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The Blow-Up Rate for a Strongly Coupled System of Semilinear Heat Equations with Nonlinear Boundary Conditions

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The paper deals with the blow-up rate of positive solutions to the system $u_t = u_{xx} + u^{l_{11}}v^{l_{12}}$, $v_t = v_{xx} + u^{l_{21}}v^{l_{22}}$ with boundary conditions $u_x(1,t) = (u^{p_{11}}v^{p_{12}})(1,t)$ and $v_x(1,t) = (u^{p_{21}}v^{p_{22}})(1,t)$. Under some assumptions on the matrices $L = (l_{ij})$ and $P = (p_{ij})$ and on the initial data u_0, v_0, the solution (u,v) blows up at finite time T, and we prove that $\max_{x\in[0,1]} u(x,t)$ (resp. $\max_{x\in[0,1]} v(x,t)$) goes to infinity as $(T-t)^{\alpha_1/2}$ (resp. $(T-t)^{\alpha_2/2}$), where $\alpha_i < 0$ are the solutions of $(P - \mathrm{Id})(\alpha_1, \alpha_2)^t = (-1,-1)^t$. © 2001 Academic Press

1. INTRODUCTION

In this paper we consider the blow-up rate for the following system of semilinear heat equations with nonlinear boundary conditions

$$
\begin{aligned}
&u_t = u_{xx} + u^{l_{11}}v^{l_{12}}, \quad v_t = v_{xx} + u^{l_{21}}v^{l_{22}}, \\
&\qquad\qquad (x,t) \in (0,1)\times(0,T), \\
&u_x(0,t) = 0, \quad v_x(0,t) = 0, \qquad t\in(0,T), \\
&u_x(1,t) = (u^{p_{11}}v^{p_{12}})(1,t), \quad v_x(1,t) = (u^{p_{21}}v^{p_{22}})(1,t), \\
&\qquad\qquad t \in (0,T), \\
&u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x\in(0,1).
\end{aligned}
\tag{1.1}
$$

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Here the matrices $L = (l_{ij})$ and $P = (p_{ij})$ satisfy the following assumption

(A) P and L are two matrices with non-negative entries such that $\max\{l_{11}, l_{22}\} < 1$, $\max\{p_{11}, p_{22}\} < 1$, $\det(L - \mathrm{Id}) \neq 0$, and $\det(P - \mathrm{Id}) < 0$.

Under these hypotheses, there exist two unique vectors (α_1, α_2) and (β_1, β_2) with $\alpha_i < 0$ and $\beta_i < 0$ (or $\beta_i > 0$) such that

$$(P - \mathrm{Id})\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \qquad (L - \mathrm{id})\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}. \tag{1.2}$$

Here, without loss of generality, we assume that $\alpha_1 \leq \alpha_2 < 0$ and $\beta_1 \geq \beta_2 > 0$ (or $\beta_1 \leq \beta_2 < 0$). Further, we suppose that l_{ij}, α_i, and β_i satisfy the following hypotheses:

(B)

$$l_{11} \geq l_{21}, \qquad \beta_1/\beta_2 \geq \alpha_1/\alpha_2 > 1, \qquad \text{and} \qquad (L - \mathrm{Id})\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} > \begin{pmatrix} -2 \\ -2 \end{pmatrix}.$$

EXAMPLE. Let $l_{11} = 1/2$, $l_{21} = 1/3$, $l_{12} = 6/7$, $l_{22} = 1/7$, $p_{11} = 1/2$, $p_{12} = 2$, $p_{21} = 3/4$, and $p_{22} = 1/2$. Then we get $\alpha_1 = -2$, $\alpha_2 = -1$, $\beta_1 = 12$, $\beta_2 = 35/6$, and (α_1, α_2), (β_1, β_2) satisfy conditions (A) and (B).

We also suppose that the initial data satisfy the following conditions

(C) $u_0(x), v_0(x) \in C^3([0,1])$, $u_0''' \geq 0$, $u_0'' \geq 0$, $u_0' \geq 0$, $v_0''' \geq 0$, $v_0'' \geq 0$, $v_0' \geq 0$, $u_0(x) \geq 1$, and $v_0(x) \geq 1$ for any $x \in (0,1)$.

Under condition (C), by the minimum principle we have $u(x,t) \geq 1$ and $v(x,t) \geq 1$ for any $(x,t) \in [0,1] \times [0,T)$.

Under hypothesis (A), it is proved in [15] that the solution $(u(x,t), v(x,t))$ of (1.1) blows up in finite time T. As $t \to T$ we have

$$\limsup_{t \to T} \left\{ \| u(\,.\,, t) \|_{L^\infty([0,1])} + \| v(\,.\,, t) \|_{L^\infty([0,1])} \right\} = +\infty.$$

We can also prove that both functions $u(x,t)$ and $v(x,t)$ go to infinity as $t \to T$. In fact, assume that $u(x,t)$ remains bounded in $[0,1] \times [0,T)$. Then $v(x,t)$ satisfies the relations

$$\begin{gathered} v_t = v_{xx} + K v^{l_{22}} \qquad \text{in } (0,1) \times (0,T), \\ v_x(0,t) = 0, \qquad v_x(1,t) \leq K v^{p_{22}}(1,t), \\ v(x,0) = v_0(x), \qquad \text{in } (0,1), \end{gathered} \tag{1.3}$$

where K is a bound for $\max\{u^{l_{21}}, u^{p_{21}})$. Since $\max\{l_{22}, p_{22}\} \leq 1$, it is well known that $v(x,t)$ remains bounded up to time T (see [13]). Hence, T is not the blow-up time; this is a contradiction to our assumption.

Over the past two decades the blow-up problem for the solutions of nonlinear parabolic equations with nonlinear boundary conditions has deserved a great deal of interest (see [2, 3, 5, 7, 8, 11–14]). For these kinds of problems, in particular, the blow-up rate and the localization of blow-up points are not well known even in the case of a single parabolic equation with a nonlinear boundary condition. Some of those results closely related to ours are as follows.

In [1, 10] the authors studied the problem

$$\begin{aligned} &u_t = \Delta u, \quad v_t = \Delta v, \qquad (x,t) \in B_R(0) \times (0,T), \\ &\frac{\partial u}{\partial n} = v^p, \quad \frac{\partial v}{\partial n} = u^q, \qquad (x,t) \in \partial B_R(0) \times (0,T), \\ &u_0(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in B_R(0), \end{aligned} \tag{1.4}$$

where $pq > 1$, $u_0(x)$, $v_0(x) \in C^2$ are radially symmetric and satisfy the boundary conditions, and $\Delta u_0 \geq \epsilon > 0$, $\Delta v_0 \geq \epsilon > 0$. They proved that there exist two positive constants c and C such that

$$\begin{aligned} &c \leq \max_{x \in B_R(0)} u(x,t)(T-t)^{\alpha/2} \leq C \qquad \text{for } 0 < t < T, \\ &c \leq \max_{x \in B_R(0)} v(x,t)(T-t)^{\beta/2} \leq C \qquad \text{for } 0 < t < T, \end{aligned} \tag{1.5}$$

where T is the blow-up time, $\alpha = (p+1)/(pq-1)$, and $\beta = (q+1)/(pq-1)$.

In [12] Rossi considered the problem

$$\begin{aligned} &u_t = \Delta u, \quad v_t = \Delta v, \qquad (x,t) \in B_1(0) \times (0,T), \\ &\frac{\partial u}{\partial n} = u^{p_{11}} v^{p_{12}}, \quad \frac{\partial v}{\partial n} = u^{p_{21}} v^{p_{22}}, \qquad (x,t) \in \partial B_1(0) \times (0,T), \\ &u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) > 0, \qquad x \in B_1(0), \end{aligned} \tag{1.6}$$

where the matrix $P = (p_{ij})$ satisfies hypothesis (A), the initial functions $u_0, v_0 \in C^3(\overline{B}_1(0))$ are radially symmetric and satisfy the boundary conditions, and the first three derivatives of $u_0(r)$, $v(r)$ ($r = \|x\|$) are non-negative. In [12] the author proved that there exist positive constants c and C such that

$$c \leq \max_{x \in B_R(0)} u(x,t)(T-t)^{-\alpha_1/2} \leq C \qquad \text{for } 0 < t < T,$$

$$c \le \max_{x \in B_R(0)} v(x,t)(T-t)^{-\alpha_2/2} \le C \qquad \text{for } 0 < t < T, \tag{1.7}$$

where α_1 and α_2 are given by (1.2).

In [11] the author considered the problem (1.1) for the case $l_{12} = l_{21} = 0$. The same estimates as (1.7) were obtained.

Similar results on blow-up rate were obtained in [2–4, 6, 9] for some single equations.

In this paper, by a modification of the method given in [11, 12], we establish the following results.

THEOREM 1.1. *If assumptions* (A), (B), *and* (C) *hold*, *then the solution* $(u(x,t), v(x,t))$ *of* (1.1) *blows up at finite time* T *and there exist positive constants* c *and* C *such that*

$$\begin{aligned} c \le \max_{x \in [0,1]} u(x,t)(T-t)^{-\alpha_1/2} \le C \qquad \text{for } 0 < t < T, \\ c \le \max_{x \in [0,1]} v(x,t)(T-t)^{-\alpha_2/2} \le C \qquad \text{for } 0 < t < T, \end{aligned} \tag{1.8}$$

where α_i $(i = 1, 2)$ *are given by* (1.2).

THEOREM 1.2. *If assumptions* (A), (B), *and* (C) *hold*, *then for any* $r \in [0,1)$ *there exists a constant* $C = C(r)$ *such that*

$$\max_{x \in [0,r]} u(x,t) < C, \qquad t \in [0,T),$$

$$\max_{x \in [0,r]} v(x,t) < C, \qquad t \in [0,T)$$

($i.e.$, *the blow-up set is localized in the boundary* $x = 1$).

To prove Theorem 1.1 we need a result for a single equation that has independent interest.

THEOREM 1.3. *Let* $u(x,t)$ *be a positive solution of the problem*

$$\begin{aligned} u_t = u_{xx} + \tilde{C}_0 \frac{u^{\tilde{l}}(x,t)}{(T-t)^{\tilde{s}}}, \qquad \text{in } (0,1) \times (0,T), \\ u_x(0,t) = 0, \quad u_x(1,t) = C_0 \frac{u^r(1,t)}{(T-t)^s}, \qquad t \in (0,T), \\ u(x,0) = u_0(x), \qquad \text{in } (0,1), \end{aligned} \tag{1.9}$$

where $0 < \tilde{l} < 1$, $s > 1/2$, $0 < r < 1$, $0 < \tilde{s} < 1 + (1-\tilde{l})(2s-1)/(2(1-r))$, *and the initial function* $u_0(x) \in C^3$. *Then* $u(x,t)$ *blows up as* $t \to T$ *and*

$$\tilde{c} \le \max_{x \in [0,1]} u(x,t)(T-t)^{\beta} \le \tilde{C}, \qquad t \in (0,T),$$

where $\beta = (s - 1/2)/(1-r)$.

The paper is organized as follows. In Section 2, we give some auxiliary propositions and prove Theorem 1.1. In Section 3, which deals with the blow-up rates, we prove our main results.

2. AUXILIARY PROPOSITIONS

In this section, we state some propositions that play an important role in Section 3. We begin with a result of [12] (see also [4, 6]).

PROPOSITION 2.1. *Let z be the positive solution of the problem*

$$\begin{gathered} z_t = z_{xx}, \qquad (x,t) \in (0,1) \times (0,T), \\ z_x(0,t) = 0, \quad z_x(1,t) = z^k(1,t), \qquad t \in (0,T), \\ z(x,0) = z_0(x) > 0, \qquad x \in \Omega, \end{gathered} \tag{2.1}$$

where $k > 1$, $z_0 \in C^3$ *satisfies the inequalities* $z_0' \geq 0$, $z_0'' \geq 0$, $z_0''' \geq 0$ *and boundary conditions. Then there exist positive constants c and C such that*

$$c \leq \max_{x \in [0,1]} u(x,t)(T-t)^{\alpha} = u(1,t)(T-t)^{\alpha} \leq C, \qquad \text{for } 0 < t < T, \tag{2.2}$$

where $\alpha = 1/(2(k-1))$.

Next we state two results due to [9, 12].

PROPOSITION 2.2 (see [9]). *Let* $w(x,t)$ *be the positive solution of the problem*

$$\begin{gathered} w_t = w_{xx} + w^l, \qquad \text{in } (0,1) \times (0,T), \\ w_x(0,t) = 0, \quad w_x(1,t) = w^q(1,t), \qquad t \in (0,T), \\ w(x,0) = w_0(x) > 0, \qquad \text{in } [0,1], \end{gathered} \tag{2.3}$$

where $l > 0$, $q > 0$, $\max\{l,q\} > 1$, *the initial function* $w_0(x)$ *satisfies the inequalities* $w_0'' + w_0^l \geq 0$ *and* $w_0' \geq 0$, *and T is the blow-up time. Then blow-up occurs only at* $x = 1$ *and there exist positive constants c and C such that*

$$c \leq \max_{x \in [0,1]} w(x,t)(T-t)^{\alpha} = w(1,t)(T-t)^{\alpha} \leq C \qquad \text{for } 0 < t < T, \tag{2.4}$$

where $\alpha = 1/(l-1)$ *if* $l \geq 2q-1$, $\alpha = 1/(2(q-1))$ *if* $l < 2q-1$, *and T is blow-up time.*

PROPOSITION 2.3 (see [12]). *Let $u(x,t)$ be the positive solution of the problem*

$$\begin{aligned} u_t &= u_{xx}, \qquad \text{in } (0,1)\times(0,T), \\ u_x(0,t) &= 0, \quad u_x(1,t) = \frac{Cu^r(1,t)}{(T-t)^s}, \qquad t\in(0,T), \\ u(x,0) &= u_0(x) > 0, \qquad \text{in } (0,1), \end{aligned} \tag{2.5}$$

where $s > 1/2$, $0 < r < 1$, and C is an arbitrary constant. Then $u(x,t)$ blows up at time T and

$$c \le \max_{x\in[0,1]} u(x,t)(T-t)^{(s-1/2)/(1-r)} \le \overline{C}, \qquad t\in(0,T).$$

Proof of Theorem 1.1.

Step 1. Let $k = (2s-r)/(2s-1)$. Since $\tilde{s} < 1 + (1-\tilde{l})(2s-1)/(2(1-r))$, we can take a constant $\bar{l}$ such that $\tilde{l} < \bar{l} < 2k-1$ and $(\bar{l}-\tilde{l})/(2(k-1)) = \tilde{s}$. Denote by $\overline{w}(x,t)$ the solution of the problem

$$\begin{aligned} \overline{w}_t &= \overline{w}_{xx} + \overline{w}^{\bar{l}}, \qquad \text{in } (0,1)\times(0,T), \\ \overline{w}_x(0,t) &= 0, \quad \overline{w}_x(1,t) = \overline{w}^k(1,t), \qquad t\in(0,T), \\ \overline{w}(x,0) &= \overline{w}_0(x) < u_0(x), \qquad \text{in } (0,1), \end{aligned} \tag{2.6}$$

where the initial function $\overline{w}_0(x)$ satisfies the conditions $\overline{w}_0'' + \overline{w}_0^{\tilde{l}} > 0$ and $\overline{w}_0' \ge 0$. Since $s > 1/2$ and $r < 1$, we have $k > 1$. By $\bar{l} < 2k-1$ and Proposition 2.2 we know that the solution $\overline{w}(x,t)$ of (2.6) blows up in finite time T and

$$c_0 \le \max_{x\in[0,1]} \overline{w}(x,t)(T-t)^{1/(2(k-1))} \le \hat{C}_0 \qquad \text{for } 0 < t < T. \tag{2.7}$$

We are going to prove that $u(x,t) > \overline{w}(x,t)$. To this end, we consider two cases.

(I) Assume that $u_0(x)$, $\tilde{C}_0$, and C_0 are large enough. Suppose that there exist a first time $t_0 \in (0,T)$ and a point x_0 such that $(\overline{w} - U)(x_0,t_0) = 0$ and $(u-\overline{w})(x,t) > 0$ for all $(x,t) \in [0,1]\times[0,t_0)$. Then we easily deduce that x_0 must belong to the half-interval $(0,1]$. If $x_0 \in (0,1)$, then taking into account (2.5), (2.6), by (2.7), $(\bar{l} - \tilde{l})/(2(k-1)) = \tilde{s}$, and the fact that $u_0(x)$, $\tilde{C}_0$, and C_0 are large enough we get

$$\begin{aligned} &(\overline{w} - u)_t = (\overline{w} - u)_{xx} + \overline{w}^{\tilde{l}} - \tilde{C}_0 \frac{u^{\tilde{l}}}{(T-t)^{\tilde{s}}} \leq (\overline{w} - u)_{xx}, \\ &\qquad\qquad \text{in } (0,1) \times (0, t_0), \\ &(\overline{w} - u)_x(0,t) = 0, \\ &(\overline{w} - u)_x(1,t) = \overline{w}^k(1,t) - C_0 \frac{u^r(1,t)}{(T-t)^s} < 0, \\ &\qquad\qquad t \in (0, t_0), \\ &(\overline{w} - u)(x,0) < 0, \qquad \text{in } (0,1). \end{aligned} \tag{2.8}$$

By the minimum principle we have $(\overline{w} - u)(x_0, t_0) < 0$. This is a contradiction with our assumption.

If $x_0 = 1$, then we have

$$(\overline{w} - u)_x(1,t) = \overline{w}^r(1,t)\left(\overline{w}^{k-r}(1,t) - \frac{C_0}{(T-t)^s}\right). \tag{2.9}$$

By (2.7) and the fact that C_0 is large enough, we know that

$$\overline{w}^{k-r}(1,t) \leq \frac{\overline{C}_0}{(T-t)^{(k-r)/(2(k-1))}} < \frac{C_0}{(T-t)^s}. \tag{2.10}$$

On the other hand, we have $\overline{w}(x,t) < u(x,t)$ for any $(x,t) \in (0,1) \times [0, t_0)$. Thus we also get $(\overline{w} - u)_t \leq (\overline{w} - u)_{xx}$ in $(0,1) \times (0, t_0)$. By the minimum principle and (2.8)–(2.10) we have $(\overline{w} - u)(x_0, t_0) < 0$. This is a contradiction with our assumption. Therefore, from (2.7) we obtain

$$\max_{x \in [0,1]} u(x,t)(T-t)^{\beta} \geq c \qquad \text{for } 0 < t < T. \tag{2.11}$$

(II) Let $u(x,t)$ be the solution of (1.9) with arbitrary $u_0(x)$, $\tilde{C}_0$, and C_0. We take a constant M such that $Mu_0(x) > \overline{w}_0(x)$, $M^{1-\tilde{l}}\tilde{C}_0$ and

$M^{1-r}C_0$ are large enough, and $\overline{U} = Mu$ satisfies the following relations:

$$\begin{aligned} \overline{U}_t &= \overline{U}_{xx} + M^{1-\tilde{l}}\tilde{C}_0 \frac{\overline{U}^{\tilde{l}}}{(T-t)^{\tilde{s}}}, \qquad \text{in } (0,1)\times(0,T), \\ \overline{U}_x(0,t) &= 0, \quad \overline{U}_x(1,t) = M^{1-r}C_0 \frac{\overline{U}^r(1,t)}{(T-t)^s}, \qquad t \in (0,T), \\ \overline{U}(x,0) &= Mu_0(x) > \overline{w}_0(x), \qquad \text{in } (0,1), \end{aligned} \tag{2.12}$$

By the previous result (2.11) we have

$$\max_{x\in[0,1]} \overline{U}(x,t)(T-t)^{\beta} \geq c \qquad \text{for } 0<t<T. \tag{2.13}$$

This completes the first step of the proof.

Step 2. Let $q = (2s-r)/(2s-1)$. By $\tilde{s} < 1 + (1-\tilde{l})(2s-1)/(2(1-r))$, we can also take a constant l such that $l < 2q-1$ and $(l-\tilde{l})/(2(q-1)) = \tilde{s}$. Let $\tilde{w}(x,t)$ be the solution of the problem

$$\begin{aligned} \tilde{w}_t &= \tilde{w}_{xx} + \tilde{w}^l, \qquad \text{in } (0,1)\times(0,T), \\ \tilde{w}_x(0,t) &= 0, \quad \tilde{w}_x(1,t) = \tilde{w}^q(1,t), \qquad t\in(0,T), \\ \tilde{w}(x,0) &= \tilde{w}_0(x) > u_0(x), \qquad \text{in } (0,1), \end{aligned} \tag{2.14}$$

where $\tilde{w}_0 \in C^2$, $\tilde{w}_0'' + \tilde{w}_0^l > 0$, and $\tilde{w}_0' \geq 0$. By Proposition 2.2, $\tilde{w}(x,t)$ blows up in finite time T and there exist positive constants c and C such that

$$c \leq \max_{x\in[0,1]} \tilde{w}(x,t)(T-t)^{\beta} \leq C \qquad \text{for } 0<t<T. \tag{2.15}$$

We shall prove that $\tilde{w}(x,t) > u(x,t)$. Consider two cases.

(III) Assume that $u_0(x)$, $\tilde{C}_0$, and C_0 are small enough. Arguing as in (I), we obtain $\tilde{w}(x,t) > u(x,t)$ and

$$\max_{x\in[0,1]} u(x,t)(T-t)^{\beta} \leq C \qquad \text{for } 0<t<T. \tag{2.16}$$

(IV) Let the constants $\tilde{C}_0, C_0$ and the initial data be arbitrary. We take a constant m such that $mu_0 < \tilde{w}_0$, $m^{1-\tilde{l}}\tilde{C}_0$ and $m^{1-r}C_0$ are small enough, and $\tilde{U} = mu$ satisfies the following relations:

$$
\begin{aligned}
\tilde{U}_t &= \tilde{U}_{xx} + m^{1-\tilde{l}}\tilde{C}_0 \frac{\tilde{U}^{\tilde{l}}}{(T-t)^{\tilde{s}}}, \qquad \text{in } (0,1)\times(0,T),\\
\tilde{U}_x(0,t) &= 0, \quad \tilde{U}_x(1,t) = m^{1-r}C_0 \frac{\tilde{U}^r(1,t)}{(T-t)^{s}}, \qquad t\in(0,T),\\
\tilde{U}(x,0) &= mu_0(x) < \tilde{w}_0(x), \qquad \text{in } (0,1).
\end{aligned} \tag{2.17}
$$

By (III) we have

$$
\max_{x\in[0,1]} \tilde{U}(x,t)(T-t)^{\beta} \le C \qquad \text{for } 0<t<T. \tag{2.18}
$$

From (2.13) and (2.18) it follows that the proof of Theorem 1.3 is completed.

3. BLOW-UP RATE FOR THE SYSTEM

In this section, we prove Theorems 1.1 and 1.2. To this end, we start with a result of a comparison of the functions $u(x,t)$ and $v^r(x,t)$ (where (u,v) is the solution of (1.1)). This result allows us to reduce in a sense the case of a system to the case of a single equation.

LEMMA 3.1. *Under assumptions* (A), (B), *and* (C), *there exists a constant* $C>0$ *such that* $Cu \ge v^r$, *where* $r = \alpha_1/\alpha_2 > 1$ *and* (u,v) *is the solution of* (1.1).

Proof.

Step 1. We choose a constant $C_1 \ge 1$ large enough such that

$$
v^{\alpha_1/\alpha_2}(x,0) \le C_1 u(x,0) \qquad \text{for } x\in[0,1]. \tag{3.1}
$$

Step 2. By condition (C), we have $u(x,t)\ge 1$, $v(x,t)\ge 1$ for any $(x,t)\in[0,1]\times[0,T)$. Moreover, $C_2 = r^{1/(1-l_{11}+l_{21})} \ge 1$. Therefore, taking into account (1.1), for any constant $C\ge C_2$ we get

$$
\begin{aligned}
(Cu)_t &= (Cu)_{xx} + C^{1-l_{11}}(Cu)^{l_{11}}(v^r)^{l_{12/r}}, \qquad \text{in } (0,1)\times(0,T),\\
(v^r)_t &\le (v^r)_{xx} + C^{1-l_{11}}(Cu)^{l_{21}}(v^r)^{(r-1+l_{22})/r},\\
&\qquad \text{in } (0,1)\times(0,T).
\end{aligned} \tag{3.2}
$$

Step 3. Fix a constant $C > \max\{C_1, C_2, r^{1/(1+p_{21}-p_{11})}\}$. We prove that $Cu > v^r$ for any $(x,t) \in [0,1] \times [0,T)$. Set $t_0 = \sup\{t \mid Cu(x,\tau) > v^{\alpha_1/\alpha_2}(x,\tau)$ in $(0,1) \times (0 \le \tau < t)\}$. Then we have $t_0 > 0$. Suppose that $t_0 < T$ and there exists a point $x_0 \in [0,1]$ such that $Cu(x_0,t_0) = v^{\alpha_1/\alpha_2}(x_0,t_0)$. Likewise, in Step 1 of the proof of Theorem 1.3, by (1.1), (3.1), (3.2), the assumption (B), and the definition of t_0, we deduce easily that x_0 cannot belong to the half-interval $[0,1)$. Thus, we have $x_0 = 1$, and at the point $(1,t_0)$ we get

$$\begin{aligned}(Cu - v^{\alpha_1/\alpha_2})_x(1,t_0)& \\ &= C^{1-p_{11}}(Cu)^{p_{12}\alpha_2/\alpha_1+p_{11}} - \frac{\alpha_1}{\alpha_2 C^{p_{21}}}(Cu)^{(p_{22}-1)\alpha_1/\alpha_2+p_{21}+1}. \end{aligned}\tag{3.3}$$

By assumption (A) and the choice of the constant C, we have

$$\begin{gathered} p_{11} + p_{12}\alpha_2/\alpha_1 = p_{21} + 1 + (p_{22}-1)\alpha_2/\alpha_1, \\ C^{1-p_{11}} - \frac{\alpha_1}{C^{p_{21}}\alpha_2} > 0. \end{gathered}\tag{3.4}$$

From (3.3), (3.4), and (1.1) we obtain

$$(Cu - v^{\alpha_1/\alpha_2})_x(1,t_0) > 0, \tag{3.5}$$

On the other hand, by (3.2), (B), and $(Cu - v^r)(x,t) > 0$ in $(0,1) \times (0,t_0)$, we have

$$(Cu - v^{\alpha_1/\alpha_2})_t \ge (Cu - v^{\alpha_1/\alpha_2})_{xx} \qquad \text{in } (0,1) \times (0,t_0). \tag{3.6}$$

From (1.1) it follows that

$$(Cu - v^{\alpha_1/\alpha_2})_x(0,t) = 0 \qquad \text{for } 0 < t < t_0. \tag{3.7}$$

Therefore, from (3.1), (3.5)–(3.7) we obtain $(Cu - v^{\alpha_1/\alpha_2})(x_0,t_0) > 0$. This is a contradiction with our assumption. The proof of Lemma 3.1 is completed.

Now, let us prove Theorem 1.1.

Proof of Theorem 1.1. We begin with estimating $v(x,t)$ from below. By Lemma 3.1, we obtain

$$\begin{gathered} v_t = v_{xx} + u^{l_{21}}v^{l_{22}} \ge v_{xx} \qquad \text{in } (0,1) \times (0,T), \\ v_x(0,t) = 0, \quad v_x(1,t) = u^{p_{21}}(1,t)v^{p_{22}}(1,t) \ge cv^{p_1}(1,t), \\ v(x,0) = v_0(x), \qquad \text{in } (0,1), \end{gathered}\tag{3.8}$$

where $p_1 = (\alpha_1/\alpha_2)p_{21} + p_{22} = 1 - 1/\alpha_2 > 1$. By Proposition 2.1, we can conclude that there exists a constant c_1 such that

$$\max_{x \in [0,1]} v(x,t) \geq \frac{c_1}{(T-t)^{1/(2(p_1-1))}} \qquad (0 < t < T).$$

But $1/(2(p_1 - 1)) = -\alpha_2/2$, and therefore, we have

$$\max_{x \in [0,1]} v(x,t) \geq c_1(T-t)^{\alpha_2/2} \qquad (0 < t < T). \tag{3.9}$$

Now we pass to $u(x,t)$. From (3.9) we get

$$\begin{gathered} u_t \geq u_{xx}, \qquad \text{in } (0,1) \times (0,T), \\ u_x(0,t) = 0, \qquad u_x(1,t) = u^{p_{11}}(1,t)v^{p_{12}}(1,t) \geq c_2 \frac{u^{p_{11}}(1,t)}{(T-t)^{s_1}}, \\ u(x,0) = u_0(x), \qquad \text{in } (0,1), \end{gathered} \tag{3.10}$$

where $0 < p_{11} < 1$ and $s_1 = -(\alpha_2 p_{12})/2$. By hypothesis (A), we have $s_1 > 1/2$. Therefore, by Proposition 2.3, we obtain

$$\max_{x \in [0,1]} u(x,t) \geq c_3(T-t)^{-(s_1 - 1/2)/(1-p_{11})}.$$

We remark that

$$\frac{s_1 - 1/2}{1 - p_{11}} = -\alpha_1/2.$$

Thus we have obtained the lower bound for $u(x,t)$:

$$\max_{x \in [0,1]} u(x,t) \geq c_3(T-t)^{\alpha_1/2} \qquad (0 < t < T). \tag{3.11}$$

Next, we pass to the reverse inequalities in Theorem 1.1. Now, we start with $u(x,t)$. By Lemma 3.1, we have

$$\begin{gathered} u_t \leq u_{xx} + \tilde{C}_1 u^{l_{11} + l_{12}/r}, \qquad \text{in } (0,1) \times (0,T), \\ u_x(0,t) = 0, \quad u_x(1,t) = u^{p_{11}}(1,t)v^{p_{12}}(1,t) \leq C_1 u^{p_2}(1,t), \\ u(x,0) = u_0(x), \qquad \text{in } (0,1), \end{gathered}$$

where $\min\{\tilde{C}_1, C_1\} > 1$, $r = \alpha_1/\alpha_2$, and $p_2 = \{\alpha_1 p_{11} + \alpha_2 p_{12}\}/\alpha_1 = -1/\alpha_1 + 1 > 1$. By assumption (A), we have $l_{11} + l_{12}/r < 2p_2 - 1$. Thus, we can take a constant l_2 such that $l_{11} + l_{12}/r < l_2 < 2p_2 - 1$ and $l_2 > 1$.

By assumption (C), we have $u(x,t) \geq 1$. On the other hand, we can take a constant M large enough such that $\max\{\tilde{C}_1 M^{1-l_2}, C_1 M^{1-p_2}\} \leq 1$. Set $\tilde{u} = Mu$; then we get

$$\begin{gathered} \tilde{u}_t = \tilde{u}_{xx} + \tilde{C}_1 \tilde{u}^{l_{11}+l_{12}/r} \leq \tilde{u}_{xx} + \tilde{u}^{l_2}, \qquad \text{in } (0,1) \times (0,T), \\ \tilde{u}_x(0,t) = 0, \quad \tilde{u}_x(1,t) \leq C_1 \tilde{u}^{p_2}(1,t) \leq \tilde{u}^{p_2}(1,t), \\ \tilde{u}(x,0) = Mu_0(x), \qquad \text{in } (0,1). \end{gathered}$$

Thus, by Proposition 2.2, we obtain

$$\max_{x \in [0,1]} Mu(x,t) \leq \frac{C_2}{(T-t)^{1/(2(p_2-1))}} \qquad (0 < t < T).$$

But $1/(2(p_2 - 1)) = -\alpha_1/2$, whence

$$\max_{x \in [0,1]} u(x,t) \leq \frac{\overline{C}_2}{(T-t)^{-\alpha_1/2}} \qquad (0 < t < T). \tag{3.12}$$

By the above estimate for $u(x,t)$, we have

$$\begin{gathered} v_t \leq v_{xx} + \tilde{C}_3 \frac{v^{l_{22}}}{(T-t)^{\tilde{s}_2}} \qquad \text{in } (0,1) \times (0,T), \\ v_x(0,t) = 0, \quad v_x(1,t) = u^{p_{21}}(1,t) v^{p_{22}}(1,t) \leq C_3 \frac{v^{p_{22}}(1,t)}{(T-t)^{s_2}}, \\ v(x,0) = v_0(x), \qquad \text{in } (0,1), \end{gathered} \tag{3.13}$$

where $0 < l_{22} < 1$, $0 < p_{22} < 1$, $\tilde{s}_2 = -\alpha_1 l_{21}/2$, and $s_2 = (-\alpha_1 p_{21})/2$. Using again assumption (A), we get $s_2 > 1/2$ and $\tilde{s}_2 < 1 + (1 - l_{22})(2s_2 - 1)/(2(1 - p_{22}))$. Thus, by Theorem 1.3 we obtain

$$\max_{x \in [0,1]} v(x,t) \leq \frac{C_4}{(T-t)^{(s_2 - 1/2)/(1-p_{22})}}.$$

We observe that $(s_2 - 1/2)/(1 - p_{22}) = (-\alpha_2)/2$. Therefore, we have

$$\max_{x \in [0,1]} v(x,t) \leq \frac{C_5}{(T-t)^{-\alpha_2/2}} \qquad (0 < t < T). \tag{3.14}$$

Combining (3.9), (3.11), (3.12), with (3.14), we complete the proof of Theorem 1.1.

Proof of Theorem 1.2. We begin with $u(x,t)$. By Lemma 3.1, we have

$$\begin{gathered} u_t \le u_{xx} + \tilde{C}_1 u^{l_{11}+l_{12}/r} \qquad \text{in } (0,1)\times(0,T), \\ u_x(0,t) = 0, \quad u_x(1,t) = u^{p_{11}}(1,t)v^{p_{12}}(1,t) \le C_1 u^{p_2}(1,t), \\ u(x,0) = u_0(x), \qquad \text{in } (0,1), \end{gathered} \tag{3.15}$$

where $r = \alpha_1/\alpha_2$, $\max\{\tilde{C}_1, C_1\} > 1$, and $p_2 = \{\alpha_1 p_{11} + \alpha_2 p_{12}\}/\alpha_1 > 1$. By (A), we can take a constant l_3 such that $l_{11} + l_{12}/r < l_3 < 2p_2 - 1$ and $l_3 > 1$. We can also take another constant K large enough such that $\max\{\tilde{C}_1 K^{1-l_3} C_1 K^{1-p_2}\} \le 1$. Let $\bar{u} = Ku$. Taking into account (3.15), by (C) we get

$$\begin{gathered} \bar{u}_t \le \bar{u}_{xx} + \bar{u}^{l_3} \qquad \text{in } (0,1)\times(0,T), \\ \bar{u}_x(0,t) = 0, \qquad \bar{u}_x(1,t) \le \bar{u}^{p_2}(1,t), \\ \bar{u}(x,0) = Ku_0(x), \qquad \text{in } (0,1), \end{gathered} \tag{3.16}$$

By Proposition 2.2 and condition (C), for any $0 \le r < 1$ there exists a constant $C_7 = C_7(r)$ such that

$$\max_{x\in[0,r]} u(x,t) \le C_7(r), \qquad t \in [0,T). \tag{3.17}$$

By Lemma 3.1 and (3.17), we obtain

$$\max_{x\in[0,r]} v(x,t) \le C_8(r), \qquad t \in [0,T). \tag{3.18}$$

This completes the proof of Theorem 1.2.

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