (Malliavin, Stroock). For \( a \in M, b \in M \setminus \text{Cut}(a), v \in T_aM, \) and \( T > 0 \) let \( X_{a,b}^T,v \) be \( P^T_{a,b} \)-a.s. defined by
\[
X_{a,b}^{T,a,v}(\omega) = \tau_s(\omega)A_{s}^{T,a}(\omega)\xi_{a,b}^v(s), \quad 0 \leq s \leq 1.
\]
(3.15)
Then for every $T_0 \in (0, \infty)$ and every compact subset $K \subset \{(a, b) \in M \times M; b \notin \text{Cut}(a)\}$,

$$\sup_{T \in (0, T_0]} \sup_{(a, b) \in K} \sup_{v \in S_a M} \text{Var}(\delta^T X^{T, a, b, v}, P_{a, b}^T) < \infty. \quad (3.16)$$

**Remarks.**

(1) By (3.4) and (3.14),

$$\delta^T X^{T, a, b, v} = T^{-1} \cdot \int_0^1 \left(A_{a, b}^T (\xi_{a, b}^v)'(s)\right) \cdot dB_s. \quad (3.17)$$

Note that $A_{a, b}^T (\xi_{a, b}^v)'$ is continuously differentiable, whence the Itô integral can be replaced by a Riemann–Stieltjes integral. This is the main reason for using the function $A_{a, b}^T$ in the definition of $X^{T, a, b, v}$.

(2) The key observation that explains why $\text{Var}(\delta^T X^{T, a, b, v}; P_{a, b}^T)$ can be expected to be bounded for small $T$ is a different one: let $F : \Omega_{a, b} \cap H^{1,2}([0, 1], M) \to \mathbb{R}$ be given by:

$$F(\omega) = \int_0^1 \langle \tau_s(\omega)(\xi_{a, b}^v)'(s), \omega'(s) \rangle ds = \int_0^1 (\xi_{a, b}^v)'(s) \cdot b'(s) ds,$$

where $b$ denotes the anti-development of $\omega$. Note that up to the factor $A_{a, b}^T$, which is close to the identity for small $T$, $\delta^T X^{T, a, b}$ is a natural extension of $F$ to $\Omega_{a, b}$. Moreover, by our choice of $\xi_{a, b}^v$, the geodesic $\gamma_{a, b}$ (which is where the measure $P_{a, b}^T$ concentrates as $T \downarrow 0$) is a critical point for $F$. In fact, $F(\omega)$ is the directional derivative of the energy functional $E(\omega) = \frac{1}{2} \int_0^1 |\omega'(s)|^2 ds$ in direction $Y(\omega) = \tau(\omega)\xi_{a, b}^v$. Since $Y(\gamma_{a, b})$ is the Jacobi field $Y_{a, b, v}$,

$$ZF = \text{Hess}_{\gamma_{a, b}} E(Z, Y_{a, b, v}) = I_{\gamma_{a, b}}(Z, Y_{a, b, v}) = 0$$

for all $Z \in T^1_{\gamma_{a, b}} \Omega_{a, b}$. Hence $F$ varies at most quadratically in directions $Z \in T^1_{\gamma_{a, b}} \Omega_{a, b}$. This gives an idea why $\text{Var}(\delta^T X^{T, a, b, v}; P_{a, b}^T)$ can be of order $O(1)$ instead of the order $O(T^{-1/2})$ which could be naïvely expected. To make these considerations rigorous is highly nontrivial. The necessary tools from Malliavin calculus and large deviation theory have been developed in [31] and are applied to the concrete problem in [33].

(3) The considerations in [33] indicate that the proof of Theorem 3.3 can be simplified, if it can be shown by other means that

$$\text{Hess}_a \log p_T(a, b) + T^{-1} \text{Hess}_a (d(\cdot, b)^2 / 2)$$

is uniformly bounded for $T \in (0, 1]$ on compact subsets of $\{(a, b) \in M \times M; b \notin \text{Cut}(a)\}$. For example, on $S^3$, the uniform boundedness can be shown by using the explicit representation of the heat kernel, cf. [20].
Proof of Theorem 3.3. By (3.17), the corresponding statement with \(a\) and \(b\) fixed, and the supremum only taken over \(T \in (0,1]\) and \(b\) varying in a compact subset of \(M \setminus \text{Cut}(a)\) is an immediate consequence of Theorem 2.25 in [33], cf. also the last part of the proof of Corollary 2.29 in [33]. The key ingredient in the proof of Theorem 2.25 in [33] is Theorem 4.21 in [31], which is itself motivated by [4]. The joint uniformity in \((a,b)\) of Corollary 2.29 in [33] holds if and only if the corresponding approximations for the parallel transport along \(\gamma\) converge uniformly, if and only if the corresponding approximations for the parallel transports along \(\omega\) and \(\bar{\omega}\) converge both uniformly. In this case, 

\[
\tau_{1/2,s}(\omega \vee \bar{\omega}) = \begin{cases} 
\tau_{1-2s}(\omega) & \text{for } s \in [0,1/2], \\
\tau_{2s-1}(\bar{\omega}) & \text{for } s \in [1/2,1].
\end{cases}
\] (4.1)

Let \(T > 0\), \(a, b \in M\), and \(\sigma \in \Omega_{a,b}\) with \(a, b \notin \text{Cut}(\sigma(1/2))\). For \(v \in T_{\sigma(1/2)}M\), we define:

\[
\tilde{X}^{T,v}_{\omega}(\sigma) = \begin{cases} 
X_{1-2s}^{T,\sigma(1/2),\sigma(0),v}(\omega) & \text{for } s \in [0,1/2], \\
X_{2s-1}^{T,\sigma(1/2),\sigma(1),v}(\bar{\omega}) & \text{for } s \in [1/2,1],
\end{cases}
\] (4.2)

where \(\omega, \bar{\omega} \in \Omega\) are given by \(\sigma = \omega \vee \bar{\omega}\), and the vector fields on the right are defined by (3.15).

Remark. If \(\sigma\) is a minimal geodesic both on \([0,1/2]\) and on \([1/2,1]\), then \(\tilde{X}^{T,v}_{\omega}(\sigma)\) is a perturbation of the unique continuous vector field \(Y\) along \(\sigma\) which is Jacobi both on \((0,1/2]\) and on \((1/2,1]\), and satisfies \(Y(0) = 0\), \(Y(1/2) = v\), and \(Y(1) = 0\). For general \(\sigma\), let \(\gamma\) be the piecewise minimal geodesic connecting \(\gamma(0) = \sigma(0)\), \(\gamma(1/2) = \sigma(1/2)\), and \(\gamma(1) = \sigma(1)\). Then \(\tilde{X}^{T,i}_{\sigma}(\sigma)\) is a perturbation of the vector field obtained by first parallel transporting the piecewise Jacobi field along \(\gamma\) to \(T_{\gamma(1/2)}M\) (= \(T_{\sigma(1/2)}M\)), and then parallel transporting it along \(\sigma\).
Since $\hat{X}^{T,v}(\sigma)$ is a piecewise smooth, continuous vector field along $\sigma$ that vanishes at 0 and 1, it is contained in $T_{\sigma}^1\Omega_{a,b}$. For a smooth cylinder function $F : \Omega_{a,b} \to \mathbb{R}$ let
\begin{equation}
I_r^T(F)(\sigma) = \sup\{ (\hat{X}^{T,v} F)^2(\sigma); \ v \in S_{(1/2)} \}
\end{equation}
equivalently,
\begin{equation}
I_r^T(F)(\sigma) = \sum_{i=1}^{d} (\hat{X}^{T,e_i(\sigma(1/2))} F^2(\sigma),
\end{equation}
where $e_i$, $1 \leq i \leq d$, are arbitrary measurable vector fields on $M$ such that $\{e_i(z); 1 \leq i \leq d\}$ is an orthonormal basis of $T_zM$ for every $z \in M$. For $\sigma \in \Omega_{a,b}$ with $a \in \text{Cut}(\sigma(1/2))$ or $b \in \text{Cut}(\sigma(1/2))$, let $I_r^T(F)(\sigma) = 0$. By (4.4), one easily verifies that $I_r^T(F) : \Omega_{a,b} \to \mathbb{R}$ is measurable. A general estimate of $I_r^T(F)$ in terms of more standard objects will be given in Lemma 4.3 below.

**Example.** Suppose that $M = \mathbb{R}^d/\mathbb{Z}^d$. Then the parallel transport $\tau_{a,b}$ between two points $a$ and $b$ is independent of the path. For all $T > 0$ and $v \in \mathbb{R}^d$, $\hat{X}^{T,v}(\sigma) = 2 \cdot (s \wedge (1 - s)) \cdot v$, $0 \leq s \leq 1$. Hence for a smooth cylinder function $F(\sigma) = f(\sigma(s_1), \ldots, \sigma(s_m))$ and $\sigma$ with $a, b \notin \text{Cut}(\sigma(1/2))$,
\begin{equation}
I_r^T(F)(\sigma) = 4 \cdot \sum_{j,l=1}^{m} (s_j \wedge (1 - s_j)) \cdot (s_l \wedge (1 - s_l))
\end{equation}
\begin{equation}
\times \langle \text{grad}^{(j)} f(\sigma(s_1), \ldots, \sigma(s_m)), \tau_{\sigma(s_j),\sigma(s_j)} \text{grad}^{(l)} f(\sigma(s_1), \ldots, \sigma(s_m)) \rangle.
\end{equation}
Recall the definitions (2.1) of $U_{a,b}^r$ and (2.3) of $c_T(r)$ for $r, T > 0$ and $a, b \in M$. For $r \in (0, \text{inj}(M))$ let
\begin{equation}
\rho_T(r) = \sup\{ \text{Var}(\delta_T X^{T,a,b,v}; P_{a,b}^T); \ a, b \in M \text{ with } d(a, b) \leq r, v \in S_{a} \}
\end{equation}
Note that by Theorem 3.3,
\begin{equation}
\sup_{T \in (0,T_0)} \rho_T(r) < \infty \quad \text{for every } T_0 \in (0, \infty).
\end{equation}
Recall also that in Theorem 2.1, we have proven an asymptotic estimate for $c_T(r)$ as $T \downarrow 0$.

For a function $F : \Omega_{a,b} \to \mathbb{R}, a, b \in M$, let $\widetilde{F}$ denote the function defined on $\Omega_{z,a} \times \Omega_{z,b}$ for every $z \in M$ by
\begin{equation}
\widetilde{F}(\omega, \overline{\omega}) = F(\omega \vee \overline{\omega}).
\end{equation}
Let $W : \Omega_{z,a} \times \Omega_{z,b} \to \Omega_{z,a}$ and $\overline{W} : \Omega_{z,a} \times \Omega_{z,b} \to \Omega_{z,b}$ denote the canonical projections on the first and second component in the product space.
Lemma 4.1 (Key estimate). Let $r, \delta \in (0, \infty)$. Then

$$\text{Var}(F; P_{a,b}^T) \leq \left(1 + \delta\right) c_T(r) \cdot \mathbf{E}_{a,b}^T [\Gamma^T(F)] + \left(1 + 2 \cdot (1 + \delta^{-1}) c_T(r) \rho_T(r)\right)$$

$$\times \int_{U_{a,b}^r} \left\{ \mathbf{E}_{z,a}^{T/2} \left[ \text{Var}\left(\tilde{F}(W, \overline{W})\right); P_{z,b}^{T/2}\right] \right\}$$

$$+ \mathbf{E}_{z,b}^{T/2} \left[ \text{Var}(\tilde{F}(W, \overline{W}); P_{z,a}^{T/2})\right] \mu_{a,b}^T(\dd z).$$

for all $T > 0$, $a, b \in M$, and every smooth cylinder function $F : \Omega \to \mathbb{R}$ such that $F(\omega) = 0$ if $\omega(1/2) = 0$. Hence $\overline{F}$, $\text{Var}$ means that the corresponding expectation or variance is taken w.r.t. $\overline{W}$, whereas otherwise it is taken w.r.t. $W$.

Proof. Fix $T, a, b$, and $F$ as in the assertion. One easily verifies that for a smooth cylinder function $G$ on $\Omega_{a,b}$, the function $z \mapsto \mathbf{E}_{z,a}^{T/2} [\mathbf{E}_{z,b}^{T/2} G(W, \overline{W})]$ is a $C^\infty$ version of the conditional expectation $\mathbf{E}_{a,b}^T G|\Pi_{1/2} = z|$ of $G$ given $\Pi_{1/2}(\omega) = \omega(1/2) = z$. Hence

$$\text{Var}(F; P_{a,b}^T) = \mathbf{E}_{a,b}^T \left[ (F - \mathbf{E}_{a,b}^T[F])^2 \right]$$

$$= \mathbf{E}_{a,b}^T \left[ (F - \mathbf{E}_{a,b}^T[F | \Pi_{1/2}])^2 \right] + \mathbf{E}_{a,b}^T \left[ \left( \mathbf{E}_{a,b}^T[F | \Pi_{1/2}] - \mathbf{E}_{a,b}^T[F] \right)^2 \right]$$

$$= \int \text{Var}(\tilde{F}; P_{z,a}^{T/2} \otimes P_{z,b}^{T/2}) \mu_{a,b}^T(\dd z)$$

$$+ \text{Var}(\mathbf{E}_{z,a}^{T/2} [\mathbf{E}_{z,b}^{T/2} \tilde{F}(W, \overline{W})]; \mu_{a,b}^T). \quad (4.7)$$

The variance w.r.t. the product measure can be estimated by:

$$\text{Var}(\tilde{F}; P_{z,a}^{T/2} \otimes P_{z,b}^{T/2}) = \mathbf{E}_{z,a}^{T/2} \left[ \text{Var}(\tilde{F}(W, \overline{W}); P_{z,b}^{T/2}) \right] + \text{Var}(\mathbf{E}_{z,b}^{T/2} \tilde{F}(W, \overline{W}); P_{z,a}^{T/2})$$

$$\leq \mathbf{E}_{z,a}^{T/2} \left[ \text{Var}(\tilde{F}(W, \overline{W}); P_{z,b}^{T/2}) \right] + \mathbf{E}_{z,b}^{T/2} \left[ \text{Var}(\tilde{F}(W, \overline{W}); P_{z,a}^{T/2}) \right]. \quad (4.8)$$

To estimate the second summand on the right-hand side of (4.7) let

$$f_T(z) = \mathbf{E}_{z,a}^{T/2} \left[ \mathbf{E}_{z,b}^{T/2} \tilde{F}(W, \overline{W}) \right].$$

For $z \in M \setminus U_{a,b}^r$, $\tilde{F}$ vanishes on $\Omega_{z,a} \times \Omega_{z,b}$ because $F(\omega) = 0$ if $\omega(1/2) \notin U_{a,b}^r$. Hence $f_T$ vanishes outside $U_{a,b}^r$, and thus

$$\text{Var}(f_T; \mu_{a,b}^T) \leq c_T(r) \cdot \int_{U_{a,b}^r} |\text{grad } f_T|^2 \dd \mu_{a,b}^T. \quad (4.9)$$
Now fix \( z \in U'_{a,b} \). Since \( F \) is a smooth cylinder function on \( \Omega \), \( \tilde{F} \) can be extended to a smooth cylinder function on \( \Omega \times \Omega \) as well. For this extension, the map \((\xi, \eta) \mapsto E^{T/2}_{\tilde{z},a} [F(W, \tilde{W})] \) is smooth, whence we can apply the product rule and Theorem 3.2 to conclude:

\[
(d_z f_T)[v] = E^{T/2}_{\tilde{z},b} [d_z (E^{T/2}_{\tilde{z},a} [\tilde{F}(W, \tilde{W})])[v]] + E^{T/2}_{\tilde{z},a} [d_z (E^{T/2}_{\tilde{z},b} [\tilde{F}(W, \tilde{W})])[v]]
\]

\[
= E^{T/2}_{\tilde{z},a} [X^{T/2,\tilde{z},a,v} \tilde{F}(W, \tilde{W}) + X^{T/2,\tilde{z},b,v} \tilde{F}(W, \tilde{W})]
\]

\[
- E^{T/2}_{\tilde{z},b} [\text{Cov}((\delta^{T/2} X^{T/2,\tilde{z},a,v})(W), \tilde{F}(W, \tilde{W}); F^{T/2}_{a})]
\]

\[
- E^{T/2}_{\tilde{z},a} [\text{Cov}((\delta^{T/2} X^{T/2,\tilde{z},b,v})(W), \tilde{F}(W, \tilde{W}); F^{T/2}_{b})]
\]

\[
\leq E^{T/2}_{\tilde{z},a} [E^{T/2}_{\tilde{z},b} [(\delta^{T/2} F)(W \times \tilde{W})]]
\]

\[
+ \rho_{T/2}(r)^{1/2} \cdot E^{T/2}_{\tilde{z},b} [\text{Var}(\tilde{F}(W, \tilde{W}); F^{T/2}_{a})]
\]

\[
+ \rho_{T/2}(r)^{1/2} \cdot E^{T/2}_{\tilde{z},a} [\text{Var}(\tilde{F}(W, \tilde{W}); F^{T/2}_{b})]
\]

for every \( v \in T_M \). Here \( X^{T/2,\tilde{z},b,v} \) means that the vector field \( X^{T/2,\tilde{z},b,v} \) is applied to the second variable \((\tilde{W})\), whereas otherwise it is applied to the first variable. Thus by (4.3),

\[
|\text{grad}_z f_T|^2 = \sup \{(d_z f_T)[v]^2; \ v \in S_M\}
\]

\[
\leq (1 + \delta) \cdot E^{T/2}_{\tilde{z},a} [E^{T/2}_{\tilde{z},b} [(\delta^{T} F)(W \times \tilde{W})]]
\]

\[
+ 2 \cdot (1 + \delta^{-1}) \cdot \rho_{T/2}(r) \cdot \left[ E^{T/2}_{\tilde{z},b} [\text{Var}(\tilde{F}(W, \tilde{W}); F^{T/2}_{a})]
\right.
\]

\[
\left. + E^{T/2}_{\tilde{z},a} [\text{Var}(\tilde{F}(W, \tilde{W}); F^{T/2}_{b})]\right]. \quad (4.10)
\]

The assertion follows by combining (4.7)–(4.10). \( \square \)

We will now derive an estimate for \( \Gamma^T(F) \) in terms of simpler objects. Let \( \{h_{n,k}; \ n \geq 0, \ 1 \leq k \leq 2^n\} \) denote the orthonormal basis of \( H^{1,2}_0([0,1], \mathbb{R}) \) with inner product \((h, g) = \int_0^1 h g' ds\) consisting of Schauder functions, i.e., \( h_{0,1}(s) = s \land (1-s) \),

\[
h_{n,k}(s) = 2^{-n/2} h_{0,1}(2^n s - (k - 1)) \quad \text{for} \ s \in [(k - 1) \cdot 2^{-n}, (k - 1)], \quad (4.11)
\]

and \( h_{n,k}(s) = 0 \) else, for \( n \geq 1 \) and \( 1 \leq k \leq 2^n \). We fix measurable vector fields \( e_i, \ 1 \leq i \leq d \), on \( M \) such that \( \{e_i(z); 1 \leq i \leq d\} \) is an orthonormal basis for \( T_z M \) for every \( z \in M \). Then for every \( \sigma \in \Omega_{a,b} \), the vector fields

\[
Z^{n,k,j}_z (\sigma) = h_{n,k}(s) \tau_{1/2, s}(\sigma) e_j(\sigma(1/2)), \quad s \in [0,1],
\]
\[ n \geq 0, \ 1 \leq k \leq 2^n, \ 1 \leq j \leq d, \] form an orthonormal basis of \( T^1_a \Omega_{a,b} \). In particular, for a smooth cylinder function \( F : \Omega_{a,b} \to \mathbb{R} \),

\[
(D^0 F, D^0 F)_\sigma = \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} \Gamma_{n,k}(F)(\sigma) \quad \forall \sigma \in \Omega_{a,b}, \tag{4.12}
\]

where we have set

\[
\Gamma_{n,k}(F) = \sum_{j=1}^{d} (Z^{n,k,j} F)^2. \tag{4.13}
\]

Explicitly, if \( F(\sigma) = f(\sigma(s_1), \ldots, \sigma(s_m)), m \in \mathbb{N}, s_1, \ldots, s_m \in (0, 1), f \in C^\infty(\mathbb{M}^m) \), then

\[
\Gamma_{n,k}(F)(\sigma) = \sum_{j,l=1}^{m} h_{n,k}(s_j) h_{n,k}(s_l) \langle \text{grad}^{(j)} f(\sigma(s_1), \ldots, \sigma(s_m)), \tau_{n,l}(\sigma) \text{grad}^{(l)} f(\sigma(s_1), \ldots, \sigma(s_m)) \rangle. \tag{4.14}
\]

To control \( \Gamma^T(F)(\sigma) \), we will estimate the coefficients of the tangent vector \( \hat{X}^T,v(\sigma) \in T^1_a \Omega_{a,b} \) w.r.t. the basis \( \{Z^{n,k,j}(\sigma)\} \). We first note:

**Lemma 4.2.** Let

\[
v(s) = \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} \alpha_{n,k} h_{n,k}(s), \quad \alpha_{n,k} \in \mathbb{R}^d,
\]

be the representation of a function \( v \in H^{1,2}_0([0,1], \mathbb{R}^d) \) w.r.t. the Schauder basis of \( H^{1,2}_0([0,1], \mathbb{R}) \). Let \( n \geq 0 \) and \( 1 \leq k \leq 2^n \). If \( v \) is twice differentiable on \(((k-1)\cdot 2^{-n}, k \cdot 2^{-n})\), then

\[
|\alpha_{n,k}| \leq \frac{1}{8} \cdot 2^{-3n/2} \sup_{s \in ((k-1)\cdot 2^{-n}, k \cdot 2^{-n})} |v''(s)|.
\]

**Proof.** For \( n \geq 0, v_n(s) = \sum_{m=0}^{n} \sum_{l=1}^{2^m} \alpha_{m,l} h_{m,l}(s) \) is the linear interpolation of \( v \) w.r.t. the dyadic partition \( \{j \cdot 2^{-(n+1)}; \ 0 \leq j \leq 2^{n+1}\} \) of \([0,1]\). Since the functions \( h_{n,l}, 1 \leq l \leq 2^n \), have disjoint supports, and

\[
\alpha_{n,k} = 2^{n/2} \sum_{l=1}^{2^n} \alpha_{n,l} h_{n,l} \left( \left( k - \frac{1}{2} \right) \cdot 2^{-n} \right)
\]

\[
= 2^{n/2} \cdot v_n \left( \left( k - \frac{1}{2} \right) \cdot 2^{-n} \right) - v_{n-1} \left( \left( k - \frac{1}{2} \right) \cdot 2^{-n} \right)
\]
\[ \omega = 2^{\lfloor k/2 \rfloor} \cdot \left( v\left( (k - 1) \cdot 2^{-n} \right) - \frac{1}{2} \right) . \]

Now consider the function \( f(s) = v(s) - (v((k - 1) \cdot 2^{-n}) + v(k \cdot 2^{-n})) / 2 \). Let \( I_{n,k} = ((k - 1) \cdot 2^{-n}, k \cdot 2^{-n}) \), and let \( s_0 \in I_{n,k} \) with \( |f(s_0)| = \max \{|f(s)|; s \in I_{n,k}\} \). W.l.o.g. we assume \( s_0 \leq (k - 1/2) \cdot 2^{-n} \). If \( v \) is twice differentiable on \( I_{n,k} \) then \( f'(s_0) = 0 \), and

\[ |f'(s)| \leq \int_s^{s_0} |f''| \, ds \leq (s_0 - s) \cdot \sup_{I_{n,k}} |v''| \]

for all \( s \in I_{n,k} \). Hence

\[ |f(s_0)| \leq \int_{(k-1)2^{-n}}^{s_0} |f'| \, ds \leq \frac{1}{2} (s_0 - (k - 1) \cdot 2^{-n})^2 \cdot \sup_{I_{n,k}} |v''| \leq \frac{1}{8} \cdot 2^{-2n} \cdot \sup_{I_{n,k}} |v''|, \]

and thus \( |a_{n,k}| \leq \frac{1}{8} \cdot 2^{-3n/2} \cdot \sup_{I_{n,k}} |v''| \). □

**Lemma 4.3.** Let \( r \in (0, \pi/(2\sqrt{2})) \), \( T > 0 \), and \( a, b \in M \). Then for all \( \sigma \in \Omega_{a,b} \), such that \( \sigma(1/2) \in U'_{a,b} \), and for all smooth cylinder functions \( F \) on \( \Omega_{a,b} \),

\[ I^T(F)(\sigma) \leq 4 \cdot (1 + \kappa T) \cdot \Gamma_{0,1}(F)(\sigma) + \sum_{n=0}^{\infty} g_n(T, r) \sum_{k=1}^{2^n} \Gamma_{n,k}(F)(\sigma), \]

where

\[ g_n(T, r) = 2^{-n} \kappa r^2 \cdot \left( 1 + \frac{1}{2} \kappa r^2 \right) + (d - 1)^2 \kappa T \cdot (1 + 2\kappa T) e^{(d-1)\kappa T/2} (1 + \kappa T)^2. \text{ (4.15)} \]

**Proof.** Fix \( \sigma \) and \( F \) as in the assertion, and let \( z = \sigma(1/2) \) and \( \omega \in \Omega_{z,a} \) and \( \overline{\omega} \in \Omega_{z,b} \) with \( \sigma = \omega \lor \overline{\omega} \). Let \( t \in S, M \). By definition of \( \tilde{X}^T,v(\sigma) \), (3.15) and (4.1),

\[ t_{3/2,1}(\sigma) \tilde{A}^T,v(\sigma) = \tilde{A}^T,z(\sigma) \tilde{v}^v_{a,z,b}(s) \quad \text{for} \ s \in [0,1], \]

where

\[ \tilde{A}^T,z(\sigma) = \begin{cases} A_{1/2}^{T,2}((\omega), & \text{and} \ \tilde{v}^v_{a,z,b} = \begin{cases} \xi^v_{z,a}(1 - 2s) \quad \text{for} \ s \in [0,1/2], \\ \xi^v_{z,b}(2s - 1) \quad \text{for} \ s \in [1/2,1]. \end{cases} \end{cases} \]

We decompose

\[ \tilde{X}^T,v(\sigma) = \tilde{X}^{0,v}(\sigma) + (\tilde{X}^T,v(\sigma) - \tilde{X}^{0,v}(\sigma)), \]
where $\hat{X}^{0,v}(\sigma) := \tau_{1/2,s}(\sigma)\hat{\xi}^v_{a,z,b}(s)$ for $s \in [0, 1]$. Let $\alpha_{n,k,j} \in \mathbb{R}$ be the coefficients in the representation

$$
\hat{X}^{0,v}(\sigma) = \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} \sum_{j=1}^{d} \alpha_{n,k,j} Z^{n,k,j}(\sigma),
$$

(4.16)
i.e.,

$$
\hat{\xi}^v_{a,z,b}(s) = \sum_{n=0}^{2^n} \sum_{k=1}^{2^n} \alpha_{n,k} h_{n,k}(s), \quad \alpha_{n,k} := \sum_{j=1}^{d} \alpha_{n,k,j} \epsilon_j(z) \in TzM.
$$

(4.17)

For $0 \leq s \leq 1$, $\xi^v_{z,a}(s) = \tau_s(\gamma_{z,a})^{-1} Y^v_{z,a}(s)$ where $Y^v_{z,a}$ is the Jacobi field along $\gamma_{z,a}$ with boundary values $Y^v_{z,a}(0) = v$ and $Y^v_{z,a}(1) = 0$. Since $d(z, a) < r \leq \pi/(2\sqrt{\kappa})$, we have:

$$
\bigg|\frac{d^2}{ds^2} \xi^v_{z,a}(s)\bigg| \leq \kappa r^2 \quad \forall s \in [0, 1],
$$

(4.18)

by the Jacobi equation, cf. Appendix A. Similarly, $|d^2/ds^2 \xi^v_{z,b}(s)| \leq \kappa r^2$, whence

$$
\bigg|\frac{d^2}{ds^2} \hat{\xi}^v_{a,z,b}(s)\bigg| \leq 4\kappa r^2 \quad \text{for} s \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right).
$$

(4.19)

Thus by Lemma 4.2,

$$
|\alpha_{n,k}| \leq \frac{1}{2} \kappa r^2 2^{-3n/2} \quad \text{for all} \ n \geq 1 \text{ and } 1 \leq k \leq 2^n.
$$

(4.20)

Moreover, since $h_{0,1}(1/2) = 1/2$ and $h_{n,k}(1/2) = 0$ for $n \geq 1$ and $1 \leq k \leq 2^n$,

$$
\alpha_{0,1} = 2 \cdot \hat{\xi}^v_{a,z,b}(1/2) = 2v.
$$

(4.21)

Since $|v| = 1$, we obtain by (4.16),

$$
|\langle \hat{X}^{0,v}F\rangle(\sigma)| \leq \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} |\alpha_{n,k}| \cdot \left(\sum_{j=1}^{d} (Z^{n,k,j} F(\sigma))^2\right)^{1/2}
$$

$$
\leq 2 \cdot \left(F_{0,1}(\sigma)\right)^{1/2} + \frac{1}{2} \kappa r^2 \sum_{n=1}^{\infty} 2^{-3n/2} \sum_{k=1}^{2^n} (\Gamma_{n,k}(F)(\sigma))^{1/2}.
$$

(4.22)

Moreover, the lemma in Appendix A implies $|d/ds \xi^v_{z,a}(s)| \leq 1 + \kappa r^2$ and $|\xi^v_{z,a}(s)| \leq 1$ for all $s \in [0, 1]$, whence

$$
\bigg|\frac{d}{ds} \hat{\xi}^v_{a,z,b}(s)\bigg| \leq 2 + 2\kappa r^2 \quad \text{and} \quad \hat{\xi}^v_{a,z,b}(s) \leq 1
$$

(4.23)
for all $s \in (0, 1/2)$, and, by a similar argument, for $s \in (1/2, 1)$ as well. Let $C = (d - 1)/4 \cdot \kappa T$. By (3.14), $|A_s^{T/2, z}(\omega)| \leq e^{Cs}$,

$$
|A_s^{T/2, z}(\omega) - \text{id}_{T_s M}| \leq C \cdot \int_0^s |A_t^{T/2, z}(\omega)| \, dt \leq e^{Cs} - 1 \leq C \cdot e^{Cs},
$$

and

$$
\left| \frac{d}{ds} A_s^{T/2, z}(\omega) \right| \leq C \cdot |A_s^{T/2, z}(\omega)| \leq C \cdot e^{Cs} \quad \text{for all } s \in [0, 1].
$$

Hence

$$
|\hat{\lambda}_s^{T, z}(\sigma) - \text{id}_{T_s M}| \leq C \cdot e^{C} \quad \text{and} \quad \left| \frac{d}{ds} \hat{\lambda}_s^{T, z}(\sigma) \right| \leq 2C \cdot e^{C} \quad (4.24)
$$

for $s \in (0, 1/2)$, and, by a similar argument for $s \in (1/2, 1)$. By (4.23) and (4.24),

$$
\left| \nabla \frac{d}{ds} (\hat{X}_s^{T, v}(\sigma) - \hat{X}_s^{0, v}(\sigma)) \right| = \left| \frac{d}{ds} ((\hat{A}_s^{T, z}(\sigma) - \text{id}_{T_s M}) \hat{\xi}_v^{l, z, a, b}(s)) \right|
\leq C \cdot e^{C} \cdot (4 + 2\kappa r^2)
\leq (d - 1) \cdot \kappa T e^{(d-1)\kappa T/4} \cdot (1 + \kappa r^2) \quad (4.25)
$$

for a.e. $s \in (0, 1)$. Thus the norm of $\hat{X}^{T, v}(\sigma) - \hat{X}^{0, v}(\sigma)$ in $T^{1}_{\sigma} \Omega_{a, b}$ is bounded by the right-hand side as well, and

$$
\left| (\hat{X}^{T, v} F)(\sigma) - (\hat{X}^{0, v} F)(\sigma) \right| \leq (d - 1) \cdot \kappa T e^{(d-1)\kappa T/4} \cdot (1 + \kappa r^2) \cdot \left| D^0 F \right|_{\sigma}. \quad (4.26)
$$

By (4.22) and (4.26),

$$
\left| (\hat{X}^{T, v} F)(\sigma) \right|^2 \leq 4 \cdot \left( 1 + \frac{1}{4} \kappa r^2 + \kappa T \right) \cdot \Gamma_{0,1}(F)(\sigma)
+ \frac{1}{4} \cdot (4\kappa r^2 + 2\kappa^2 r^4) \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} 2^{-n} \Gamma_{n,k}(F)(\sigma)
+ (d - 1)^2 \kappa T \cdot (1 + 2\kappa T) \cdot e^{(d-1)\kappa T/2} \cdot (1 + \kappa r^2)^2 \cdot \left| D^0 F \right|_{\sigma}^2. \quad (4.27)
$$

Here we have used the estimates:

$$(x + \kappa r^2 y + \kappa T z)^2 \leq x^2 \cdot \left( 1 + \frac{\kappa r^2}{4} + \kappa T \right) + y^2 \cdot (4\kappa r^2 + 2\kappa^2 r^4) + z^2 \cdot (\kappa T + 2\kappa^2 T^2)$$
for }x, y, z \in \mathbb{R}, \text{ and }
\left( \sum_{n=1}^{\infty} 2^{-3n/2} \sum_{k=1}^{2^n} p_{n,k} \right)^2 \leq \left( \sum_{n=1}^{\infty} 2^{-n} \right)^2 \cdot \left( \sum_{n=1}^{\infty} 2^{-n} \sum_{k=1}^{2^n} p_{n,k}^2 \right), \quad p_{n,k} \in \mathbb{R}.
\]

Since (4.27) holds for every }v \in T_zM \text{ with } |v| = 1, \Gamma_T (F)(\sigma) \text{ is dominated by the right-hand side of (4.27) as well. The assertion now follows by (4.12).} \quad \square

5. Poincaré inequalities on small balls in pinned path spaces

In this section, we prove Theorem 1.1. Let }a, b \in M. \text{ For a smooth cylinder function } F \text{ on } \Omega_{a,b}, \text{ we define inductively measurable functions } \Gamma_T^{n,k}(F) \text{ on } \Omega_{a,b}, T > 0, n \in \mathbb{N} \cup \{0\}, k = 1, 2, \ldots, 2^n, \text{ by:}

\[
\Gamma_T^{0,1}(F) = \Gamma_T (F),
\]

\[
\Gamma_T^{n+1,k}(F)(\omega \lor \omega) = \frac{1}{2} \Gamma_T^{n/2,k} \tilde{F}(\cdot, \omega)(\omega),
\]

\[
\Gamma_T^{n+1,2^n+k}(F)(\omega \lor \omega) = \frac{1}{2} \Gamma_T^{n/2,k} \tilde{F}(\omega, \cdot)(\omega)
\]

for all }n \geq 0, 1 \leq k \leq 2^n, z \in M, \omega \in \Omega_{z,a}, \text{ and } \overline{\omega} \in \Omega_{z,b}. \text{ Here } \omega \lor \overline{\omega} \text{ and } \tilde{F} \text{ are defined as in Section 4. Recall the definition of } \rho_t(r) \text{ from (4.5). For } r, T > 0 \text{ let}

\[
\hat{c}_T(r) = \sum_{n=0}^{\infty} c_{2^n} T(r) \rho_{2^{n+1}} T(r).
\]

Note that by Theorem 2.1 and (4.6), }\hat{c}_T(r) < \infty \text{ for all } T > 0, \text{ and}

\[
\limsup_{T \downarrow 0} T^{-1} \hat{c}_T(r) < \infty \quad \text{for every } r \in (0, r_0],
\]

where }r_0 > 0 \text{ is chosen as in Theorem 2.1. By iterating the key estimate from Lemma 4.1, we obtain:}

**Lemma 5.1.** Let }r \in (0, \infty) \text{ and } n \in \mathbb{N}. \text{ Then,}

\[
\text{Var}(F; P_{a,b}^T) \leq e^{2\hat{c}_T(r)} \cdot \sum_{j=0}^{n-1} 2^j c_{2^{j-T}} T(r) E_{a,b}^T \left[ \sum_{k=1}^{2^j} \Gamma_{j,k}^T(F) \right]
\]

for all }T > 0, a, b \in M, \text{ and every smooth cylinder function } F : \Omega_{a,b} \to \mathbb{R} \text{ of type}

\[
F(\omega) = f(\omega(2^{-n}), \omega(2 \cdot 2^{-n}), \ldots, \omega(1 - 2^{-n})) \quad \text{for some } f \in C^\infty(M^{2^n-1})
\]

such that }F(\omega) = 0 \text{ if } \max(d(\omega(k \cdot 2^{-n}), \omega(l \cdot 2^{-n})); 0 \leq k, l \leq 2^n) \geq r.

Proof. Let \( \delta > 0 \). We first show by induction on \( n \) that for \( F \) as in the assertion,

\[
\text{Var}(F; P_{x,b}^T) \leq (1 + \delta) \cdot \sum_{j=0}^{n-1} p_{j,T}(r, \delta) 2^j c_{2-j,T}(r) E_{x,b}^{T/2} \left[ \sum_{k=1}^{2^j} \Gamma_{j,k}^T(F) \right].
\]

(5.7)

where

\[
p_{j,T}(r, \delta) = \prod_{i=1}^{j} \left( 1 + 2 \cdot (1 + \delta^{-1}) \cdot c_{2-j,T}(r) \rho_{2-j,T}(r) \right).
\]

For \( n = 1 \), this follows directly from the assertion of Lemma 4.1, because

\[ p_{0,T}(r, \delta) = 1, \quad \Gamma_{0,1}^T(F) = \Gamma^T(F), \]

and \( \tilde{F} \) is constant on \( \Omega_{z,a} \times \Omega_{z,b} \) for every \( z \in \mathcal{M} \), if \( F(\omega) \) depends only on \( \omega(1/2) \).

Now suppose (5.7) holds for some \( n \geq 0 \), and let \( F \) be a function of type (5.6) with \( n \) replaced by \( n + 1 \). Then for \( z \in U_{a,b}, \omega \in \Omega_{z,a}, \) and \( \bar{\omega} \in \Omega_{z,b}, \tilde{F}(\omega, \cdot) \) and \( \tilde{F}(\cdot, \bar{\omega}) \) are smooth cylinder functions of type (5.6) on \( \Omega_{z,b}, \Omega_{z,a} \), respectively. Hence by the induction hypotheses and (5.3),

\[
\text{Var}(\tilde{F}(\omega, \cdot); P_{z,b}^{T/2}) \leq (1 + \delta) \cdot \sum_{j=0}^{n-1} p_{j,T/2}(r, \delta) 2^j c_{2-j,T}(r)
\times E_{z,b}^{T/2} \left[ \sum_{k=1}^{2^j} \Gamma_{j,k}^{T/2}(\tilde{F}(\omega, \cdot)) \right]
\]

\[ = (1 + \delta) \cdot \sum_{j=1}^{n} p_{j-1,T/2}(r, \delta) 2^j c_{2-j,T}(r)
\times E_{z,b}^{T/2} \left[ \sum_{k=1}^{2^{j-1}} \Gamma_{j-1,2-k}^T(F)(\omega \lor \cdot) \right],
\]

and, similarly, by (5.2),

\[
\text{Var}(\tilde{F}(\cdot, \bar{\omega}); P_{z,a}^{T/2}) \leq (1 + \delta) \cdot \sum_{j=1}^{n} p_{j-1,T/2}(r, \delta) 2^j c_{2-j,T}(r)
\times E_{z,a}^{T/2} \left[ \sum_{k=1}^{2^{j-1}} \Gamma_{j,k}^T(F)(\cdot \lor \bar{\omega}) \right]
\]

for all \( T > 0 \). Moreover, for all \( j \),
\[ p_{j-1, T/2}(r, \delta) = \prod_{i=1}^{j-1} \left( 1 + 2 \cdot (1 + \delta^{-1}) \cdot c_{2^{-i}T}(r) \rho_{2^{-i}T}(r) \right) = \prod_{i=2}^{j} \left( 1 + 2 \cdot (1 + \delta^{-1}) \cdot c_{2^{-i}T}(r) \rho_{2^{-i}T}(r) \right), \]

whence

\[ (1 + 2 \cdot (1 + \delta^{-1}) c_T(r) \rho_{T/2}(r)) \cdot p_{j-1, T/2}(r, \delta) = p_{j, T}(r, \delta). \]

Thus by Lemma 4.1 and (5.1),

\[
\text{Var}(F; P_{a,b}^T) \leq (1 + \delta)c_T(r) E_{a,b}^T \left[ \Gamma_{0,1}^T(F) \right] \\
+ (1 + \delta) \sum_{j=1}^{n} p_{j, T}(r, \delta) 2^j c_{2^{-j}T}(r) \\
\times \int_M E_{a,a}^{T/2} \left[ E_{b,b}^{T/2} \left[ \sum_{k=1}^{\infty} \Gamma_{j,k}^T(F)(W \lor \overline{W}) \right] \right] \mu_{a,b}(dz)
\]

for all \( T > 0 \), where we have used the notation from Lemma 4.1. Since

\[
E_{a,b}^T \left[ \Gamma_{j,k}^T(F) \right] = \int_M E_{a,a}^{T/2} \left[ E_{b,b}^{T/2} \left[ \sum_{k=1}^{\infty} \Gamma_{j,k}^T(F)(W \lor \overline{W}) \right] \right] \mu_{a,b}(dz),
\]

and \( p_{0,T}(r, \delta) = 1 \), the last estimate implies (5.7) with \( n \) replaced by \( n + 1 \).

To obtain the assertion from (5.7) note that

\[
p_{j, T}(r, \delta) \leq \exp \left( 2 \cdot (1 + \delta^{-1}) \sum_{i=1}^{\infty} c_{2^{-i}T}(r) \rho_{2^{-i}T}(r) \right) = \exp \left( 2 \cdot (1 + \delta^{-1}) \hat{c}_T(r) \right)
\]

for all \( j \geq 0 \) and \( T > 0 \). Hence

\[
(1 + \delta) p_{j, T}(r, \delta) \leq \exp \left( 2 \cdot (1 + \delta) \cdot \hat{c}_T(r) \right) \tag{5.8}
\]

for all \( j \geq 0, T > 0 \) and \( \delta > 0 \). The assertion follows from (5.5) and (5.8) when letting \( \delta \) tend to 0. \qed

The next estimate is an immediate consequence of Lemma 4.3. We define the functions \( g_m, m \in \mathbb{N} \cup \{0\} \), as in the lemma.
Lemma 5.2. Let \( r \in (0, \pi/(2\sqrt{\kappa})) \), \( T > 0 \), and \( a, b \in M \). Then for all \( n \geq 0 \), \( \omega \in \Omega_{a,b} \) such that
\[
\max(d(\omega(k \cdot 2^{-(n+1)}), \omega(l \cdot 2^{-(n+1)})) ; 0 \leq k, l \leq 2^n + 1) < r
\]
and for all smooth cylinder functions \( F \) on \( \Omega_{a,b} \),
\[
\sum_{k=1}^{2^n} \Gamma_{n,k}^T(F)(\omega) \leq 4 \cdot (1 + 2^{-n}kT) \cdot \sum_{k=1}^{2^n} \Gamma_{n,k}(F)(\omega)
\]
\[
+ \sum_{m=0}^{\infty} g_m(2^{-n}T, r) \sum_{k=1}^{2^{n+m}} \Gamma_{n+m,k}(F)(\omega).
\]

Proof. For \( n = 0 \) and \( k = 1 \) this is precisely the assertion of Lemma 4.3, cf. (5.1). Moreover, for all \( n \geq 0 \) and \( 1 \leq k \leq 2^n \),
\[
h_{n+1,k}(s) = \begin{cases} 2^{-1/2}h_{n,k}(2s) & \text{for } s \in [0, 1/2], \\ 0 & \text{else,} \end{cases}
\]
\[
h_{n+1,2^n+k}(s) = \begin{cases} 2^{-1/2}h_{n,k}(2s - 1) & \text{for } s \in [1/2, 1], \\ 0 & \text{else,} \end{cases}
\]
cf. (4.11). Since \( h_{n,k}(1 - s) = h_{n,2^n+1-k}(s) \) for all \( s \in [0, 1] \), (4.14) implies:
\[
\Gamma_{n+1,k}(F)(\omega \lor \overline{\omega}) = \frac{1}{2} \cdot \Gamma_{n,2^n+1-k}(\widetilde{F}(\cdot, \overline{\omega}))(\omega),
\]
\[
\Gamma_{n+1,2^n+k}(F)(\omega \lor \overline{\omega}) = \frac{1}{2} \cdot \Gamma_{n,k}(\widetilde{F}(\omega, \cdot))(\overline{\omega})
\]
for every smooth cylinder function \( F \) on \( \Omega_{a,b} \), \( z \in M \), \( \omega \in \Omega_{z,a} \), and \( \overline{\omega} \in \Omega_{z,b} \). In particular,
\[
\sum_{k=1}^{2^{n+1}} \Gamma_{n+1,k}(F)(\omega \lor \overline{\omega}) = \frac{1}{2} \sum_{k=1}^{2^n} \Gamma_{n,k}(\widetilde{F}(\cdot, \overline{\omega}))(\omega) + \frac{1}{2} \sum_{k=1}^{2^n} \Gamma_{n,k}(\widetilde{F}(\omega, \cdot))(\overline{\omega}).
\]
Similarly by (5.2) and (5.3),
\[
\sum_{k=1}^{2^{n+1}} \Gamma_{n+1,k}(F)(\omega \lor \overline{\omega}) = \frac{1}{2} \sum_{k=1}^{2^n} \Gamma_{n,k}^{T/2}(\widetilde{F}(\cdot, \overline{\omega}))(\omega) + \frac{1}{2} \sum_{k=1}^{2^n} \Gamma_{n,k}^{T/2}(\widetilde{F}(\omega, \cdot))(\overline{\omega})
\]
for all \( T > 0 \). If \( \omega \lor \overline{\omega} \) satisfies (5.9) with \( n \) replaced by \( n + 1 \), then both \( \omega \) and \( \overline{\omega} \) satisfy (5.9) with \( n \). Moreover, \( \widetilde{F}(\omega, \cdot) \) and \( \widetilde{F}(\cdot, \overline{\omega}) \) are again smooth cylinder functions. Therefore, by using the last two equations above, the assertion follows by induction on \( n \). \( \square \)
We choose $r_0 > 0$ as in Theorem 2.1. By combining the estimates from Lemmas 5.1 and 5.2, Theorem 2.1, and (5.5), we obtain:

**Lemma 5.3.** There exists a finite constant $C_1$ and a function $u : (0, \infty) \times (0, \infty) \to (0, \infty)$ with

$$\limsup_{T \downarrow 0} u(T, r) \leq 1 + C_1 r^2$$

for every $r > 0,

such that

$$\text{Var}(F; P_{a,b}^T) \leq u(T, r) \cdot T \cdot E_{a,b}^{T}((D^0 F, D^0 F))$$

holds for all $a, b \in M$, $T > 0$, $r \in (0, \min(r_0, \pi/(2\sqrt{\kappa}))$, $n \in \mathbb{N}$, and all smooth cylinder functions $F : \Omega_{a,b} \to \mathbb{R}$ depending only on $\omega(k \cdot 2^{-n})$, $1 \leq k < 2^n$, such that $F(\omega) = 0$ if $\max(d(\omega(k \cdot 2^{-n}), \omega(l \cdot 2^{-n}))) \leq r$.

**Proof.** Fix $r \in (0, \pi/(2\sqrt{\kappa}))$, $T > 0$, $a, b \in M$, and $n \in \mathbb{N}$. For a cylinder function $F$ as in the assertion, we obtain by Lemmas 5.1 and 5.2:

$$\text{Var}(F; P_{a,b}^T) \leq e^{2\hat{c}_T(r)} \cdot \sum_{j=0}^{n-1} 2^j c_{2^{-j}T}(r) \cdot \left\{ 4 \cdot (1 + 2^{-j} \kappa T) \cdot E_{a,b}^{T} \left[ \sum_{k=1}^{2^j} \Gamma_{j,k}(F) \right] + \sum_{m=0}^{\infty} g_m(2^{-j}T, r) E_{a,b}^{T} \left[ \sum_{k=1}^{2^{j+m}} \Gamma_{n+m,k}(F) \right] \right\}$$

$$= e^{2\hat{c}_T(r)} \cdot \sum_{j=0}^{n-1} A_j(T, r) \cdot E_{a,b}^{T} \left[ \sum_{k=1}^{2^j} \Gamma_{j,k}(F) \right],

(5.12)$$

where

$$A_j(T, r) = 4 \cdot (1 + 2^{-j} \kappa T) 2^j c_{2^{-j}T}(r) + \sum_{m=0}^{j} g_m(2^{m-j}T, r) 2^{j-m} c_{2^{m-j}T}(r).$$

By definition of the functions $g_m$, $m \in \mathbb{N}$,

$$\sum_{m=0}^{j} g_m(2^{m-j}T, r) \leq 2 \kappa r^2 \cdot \left( 1 + \frac{1}{2} \kappa r^2 \right) + 2(d - 1)^2 \kappa T (1 + 2\kappa T) e^{(d-1)\kappa T/2} (1 + \kappa r^2)^2$$

for every $j \in \mathbb{N} \cup \{0\}$, cf. (4.15). Hence for all $j,$
By combining (5.12) and (5.13) with the estimates for $c_T(r)$ in Theorem 2.1 and $\hat{c}_T(r)$ in (5.5), we obtain the claimed assertion. □

**Proof of Theorem 1.1.** Fix $\xi \in (0, 1)$ and let $R_0 = \xi \cdot \min(r_0, \pi/(2\sqrt{T}))$. Moreover, fix $a, b \in M$, $T > 0$, and $r \in (0, R_0)$, and let $F$ be an arbitrary function in $H^{1,2}(\Omega_{a,b}; \mathbb{P}_r^\alpha)$ that vanishes outside $\Omega_{a,b}$. There exists a sequence $F_n, n \in \mathbb{N}$, of smooth cylinder functions on $\Omega_{a,b}$ such that $F_n \to F$ in $H^{1,2}(\Omega_{a,b}; \mathbb{P}_r^\alpha)$. Moreover, one easily verifies that every smooth cylinder function can be approximated w.r.t. the $H^{1,2}$ norm by dyadically based smooth cylinder functions. Hence we may assume w.l.o.g. that for every $n \in \mathbb{N}$,

$$F_n(\omega) = f_n(\omega(2^{-n}), \omega(2 \cdot 2^{-n}), \ldots, \omega(1 - 2^{-n})) \quad \forall \omega \in \Omega_{a,b}$$

(5.14)

with some function $f_n \in C^\infty(M^{2^n})$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth decreasing function such that $\varphi(t) = 1$ for $t \leq r$, $\varphi(t) = 0$ for $t \geq r/\xi$ and $|\varphi'| \leq 2r^{-1} \cdot (1 - \xi)^{-1}$. For $n \in \mathbb{N}$ and $\omega \in \Omega_{a,b}$ let

$$\Psi_n(\omega) = \varphi \left( \max_{0 \leq i,j \leq 2^n} d(\omega(i \cdot 2^{-n}), \omega(j \cdot 2^{-n})) \right).$$

Then $\Psi_n$ is in $H^{1,2}(\Omega_{a,b}; \mathbb{P}_r^\alpha)$ and

$$\left( D^0 \Psi_n, D^0 \Psi_n \right)_{\omega}^{1/2} \leq 4r^{-1} (1 - \xi)^{-1} \quad \text{for } \mathbb{P}_r^\alpha \text{-a.e. } \omega.$$ (5.15)

Moreover, $\Psi_n = 1$ on $\Omega_{a,b}^\alpha$, and thus

$$D^0 \Psi_n \equiv 0 \quad \text{on } \Omega_{a,b}^\alpha.$$ (5.16)

Let $\tilde{F}_n = F_n \cdot \Psi_n$. Then $\tilde{F}_n$ is again of type (5.14) with $f_n$ replaced by a Lipschitz continuous function $\tilde{f}_n : M^{2^n} \to \mathbb{R}$. Moreover, $\tilde{F}_n(\omega)$ vanishes if $\max(d(\omega(k \cdot 2^{-n}), \omega(l \cdot 2^{-n}))) \leq r/\xi$. Note that $r/\xi \leq R_0/\xi = \min(r_0, \pi/(2\sqrt{T}))$. Approximating $\tilde{f}_n$ by smooth functions in a suitable way shows that (5.11) with $r$ replaced by $r/\xi$ holds for $\tilde{F}_n$, i.e.,

$$\text{Var}(\tilde{F}_n; \mathbb{P}_r^\alpha) \leq u(T, r/\xi) \cdot T \cdot E_{a,b}^{\mathbb{P}_r^\alpha}(D^0 \tilde{F}_n, D^0 \tilde{F}_n)$$

for all $n \in \mathbb{N}$, (5.17)

where $u$ is chosen as in Lemma 5.3. Furthermore, $\tilde{F}_n \to F$ in $H^{1,2}(\Omega_{a,b}; \mathbb{P}_r^\alpha)$ as $n \to \infty$. In fact, for $n \in \mathbb{N}$,

$$D^0 \tilde{F}_n = \Psi_n \cdot D^0 F_n + F_n D^0 \Psi_n \quad \text{on } \mathbb{P}_r^\alpha \text{-a.e.}$$ (5.18)
6. Poincaré inequalities for pinned paths with jumps of limited size

Since \( \Psi_n = 1 \) on \( \Omega_{a,b}^r \), and \( |\Psi_n| \leq 1 \), \( F = 0 \) and \( D^0 F = 0 \) \( P_{a,b}^T \) a.e. outside \( \Omega_{a,b}^r \), we have \( |F_n - F| \leq |F_n - F| \) and \( |\Psi_n \cdot D^0 F_n - D^0 F| \leq |D^0 F_n - D^0 F| \) \( P_{a,b}^T \) a.e. on \( \Omega_{a,b} \).

Hence both terms converge to 0 in \( L^2(\Omega_{a,b}^r; P_{a,b}^T) \). Moreover, by (5.15) and (5.16), \( |F_n D^0 \Psi_n| \leq 4 \cdot r^{-1} \cdot (1 - \xi)^{-1} \cdot F_n \cdot \chi_{\Omega_{a,b}^r \setminus \Omega_{a,b}^r} \). Since \( F_n \rightarrow F \) in \( L^2(\Omega_{a,b}^r; P_{a,b}^T) \), and \( F \) vanishes outside \( \Omega_{a,b}^r \), \( |F_n \cdot D^0 \Psi_n| \), and thus by (5.18), \( |D^0 F_n - D^0 F| \) converge to 0 in \( L^2(\Omega_{a,b}^r; P_{a,b}^T) \) as well. Hence \( F_n \rightarrow F \) in \( H^{1,2}(\Omega_{a,b}^r; P_{a,b}^T) \). By letting \( n \) tend to infinity in (5.17), we obtain:

\[
\text{Var}(F; P_{a,b}^T) \leq u(T, r/\xi) \cdot T \cdot E_{a,b}^T[(D^0 F, D^0 F)].
\]

Since this estimate holds for every function \( F \in H^{1,2}_0(\Omega_{a,b}^r; P_{a,b}^T) \),

\[
T \cdot \lambda(\Omega_{a,b}^r; P_{a,b}^T) \geq 1/u(T, r/\xi).
\]

The assertion of Theorem 1.1 follows by (5.10), because \( u \) does not depend on \( a \) and \( b \).

Proof of Lemma 1.2. Let \( \varphi : R \rightarrow R \) be decreasing and smooth with \( \varphi(t) = 1 \) for \( t \leq R/2 \), \( \varphi(t) = 0 \) for \( t \geq R \), and \( |\varphi| \leq 4/R \). We define cut-off functions \( \Psi_N : \Omega_{x,y} \rightarrow R \), \( N \in \mathbb{N} \), by:

\[
\Psi_N(\omega) = \varphi(\sup\{d(\omega(s), \omega(t)); s, t \in [0, 1] \cap Q, |s - t| \leq N^{-1}\}).
\]

Clearly, \( \Psi_N = 1 \) on \( \Omega_{x,y}^{R/2,N} \) and \( \Psi_N = 0 \) on \( \Omega_{x,y} \setminus \Omega_{x,y}^{R,N} \). Since \( R < \text{inj}(M) \), the functions

\[
\Phi_{x,t}(\omega) = \varphi \circ d(\omega(s), \omega(t)), \quad s, t \in [0, 1],
\]

are smooth cylinder functions on \( \Omega_{x,y} \). We have:

\[
(D^0 \Phi_{x,t}, D^0 \Phi_{x,t})_{\omega}^{1/2} \leq 8/R \quad \text{for all } \omega \in \Omega_{x,y}.
\]

For every \( N \in \mathbb{N} \), \( \Psi_N \) is an infimum of such functions. Thus \( \Psi_N \) is in \( H^{1,2}(\Omega_{x,y}; P_{x,y}^T) \) and \( |D^0 \Psi_N| \leq 8/R \) \( P_{x,y}^T \) a.s. as well. Moreover, \( D^0 \Psi_N \) vanishes \( P_{x,y}^T \) a.s. on \( \Omega_{x,y}^{R/2,N} \).

Now let \( F \) be a function in \( H^{1,2}(\Omega_{x,y}; P_{x,y}^T) \). Then \( F \cdot \Psi_N \) is contained in \( H^{1,2}_0(\Omega_{x,y}; P_{x,y}^T) \) for every \( N \in \mathbb{N} \). The sequence \( F - F \cdot \Psi_N \) converges to 0 in
to complete the proof it only remains to show inf

i.e., (6.1) holds. ✷

In fact, let

for every

Then the modified sequence

Moreover,

for every

Let

Clearly, this is a decreasing sequence. To complete the proof it only remains to show

i.e.,

for every

To show this, we may assume

Let

Then in particular,

W.l.o.g. we may assume

In fact, let

such that

is not empty, and let

with

Then the modified sequence

converges to

as well. Moreover,

is in

and

for every

Hence we may replace

by

Now assume (6.2). Then

i.e., (6.1) holds. □

Now fix

We want to extend the key estimate from Lemma 4.1 to the case where the interval

is divided into

intervals of equal size. For a path

let

denote the “segments” defined by:
\[ p_k(\omega)(s) = \omega \left( (k - 1 + s)/N \right), \quad s \in [0, 1], \; 1 \leq k \leq N. \]

Moreover, we set:
\[ r(s) = 1 - s, \quad 0 \leq s \leq 1 \quad \text{(time reversal)}. \]

For \( T > 0, \omega \in \Omega_{x,y} \) such that \( \omega(k/N) \notin \text{Cut}(\omega((k - 1)/N)) \) for \( 1 \leq k \leq N \), and a tangent vector \( v = (v_1, \ldots, v_{N-1}) \in T_{(\omega(1/N), \omega(2/N), \ldots, \omega(1-1/N))}M^{N-1} \) let
\[
\hat{X}_{T,N,v}(\omega) = X_{N-(k-1)}^{T/N,\omega((k-1)/N),\omega(k/N),v_k}(p_k(\omega)) \\
+ X_{k-N}^{T/N,\omega((k-1)/N),\omega(k/N),v_k}(p_k(\omega) \circ r)
\]
for \( s \in [(k-1)/N, k/N] \), \( 1 \leq k \leq N \), where we set \( v_0 = 0 \) and \( v_N = 0 \). With the notation from Section 4,
\[
\hat{X}_{T,N,v}(\omega) = \sum_{k=1}^{N-1} \tilde{X}^{(k)}_{s}(\omega), \quad (6.5)
\]
where
\[
\tilde{X}^{(k)}_{s}(\omega) = \begin{cases} 
\hat{X}_{(N-(k-1))/2}^{T/N,(p_k(\omega) \circ r) \lor p_{k+1}(\omega))} \\
0 & \text{for } s \in \left[ \frac{k-1}{N}, \frac{k+1}{N} \right], \quad (6.6)
\end{cases}
\]
This shows in particular that \( \hat{X}_{T,N,v}(\omega) \) is a piecewise smooth, continuous vector field along \( \omega \). Since \( \hat{X}_{T,N,v}(0) = 0 \) and \( \hat{X}_{T,N,v}(\omega) = 0 \), \( \hat{X}_{T,N,v}(\omega) \) is contained in \( T_0^1 \Omega_{x,y} \). If the segments \( p_k(\omega), 1 \leq k \leq n \), are minimal geodesics, then \( \hat{X}_{T,N,v}(\omega) \) is a perturbation of the piecewise Jacobi field \( Y \) along \( \omega \) which is Jacobi on \( [(k-1)/N, k/N] \) for all \( 1 \leq k \leq N \), and satisfies \( Y(k/N) = v_k \) for \( 0 \leq k \leq N \). For \( F \in \mathcal{F}C^\infty \), let
\[
\Gamma_{T,N}(F)(\omega) = \sup_{v \in T_{(\omega(1/N), \ldots, \omega(1-1/N))}M^{N-1}} \left\{ \langle F, v, v \rangle \right\}. \quad (6.7)
\]
Here \( \langle \cdot, \cdot \rangle \) denotes the product metric on \( M^{N-1} \). For \( \omega \in \Omega_{x,y} \) with \( \omega(k/N) \in \text{Cut}(\omega((k-1)/N)) \) for some \( 1 \leq k \leq N \), we set \( \Gamma_{T,N}(F)(\omega) = 0 \). As in Section 4, one easily verifies that \( \Gamma_{T,N}(F) : \Omega_{x,y} \to \mathbb{R} \) is measurable.

Now, let
\[
U_{x,y}^{R,N} = \left\{ z \in M^{N-1} ; \; d(x, z_i) < R, \quad d(z_i, z_{i+1}) < R \; \forall 1 \leq i \leq N - 1, \; d(z_{N-1}, y) < R \right\},
\]
and
\[
\mu^{T,N}_{x,y} = \frac{\mu^{T/N}(x, z_1) \mu^{T/N}(z_1, z_2) \cdots \mu^{T/N}(z_{N-2}, z_{N-1}) \mu^{T/N}(z_{N-1}, y)}{\mu^T(x, y)} V^{N-1}(dz)
\]
denote the distribution of $\omega \mapsto (\omega(i/N); 1 \leq i \leq N - 1)$ w.r.t. $P_{x,y}^T$. Similarly as above, we define:

$$\lambda(U_{x,y}^{R,N}; \mu_{x,y}^T) = \inf \left( \int_{U_{x,y}^{R,N}} |\nabla f|^2 \, d\mu_{x,y}^T / \int_{U_{x,y}^{R,N}} (f - \frac{1}{T} \int_{U_{x,y}^{R,N}} f \, d\mu_{x,y}^T)^2 \, d\mu_{x,y}^T \right),$$

where the infimum is taken over all $f \in \mathbb{C}_\infty(\overline{M} - 1)$, and $\int$ denotes integration w.r.t. the normalized measure $\mu_{x,y}^T/\mu_{x,y}(U_{x,y}^{R,N})$. $\lambda^*(U_{x,y}^{R,N}; \mu_{x,y}^T)$ is defined correspondingly with the infimum taken only over $f \in \mathbb{C}_0(\overline{M} - 1)$ with $\int f \, d\mu_{x,y}^T = 0$. The asymptotics of these quantities as $T \downarrow 0$ has been studied in [17].

For a function $F: \Omega_{x,y} \to \mathbb{R}$ let $F[N]$ denote the unique function defined on $\prod_{i=1}^N \Omega_{z_i-1,z_i}$ for all $z_1, \ldots, z_{N-1} \in \mathbb{M}$, $z_0 := x$, and $z_N := y$ such that $F[N](p_1(\omega), \ldots, p_N(\omega)) = F(\omega)$ for all $\omega \in \Omega_{x,y}$.

Similarly to Lemma 4.1, we obtain:

**Lemma 6.1.** Let $\delta \in (0, \infty)$. We set $z_0 = x$ and $z_N = y$. Then,

$$\text{Var}(F; P_{x,y}^T) \leq (1 + \delta) \cdot \lambda(U_{x,y}^{R,N}; \mu_{x,y}^T)^{-1} \cdot \mathbb{E}_{x,y}^T \left( \Gamma_{T,N}(F) \right)$$

$$+ \left( 1 + (1 + \delta)^{-1} \cdot 4N \lambda(U_{x,y}^{R,N}; \mu_{x,y}^T)^{-1} \cdot \rho_{T,N}(R) \right)$$

$$\times \left( \sum_{k=1}^N \int_{U_{x,y}^{R,N}} \text{Var}(F[N](\omega_1, \ldots, \omega_k-1, \cdot, \omega_{k+1}, \ldots, \omega_N); P_{z_i-1,z_i}^T) \right)$$

$$\times \prod_{i \neq k} P_{z_i-1,z_i}^T (d\omega_i) \right) \mu_{x,y}^T (dz_1 \cdots dz_{N-1}) \quad (6.8)$$

for all $T > 0$ and every smooth cylinder function $F: \Omega_{x,y} \to \mathbb{R}$ such that $F(\omega) = 0$ if $(\omega(1/N), \omega(2/N), \ldots, \omega((1-1/N)))$ is not in $U_{x,y}^{R,N}$. Moreover, the same estimate with $\lambda(U_{x,y}^{R,N}; \mu_{x,y}^T)$ replaced by $\lambda^*(U_{x,y}^{R,N}; \mu_{x,y}^T)$ holds for all $F$ as above with $\mathbb{E}_{x,y}^T[F] = 0$.

**Proof.** Fix $T > 0$, and let $\lambda_T = \lambda(U_{x,y}^{R,N}; \mu_{x,y}^T)$. We first remark that the Poincaré inequality

$$\text{Var}(f; \mu_{x,y}^T) \leq \lambda_T^{-1} \cdot \int_{U_{x,y}^{R,N}} (\nabla f, \nabla f) \, d\mu_{x,y}^T$$

extends from functions $f \in \mathbb{C}_\infty(\mathbb{M}^{N-1})$ with supp $f \subset U_{x,y}^{R,N}$ to arbitrary $f \in \mathbb{C}_\infty(\mathbb{M}^{N-1})$ that vanish outside $U_{x,y}^{R,N}$. In fact, let $\psi_e: \mathbb{R} \to [0, 1]$, $e > 0$, be smooth and increas-
ing functions with $\psi_\varepsilon(x) = 0$ for $x \leq \varepsilon/2$, $\psi_\varepsilon(x) = 1$ for $x \geq \varepsilon$, and $\psi'_\varepsilon \leq 4\varepsilon^{-1}$. For $f \in C^\infty(M^{N-1})$ with $\text{supp}(f) \subset \overline{U}^R_{x,y}$ let $f_\varepsilon = f \cdot \psi_\varepsilon \circ \text{dist}(\cdot, \partial U^R_{x,y})$, $\varepsilon > 0$. Clearly, $f_\varepsilon \in C^\infty(U^R_{x,y})$ for all $\varepsilon > 0$, and $f_\varepsilon \to f$ in $L^2(U^R_{x,y}; \mu^T_{x,y})$ as $\varepsilon \downarrow 0$. Moreover, $\text{grad}(f_\varepsilon - f) = 0$ for $z \in U^R_{x,y}$ with $d(x,z) \geq \varepsilon$, and

$$|\text{grad}(f_\varepsilon - f)| \leq |\text{grad } f| + 4\varepsilon^{-1} \cdot \sup |\text{grad } f| \cdot \varepsilon \leq 5 \sup |\text{grad } f|$$

else. Hence (6.9) for $f$ follows from (6.9) for $f_\varepsilon$ as $\varepsilon \downarrow 0$.

Now fix $F$ as in the assertion. Note that $F^{[N]}$ vanishes on $\prod_{i=1}^N \Omega_{z_i,1-z_i}$ for $(z_1, \ldots, z_{N-1}) \notin U^R_{x,y}$. Let $f_T^N(z_1, \ldots, z_{N-1}) = \int F^{[N]} \prod_{i=1}^N dP_{z_{i-1},z_i}^{T/N}$. Then $f_T^N$ is smooth and vanishes outside $U^R_{x,y}$. Similarly to (4.7) we obtain:

$$\text{Var}(F; P_T^{T_{x,y}}) \leq \text{Var}(f_T^N; \mu^T_{x,y}) + \int_{U^R_{x,y}} \text{Var}(F^{[N]}, \prod_{i=1}^N P_{z_{i-1},z_i}^{T/N}) \mu^T_{x,y} (dz_1 \cdots dz_{N-1}). \quad (6.10)$$

Similarly to (4.8), the second term can be estimated by the outer integral on the right-hand side of (6.8). Moreover, (6.9) holds for $f_T^N$. It remains to estimate $\langle \text{grad } f_T^N, \text{grad } f \rangle$. Fix $z = (z_1, \ldots, z_{N-1}) \in U^R_{x,y}$ and $v = (v_1, \ldots, v_{N-1}) \in T_{z_1, \ldots, z_{N-1}} M^{N-1}$, and let $v_0 = 0 \in T_x M$ and $v_N = 0 \in T_y M$. Then similarly as in the proof of Lemma 4.1, we obtain:

$$(d_z f_T^N)[v]
= \sum_{k=1}^{N-1} (d_z^{(k)} f_T^N)[v_k]
= \sum_{k=1}^{N-1} \left( d_z \int F^{[N]}(\omega_1, \ldots, \omega_N) P_{z_{k-1},z_k}(d\omega_k) \right) [v_k] \prod_{i \neq k} P_{z_{i-1},z_i}(d\omega_i)
+ \sum_{k=1}^{N-1} \left( d_z \int F^{[N]}(\omega_1, \ldots, \omega_N) P_{z_{k-1},z_k}(d\omega_k+1) \right) [v_k] \prod_{i \neq k+1} P_{z_{i-1},z_i}(d\omega_i)
\leq \int (\tilde{\chi}^{T,N,v} F)(\tilde{\rho}(\omega_1, \ldots, \omega_N)) \prod_{i=1}^{N-1} P_{z_{i-1},z_i}(d\omega_i)
+ \rho_{T/N}(R)^{1/2} \sum_{k=1}^N (|v_k| + |v_{k-1}|)
\times \int \text{Var}(F^{[N]}(\omega_1, \ldots, \omega_{k-1}, \omega_{k+1}, \ldots, \omega_N); P_{z_{k-1},z_k}^{T/N})^{1/2} \prod_{i \neq k} P_{z_{i-1},z_i}(d\omega_i), \quad (6.11)$$
where \( \hat{p}(\omega_1, \ldots, \omega_N) \) denotes the unique path in \( \Omega_{x,y} \) such that \( p_k(\hat{p}(\omega_1, \ldots, \omega_N)) = \omega_k \) for all \( 1 \leq k \leq N \). Here we have applied Theorem 3.2 with the vector fields \( X_{T/N; \omega, z_k-1, \omega_k} \) and \( X_{T/N; \omega, z_k+1, \omega_k} \) to estimate the summands in the first and second sum, respectively, and we have used that for \( a \in M \), \( P_{z_k-1, a}^T \) is the image of \( P_{z_k-1}^T \) under the time reversal \( \omega \mapsto \omega \circ r \). Moreover,

\[
\left( \sum_{k=1}^{N} (|v_k| + |v_{k-1}|) \right)^2 = 4 \cdot \left( \sum_{k=1}^{N-1} |v_k| \right)^2 \leq 4N \cdot \sum_{k=1}^{N-1} |v_k|^2 = 4N(v, v).
\]

Since

\[
|\nabla f_N^T, \nabla f_N^T| = \sup \{|(d_z f_N^T)[v]|^2 : v \in T_z M^{N-1} \}, \quad (v, v) = 1,
\]

(6.8) now follows from (6.10), (6.9) for \( f_N^T \), (6.11), and (6.7).

If, moreover, \( E_{x,y}^T[F] = 0 \) then \( \int f_N^T d\mu_{x,y} = 0 \) as well. Hence (6.9) holds with \( f \) replaced by \( f_N^T \) and \( \lambda_T = \lambda^*(U_{x,y}, \mu_{x,y}^{T/N}) \). Thus we obtain (6.8) with \( \lambda \) replaced by \( \lambda^* \) in the same way as before. \( \square \)

Similarly to Lemma 4.3, we can derive an upper bound for \( \Gamma_{T,N}(F) \):

**Lemma 6.2.** Suppose that \( R < \pi/(2\sqrt{R}) \). Then for \( T > 0 \) and \( \omega \in \Omega_{x,y} \), with \( \omega(1/N), \omega(2/N), \ldots, \omega((N-1)/N) \) in \( U_{x,y}^{R,N} \),

\[
\Gamma_{T,N}(F)(\omega) \leq 6N \cdot (1 + g(2T/N, R)) \cdot (D^0, D^0 F)_{\omega} \quad \forall F \in FC^\infty,
\]

where

\[
g(T, R) = \frac{k^2 R^4}{48} + \left( d - 1 \right) \kappa T \cdot \left( 1 + \frac{d - 1}{4 \kappa T} \right) e^{(d - 1) \kappa T/2} \left( 1 + \kappa R^2 \right)^2.
\]

**Proof.** Fix \( T > 0 \), \( \omega \in \Omega_{x,y}^{R,N} \), and \( v \in T_{\omega(1/N), \omega(2/N), \ldots, \omega((N-1)/N)} M^{N-1} \). With the notation from (6.5),

\[
|\tilde{X}_{T,N,v}(\omega)|^2 = \sum_{k,l=1}^{N-1} (\tilde{X}^{(k)}(\omega), \tilde{X}^{(l)}(\omega)) \leq 3 \sum_{k=1}^{N-1} |\tilde{X}^{(k)}(\omega)|^2,
\]

(6.12) because \( (\tilde{X}^{(k)}(\omega), \tilde{X}^{(l)}(\omega)) = 0 \) if \( |k - l| > 1 \). Moreover, with the notation from Lemma 4.3, we have:

\[
|\tilde{R}^{0,w}(\sigma)|^2 = \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} |\sigma_{n,k}|^2 \leq 4 \cdot \frac{k^2 R^4}{4} \cdot \sum_{n=1}^{\infty} 2^{-2n} = 4 + \frac{k^2 R^4}{12}.
\]
and

$$|\tilde{X}^{T,w}(\sigma) - \tilde{X}^{0,w}(\sigma)| \leq (d-1)\kappa T \cdot e^{(d-1)\kappa T/4} \cdot (1 + \kappa R^2)$$

for all $T > 0$, $\sigma \in C([0, 1], M)$ with $d(\sigma(1/2), \sigma(0)) < R$ and $d(\sigma(1/2), \sigma(1)) < R$ and all $\omega \in T_{\sigma(1/2)}M$, cf. (4.20), (4.21) and (4.25). Combining these estimates implies

$$|\tilde{X}^{T,w}(\sigma)|^2 \leq 4 \cdot (1 + g(T, R))$$

for $T$, $\omega$, and $\sigma$ as above. Hence by (6.5),

$$|\tilde{X}^{(k)}(\omega)|^2 = N^2 \cdot |\tilde{X}^{2T/N,v}((p_k(\omega) \circ r) \vee p_{k+1}(\omega))|^2$$

$$\leq 2, N \cdot (1 + g(2T/N, R)) \cdot |v_k|^2$$

(6.13)

for $1 \leq k \leq N - 1$, and thus by (6.12),

$$|\tilde{X}^{T,N,v}(\omega)|^2 \leq 6N \cdot (1 + g(2T/N, R)) \cdot \langle v, v \rangle.$$

The assertion now follows by definition of $\Gamma^{T,N}$. \qed

We choose $r_0 > 0$ as in Theorem 2.1, and $u, g : (0, \infty) \times (0, \infty) \to (0, \infty)$ as in Lemmas 5.3, 6.2, respectively. By combining Lemmas 6.1, 6.2 and 5.3, we obtain:

**Lemma 6.3.** Let $R \in (0, \min(r_0, \pi/(2\sqrt{R})))$, and let

$$C_N(T, R) = 12N \cdot \chi(U_{x,y}^{R,N}; \mu_{x,y}^{T,N})^{-1} \left( 1 + g(2T/N, R) + T \rho_{T/N}(R) \cdot u(T/N, R) \right)$$

$$+ T \cdot u(T/N, R).$$

Then

$$\text{Var}(F; P_{T,x,y}^T) \leq C_N(T, R) \cdot E_{T,x,y}^T \left[ (D^0 F, D^0 F) \right]$$

(6.14)

holds for all $T > 0$, $n \in N$, and every smooth cylinder function $F : \Omega_{x,y} \to R$ depending only on $\omega(k/(N \cdot 2^n))$, $1 \leq k < N \cdot 2^n$, such that $F(\omega) = 0$ if

$$\max \left\{ d \left( \omega \left( \frac{l + i \cdot 2^{-n}}{N} \right), \omega \left( \frac{l + j \cdot 2^{-n}}{N} \right) \right) : 0 \leq i, j \leq 2^n, \ 0 \leq l < N \right\} \geq R.$$

Moreover, let $C_N^*(T, R)$ be defined as $C_N(T, R)$ but with $\chi(U_{x,y}^{R,N}; \mu_{x,y}^{T,N})$ replaced by $\lambda^*(U_{x,y}^{R,N}; \mu_{x,y}^{T,N})$. Then (6.14) with $C_N(T, R)$ replaced by $C_N^*(T, R)$ holds for all $T > 0$, $n \in N$, and $F$ as above with $E_{T,x,y}^T [F] = 0$. 
Proof. Fix $T > 0$ and $n \in \mathbb{N}$, and let $F$ be a cylinder function as in the assertion. Then $F$ satisfies the assumptions from Lemma 6.1, and $F^{[N]}$ satisfies the assumptions from Lemma 5.3 as a function in each of its components. Hence by Lemma 6.1 with $\delta = 1$, Lemmas 6.2 and 5.3,

$$
\text{Var}(F; P^T_{x,y}) \\
\leq 12 \cdot N(1 + g(2T/N, R)\lambda(U^{R,N}_{x,y}; \mu_{x,y}^{T,N} \right)^{-1} \cdot E^T_{x,y}[|D^0 F|^2] \\
+ (1 + 8N\lambda(U^{R,N}_{x,y}; \mu_{x,y}^{T,N})^{-1}) \cdot u(T/N, R) \\
\times \frac{T}{N} \cdot \sum_{k=1}^N |D^0_{(k)} (p_1(\omega), p_2(\omega), \ldots, p_N(\omega))|_{P_t(\omega)} P^T_{x,y}(d\omega). \quad (6.15)
$$

Here $D^0_{(k)}$ means that the gradient $D^0$ is applied to $F^{[N]}$ as a function in its $k$th variable, and we have used that $F^{[N]}$ vanishes on $\prod_{z=1}^N \Omega_{z-1,z}$ for $z \in M^{N-1} \setminus U^{R,N}_{x,y}$.

Now fix $\omega \in \Omega_{x,y}$. Let $X_k \in T^1_{p_t(\omega)} \Omega_{w((k-1)/N),w(k/N)}$, $1 \leq k \leq N$, and let $X \in T^1_{w} \Omega_{x,y}$ with $X((k-1+s)/N) = X_k(s)$ for all $1 \leq k \leq N$ and $s \in [0, 1]$. We have:

$$
\sum_{k=1}^N (X^{(k)} F^{[N]})(p_1(\omega), \ldots, p_N(\omega)) = (XF)(\omega) \leq |X|_{\omega} \cdot |D^0 F|_{\omega}
\leq \left( N \cdot \sum_{k=1}^N |X^{(k)}|^2_{p_t(\omega)} \right)^{1/2} \cdot |D^0 F|_{\omega},
$$

where $X^{(k)}$ denotes the directional derivative in direction $X_k$ in the $k$th component of $F^{[N]}$. By taking the supremum over all $X_k$, $1 \leq k \leq N$, $N$ as above with $\sum |X^{(k)}|^2_{p_t(\omega)} \leq 1$, we obtain:

$$
\sum_{k=1}^N |D^0_{(k)} (p_1(\omega), \ldots, p_N(\omega))|_{P_t(\omega)} \leq N \cdot |D^0 F|_{\omega}^2, \quad (6.16)
$$

(6.14) follows from (6.15) and (6.16). The proof of the corresponding estimate with $C_N(T, R)$ replaced by $C^*_N(T, R)$ for $F$ with $F^{T}_{x,y}[F] = 0$ is similar. $\square$

Let $\overline{C}_N(T, R) = \limsup_{r \downarrow R} C_N(T, R)$ and $\overline{C}^*_N(T, R) = \limsup_{r \downarrow R} C^*_N(T, R)$ with $C_N(T, R)$ and $C^*_N(T, R)$ as in Lemma 6.3.

Corollary 6.4. Let $R \in (0, \min(r_0, \pi/(2\sqrt{K}))$. Then

$$
\lambda(U^{R,N}_{x,y}; P^{T}_{x,y} ; \mu_{x,y}) \geq \overline{C}_N(T, R), \quad (6.17)
$$

$$
\lambda^*(U^{R,N}_{x,y}; P^{T}_{x,y} ; \mu_{x,y}) \geq \overline{C}^*_N(T, R) \quad \text{for all } T > 0. \quad (6.18)
$$
Proof. Let \( r \in (R, \min(r_0, \pi/(2\sqrt{\kappa}))) \), and let \( F \in H^{1,2}_0(\Omega_{x,y}; P^T_{x,y}) \). Then Lemma 6.3 implies:

\[
\text{Var}(F; P^T_{x,y}) \leq C_N(T, R) \cdot E^T_{x,y}[(D^0F, D^0F)].
\]

This can be shown similarly to the proof of (5.19)—we just have to use the cut-off functions \( \Psi_n(\omega) = \phi(\max_{0 \leq i \leq N} 0 \leq j \leq 2^n d(\omega((l+i\cdot2^{-n}), \omega((l+j\cdot2^{-n}))) \]

instead of those used in the proof of (5.19). The lower bound for \( \lambda(\Omega_{x,y}; P^T_{x,y}) \) follows as \( r \downarrow R \).

The lower bound for \( \lambda^*(\Omega_{x,y}; P^T_{x,y}) \) can be shown similarly—with one slight difference: the approximating sequence \( \tilde{F}_n \) for a given function \( F \in H^{1,2}_0(\Omega_{x,y}; P^T_{x,y}) \) with \( E^T_{x,y}[F] = 0 \) that is constructed as in the proof of (5.19) does not necessarily satisfy the condition \( E^T_{x,y}[\tilde{F}_n] = 0 \) that is needed to apply Lemma 6.3 to \( \tilde{F}_n \). Instead, we replace \( \tilde{F}_n \) by \( \hat{F}_n := \tilde{F}_n - \left( E^T_{x,y}[\tilde{F}_n]/E^T_{x,y}[\Psi_n] \right) \cdot \Psi_n \), which has expectation 0. Since \( \tilde{F}_n \to F \) in \( H^{1,2}(\Omega_{x,y}; P^T_{x,y}) \), we also have \( E^T_{x,y}[\hat{F}_n] \to E^T_{x,y}[F] = 0 \). Since the sequence \( \Psi_n \) is bounded in \( H^{1,2}(\Omega_{x,y}; P^T_{x,y}) \) and \( \inf_{n \in \mathbb{N}} E^T_{x,y}[\Psi_n] > 0 \), we obtain \( \lim_{n \to \infty} \hat{F}_n = \lim_{n \to \infty} \tilde{F}_n = F \) in \( H^{1,2}(\Omega_{x,y}; P^T_{x,y}) \). Now the argument can be carried out as before. \( \square \)

The assertion of Corollary 1.3 with \( R_0 := \min(r_0, \pi/(2\sqrt{\kappa})) \) follows immediately from Corollary 6.4.

Proof of Theorem 1.4. By (4.6) and (5.10),

\[
\widetilde{C}_N(T, R)^{-1} \geq c_0(R)/(N\lambda(U_{x,y}^{R,N}; \mu_{x,y}^{T,N})^{-1} + T)
\]

\[
= c_0(R) \cdot \lambda(U_{x,y}^{R,N}; \mu_{x,y}^{T,N})/(N + T\lambda(U_{x,y}^{R,N}; \mu_{x,y}^{T,N}))
\]

for some strictly positive constant \( c_0(R) \). It has been shown in [17], Corollary 1.3(i), that

\[
\lim_{T \downarrow 0} T \log \lambda(U_{x,y}^{R,N}; \mu_{x,y}^{T,N}) = -m_{x,y}^{R,N}.
\]

Hence by Corollary 6.4,

\[
\liminf_{T \downarrow 0} T \log \lambda(\Omega_{x,y}^{R,N}; P^T_{x,y}) \geq -m_{x,y}^{R,N}.
\]

Similarly, the corresponding estimate for \( \widetilde{C}_N^*(T, R)^{-1} \) with \( \lambda(U_{x,y}^{R,N}; \mu_{x,y}^{T,N}) \) replaced by \( \lambda^*(U_{x,y}^{R,N}; \mu_{x,y}^{T,N}) \) and Corollary 1.3(ii) and Lemma 1.2 in [17] imply

\[
\liminf_{T \downarrow 0} T \lambda^*(\Omega_{x,y}^{R,N}; P^T_{x,y}) > 0
\]
provided \( y \) is not conjugate to \( x \), \( \Omega_{x,y} \) does not contain a geodesic of length \( NR \), and \( \Omega_{x,y} \) contains only one local minimum of the energy functional of length \( \leq NR \). Note that the metric on \( M^{N-1} \) used in Lemma 1.2 in [17] is different from the metric used here. Since all metrics on a compact finite-dimensional Riemannian manifold are equivalent, the lemma can nevertheless be applied. Furthermore, since the set of all lengths of geodesics in \( \Omega_{x,y} \) has measure 0, and \( \lambda^*(\Omega_{x,y}^{R,N}, P_{x,y}^{T}) \) is decreasing in \( R \), the condition that \( \Omega_{x,y} \) does not contain a geodesic of length \( NR \) can be dropped. This proves Theorem 1.4(ii).

For the proof of (i) it remains to show the upper bound

\[
\limsup_{T \downarrow 0} T \log \lambda(\Omega_{x,y}^{R,N}, P_{x,y}^{T}) \leq -\tilde{m}_{x,y}^{R,N}.
\]

Suppose that \( U \) and \( V \) are nonempty disjoint open subsets of \( \Omega_{x,y}^{R,N} \), and there exist a constant \( L \in (0, \infty) \) and a function \( F \in \bigcap_{T > 0} H^{1,2}(\Omega_{x,y}^{R,N}, P_{x,y}^{T}) \) with \( 0 \leq F \leq 1 \), \( F = 1 \) on \( U \), \( F = 0 \) on \( V \), and \( |D^0 F| \leq L \) \( P_{x,y}^{T} \)-a.e. for all \( T > 0 \). Then for every \( T > 0 \), \( D^0 F \) vanishes \( P_{x,y}^{T} \)-a.e. on \( \Omega_{x,y}^{R,N} \setminus (U \cup V) \).

Thus

\[
\int |D^0 F|^2 dP_{x,y}^{T} \leq L^2 \cdot P_{x,y}^{T}(\Omega_{x,y}^{R,N} \setminus (U \cup V)),
\]

whereas

\[
\int (F - E_{x,y}^{T}[F|\Omega_{x,y}^{R,N}])^2 dP_{x,y}^{T} \geq \int_{V} P_{x,y}^{T}[U|\Omega_{x,y}^{R,N}]^2 dP_{x,y}^{T}
\]

\[
\geq P_{x,y}^{T}(V) \cdot P_{x,y}^{T}(U)^2.
\]

In particular,

\[
\log \lambda(\Omega_{x,y}^{R,N}, P_{x,y}^{T}) \leq \log \frac{L^2 \cdot P_{x,y}^{T}(\Omega_{x,y}^{R,N} \setminus (U \cup V))}{P_{x,y}^{T}(V) \cdot P_{x,y}^{T}(U)^2}
\]

\[
= 2 \log L + \log P_{x,y}^{T}(\Omega_{x,y}^{R,N} \setminus (U \cup V)) - \log P_{x,y}^{T}(V) - 2 \log P_{x,y}^{T}(U).
\]

The terms on the right-hand side can be estimated by the large deviation principle for the Brownian bridge, which has been established by E. Hsu in [25]. Suppose that \( U \) contains a minimal geodesic. Since \( U \) and \( V \) are open, we then obtain:

\[
\liminf_{T \downarrow 0} T \log P_{x,y}^{T}(V) \geq \frac{1}{2} d(x, y)^2 - \inf_{V} E,
\]

\[
\liminf_{T \downarrow 0} T \log P_{x,y}^{T}(U) \geq \frac{1}{2} d(x, y)^2 - \inf_{U} E = 0,
\]

\[
\limsup_{T \downarrow 0} T \log P_{x,y}^{T}(\Omega_{x,y}^{R,N} \setminus (U \cup V)) \leq \frac{1}{2} d(x, y)^2 - \inf_{V} E.
\]
where \( \mathcal{W} \) denotes the closure of \( \Omega_{x,y}^{R,N} \setminus (U \cup V) \) in \( \Omega_{x,y} \). Hence

\[
\limsup_{T \downarrow 0} T \log \lambda(\Omega_{x,y}^{R,N}, P_{x,y}^T) \leq \inf_{\mathcal{V}} E - \inf_{\mathcal{W}} E. \tag{6.22}
\]

We now make a special choice for \( U \), \( V \) and \( F \), cf. also the proof of Theorem 2.2 in [17], Steps 1 and 2, where the discrete counterpart to the estimate (6.19) has been proven. As in [17] we consider first the case where \( \tilde{m}_{x,y}^{R,N} > 0 \). Fix a minimal geodesic \( \gamma_{x,y} \in \Omega_{x,y} \).

Let \( \tilde{m}_{x,y}^{R,N}(\zeta) = \inf \{ \sup(EN_{x,y} \circ p); p \in \tilde{C}([0, 1], \overline{U}_{x,y}^{R,N}), p(0) = \zeta, p(1) = z(0) \} \),

where \( \tilde{C}([0, 1], \overline{U}_{x,y}^{R,N}) \) denotes the space of paths \( \omega: [0, 1] \to \overline{U}_{x,y}^{R,N} \) that project to a continuous path if the boundary \( \partial U_{x,y}^{R,N} \) is identified to a single point. Clearly, \( E(\gamma) = E_{x,y}^{R,N}(z) \) and \( E(\gamma_{x,y}) = E_{x,y}^{R,N}(z(0)) \). Moreover, by Lemma 3.5 in [17],

\[
M_{x,y}^{R,N}(\gamma) = \tilde{m}_{x,y}^{R,N}(z).
\]

Suppose that \( M_{x,y}^{R,N}(\gamma) > E(\gamma) \). We set:

\[ A = \{ \zeta \in \overline{U}_{x,y}^{R,N}; \quad \tilde{m}_{x,y}^{R,N}(\zeta) < \tilde{M}_{x,y}^{R,N}(z) \}. \]

Let \( \varepsilon > 0 \). For \( \omega \in \overline{U}_{x,y}^{R,N} \) let \( \pi_N(\omega) = (\omega(i/N); 1 \leq i \leq N - 1) \in \overline{U}_{x,y}^{R,N} \).

**Case (i).** \( A \cap \partial U_{x,y}^{R,N} = \emptyset \). In this case we set:

\[
\mathcal{U} = \{ \omega \in \Omega_{x,y}^{R-e,N}; \quad \pi_N(\omega) \in \mathcal{A} \text{ and } E_{x,y}^{N} \left( \pi_N(\omega) \right) \leq \tilde{M}_{x,y}^{R,N}(z) - 2\varepsilon \},
\]

\[
\mathcal{V} = \{ \omega \in \Omega_{x,y}; \pi_N(\omega) \notin \tilde{A} \}.
\]

Both sets are open. Since

\[
\tilde{M}_{x,y}^{R,N}(z) = M_{x,y}^{R,N}(\gamma) > E(\gamma) \geq E(\gamma_{x,y}) = E_{x,y}^{N}(\pi_N(\gamma_{x,y})),
\]

\( \gamma_{x,y} \) is contained in \( \mathcal{U} \) if \( \varepsilon \) is chosen sufficiently small. On the other hand, \( \gamma \) is in \( \mathcal{V} \). The set \( A \cap U_{x,y}^{R,N} \) is open. Since \( \tilde{M}_{x,y}^{R,N}(z) > E_{x,y}^{N}(z) \geq 0 = \tilde{M}_{x,y}^{R,N}(z(0)), z(0) \) is in \( A \).
On \( \{ \zeta \in U_{x,y}^{R,N}; E_{x,y}^N(\zeta) < \tilde{M}_{x,y}^R(\zeta) \} \), \( \tilde{M}_{x,y}^R \) is locally constant by definition. Hence

\[ E_{x,y}^N = \tilde{M}_{x,y}^R(\zeta) \] on \( \partial A \cap U_{x,y}^{R,N} \). For \( \omega \in \mathcal{V} \), we obtain:

\[
E(\omega) \geq E_{x,y}^N(\pi_N(\omega)) > \tilde{M}_{x,y}^R(\zeta) - 2\varepsilon = M_{x,y}^R(\gamma) - 2\varepsilon, \quad \text{or} \quad (6.23)
\]

\[
E(\omega) \geq \frac{N}{2} \cdot (R - \varepsilon)^2 \quad \text{(if \( \omega \notin \Omega_{x,y}^{R-\varepsilon,N} \)).} \quad (6.24)
\]

Hence

\[
\inf_{\mathcal{V}} E - \inf_{\mathcal{W}} E \leq E(\gamma) - \min\left(M_{x,y}^R(\gamma) - 2\varepsilon, \frac{N}{2} \cdot (R - \varepsilon)^2\right) \quad (6.25)
\]

For \( \varepsilon < (\tilde{M}_{x,y}^R(\zeta) - E_{x,y}^N(\zeta))/3 \) let \( \psi_\varepsilon \) be the function in \( C^\infty_0(U_{x,y}^{R,N}) \) defined by \( \psi_\varepsilon(\zeta) = \phi_\varepsilon(E_{x,y}^N(\zeta)) \) for \( \zeta \in A \), 0 else, where \( \phi_\varepsilon : \mathbb{R} \rightarrow \mathbb{R} \) is smooth with \( \phi_\varepsilon(t) = 1 \) for \( t < \tilde{M}_{x,y}^R(\zeta) - 2\varepsilon \), \( \phi_\varepsilon(t) = 0 \) for \( t \geq \tilde{M}_{x,y}^R(\zeta) - \varepsilon \), \( 0 \leq \phi_\varepsilon \leq 1 \), and \( |\phi_\varepsilon'| \leq 2/\varepsilon \). For \( \omega \in \Omega_{x,y} \) we set:

\[
F(\omega) = \tilde{\phi}_\varepsilon G(\omega) \cdot \psi_\varepsilon(\pi_N(\omega))
\]

where \( \psi_\varepsilon \) is extended trivially to \( M^{N-1} \), \( \tilde{\phi}_\varepsilon = ((R - t)/\varepsilon)^+ \wedge 1 \), and

\[
G(\omega) = \max_{0 \leq i < N-1, s,t \in [i/N, (i+1)/N]} \sup d(\omega(s), \omega(t)).
\]

Then \( 0 \leq F \leq 1 \), \( F = 1 \) on \( \mathcal{U} \), and \( F = 0 \) on \( \mathcal{V} \). Moreover, \( F \) is contained in \( H^{1,2}(\Omega_{x,y}; P_{x,y}^T) \) for all \( T > 0 \), and \( |D^0 F| \) is bounded. Since \( \tilde{\phi}_\varepsilon \circ G \) vanishes outside \( \Omega_{x,y} \), \( F \) is even contained in \( \bigcap_{T>0} H^{1,2}(\Omega_{x,y}^{T,N}; P_{x,y}^T) \). Hence (6.22) holds for sufficiently small \( \varepsilon \). By combining (6.22) and (6.25) and letting \( \varepsilon \) go to 0, we obtain:

\[
\limsup_{T \downarrow 0} T \log \lambda(\Omega_{x,y}^{R,N}; P_{x,y}^T) \leq E(\gamma) - \min(M_{x,y}^R(\gamma), N R^2/2). \quad (6.26)
\]

**Case (ii).** \( A \cap \partial U_{x,y}^{R,N} \neq \emptyset \). We first remark that (6.20), (6.21) and (hence) (6.22) also hold if \( F = 0 \) on \( \mathcal{U} \) and \( F = 1 \) on \( \mathcal{V} \). In fact, (6.20) is obvious, and (6.21) follows from the corresponding estimate with \( F \) replaced by \( 1 - F \). For \( \varepsilon < \tilde{M}_{x,y}^R(\zeta) \) we set:

\[
\mathcal{U} = \{ \omega \in \Omega_{x,y}^{R,N}; \pi_N(\omega) \in A \},
\]

\[
\mathcal{V} = \{ \omega \in \Omega_{x,y}^{R-\varepsilon,N}; \pi_N(\omega) \notin A \text{ and } E_{x,y}^N(\pi_N(\omega)) \leq \tilde{M}_{x,y}^R(\zeta) - 2\varepsilon \},
\]

\[
F(\omega) = \tilde{\phi}_\varepsilon G(\omega) \cdot \psi_\varepsilon(\pi_N(\omega)),
\]

where \( \psi_\varepsilon : M^{N-1} \rightarrow \mathbb{R} \) is now defined by \( \psi_\varepsilon(\zeta) = \phi_\varepsilon(E_{x,y}^N(\zeta)) \) for \( \zeta \in U_{x,y}^{R,N} \setminus A \), 0 else. To show \( \text{supp} \psi_\varepsilon \subset U_{x,y}^{R,N} \), we fix \( \bar{z} \in A \cap \partial U_{x,y}^{R,N} \). Then for \( \zeta \in \partial U_{x,y}^{R,N} \setminus A \), we have \( \tilde{M}_{x,y}^R(\zeta) \geq \tilde{M}_{x,y}(\zeta) > \tilde{M}_{x,y}(\bar{z}) \), and thus \( E_{x,y}^N(\zeta) = \tilde{M}_{x,y}^R(\zeta) \geq \tilde{M}_{x,y}(\zeta) \geq \tilde{M}_{x,y}(\bar{z}) \).
by definition of \( \tilde{M}^{R,N}_{x,y} \). Hence \( \psi_{\varepsilon} \) vanishes in a neighborhood of the compact subset \( \partial U_{R,N}^{x,y} \setminus A \) of \( \nabla U_{R,N}^{x,y} \). Since \( \psi_{\varepsilon} \) also vanishes on \( A \), it vanishes on a neighborhood of \( \partial U_{R,N}^{x,y} \), i.e., \( \text{supp} \psi_{\varepsilon} \subset U_{R,N}^{x,y} \). Similarly as above, we obtain \( 0 \leq F \leq 1 \), \( F \equiv 0 \) on \( U \), \( F \equiv 1 \) on \( V \), and \( F \in \bigcap_{T>0} H_{1,2}^{0}(\Omega_{R,N}^{x,y}; P^{T}_{x,y}) \). Furthermore, \( |D^{0}F| \) is again bounded, \( \gamma_{x,y} \) is contained in \( U \), \( \gamma \) is contained in \( V \), and (6.23) and (6.24) hold as before. Hence we again obtain (6.26).

Both in Cases (i) and (ii), (6.26) holds for every \( \gamma \in \Gamma_{\min}^{x,y} \) with \( L(\gamma) \leq NR \) and \( M^{R,N}_{x,y}(\gamma) > E(\gamma) \). Hence we obtain (6.19) provided \( \tilde{m}^{R,N}_{x,y} > 0 \).

Finally, to prove (6.19) in the case \( \tilde{m}^{R,N}_{x,y} = 0 \) we fix \( \varepsilon > 0 \), and an open set \( O \subset \Omega_{R,N}^{x,y} \) with \( \inf_{O} E > d(x,y)^{2}/2 \). Let \( \omega_{0} \in O \) with \( E(\omega_{0}) < \varepsilon + \inf_{O} E \), and let \( r_{0} > 0 \) such that \( \{ \omega \in \Omega_{x,y}^{R,N} ; d_{\infty}(\omega, \omega_{0}) \leq 2r_{0} \} \subset O \). Let \( V = \{ \omega \in \Omega_{x,y}^{R,N} ; d_{\infty}(\omega, \omega_{0}) < r_{0} \} \), \( U = \{ \omega \in \Omega_{x,y}^{R,N} ; d_{\infty}(\omega, \omega_{0}) > 2r_{0} \} \), and \( F(\omega) = (2 - d_{\infty}(\omega, \omega_{0})/r_{0})^{+} \). Clearly, \( F \in \bigcap_{T>0} H_{1,2}^{0}(\Omega_{x,y}^{R,N}; P_{x,y}^{T}) \), and \( |D^{0}F| \) is bounded. Since \( \omega_{0} \in V \) and \( W \subset \Omega_{R,N}^{x,y} \setminus U \subset O \), we have

\[
\inf_{V} E - \inf_{W} E < \varepsilon.
\]

Hence by (6.22),

\[
\limsup_{T \downarrow 0} T \log \lambda(\Omega_{x,y}^{R,N}, P_{x,y}^{T}) < \varepsilon.
\]

The assertion follows again as \( \varepsilon \downarrow 0 \). \( \square \)

Acknowledgements

I thank Bruce Driver for many very helpful discussions during my stay in San Diego, where underlying ideas of this work have been developed. Financial support for this stay has been provided by Deutsche Forschungsgemeinschaft.

Appendix A. Jacobi field estimates

Let \( M \) be a compact Riemannian manifold, and \( \gamma : [0, 1] \rightarrow M \) a geodesic. Let \( \kappa \) and \( \bar{\kappa} \) denote the maximum of the absolute value and of the positive part of the sectional curvature along \( \gamma \). We derive some straightforward estimates for Jacobi fields along \( \gamma \).

Lemma. Suppose that \( L(\gamma) \leq \pi/(2\sqrt{\bar{\kappa}}) \). Then for every Jacobi field \( Y \) along \( \gamma \), and all \( s \in [0, 1] \).
(i) \[ |Y(s)| \leq |Y(0)| + |Y(1)|, \]

(ii) \[ \left| \frac{\nabla^2 Y}{d^2 s} (s) \right| \leq \kappa \cdot L(\gamma)^2 \cdot (|Y(0)| + |Y(1)|), \]

(iii) \[ \left| \frac{\nabla Y}{d s} (s) \right| \leq (1 + \kappa \cdot L(\gamma)^2) \cdot (|Y(0)| + |Y(1)|), \]

(iv) \[ |Y(s)| \geq |Y(0)| - s \cdot (1 + \kappa \cdot L(\gamma)^2) \cdot (|Y(0)| + |Y(1)|), \]

\[ |Y(s)| \geq |Y(1)| - (1 - s) \cdot (1 + \kappa \cdot L(\gamma)^2) \cdot (|Y(0)| + |Y(1)|), \]

(v) \[ \int_0^1 |Y(s)|^2 \, ds \geq \frac{1}{6 \cdot (1 + \kappa L(\gamma)^2)} \cdot (|Y(0)|^2 + |Y(1)|^2). \]

**Proof.** (i) Let \( Y^{(0)} \) and \( Y^{(1)} \) denote the Jacobi fields along \( \gamma \) with boundary conditions \( Y^{(0)}(0) = Y(0), \quad Y^{(1)}(0) = 0, \quad Y^{(1)}(1) = Y(1). \) By linearity of the Jacobi equation, \( Y = Y^{(0)} + Y^{(1)}. \) Since \( L(\gamma) \leq \pi/(2 \cdot \sqrt{K}), \ s \mapsto |Y^{(0)}(s)| \) is decreasing on \([0, 1]\) by Rauch’s comparison theorem, cf., e.g., [6], (2.59). Hence \( |Y^{(0)}(s)| \leq |Y^{(0)}(0)| = |Y(0)| \) for all \( s \in [0, 1]. \) Similarly, \( |Y^{(1)}(s)| \leq |Y(1)| \) for all \( s \in [0, 1], \) and thus \( |Y(s)| \leq |Y^{(0)}(s)| + |Y^{(1)}(s)| \leq |Y(0)| + |Y(1)|. \)

(ii) Is an immediate consequence of (i) and the Jacobi equation.

(iii) Let \( E(s), \ 0 \leq s \leq 1, \) be a parallel unit vector field along \( \gamma. \) By the mean value theorem, there exists \( s_0 \in (0, 1) \) with

\[ \left\langle E(s_0), \frac{\nabla Y}{d s} (s_0) \right\rangle = \left\langle E, Y \right\rangle'(s_0) = \left\langle E(1), Y(1) \right\rangle - \left\langle E(0), Y(0) \right\rangle \leq |Y(1)| + |Y(0)|. \]

Hence by (ii),

\[ \left\langle E(s), \frac{\nabla Y}{d s} (s) \right\rangle = \left\langle E(s_0), \frac{\nabla Y}{d s} (s_0) \right\rangle + \int_{s_0}^s \left\langle E(t), \frac{\nabla^2 Y}{d^2 t} (t) \right\rangle \, dt \]

\[ \leq (1 + \kappa L(\gamma)^2) \cdot (|Y(0)| + |Y(1)|) \]

for all \( s \in [0, 1]. \) Since this holds for every parallel unit vector field \( E \) along \( \gamma, \) it implies (iii).

(iv) Follows from (iii) by integration.

(v) If \( Y(0) \) and \( Y(1) \) vanish there is nothing to prove. Otherwise let \( s_i = (1 + \kappa L(\gamma)^2)^{-1}(|Y(0)| + |Y(1)|)^{-1} \cdot |Y(i)|, \ i = 0, 1. \) Clearly, \( s_0 + s_1 \leq 1. \) By (iv),

\[ \int_0^{s_0} |Y(s)|^2 \, ds \geq \int_0^{s_0} (|Y(0)| - s \cdot (1 + \kappa L(\gamma)^2) \cdot (|Y(0)| + |Y(1)|))^2 \, ds \]
\[
\begin{align*}
&= (1 + \kappa L(\gamma)^2)^{-1} \left( [Y(0)] + [Y(1)] \right)^{-1} \cdot \int_0^1 [Y(0) - t]^2 \, dt \\
&= (1 + \kappa L(\gamma)^2)^{-1} \left( [Y(0)] + [Y(1)] \right)^{-1} \cdot |Y(0)|^3 / 3.
\end{align*}
\]

Similarly,
\[
\int_0^1 \bigl| Y(s) \bigr|^2 \, ds \geq (1 + \kappa L(\gamma)^2)^{-1} \left( [Y(0)] + [Y(1)] \right)^{-1} \cdot |Y(1)|^3 / 3.
\]

Since \((a + b)(a^2 + b^2) = a^3 + b^3 + a^2 b + ab^2 \leq 2(a^3 + b^3)\) for all positive reals \(a, b\), we obtain
\[
\int_0^1 \bigl| Y(s) \bigr|^2 \, ds \geq \frac{1}{3 \cdot (1 + \kappa L(\gamma)^2)} \cdot \frac{[Y(0)]^3 + [Y(1)]^3}{[Y(0)] + [Y(1)]} \geq \frac{[Y(0)]^2 + [Y(1)]^2}{6 \cdot (1 + \kappa L(\gamma)^2)}. \quad \square
\]

References