Internal Stabilizability of Some Diffusive Models

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Submitted by Hélène Frankowska

Received February 6, 2001

We consider a single species population dynamics model with age dependence, spatial structure, and a nonlocal birth process arising as a boundary condition. We prove that under a suitable internal feedback control, one can improve the stabilizability results given in Kubo and Langlais [J. Math. Biol. 29 (1991), 363–378]. This result is optimal.

Our proof relies on an identical stabilizability result of independent interest for the heat equation, that we state and prove in Section 3.

1. INTRODUCTION AND MAIN RESULTS

We consider a linear model describing the dynamics of a single species population with age dependence and spatial structure.

Let $p = p(a, t, x)$ be the distribution of individuals having age $a \geq 0$ at time $t \geq 0$ and location $x$ in $\Omega$, a bounded open set in $\mathbb{R}^d$, $d \in \{1, 2, 3\}$, having a suitably smooth boundary $\Gamma$. Thus

$$P(t, x) = \int_0^{A_1} p(a, t, x) da \quad (1)$$
is the total population density at time $t$ and location $x$, $A_1$ being the maximum life expectancy of an individual. Let $\beta(a) \geq 0$ be the natural fertility rate and let $\mu(a) \geq 0$ be the natural death rate of individuals having age $a$. Under these conditions the evolution of the distribution $p$ is governed by the partial differential equation

$$\partial_t p + \partial_a p - \Delta_x p + \mu(a) p = 0, \quad a \in (0, A_1), \quad t \in (0, +\infty), \quad x \in \Omega;$$

see [6, 7, 11]. The birth process is given by the renewal equation

$$p(0, t, x) = \int_0^{A_1} \beta(a) p(a, t, x) da, \quad t \in (0, +\infty), \quad x \in \Omega. \quad (3)$$

We assume hostile ends corresponding to homogeneous Dirichlet boundary conditions

$$p(a, t, x) = 0, \quad a \in (0, A_1), \quad t \in (0, +\infty), \quad x \in \Gamma. \quad (4)$$

The dynamics of the solution to (2)–(4) starting at time $t = 0$ from

$$p(a, 0, x) = p_0(a, x), \quad a \in (0, A_1), \quad x \in \Omega, \quad (5)$$

depends on the root $r^*$ of the characteristic equation

$$1 = \int_0^{A_1} \beta(a) \pi(a) \exp(-ra) da, \quad \pi(a) = \exp\left(-\int_0^a \mu(s) ds\right), \quad (6)$$

as well as on $\lambda_1$, the first eigenvalue of the Dirichlet problem for $-\Delta$ in $\Omega$. When $\beta$ satisfies $({\mathbf{H}_1})$ below and $p_0(a, x) \not\equiv 0$ on $(0, a_1) \times \Omega$, the large time behavior of the solutions to (2)–(5) is the following: for each finite $A, 0 < A < A_1$, one has

1. If $r^* < \lambda_1$, then $\|p(\cdot, t, \cdot)\|_{L^2((0, A) \times \Omega)} \to 0$ as $t \to +\infty$.
2. If $r^* > \lambda_1$, then $\|p(\cdot, t, \cdot)\|_{L^2((0, A) \times \Omega)} \to +\infty$ as $t \to +\infty$.
3. If $r^* = \lambda_1$, then $p(\cdot, t, \cdot) \to C(p_0)\pi(a)$ as $t \to +\infty$, for some constant $C(p_0)$.

Herein $\|\cdot\|_{L^2((0, A) \times \Omega)}$ is the norm in $L^2((0, A) \times \Omega)$. When $p_0(x, a) \equiv 0$ on $(0, a_1) \times \Omega$, then $p(a, t, x) \equiv 0$ for $t > a$. A quick proof can be found in [9].

A natural question to address is the following: is it possible to stabilize in a suitable fashion the dynamics of $p$, a solution to (2)–(4)? An answer in this direction is found in [8] for nonnegative data. Assuming that an input $u(a, t, x)$ is supplied, the new system becomes

$$\partial_t p + \partial_a p - \Delta_x p + \mu(a) p = u(a, t, x), \quad \text{in} \quad (0, A_1) \times (0, T) \times \Omega,$$

$$p(a, 0, x) = p_0(a, x), \quad \text{on} \quad (0, A_1) \times \Omega,$$

$$p(0, t, x) = \int_0^{A_1} \beta(a) p(a, t, x) da, \quad \text{on} \quad (0, T) \times \Omega,$$

$$p(a, t, x) = 0, \quad \text{on} \quad (0, A_1) \times (0, T) \times \Gamma. \quad (7)$$
First, when \( r^* > \lambda_1 \), assuming

\[
    u(a, t, x) \geq 0 \quad \text{and} \quad p_0(a, x) \geq 0,
\]

and using a comparison principle \([2, 5]\), then the solution to (7) is bounded from below by the nonnegative solution to (2)–(5); therefore its \( L^2((0, A) \times \Omega) \) norm is still diverging. Next, when \( r^* < \lambda_1 \), in order to prevent the decay to 0 of the solution to (2)–(5), a basic idea in \([8]\) is to supply a nonnegative and periodic, in time, input of individuals. Let \( T^* > 0 \) be fixed; when

\[
    r^* < \lambda_1, \quad p_0(a, x) \geq 0, \quad u(a, t + T^*, x) = u(a, t, x) \geq 0, \quad (9)
\]

then the solution to (7) stabilizes as \( t \to +\infty \) toward a unique (stable) nonnegative \( T^* \) periodic solution. Last, in the limiting case

\[
    r^* = \lambda_1, \quad p_0(a, x) \geq 0, \quad u(a, t + T^*, x) = u(a, t, x) \geq 0, \quad (10)
\]

the \( L^2((0, A) \times \Omega) \) norm of solutions to (7) is still diverging.

In this paper we are interested in the internal stabilizability of (7) under feedback control lying on \( \bar{\omega} \), where \( \omega \) is a subdomain of \( \Omega \); that is, we have to find a feedback control \( u \) such that \( \lim_{t \to \infty} p(a, t, x) = 0 \) a.e. \( (a, x) \in (0, A_t) \times \bar{\Omega} \). Another approach, i.e., approximate controllability, is devised in \([1]\); see also \([2]\). Let \( \omega \) be a nonempty subdomain of \( \Omega \) satisfying \( \omega \subset \bar{\omega} \subset \Omega \).

We denote by \( \lambda_1^\omega \) the first eigenvalue of the Dirichlet problem for \( -\Delta \) in \( \Omega \setminus \bar{\omega} \),

\[
    -\Delta w = \lambda w, \quad \text{on} \ \Omega \setminus \bar{\omega},
\]

\[
    w(x) = 0, \quad \text{on} \ \Gamma \cup \partial \omega. \quad (11)
\]

It follows that \( \lambda_1 < \lambda_1^\omega \).

Our main result will be proved in Section 4.

**Theorem 1.1.** If \( r^* < \lambda_1^\omega \), then there exists a feedback control \( u(a, t, x) \) with support in \( [0, A_t) \times [0, \infty) \times \bar{\omega} \) that stabilizes (7).

**Remark 1.2.** Note that when \( r^* < \lambda_1 \), this can be done with \( u(a, t, x) = 0 \), while for \( \lambda_1 \leq r^* < \lambda_1^\omega \) a nontrivial control is required. On the other hand, if \( r^* \geq \lambda_1^\omega > \lambda_1 \) and \( p_0(a, x) \geq 0, \ p_0 \neq 0 \), then there is no control with support in \( \bar{\omega} \) such that \( \lim_{t \to +\infty} p(\cdot, t, \cdot) = 0 \) in an appropriate space.
2. ASSUMPTIONS

Let $\Omega$ be a bounded open set in $\mathbb{R}^d$ with a smooth boundary $\Gamma$ so that locally $\Omega$ lies on one side of $\Gamma$. Next, $\omega$ is a nonempty open subset of $\Omega$ having a positive $d$-dimensional Lebesgue measure, such that $\omega \subset \bar{\omega} \subset \Omega$, and $\chi_{\omega}$ is the characteristic function of $\omega$. We define

$$
\text{sgn}_{L^1(\omega)} u = \begin{cases} 
\frac{u}{\|u\|_{L^1(\omega)}}, & \text{if } \|u\|_{L^1(\omega)} \neq 0, \\
\{m \in L^1(\omega); \|m\|_{L^1(\omega)} \leq 1\}, & \text{if } \|u\|_{L^1(\omega)} = 0,
\end{cases}
$$

(12)

Let $A^\dagger$ be a finite positive number. One assumes the fertility rate $\beta$, the mortality rate $\mu$, and the initial data satisfy

$$(H_1) \quad \beta \in L^\infty([0, A^\dagger]), \beta(a) \geq 0 \text{ a.e. } a \in (0, A_t), \text{ and } a_1 = \max \text{supp}(\beta) < A^\dagger.$$ 

$$(H_2) \quad \text{there exists } a_0 \in (0, a_1) \text{ such that } \beta(a) = 0 \text{ a.e. } a \in (0, a_0) \cup (a_1, A^\dagger).$$

$$(H_3) \quad \mu \in L^\infty_{\text{loc}}([0, A^\dagger]), \mu(a) \geq 0 \text{ a.e. } a \in (0, A_t),$$

$$(H_4) \quad \int_0^{A^\dagger} \mu(a) \, da = +\infty,$$

$$(H_5) \quad p_0 \in L^\infty((0, A_t) \times \Omega)), p_0(a, x) \geq 0 \text{ a.e. in } (0, A_t) \times \Omega.$$ 

The divergence condition in $(H_4)$ means that each individual in the population dies before age $A_t$. For each $A \in (a_0, A^\dagger)$, we set $Q^A_t = (0, A) \times (0, T) \times \Omega$.

3. STABILIZABILITY FOR THE HEAT EQUATION

In this section $y_0$ is an initial condition and $q$ is a potential satisfying

$$(H_6) \quad y_0, q \in L^\infty(\Omega), y_0(x) \geq 0 \text{ a.e. in } \Omega.$$ 

Along the characteristic lines of $\partial_t + \partial_a$, problem (7) resembles a parabolic problem (but with the integral condition at $a = 0$). Hence we begin by proving a related stabilization result for the heat equation, having its own interest, using controls supported in $\bar{\omega}$. 
3.1. Partial Zero Exact Feedback Controllability for the Heat Equation

Consider the problem
\[
y_t - \Delta y + q(x)y \in -\rho \, \text{sgn}_{L^1(\omega)} y \chi_\omega, \quad t \in (0, T), \quad x \in \Omega,
\]
\[
y(t, x) = 0, \quad t \in (0, T), \quad x \in \Gamma,
\]
\[
y(0, x) = y_0(x), \quad x \in \Omega. \tag{13}
\]
We introduce the following definition for a solution of this parabolic problem.

**Definition 3.1.** An element \( y \) in \( C([0, T]; L^1(\Omega)) \) is a weak solution of (13) if there exists an element \( u \) in \( L^\infty([0, T]; L^1(\Omega)) \) such that \( u(t, \cdot) \in \text{sgn}_{L^1(\omega)} y(t, \cdot) \) a.e. \( t \in (0, T) \), and such that
\[
\int_\Omega y(T, x) \xi(T, x) \, dx - \int_\Omega y_0(x) \xi(0, x) \, dx
\]
\[
- \int_0^T \int_\Omega y(t, x)(\xi_t + \Delta \xi - q \xi)(t, x) \, dx \, dt
\]
\[
= -\rho \int_0^T \int_\omega u(t, x) \xi(t, x) \, dx \, dt,
\]
\[\forall \xi \in W = \{ h \in C^{1,2}([0, T] \times \overline{\Omega}); h(t, x) = 0 \text{ on } [0, T] \times \Gamma \}.
\]

**Theorem 3.2.** Given \( y_0 \) in \( L^1(\Omega) \), \( y_0(x) \geq 0 \) a.e. in \( \Omega \), and \( q \) in \( L^\infty(\Omega) \), there exists a \( \rho_0(y_0, q) \) such that, for \( \rho > \rho_0(y_0, q) \), there exists a \( T = T(\rho) > 0 \) such that problem (13) has a unique weak solution \( y \) in \( (0, T) \times \Omega \), and in addition
\[
y(T, x) = 0, \quad \text{a.e. } x \in \omega
\]
and
\[
\|y(t, \cdot)\|_{L^1(\omega)} > 0, \quad \forall t \in (0, T).
\]

**Proof.** The operator \( \mathcal{A} \) defined via
\[D(\mathcal{A}) = \{ y \in W^{1,1}_0(\Omega); \Delta y \in L^1(\Omega) \},
\]
\[\mathcal{A} y = \Delta y - q(y) y,
\]
is the generator of a compact \( C_0 \)-semigroup in \( L^1(\Omega) \); see [4].

For each \( \varepsilon > 0 \), consider the regularized problem
\[
y_t - \Delta y + q(x)y = -\rho \frac{y(t, x)}{\|y(t, \cdot)\|_{L^1(\omega)} + \varepsilon} \chi_\omega(x), \quad t > 0, \quad x \in \Omega,
\]
\[
y(t, x) = 0, \quad t > 0, \quad x \in \Gamma,
\]
\[
y(0, x) = y_0(x), \quad x \in \Omega. \tag{14}
\]
The mapping from $L^1(\Omega)$ to $L^1(\Omega)$

$$y \rightarrow \frac{y}{\|y\|_{L^1(\omega)} + \varepsilon} \chi_{\omega}$$

is locally Lipschitz continuous. As a consequence, problem (14) has a unique mild solution $y_\varepsilon \in C(\mathbb{R}^+; L^1(\Omega))$, in the sense of Pazy [10]. Since $\| - \rho y_\varepsilon(t, \cdot)/(\|y_\varepsilon(t, \cdot)\|_{L^1(\omega)} + \varepsilon) \|_{L^1(\Omega)} \leq \rho$ a.e. $t > 0$ and using the Baras compactness theorem, see [3], one gets that, for some subsequence $(\varepsilon' \searrow 0)$,

$$y_\varepsilon' \rightarrow y \quad \text{in } C([0, T]; L^1(\Omega)),$$

for any $T > 0$. Hence, as $\varepsilon' \rightarrow 0$,

$$\|y_\varepsilon(t, \cdot)\|_{L^1(\omega)} \rightarrow \|y(t, \cdot)\|_{L^1(\omega)}, \quad \forall t \in [0, T]$$

(and in $L^\infty(0, T), \forall T > 0$).

Now there exists a $T \in \mathbb{R}^+$ such that

$$\|y(t, \cdot)\|_{L^1(\omega)} > 0, \quad \forall t \in [0, T),$$

$$\|y(T, \cdot)\|_{L^1(\omega)} = 0;$$

see a proof below. Passing to the limit $(\varepsilon' \searrow 0)$ in (14), one gets that $y$ is a solution of (13) on $[0, T]$.

Let us now prove (15). Indeed, assume that $\|y(t, \cdot)\|_{L^1(\omega)} > 0$ for any $t > 0$. Then, multiplying (13) by $\text{sgn} y$ and integrating over $(0, t) \times \Omega$, one has

$$\left(\|y(t, \cdot)\|_{L^1(\omega)} - 1\right) \leq \int_0^t \|q(s, \cdot)\|_{L^1(\omega)} ds - \rho.$$

Set $z(t) = \|y(t, \cdot)\|_{L^1(\omega)}$ and $\alpha > \|q\|_{L^\infty(\Omega)}$, a constant; it follows that

$$z'(t) = \|y(t, \cdot)\|_{L^1(\omega)} \leq \|y_0\|_{L^1(\omega)} + \alpha z(t) - \rho.$$

Hence

$$(e^{-\alpha z(t)})' \leq \|y_0\|_{L^1(\omega)} e^{-\alpha t} - \rho t e^{-\alpha t}, \quad t \in [0, T),$$

and

$$e^{-\alpha z(t)} \leq \|y_0\|_{L^1(\omega)} \frac{1}{\alpha}(1 - e^{-\alpha t}) - \rho \int_0^t s e^{-\alpha s} ds$$

$$= \frac{\|y_0\|_{L^1(\omega)}}{\alpha} (1 - e^{-\alpha t}) - \frac{\rho}{\alpha^2} + \rho e^{-\alpha t} \left(\frac{t}{\alpha} + \frac{1}{\alpha^2}\right).$$

As a consequence,

$$z(t) \leq \frac{-\rho}{\alpha^2} e^{\alpha t} + \frac{\rho t}{\alpha} + \frac{\rho}{\alpha^2} + \frac{\|y_0\|_{L^1(\omega)}}{\alpha} (e^{\alpha t} - 1).$$
and
\[ \|y(t, \cdot)\|_{L^1(I)} \leq -\frac{p}{\alpha} e^{\alpha t} + \frac{p}{\alpha} + \|y_0\|_{L^1(I)} e^{\alpha t}. \]

We now observe that for \( \rho \) large enough, with respect to \( q \) and \( y_0 \), there exists a \( t_0 \) such that \( \|y(t_0, \cdot)\|_{L^1(I)} < 0 \); a contradiction. So
\[ \mathcal{B} = \{ t \in [0, +\infty), \|y(t, \cdot)\|_{L^1(\omega)} = 0 \} \]
is nonempty, and \( T = \inf \mathcal{B} \).

**Remark 3.3.** From the proof, it should be clear that this solution is non-
negative and bounded from above on \((0, T) \times \Omega \) by the classical solution of the control free problem
\begin{equation}
\begin{aligned}
y_t - \Delta y + q(x)y = 0, \quad t \in (0, +\infty), x \in \Omega, \\
y(t, x) = 0, \quad t \in (0, +\infty), x \in \Gamma, \\
y(0, x) = y_0(x), \quad x \in \Omega.
\end{aligned}
\end{equation}

### 3.2. Characterization

In order to extend after time \( T \) the solution \( y = \lim_{\varepsilon \to 0} y_\varepsilon \) constructed in the proof of Theorem 3.2, we introduce the following parabolic problem in \( \Omega \setminus \bar{\omega} \),
\begin{equation}
\begin{aligned}
y_t - \Delta y + q(x)y = 0, \quad t \in (T, +\infty), x \in (\Omega \setminus \bar{\omega}), \\
y(t, x) = 0, \quad t \in (T, +\infty), x \in (\Gamma \cup \partial \omega), \\
y(T^+, x) = y_1(x), \quad x \in \Omega \setminus \omega, \\
y_1(x) \geq 0, y_1 \in L^\infty(\Omega \setminus \bar{\omega}), \text{ supplemented by the condition in } \bar{\omega}
\end{aligned}
\end{equation}

Note that \( y = \lim_{\varepsilon \to 0} y_\varepsilon \) satisfies (16) with \( y_1(x) = y(T^-, x) \), but the boundary conditions along \((T, +\infty) \times \partial \omega\). The initial condition at \( T \) being bounded, the solution of (16) is smooth on \((T, +\infty) \times (\Omega \setminus \bar{\omega})\).

**Definition 3.4.** Given any nonnegative \( y_1 \) in \( L^\infty(\Omega \setminus \bar{\omega}) \), let \( \mu_y \) be the measure defined on \((T, +\infty) \times \Omega \) by
\[ \mu_y(\varphi) = \int_T^{+\infty} \int_{\partial \omega} \frac{\partial y^-}{\partial \nu} \varphi \, d\sigma \, dt, \quad \forall \varphi \in C_0((T, +\infty) \times \Omega), \]
where \( \nu \) is the unit outward normal to \( \omega \) and
\[ \frac{\partial y^-}{\partial \nu} (\cdot, x_0) = \lim_{\varepsilon \downarrow 0} \frac{y(\cdot, x_0 - \varepsilon \nu) - y(\cdot, x_0)}{-\varepsilon}, \quad \text{for } x_0 \in \partial \omega. \]
Then we have the following result:

**Lemma 3.5.** For \( t \geq T \) and for any \( \tau > 0 \), \( y \) satisfying (16) and (17) belongs to \( L^\infty((T, T + \tau) \times \Omega) \cap L^2(T, T + \tau; H_0^1(\Omega) \cap H^2(\Omega \setminus \tilde{\omega})) \), and is a solution to

\[
y_t - \Delta y + q(x)y = \mu_y, \quad t \in (T, +\infty), \quad x \in \Omega,
y(t, x) = 0, \quad t \in (T, +\infty), \quad x \in \Gamma,
y(T^+, x) = \begin{cases} y_1(x), & x \in \Omega \setminus \tilde{\omega}, \\ 0 & x \in \omega. \end{cases}
\]

**Proof.** First, (16) has a unique solution lying in \( L^\infty((T, T + \tau) \times \Omega \setminus \tilde{\omega}) \) and \( L^2(T, T + \tau; H_0^1(\Omega \setminus \tilde{\omega})) \). Then this solution extended by 0 on \( (T, +\infty) \times \omega \) satisfies the condition listed in the lemma.

Multiplying (16) by a smooth test function vanishing on \( (T, +\infty) \times \Gamma \) and integrating on \( (T, T) \times (\Omega \setminus \tilde{\omega}) \), one gets

\[
\int_{\Omega \setminus \tilde{\omega}} y(T, x)\varphi(T, x) dx - \int_{\Omega \setminus \tilde{\omega}} y_1(x)\varphi(T, x) dx \\
- \int_{(T, T) \times (\Omega \setminus \tilde{\omega})} y(\varphi_t + \Delta \varphi - q\varphi)(t, x) dx dt \\
- \int_{(T, T) \times \partial \omega} \frac{\partial y}{\partial \nu}(t, x) d\sigma dt = 0.
\]

One observes that, from (17), one may replace the integrals on \( \Omega \setminus \tilde{\omega} \) by integrals on \( \Omega \) in the latter relation. This shows that any \( y \) satisfying (16)–(17) is a weak solution to (18).

As a first consequence, one has

**Proposition 3.6.** Let \( y \) be the solution of (13) found in Theorem 3.2, and let \( T = T(\rho, y_0, q) \) be such that \( y(x, T) = 0 \) a.e. \( x \in \omega \). Setting \( y_1(x) = y(T^-, x) \) a.e. \( x \in \Omega \setminus \omega \), the measure \( \mu_y \) defined in Definition 3.4 stabilizes the system if and only if \( \lambda_1^{\rho, \omega} \), the first eigenvalue of

\[
-\Delta y + q(x)y = \lambda y, \quad x \in \Omega \setminus \tilde{\omega},
y(x) = 0, \quad x \in \Gamma \cup \partial \omega,
\]

is positive; \( \mu_y \) is a feedback control with support in \( [T, +\infty) \times \tilde{\omega} \).

**Proof.** Going back to (16), one may deduce that for any \( y_1 \in L^\infty(\Omega \setminus \omega) \), the solution \( y(\cdot, t) \to 0 \) as \( t \to +\infty \) in \( L^\infty(\Omega \setminus \omega) \) provided \( \lambda_1^{\rho, \omega} > 0 \).

Now, solving (13) along the lines of Theorem 3.2 on \( (0, T(\rho, y_0, q)) \), one gets \( y(\cdot, T(\rho, y_0, q)) \in L^\infty(\Omega) \). Choosing \( y_1(x) = y(x, T(\rho, y_0, q)) \), the solution of (16)–(17) goes to 0 as \( t \to +\infty \).
This can be expressed as: the corresponding measure $\mu_y$ is a feedback control with support in $[T, +\infty) \times \bar{\omega}$.

Now we are able to state our main results concerning the stabilizability of the heat equation with a bounded potential. Assuming the first eigenvalue of (19) is nonpositive, then a solution of (16) does not converge to 0 as $t \to +\infty$. It follows

**Theorem 3.7.** System (13) is stabilizable if and only if the first eigenvalue of (19) is positive. If it is stabilizable, then the feedback control $\mu_y$ in Proposition 3.6 stabilizes the system.

### 4. PROOF OF THE STABILIZABILITY RESULT FOR THE AGE AND SPACE DEPENDENT POPULATION DYNAMICS MODEL

Let us consider the problem

$$
\begin{align*}
\frac{\partial p}{\partial t} + \frac{\partial p}{\partial x} - \Delta_x p + \mu(a)p &\in -\rho \text{sgn} L_{1,(\omega)} p \chi_\omega, \quad (a, t, x) \in Q_T^A, \\
p(a, 0, x) &= p_0(a, x), \quad (a, x) \in (0, A) \times \Omega, \\
p(0, t, x) &= \int_0^A \beta(a) p(a, t, x) da, \quad (t, x) \in (0, T) \times \Omega, \\
p(a, t, x) &= 0, \quad (a, t, x) \in (0, A) (0, T) \times \Gamma,
\end{align*}
$$

(20)

where $\rho$ is a nonnegative number to be chosen later, $T$ is a positive number, and $A \in (a_1, a_2)$.

**Definition 4.1.** A function $p \in L^\infty([0, A] \times [0, T]; L^1(\Omega))$ is a solution of (20) if there exists a $u$ in $L^1_{loc}(0, T; L^1((0, A) \times \Omega))$, $u \in -\rho \text{sgn} L_{1,(\omega)} p \chi_\omega$, such that

$$
\begin{align*}
\frac{\partial p}{\partial t} + \frac{\partial p}{\partial x} - \Delta_x p + \mu(a)p &= u(a, t, x), \quad (a, t, x) \in Q_T^A, \\
\lim_{\varepsilon \to 0^+} p(a + \varepsilon, t, x) &= p_0(a, t, \cdot), \quad \text{in } L^1(\Omega), \quad \text{a.e. } a \in (0, A), \\
\lim_{\varepsilon \to 0^+} p(e, t + \varepsilon, \cdot) &= \int_0^A \beta(a) p(a, t, \cdot) da, \quad \text{in } L^1(\Omega), \quad \text{a.e. } t \in (0, T), \\
p(a, t, x) &= 0, \quad (a, t, x) \in (0, A) \times (0, T) \times \Gamma,
\end{align*}
$$

(21)

and $p \in C(\bar{S}; L^1(\Omega)) \cap AC(S; L^1(\Omega))$, for almost any characteristic line $S$ of equation

$$
S = \{(\gamma + t, \theta + t); \ t \in (0, T_0)\},
$$

$$(\gamma, \theta) \in (0, A - T_0) \times \{0\} \cup \{0\} \times (0, T - T_0).$$
4.1. Existence

**Theorem 4.2.** Under the hypotheses (H₁)–(H₅), there exists a nonnegative number ρ such that (20) has a solution.

**Proof.** Let p₁ be an arbitrary fixed nonnegative function in $L^∞((0, T) \times Ω)$, and let us study the problem

\[
\begin{align*}
p_t + p_α + μ p - Δ_x p &\in -ρ sgn L(ω) p(a, t, x) χ_ω, \quad (a, t, x) \in (0, A) × (0, T) × Ω, \\
p(a, t, x) &\equiv 0, \quad (a, t, x) \in (0, A) × (0, T) × Γ, \\
p(0, t, x) &\equiv p_1(t, x), \quad (t, x) \in (0, T) × Ω, \\
p(a, 0, x) &\equiv p_0(a, x), \quad (a, x) \in (0, A) × Ω.
\end{align*}
\]

We can view (22) as a collection of parabolic systems on the characteristic lines S. Define

\[
p_\tilde{p}(t, x) = p(γ + t, θ + t, x), \quad (t, x) \in (0, T_0) × Ω,
\]

\[
\tilde{μ}(t) = μ(γ + t), \quad (t, x) \in (0, T_0).
\]

Then, $p_\tilde{p}$ satisfies

\[
\begin{align*}
p_\tilde{p} + \tilde{μ} p_\tilde{p} - Δ_x p_\tilde{p} &\in -ρ sgn L(ω) p_\tilde{p}(t, x) χ_ω, \quad (t, x) \in (0, T_0) × Ω, \\
p_\tilde{p}(t, x) &\equiv 0, \quad (t, x) \in (0, T_0) × Γ, \\
p_\tilde{p}(0, x) &\equiv \begin{cases} 
  p_1(θ, x), & γ = 0, \\
  p_0(γ, x), & θ = 0, 
\end{cases} \quad x \in Ω.
\end{align*}
\]

Using the results from the previous section, we may conclude that for ρ large enough, system (23) has a unique weak solution on every characteristic line of equation $a - t$ = constant, because $p_0 \in L^∞([0, A]; L^1(Σ))$, $p_1 \in L^∞([0, T); L^1(Σ))$, and $μ \in L^∞(0, A)$.

Using now (H₅), we infer by an iterative procedure the existence of a unique weak and nonnegative solution of (20) with $u := -ρ sgn L(ω) p χ_ω$. For $T = a_0$, the solution of (20) on $(0, A) × (0, a_0) × Ω$ is actually given by (22) when substituting $p_1(t, x)$ by $\int_{a_0}^A β(a) p(a, t, x) da$. Note this integral depends only on $p_0$ for $0 < t < a_0$. More precisely, for $(a, t)$ satisfying $0 < t < a < A_t$ and $t < a_0$, one finds that $p(a, t, ·)$ depends only on $p_0$ and one derives from a comparison principle that $0 ≤ p(a, t, x) ≤ \|p_0\|_{L^∞((0, A) × Ω)}$. Next, in the small triangle wherein $0 < a < t < a_0$, one can check that $p(a, t, ·)$ depends only on $\int_{a_0}^A β(a) p(a, t, x) da$; again $0 ≤ p(a, t, x) ≤ C_1(\|p_0\|_{L^∞((0, A) × Ω)}; a_0)$.

Assume now that the solution of (20) is known on $(0, A) × (0, na_0) × Ω$, where $n$ is a positive integer with $0 ≤ p(a, t, x) ≤ C_n(\|p_0\|_{L^∞((0, A) × Ω)}; a_0)$.
on \((0, A) \times (0, na_0) \times \Omega\). Then, proceeding as in the first step, the solution of (20) on \((0, A) \times (na_0, (n + 1)a_0) \times \Omega\) is the solution of (22) when substituting \(p_0(a, x) = p(na_0, t, x)\) and \(p_1(t, x) = \int_{a_0}^{A} \beta(a) p(a, t, x) \, da.\) One also gets \(0 \leq p(a, t, x) \leq C_{n+1}(\|p_0\|_{L^\infty((0, A) \times \Omega)}, a_0)\) on \((0, A) \times (0, (n + 1)a_0) \times \Omega.\)

4.2. Proof of Theorem 1.1

Let \(p\) be a solution of (20). Assume that \(\|p(\cdot, t, \cdot)\|_{L^1(\omega \times (0, A))} > 0\) for any \(t > 0\). Multiplying (20) by \(\text{sgn}\, p\) and integrating over \(\Omega \times (0, t) \times (0, A)\), we obtain

\[
\int_0^t \int_0^A \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} + \mu(a) \right) \|p(a, s, \cdot)\|_{L^1(\Omega)} \, ds 
\leq -\rho A t.
\]

So

\[
\|p(\cdot, t, \cdot)\|_{L^1((0, A) \times \Omega)} \leq C + \|\beta(\cdot)\|_{L^\infty(\omega)} \int_0^t \int_0^A \int_{\overline{\Omega}} |p(a, s, x)| \, da 
\leq dx \, ds - \rho A t,
\]

and once again, for \(\rho\) large enough, as in the end of the proof for Theorem 3.2, there exists \(T > 0\) such that the solution to (20) vanishes on \(\omega \times (0, A)\) at \(t = T.\)

Now, on \(\Omega \setminus \tilde{\omega}, p\) satisfies, for \(t \geq T,\)

\[
\partial_t p + \partial_x p - \Delta_x p + \mu(a) p = 0, \quad (a, t, x) \in (0, A) \times (T, +\infty) \times (\Omega \setminus \tilde{\omega}),
\]

\[
p(a, T^+, x) = p(a, T^-, x), \quad (a, x) \in (0, A) \times (\Omega \setminus \tilde{\omega}),
\]

\[
p(0, 0, x) = \int_{0}^{A} \beta(a) p(a, t, x) \, da, \quad (t, x) \in (T, +\infty) \times (\Omega \setminus \tilde{\omega}),
\]

\[
p(a, t, x) = 0, \quad (a, t, x) \in (0, A)(T, +\infty) \times (\Gamma \cup \partial \omega).
\]

Assume for a while that \(p(a, T^-, x)\) is separable; i.e., there exists a function \(v_0\) in \(L^2(\Omega)\), and a function \(\phi\) defined below such that \(p(a, T^-, x) = v_0(x)\phi(a)\). Then one can consider separable solutions of (24); that is, solutions of the form

\[
\tilde{p}(a, t, x) = e^{\rho t} v(x, t) \phi(a),
\]

where \(v\) satisfies

\[
v_t - \Delta v = 0, \quad t \in (T, +\infty), x \in \Omega \setminus \tilde{\omega},
\]

\[
v(0, x) = v_0(x), \quad x \in \Omega \setminus \tilde{\omega},
\]

\[
v(t, x) = 0, \quad t \in (T, +\infty), x \in \Gamma \cup \partial \omega.
\]
and

\[ \phi(a) = \exp\left(-\int_0^a \mu(s) \, ds - r^* a \right). \]

Let us denote by \((\lambda_i^n, y_i)\) the eigenvalues and eigenfunctions of (11), and by \(\langle \cdot, \cdot \rangle\) the scalar product in \(L^2(\Omega \backslash \tilde{\omega})\). Then

\[ p(a, t, x) = \exp(r^* t) \sum_{i=1}^{\infty} \exp(-\lambda_i^n t) \langle v_0, y_i \rangle y_i(x) \exp\left(-\int_0^a \mu(s) \, ds - r^* a \right). \]

As a conclusion, when \(r^* - \lambda_1^n < 0\), then \(p(a, t, x) \to 0\), in \(\Omega \backslash \tilde{\omega}\), as \(t \to \infty\).

Last, from its construction, for each given initial data \(p_0(a, x)\) satisfying \((H_2)\), the solution is bounded at time \(T^-\); hence one can find a function \(v_0(x)\) and a nonnegative number \(\alpha\) sufficiently large such that

\[ p(a, T^-, x) \leq \alpha v_0(x) \phi(a), \quad (a, x) \in (0, A) \times \Omega \backslash \tilde{\omega}. \]

Using comparison results [2, 5], one has

\[ 0 \leq p(a, t, x; p(a, T^-, x)) \leq p(a, t, x; \alpha v_0(x) \phi(a)), \]

for \((a, t, x) \in (0, A) \times (T^+, +\infty) \times \Omega \backslash \tilde{\omega}\). The stabilizability result follows.

REFERENCES