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Change in autoregressive processes

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We study the detection of a possible change in a stationary autoregressive process of order r. The test statistics are based on weighted supremum and L_p-functionals of the residual sums. Some limit theorems are proven under necessary and sufficient conditions.

AMS 1980 Subject Classifications: Primary 62M10; Secondary 62F03.

autoregressive process * change-point * strong approximation * weighted suprema * weighted L_p-functionals

1. Introduction

We consider

$$X_k = \boldsymbol{\beta} \times (X_{k-1}, X_{k-2}, \ldots, X_{k-r})^{\mathrm{T}} + \varepsilon_k, \quad r < k \leq [n\lambda],$$

$$X_k^* = \boldsymbol{\beta}^* \times (X_{k-1}, X_{k-2}, \ldots, X_{k-r})^{\mathrm{T}} + \varepsilon_k, \quad [n\lambda] < k \leq n,$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_r)$, $\boldsymbol{\beta}^* = (\beta_1^*, \ldots, \beta_r^*)$ and $\boldsymbol{\beta} \neq \boldsymbol{\beta}^*$. As usual, $\boldsymbol{x}^{\mathrm{T}}$ denotes the transpose of vextor $\boldsymbol{x}$. We want to test

$$\mathrm{H}_0: \quad \lambda = 1$$

against the alternative

$$\mathrm{H}_\mathrm{A}: \quad \lambda \in (0, 1).$$

The change-point model occurs very often in econometrics and in technology. For a survey on applications in econometrics we refer to Goldfeld and Quandt (1976). Basseville and Benveniste (1986) presented a wide range of applications of the change-point analysis, from signal segmentation for pattern recognition to failure detection in dynamical controlled systems.

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We assume that under H_0 the sequence $\{X_k, r<k<\infty\}$ is a stationary autoregressive process:

under H_0, $\{X_k, r<k<\infty\}$ is a stationary sequence, $\beta_r \neq 0$ and the roots of the polynomial $t^r - \beta_1 t^{r-1} - \beta_2 t^{r-2} - \cdots - \beta_r = 0$ are less than one in absolute value. (1.1)

We also need some regularity conditions on the error terms:

$\{\varepsilon_k, 0 \leq k < \infty\}$ are independent, identically distributed random variables with

$$E\varepsilon_k = 0, \qquad 0<\sigma^2 = \operatorname{Var} \varepsilon_k < \infty \quad \text{and} \quad E\varepsilon_k^4 < \infty. \tag{1.2}$$

Under H_0 we have a stationary autoregressive process of order r, while under the alternative there is a change in the scheme after the first $[n\lambda]$ observations. Brown et al. (1975) suggested that tests for H_0 can be based on the residuals

$$\hat{\varepsilon}_{i,n} = X_i - \hat{\boldsymbol{\beta}}_n \times (X_{i-1}, X_{i-2}, \ldots, X_{i-r})^{\mathrm{T}}, \tag{1.3}$$

where $\hat{\boldsymbol{\beta}}_n$ is the least squares estimate of the coefficient based on $\{X_k, r<k\leq n\}$. Kulperger (1985) developed a family of tests based on the functionals of the sum of residuals

$$Z_n(k) = \sum_{1\leq i\leq k} \hat{\varepsilon}_{i,n}, \quad 1\leq k\leq n. \tag{1.4}$$

He showed that $n^{-1/2} Z_n((n+1)t)/\sigma$ converges weakly to a Wiener process $\{W(t), 0\leq t\leq 1\}$ in $D[0,1]$. Thus we have immediately

$$\frac{n^{-1/2}}{\sigma} \max_{1\leq k\leq n} |Z_n(k)| \xrightarrow{\mathcal{D}} \sup_{0\leq t\leq 1} |W(t)| \tag{1.5}$$

and

$$\frac{1}{\sigma^2 n^2} \sum_{1\leq k\leq n} g\left(\frac{k}{n}\right) Z_n^2(k) \xrightarrow{\mathcal{D}} \int_0^1 g(t) W^2(t)\, \mathrm{d}t, \tag{1.6}$$

if $\int_0^1 t g(t)\, \mathrm{d}t < \infty$. The statistics in (1.5) and (1.6) are not very sensitive if the change occurs at the beginning or at the end of the data, i.e., if λ is near to zero or one. We get more powerful tests, if we use the weighted functionals of the sum of the residuals. We define

$$D_n^{(1)} = \frac{n^{-1/2}}{\sigma} \sup_{0<t<1} \frac{|Z_n(nt)|}{q(t)}, \tag{1.7}$$

$$D_n^{(2)} = \frac{n^{-1/2}}{\sigma} \sup_{0<t<1} \frac{|Z_n(nt)|}{t^{1/2}} \tag{1.8}$$

and

$$D_n^{(3)} = \left(\frac{n^{-1/2}}{\sigma}\right)^p \int_0^1 \frac{|Z_n(nt)|^p}{q(t)} \mathrm{d}t, \quad 1\leq p<\infty, \tag{1.9}$$

$$D_n^{(4)} = \left(\frac{n^{-1/2}}{\sigma}\right)^p \int_0^1 \frac{|Z_n(nt)|^p}{t^{1+p/2}} \mathrm{d}t, \quad 1\leq p<\infty, \tag{1.10}$$

The tests in (1.7)-(1.10) are very sensitive if λ is near to zero, but they do not have this property if λ is large. We can get tests which are senstive to changes at both ends, using a symmetrized version of Z_n. We define

$$\hat{Z}_n(t) = Z_n((n+1)t) - tZ_n(n),$$

and similarly to (1.7) and (1.10) we consider the following functionals:

$$C_n^{(1)} = \frac{n^{-1/2}}{\sigma} \sup_{0<t<1} \frac{|\hat{Z}_n(t)|}{q(t)}, \tag{1.11}$$

$$C_n^{(2)} = \frac{n^{-1/2}}{\sigma} \sup_{0<t<1} \frac{|\hat{Z}_n(t)|}{(t(1-t))^{1/2}} \tag{1.12}$$

and

$$C_n^{(3)} = \left(\frac{n^{-1/2}}{\sigma}\right)^p \int_0^1 \frac{|\hat{Z}_n(t)|^p}{q(t)} \, dt, \quad 1 \leq p < \infty, \tag{1.13}$$

$$C_n^{(4)} = \left(\frac{n^{-1/2}}{\sigma}\right)^p \int_0^1 \frac{|\hat{Z}_n(t)|^p}{(t(1-t))^{1+p/1}} \, dt, \quad 1 \leq p < \infty. \tag{1.14}$$

We assume that our weight functions belong to one of the following classes when we consider the weighted supremum norms:

$$Q_0 = \left\{ q: q \text{ is nondecreasing in a neighbourhood of zero and } \inf_{\delta < t \leq 1} q(t) > 0 \text{ for all } 0 < \delta < 1 \right\}$$

and

$$Q_{0,1} = \left\{ q: q \text{ is nondecreasing in a neighbourhood of zero, nonincreasing in a neighbourhood of one and } \inf_{\delta \leq t \leq 1-\delta} q(t) > 0 \text{ for all } 0 < \delta < \tfrac{1}{2} \right\}.$$

Let

$$a(x) = (2 \log x)^{1/2}, \qquad b(x) = 2 \log x + \tfrac{1}{2} \log \log x - \tfrac{1}{2} \log \pi$$

and

$$I(q, c) = \int_0^1 \frac{1}{t} \exp(-cq^2(t)/t) \, dt,$$

$$I^*(q, c) = \int_0^1 \frac{1}{t(1-t)} \exp(-cq^2(t)/(t(1-t)) \, dt.$$

Throughout this paper $\{W(t), 0 \le t < \infty\}$ stands for a Wiener process. To state our results we also need

$$m = m(p) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} |x|^p \exp(-\tfrac{1}{2}x^2)\, dx$$

and

$$d = d(p) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |xy|^p \Bigg\{ \frac{1}{2\pi(1-e^{-2|u|})^{1/2}} \times \exp\left(-\frac{1}{2(1-e^{-2|u|})} (x^2+y^2-2\, e^{-|u|}xy) \right) - \frac{1}{2\pi} e^{-(x^2+y^2)/2} \Bigg\}\, dx\, dy\, du.$$

It is easy to check that $m(2)=1$ and $d(2)=2$. First we consider the asymptotics of $D_n^{(1)}, \ldots, D_n^{(4)}$.

Theorem 1.1. *We assume that* (1.1), (1.2) *and* H_0 *hold.*

(i) *Let* $q \in Q_0$. If $I(q, c) < \infty$ *for some* $c > 0$, *then we have*

$$\lim_{n\to\infty} P\{D_n^{(1)} \le x,\, a(\log n) D_n^{(2)} \le b(\log n) + y\}$$
$$= P\left\{ \sup_{0<t<1} \frac{|W(t)|}{q(t)} \le x \right\} \exp(-\exp(-y)), \tag{1.15}$$

for all x and y.

(ii) *Let q be positive on* $(0, 1]$ *and* $1 \le p < \infty$. *If*

$$\int_0^1 \frac{t^{p/2}}{q(t)}\, dt < \infty, \tag{1.16}$$

then we have

$$\lim_{n\to\infty} P\{D_n^{(3)} \le x,\, (2d \log n)^{-1/2}(D_n^{(4)} - m \log n) \le y\}$$
$$= P\left\{ \int_0^1 \frac{|W(t)|^p}{q(t)}\, dt \le x \right\} \Phi(y), \tag{1.17}$$

for all x and y where Φ stands for the standard normal distribution function.

The results in Theorem 1.1 are optimal. It is well-known (cf., for example, Csörgő et al., 1986) that $P\{\sup_{0<t<1} |W(t)|/q(t) < \infty\} = 1$ if and only if $I(q, c) < \infty$. Similarly, Csörgő, Horváth and Shao (1993) showed that the condition $\int_0^1 t^{p/2}/q(t)\, dt < \infty$ is necessary and sufficient for the almost sure finiteness of $\int_0^1 |W(t)|^p/q(t)\, dt$.

Next we state similar limit theorems for $C_n^{(1)}, \ldots, C_n^{(4)}$. Let $\{B(t), 0 \le t \le 1\}$ denote a Brownian bridge.

Theorem 1.2. *We assume* (1.1), (1.2) *and* H_0 *hold.*

(i) *Let* $q \in Q_{0,1}$. *If* $I^*(q, c) < \infty$ *for some* $c > 0$, *then we have*

$$\lim_{n\to\infty} P\{C_n^{(1)} \leq x, a(\log n)C_n^{(2)} \leq b(\log n) + y\}$$

$$= P\left\{\sup_{0<t<1} \frac{|B(t)|}{q(t)} \leq x\right\} \exp(-2\exp(-y)), \tag{1.18}$$

for all x *and* y.

(ii) *Let* q *be positive on* $(0, 1)$ *and* $1 \leq p < \infty$. *If*

$$\int_0^1 \frac{(t(1-t))^{p/2}}{q(t)}\, \mathrm{d}t < \infty, \tag{1.19}$$

then we have

$$\lim_{n\to\infty} P\{C_n^{(3)} \leq x, (4d \log n)^{-1/2}(C_n^{(4)} - 2m \log n) \leq y\}$$

$$= P\left\{\int_0^1 \frac{|B(t)|^p}{q(t)}\, \mathrm{d}t \leq x\right\} \Phi(y), \tag{1.20}$$

for all x *and* y.

By Csörgő et al. (1986) we have that the condition $I^*(q, c) < \infty$ for some $c > 0$ is necessary for (1.18). Similarly using Csörgő et al. (1993) we get that (1.20) implies (1.19). Simulations showed that tests based on (1.17) and (1.20) are not sensitive to the choice of p.

Our approach is based on the residual partial sums. Brown et al. (1975) suggested that we can use the residual partial sums to test H_0 against the change point alternative. Kulperger (1985) obtained the first results for the residual partial sums in case of an autoregressive process. A different method is based on the empirical distribution and spectral functions. Picard (1985) studied the empirical spectral function in case of stationary Gaussian sequences and Giraitis and Leipus (1990) generalized her results for moving-average sequences.

2. Preliminaries

Throughout this section we assume that H_0 holds. Using the definition of the residuals we can write

$$\hat{\varepsilon}_{i,n} = \varepsilon_i - \sum_{1 \leq j \leq r} (\hat{\beta}_{j,n} - \beta_j) X_{i-j}, \tag{2.1}$$

where $\hat{\boldsymbol{\beta}}_n = (\hat{\beta}_{1,n}, \ldots, \hat{\beta}_{r,n})$ is the least square estimator of $\boldsymbol{\beta}$ based on $\{X_k, r < k \leq n\}$. Introducing

$$R(k) = \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq r} (\hat{\beta}_{j,n} - \beta_j) X_{i-j} \tag{2.2}$$

and

$$S(k)=\sum_{1\leqslant i\leqslant k}\varepsilon_i, \tag{2.3}$$

we have

$$Z_n(k)=S(k)-R(k), \quad 1\leqslant k\leqslant n. \tag{2.4}$$

Similarly we get

$$\hat{Z}_n(t)=\hat{S}_n(t)-\hat{R}_n(t), \quad 0<t<1, \tag{2.5}$$

where

$$\hat{S}_n(t)=S((n+1)t)-tS(n), \quad 0<t<1, \tag{2.6}$$

and

$$\hat{R}_n(t)=R((n+1)t)-tR(n), \quad 0<t<1. \tag{2.7}$$

If we can show that R_n and $\hat{R}_n$ are negligible terms in (2.4) and (2.5), then it is enough to work with sums of i.i.d. r.v.'s. First we consider some technical lemmas for $R(k)$ and then we list some important properties of S_n and $\hat{S}_n$.

Lemma 2.1. *If* (1.1) *and* (1.2) *hold, then, as* $n\to\infty$ *we have*

$$n^{1/2}(\hat{\beta}_n-\beta)=O_P(1). \tag{2.8}$$

Proof. Lemma 2.1 is well-known. Its proof can be found, for example, in Kulperger (1985, Lemma 2.1). □

The following lemma is a slightly generalized version of an inequality in Kulperger (1985, p. 111).

Lemma 2.2. *If* (1.1) *and* (1.2) *hold, then we can find a constant* C *such that for all* $1\leqslant N<M$ *we have*

$$E\left(\sum_{N\leqslant i\leqslant M}X_i\right)^4\leqslant C(M-N)^2. \tag{2.9}$$

Proof. We use the moving averages representation of stationary autogressive processes. There is a sequence of i.i.d. r.v.'s $\{\delta_k, -\infty<k<\infty\}$ such that

$$X_k=\sum_{0\leqslant i<\infty}\alpha_i\delta_{k-i}, \tag{2.10}$$

δ_k and ε_0 have the same distribution. For the sake of simplicity we assume that the roots of

$$t^r-\beta_1t^{r-1}-\beta_2t^{r-2}-\cdots-\beta_r=0 \tag{2.11}$$

are real and distinct. The roots of (2.11) are denoted by $t_1, \ldots, t_r$. By Theorem 2.6.1 in Fuller (1976, p. 56) we can find constants $a_1, \ldots, a_r$ such that

$$\alpha_k = a_1 t_1^k + a_2 t_2^k + \cdots + a_r t_r^k, \quad 0 \le k < \infty. \tag{2.12}$$

With some algebra one can verify

$$\sum_{N \le i \le M} X_i = \sum_{0 \le k < \infty} \gamma_k \delta_{M-k},$$

where $\gamma_k = \gamma_k(N, M)$ and they are defined by

$$\gamma_0 = \alpha_0,$$

$$\gamma_1 = \alpha_0 + \alpha_1,$$

$$\vdots$$

$$\gamma_{M-N} = \alpha_0 + \alpha_1 + \cdots + \alpha_{M-N},$$

and

$$\gamma_l = \alpha_{l-(M-N)} + \cdots + \alpha_l, \quad M - N < l < \infty.$$

Thus we get

$$E\left(\sum_{0 \le k < \infty} \gamma_k \delta_{M-k}\right)^4 = \sigma^4 \left(\sum_{0 \le k < \infty} \gamma_k^2\right)^2 + E\varepsilon_0^4 \sum_{0 \le k < \infty} \gamma_k^4.$$

Let $\hat{t} = \max\{|t_1|, \ldots, |t_r|\}$ and $a = \max\{|a_1|, \ldots, |a_r|\}$. By (1.1) we have $0 < \hat{t} < 1$ and (2.12) gives

$$|\alpha_k| \le ra\hat{t}^k, \quad 0 \le k < \infty.$$

Using the definition of γ_k we get

$$\sum_{M-N<k<\infty} \gamma_k^4 \le \sum_{M-N<k<\infty} ((M-N)ar\hat{t}^k)^4 \le \frac{\hat{t}a^4 r^4}{1-\hat{t}} (M-N)^4.$$

Also, we obtain

$$\sum_{0 \le k \le M-N} \gamma_k^4 = \alpha_0^4 + (\alpha_0 + \alpha_1)^4 + \cdots + (\alpha_0 + \alpha_1 + \cdots + \alpha_{M-N})^4$$

$$\le (M - N + 1)(ra + ra\hat{t} + \cdots + ra\hat{t}^{M-N})^4$$

$$\le \left(\frac{rp}{1-\hat{t}}\right)^4 (M - N + 1).$$

Similar arguments give

$$\sum_{N-N<k<\infty} \gamma_k^2 \le \sum_{M-N<k<\infty} ((M-N)rp\hat{t}^k)^2 \le \frac{\hat{t}a^2 r^2}{1-\hat{t}} (M-N)^2$$

and

$$\sum_{0 \le k \le M-N} \gamma_k^2 \le \left(\frac{ar}{1-\hat{t}}\right)^2 (M - N + 1).$$

The case of multiple and complex roots can be covered similarly and therefore omitted. □

Lemma 2.3. *If* (1.1) *and* (1.2) *hold, then, as* $n\to\infty$, *we have*

$$n^{1/2}\max_{1\le k\le n}\frac{1}{k}|R(k)|=O_P(1) \tag{2.13}$$

and

$$n^{1/2}\max_{1\le k<n}\frac{1}{n-k}|R(n)-R(k)|=O_P(1). \tag{2.14}$$

Proof. By definition, we have

$$\frac{1}{k}R(k)=\sum_{1\le j\le r}(\hat\beta_{j,n}-\beta_j)\frac{1}{k}\sum_{1\le i\le k}X_{i-j}. \tag{2.15}$$

Lemma 2.1 gives

$$n^{1/2}\max_{1\le j\le r}|\hat\beta_{j,n}-\beta_j|=O_P(1). \tag{2.16}$$

Next we apply Lemma 2.2 and the Markov inequality and obtain

$$P\left\{\max_{1\le k\le n}\frac{1}{k}\left|\sum_{1\le i\le k}X_{i-j}\right|>x\right\}$$
$$\le x^{-4}\sum_{1\le k\le n}E\left(\frac{1}{k}\sum_{1\le i\le k}X_{i-j}\right)^4\le\frac{C}{x^4}\sum_{1\le k\le n}k^{-2}\le\tfrac{1}{6}C\pi^2x^{-4} \tag{2.17}$$

for all $x>0$ and $1\le j\le r$, which implies

$$\max_{1\le j\le r}\max_{1\le k\le n}\frac{1}{k}\left|\sum_{1\le i\le k}X_{i-j}\right|=O_P(1). \tag{2.18}$$

Putting together (2.15), (2.16) and (2.18) we get (2.13).

The proof of (2.14) is similar to that of (2.13) and hence omitted. □

Lemma 2.4. *If* (1.1), (1.2) *hold and* $1\le g(n)\le n$, $g(n)\to\infty$, *then, as* $n\to\infty$, *we have*

$$n^{1/2}\max_{g(n)\le k\le n}\frac{1}{k}|R(k)|=o_P(1) \tag{2.19}$$

and

$$n^{1/2}\max_{1\le k\le n-g(n)}\frac{1}{n-k}|R(n)-R(k)|=o_P(1). \tag{2.20}$$

Proof. Similarly to (2.17) we have for all $x>0$ that

$$P\left\{\max_{g(n)\le k\le n}\frac{1}{k}\left|\sum_{1\le i\le k}X_{i-j}\right|\ge x\right\}\le x^{-4}\sum_{g(n)\le k\le n}E\left(\frac{1}{k}\sum_{1\le i\le k}X_{i-j}\right)^4$$
$$\le\frac{C}{x^4}\sum_{g(n)\le k<\infty}k^{-2},$$

which immediately implies

$$\max_{1\le j\le r}\ \max_{g(n)\le k\le n}\frac{1}{k}\left|\sum_{1\le i\le k}X_{i-j}\right| = o_P(1). \tag{2.21}$$

Now (2.16) and (2.21) yield (2.19). Similar arguments give (2.20). □

Lemma 2.5. *We assume that* (1.2) *holds.*

(i) *We can define a sequence of Wiener processes* $\{W_n(t), 0\le t\le 1\}$ *such that*

$$n^{\nu}\sup_{1/n\le t\le 1}\frac{|n^{-1/2}S(nt)-\sigma W_n(t)|}{t^{1/2-\nu}} = O_P(1), \tag{2.22}$$

for all $0\le \nu\le \frac{1}{4}$ *and*

$$\sup_{0\le t\le 1}|n^{-1/2}S(nt)-\sigma W_n(t)| = o_P(1). \tag{2.23}$$

(ii) *We can define a sequence of Brownian bridges* $\{B_n(t), 0\le t\le 1\}$ *such that*

$$n^{\nu}\sup_{1/(n+1)\le t\le n/(n+1)}\frac{|n^{-1/2}\tilde S_n(t)-\sigma B_n(t)|}{(t(1-t))^{1/2-\nu}} = O_P(1), \tag{2.24}$$

for all $0\le \nu\le \frac{1}{4}$ *and*

$$\sup_{0<t<1}|n^{-1/2}\tilde S_n(t)-\sigma B_n(t)| = o_P(1). \tag{2.25}$$

Proof. (i) Komlós, Major and Tusnády (1975, 1976) constructed a Wiener process $\{W(t), 0\le t<\infty\}$ such that

$$|S(t)-\sigma W(t)| \overset{\text{a.s.}}{=} O(t^{1/4}) \quad (t\to\infty). \tag{2.26}$$

Thus we get

$$\sup_{1/n\le t\le 1}\frac{|S(nt)-\sigma W(nt)|}{(nt)^{1/2-\nu}} \overset{\text{a.s.}}{=} O(1) \quad (n\to\infty), \tag{2.27}$$

for all $0\le \nu\le \frac{1}{4}$. Since $W_n(t)=n^{-1/2}W(nt)$ is a Wiener process for each n, (2.27) implies (2.22) and (2.23) follows from (2.26).

(ii) We can write

$$\tilde S_n(t)=\begin{cases}\tilde S_n^{(1)}(t), & 0\le t\le \frac{1}{2},\\ \tilde S_n^{(2)}(t), & \frac{1}{2}\le t\le 1,\end{cases} \tag{2.28}$$

where

$$\tilde S_n^{(1)}(t) = S((n+1)t) - t(S(\tfrac{1}{2}(n+1)) + S(n) - S(\tfrac{1}{2}(n+1)))$$

and

$$\tilde S_n^{(2)}(t) = S((n+1)t) - S(n) + (1-t)(S(\tfrac{1}{2}(n+1)) + S(n) - S(\tfrac{1}{2}(n+1))).$$

Using again the Komlós, Major and Tusnády construction we can define two Wiener processes $\{W_n^{(1)}(t), 0\le t<\infty\}$ and $\{W_n^{(2)}(t), 0\le t<\infty\}$ such that $W_n^{(1)}$ and $W_n^{(2)}$ are independent for each n,

$$\sup_{1/(n+1)\le t\le 1/2}\frac{|S((n+1)t)-\sigma W_n^{(1)}(nt)|}{(nt)^{1/4}}\overset{\text{a.s.}}{=}\mathrm{O}(1)\quad (n\to\infty)\tag{2.29}$$

and

$$\sup_{1/2\le t\le n/(n+1)}\frac{|S((n+1)t)-S(n)-\sigma W_n^{(2)}(n(1-t))|}{(n(1-t))^{1/4}}\overset{\text{a.s.}}{=}\mathrm{O}(1)\quad (n\to\infty).\tag{2.30}$$

Next we define

$$B_n(t)=\begin{cases} n^{-1/2}(W_n^{(1)}(nt)-t(W_n^{(1)}(\tfrac12 n)-W_n^{(2)}(\tfrac12 n))), & 0\le t\le \tfrac12,\\ n^{-1/2}(W_n^{(2)}(n(1-t))+(1-t)(W_n^{(1)}(\tfrac12 n)-W_n^{(2)}(\tfrac12 n))), & \tfrac12\le t\le 1.\end{cases}$$

Computing the covariance function of $B_n(t)$, we can easily verify that $\{B_n(t), 0\le t\le 1\}$ is a Brownian bridge for each n. Now (2.24) follows from (2.29) and (2.30).

It is easy to see that

$$\sup_{0<t\le 1/(n+1)}|B_n(t)|=\mathrm{o}_{\mathrm{P}}(1),\qquad \sup_{n/(n+1)\le t<1}|B_n(t)|=\mathrm{o}_{\mathrm{P}}(1),$$

and the definition of $\tilde S_n(t)$ implies

$$n^{-1/2}\sup_{0<t<1/(n+1)}|\tilde S_n(t)|\le\frac{n^{-1/2}}{n+1}|S(n)|=\mathrm{o}_{\mathrm{P}}(1),$$

and similarly

$$n^{-1/2}\sup_{n/(n+1)<t<1}|\tilde S_n(t)|=\mathrm{o}_{\mathrm{P}}(1).$$

Hence (2.24) yields (2.25). □

Lemma 2.6. *We assume that* (1.2) *holds and* $1\le p<\infty$.

(i) *Let* $0\le\alpha(n)<\beta(n)\le 1$ *and*

$$\gamma(n)=\tfrac12\log\frac{\beta(n)}{\max(1/n,\alpha(n))}.$$

If $\gamma(n)\to\infty$ $(n\to\infty)$, *then we have*

$$\lim_{n\to\infty}P\left\{a(\gamma(n))\sup_{\alpha(n)\le t\le\beta(n)}\frac{1}{\sigma}\frac{|S(nt)|}{(nt)^{1/2}}\le b(\gamma(n))+y\right\}=\exp(-2\exp(-y))$$

and

$$\lim_{n\to\infty} P\left\{(4d\gamma(n))^{-1/2}\left(\sigma^{-p}\int_{\alpha(n)}^{\beta(n)}\frac{|n^{-1/2}S(nt)|^p}{t^{1+p/2}}\,\mathrm{d}t-2m\gamma(n)\right)\le y\right\}=\Phi(y),$$

for all y.

(ii) *Let* $0\le\alpha(n)<\beta(n)\le 1$ *and*

$$\gamma^*(n)=\tfrac{1}{2}\log\left\{\frac{\min(1-1/n,\beta(n))(1-\max(1/n,\alpha(n)))}{(1-\min(1-1/n,\beta(n)))\max(\alpha(n),1/n)}\right\}.$$

If $\gamma^*(n)\to\infty$ $(n\to\infty)$, *then we have*

$$\lim_{n\to\infty} P\left\{a(\gamma^*(n))\frac{1}{\sigma}\sup_{\alpha(n)\le t\le\beta(n)}\frac{n^{-1/2}|\hat{S}_n(t)|}{(t(1-t))^{1/2}}\le b(\gamma^*(n))+y\right\}$$
$$=\exp(-2\exp(-y))$$

and

$$\lim_{n\to\infty} P\left\{(4d\gamma^*(n))^{-1/2}\left(\sigma^{-p}\int_{\alpha(n)}^{\beta(n)}\frac{|n^{-1/2}\hat{S}_n(t)|^p}{t^{1+p/2}}\,\mathrm{d}t-2m\gamma^*(n)\right)\le y\right\}=\Phi(y),$$

for all y.

Proof. It follows from (2.21) and (2.22) that $(n^{-1/2}/\sigma)\sup_{\alpha(n)\le t\le\beta(n)}|S(nt)|/t^{1/2}$ and $\sup_{\alpha(n)\vee 1/n\le t\le\beta(n)}|W(t)|/t^{1/2}$ must have the same limit distribution (cf. Csörgő et al., 1986). The limit distribution of the supremum of the normalized Wiener process was calculated by Darling and Erdős (1956). Similarly, the limit distribution of $\int_{\alpha(n)}^{\beta(n)}|(n^{-1/2}/\sigma)S(nt)|^p/t^{1+p/2}\,\mathrm{d}t$ and $\int_{\alpha(n)\vee 1/n}^{\beta(n)}|W(t)|^p/t^{1+p/2}\,\mathrm{d}t$ are the same. The central limit theorem for the latter integral is proven in Csörgő and Horváth (1988). Similar arguments can be used to get proofs of the last two statements of Lemma 2.6. □

3. Proofs of Theorems 1.1 and 1.2

We start with (1.15) and (1.17).

Proof of Theorem 1.1. (i) If $I(q,c)<\infty$ for some $c>0$, then we have (cf. Csörgő et al., 1986)

$$\lim_{t\to 0}\frac{q(t)}{t^{1/2}}=\infty. \tag{3.1}$$

First we show

$$n^{-1/2}\sup_{0<t<1}\frac{|Z_n(nt)-S(nt)|}{q(t)}=o_P(1). \tag{3.2}$$

By (2.4) it suffices to prove

$$n^{-1/2} \sup_{0<t<1} \frac{|R(nt)|}{q(t)} = o_P(1). \tag{3.3}$$

By definition,

$$\sup_{0<t<1/n} \frac{|R(nt)|}{q(t)} = 0. \tag{3.4}$$

Let $0<\varepsilon<1$. Using Lemma 2.3 we obtain

$$\begin{aligned} n^{-1/2} \sup_{1/n \le t \le \varepsilon} \frac{|R(nt)|}{q(t)} &\le \sup_{0 \le t \le \varepsilon} \frac{t^{1/2}}{q(t)} \sup_{1/n \le t \le \varepsilon} \frac{|R(nt)|}{(nt)^{1/2}} \\ &= O_P(1) \sup_{0 \le t \le \varepsilon} \frac{t^{1/2}}{q(t)}. \end{aligned} \tag{3.5}$$

Applying Lemma 2.4 with $g(n) = n\varepsilon$ we obtain

$$n^{-1/2} \sup_{\varepsilon \le t < 1} \frac{|R(nt)|}{q(t)} \le \sup_{\varepsilon \le t < 1} \frac{1}{q(t)} n^{-1/2} \sup_{n\varepsilon \le k \le n} |R(k)| = o_P(1), \tag{3.6}$$

for all $0<\varepsilon<1$. Since ε can be as small as we wish, (3.3) follows from (3.1), (3.5) and (3.6).

For all $0<\varepsilon<1$ we have

$$\begin{aligned} &\sup_{1/n \le t < 1} \frac{|n^{-1/2}S(nt) - \sigma W_n(t)|}{q(t)} \\ &\le \sup_{1/n \le t \le \varepsilon} \frac{|n^{-1/2}S(nt) - \sigma W_n(t)|}{q(t)} + \sup_{\varepsilon \le t \le 1} \frac{|n^{-1/2}S(nt) - \sigma W_n(t)|}{q(t)} \\ &= O_p(1) \sup_{1/n \le t \le \varepsilon} \frac{t^{1/2}}{q(t)} + o_P(1), \end{aligned} \tag{3.7}$$

where we used (2.22) and (2.21) with $\nu = 0$. Applying

$$\sup_{1/n \le t < 1} \frac{|n^{-1/2}S(nt) - \sigma W_n(t)|}{q(t)} = o_P(1). \tag{3.8}$$

Since

$$\sup_{0<t<1/n} |S(nt)| = 0 \tag{3.9}$$

and

$$\sup_{1/n \le t < 1} \frac{|W_n(t)|}{q(t)} \xrightarrow{\mathscr{D}} \sup_{0<t<1} \frac{|W(t)|}{q(t)}, \tag{3.10}$$

we immediately obtain

$$D_n^{(1)} \xrightarrow{\mathscr{D}} \sup_{0<t<1} \frac{|W(t)|}{q(t)}. \tag{3.11}$$

Lemma 2.3 yields

$$\sup_{0<t<1} \frac{|R(nt)|}{(nt)^{1/2}} = O_P(1).$$

Lemma 2.6 gives

$$\sup_{0<t\le(\log n)/n} \frac{|S(nt)|}{(nt)^{1/2}} = O_P((\log\log\log n)^{1/2}),$$

and therefore we have

$$a(\log n) \sup_{0<t\le(\log n)/n} \frac{1}{\sigma}\frac{|Z_n(nt)|}{(nt)^{1/2}} - (b(\log n)+y) \overset{P}{\to} -\infty. \tag{3.12}$$

Using Lemma 2.4 we obtain

$$\sup_{(\log n)/n\le t\le 1/\log n} \frac{|R(nt)|}{(nt)^{1/2}} = o_P(1/(\log n)^{1/2}). \tag{3.13}$$

Similarly to (3.12) we have

$$a(\log n) \sup_{1/\log n\le t<1} \frac{1}{\sigma}\frac{|Z_n(nt)|}{(nt)^{1/2}} - (b(\log n)+y) \overset{P}{\to} -\infty \tag{3.14}$$

and

$$a(\log n) \sup_{1/\log n\le t\le 1} \frac{1}{\sigma}\frac{|S(nt)|}{(nt)^{1/2}} - (b(\log n)+y) \overset{P}{\to} -\infty.$$

Lemma 2.6 implies

$$\begin{aligned}&\lim_{n\to\infty} P\left\{a(\log n) \sup_{(\log n)/n\le t\le 1/\log n} \frac{1}{\sigma}\frac{|S(nt)|}{(nt)^{1/2}} \le b(\log n)+y\right\}\\ &= \lim_{n\to\infty} P\left\{a(\log n) \sup_{0<t<1} \frac{1}{\sigma}\frac{|S(nt)|}{(nt)^{1/2}} \le b(\log n)+y\right\}\\ &= \exp(-\exp(-y))\end{aligned} \tag{3.15}$$

for all y. Putting together (2.4) and (3.12)-(3.15), we get

$$\lim_{n\to\infty} P\{a(\log n)D_n^{(2)} \le b(\log n)+y\} = \exp(-\exp(-y)),$$

for all y.

In the light of (3.2) and (3.12)-(3.15) it is enough to establish the asymptotic independence of $n^{-1/2}\sup_{0<t<1}|S(nt)|/q(t)$ and

$$a(\log n)\sup_{0<t<1}(1/\sigma)|S(nt)|/(nt)^{1/2} - b(\log n).$$

Let

$$J(u) = \begin{cases} 0, & \text{if } 0\le u\le n/\log n,\\ S(u)-S(n/\log n), & \text{if } n/\log n\le u\le n.\end{cases} \tag{3.16}$$

Using (3.1) and the central limit theorem we get

$$n^{-1/2} \sup_{1/\log n \le t \le 1} \frac{|S(nt) - J(nt)|}{q(t)}$$

$$\le (n/\log n)^{-1/2} \left| S\left(\frac{n}{\log n}\right) \right| \sup_{0<t\le \varepsilon} \frac{t^{1/2}}{q(t)} + n^{-1/2} \left| S\left(\frac{n}{\log n}\right) \right| \sup_{\varepsilon \le t \le 1} \frac{1}{q(t)}$$

$$= o_P(1), \tag{3.17}$$

since ε can be as small as we want. Thus we get

$$n^{-1/2} \sup_{0<t<1} \frac{|S(nt)|}{q(t)} = n^{-1/2} \sup_{0<t<1} \frac{|J(nt)|}{q(t)} + o_P(1). \tag{3.18}$$

Darling and Erdős (1956) showed

$$n^{-1/2} \sup_{1/\log n \le t < 1} \frac{|S(nt)|}{t^{1/2}} = O_P((\log\log\log n)^{1/2}), \tag{3.19}$$

$$a(\log n) \sup_{1/\log n \le t < 1} \frac{1}{\sigma} \frac{|S(nt)|}{(nt)^{1/2}} - (b(\log n) + y) \overset{P}{\to} -\infty, \tag{3.20}$$

and

$$\lim_{n\to\infty} P\left\{ a(\log n) \sup_{0<t<1/\log n} \frac{1}{\sigma} \frac{|S(nt)|}{(nt)^{1/2}} \le b(\log n) + y \right\}$$

$$= \exp(-\exp(-y)), \tag{3.21}$$

for all y. Since $\{J(u), 0 \le u \le n\}$ and $\{S(t), 0 \le t \le n/\log n\}$ are independent for each n, the asymptotic independence follows from (3.18), (3.20) and (3.21).

(ii) The following elementary inequality

$$||x|^p - |y|^p| \le p2^p\{|x-y|^p + |x|^{p-1}|x-y|\}, \quad 1 \le p < \infty, \tag{3.22}$$

will be useful later on. First we prove

$$D_n^{(3)} = \left(\frac{n^{-1/2}}{\sigma}\right)^p \int_0^1 \frac{|S(nt)|^p}{q(t)} \, dt + o_P(1). \tag{3.23}$$

Let $0 < \varepsilon < 1$. By (3.22) we have

$$n^{-p/2} \int_0^\varepsilon \frac{||S(nt) - R(nt)|^p - |S(nt)|^p|}{q(t)} \, dt$$

$$\le p2^p n^{-p/2} \int_0^\varepsilon \frac{|R(nt)|^p}{q(t)} \, dt + p2^p n^{-p/2} \int_0^\varepsilon \frac{|S(nt)|^{p-1} |R(nt)|}{q(t)} \, dt$$

$$= A_n^{(1)}(\varepsilon) + A_n^{(2)}(\varepsilon). \tag{3.24}$$

Using Lemma 2.3 we get

$$A_n^{(1)}(\varepsilon) = O_P(1) \int_0^\varepsilon \frac{t^p}{q(t)} \, dt \tag{3.25}$$

and

$$A_n^{(2)}(\varepsilon)=O_P(1)n^{(1-p)/2}\int_0^{\varepsilon}\frac{t|S(nt)|^{p-1}}{q(t)}\,dt$$

$$=O_P(1)\left(\sup_{0<t\leq\varepsilon}\frac{|n^{-1/2}S(nt)|}{t^{1/2-1/(2(p-1))}}\right)^{p-1}\int_0^{\varepsilon}\frac{t^{p/2}}{q(t)}\,dt. \tag{3.26}$$

Hence by (3.8) we have

$$A_n^{(2)}(\varepsilon)=O_P(1)\int_0^{\varepsilon}\frac{t^{p/2}}{q(t)}\,dt. \tag{3.27}$$

Applying Lemma 2.4 we obtain

$$n^{-p/2}\int_{\varepsilon}^{1}\frac{||S(nt)-R(nt)|^p-|S(nt)|^p|}{q(t)}\,dt=o_P(1). \tag{3.28}$$

Now condition (1.16) and (3.25), (3.27), (3.28) yield (3.23).

Lemma 2.5 gives

$$\int_{\varepsilon}^{1}\frac{||n^{-1/2}S(nt)|^p-|\sigma W_n(t)|^p|}{q(t)}\,dt=o_P(1), \tag{3.29}$$

for all $0<\varepsilon<1$ and (3.22) implies

$$\int_{1/n}^{\varepsilon}\frac{||n^{-1/2}S(nt)|^p-|\sigma W_n(t)|^p|}{q(t)}\,dt$$

$$\leq p2^p\int_{1/n}^{\varepsilon}\frac{|n^{-1/2}S(nt)-\sigma W_n(t)|^p}{q(t)}\,dt$$

$$+p2^p\int_{1/n}^{\varepsilon}\frac{|\sigma S_n(t)|^{p-1}|n^{-1/2}S(nt)-\sigma W_n(t)|}{q(t)}\,dt$$

$$=A_n^{(3)}(\varepsilon)+A_n^{(4)}(\varepsilon). \tag{3.30}$$

Lemma 2.5 implies

$$A_n^{(3)}(\varepsilon)\leq p2^p\left(\sup_{1/n\leq t\leq 1}\frac{|n^{-1/2}S(nt)-\sigma W_n(t)|}{t^{1/2}}\right)^p\int_0^{\varepsilon}\frac{t^{p/2}}{q(t)}\,dt$$

$$=O_P(1)\int_0^{\varepsilon}\frac{t^{p/2}}{q(t)}\,dt, \tag{3.31}$$

and

$$A_n^{(4)}(\varepsilon)=O_P(1)\int_0^{\varepsilon}\frac{t^{1/2}|W_n(t)|^{p-1}}{q(t)}\,dt. \tag{3.32}$$

It is easy to see

$$E\int_0^{\varepsilon}\frac{t^{1/2}|W_n(t)|^{p-1}}{q(t)}\,dt=E|N(0,1)|^{p-1}\int_0^{\varepsilon}\frac{t^{p/2}}{q(t)}\,dt, \tag{3.33}$$

where $N(0,1)$ is a standard normal r.v. Hence we obtain

$$A_n^{(4)}(\varepsilon)=O_P(1)\int_0^{\varepsilon}\frac{t^{p/2}}{q(t)}\,dt, \tag{3.34}$$

which completes the proof of

$$\left(\frac{n^{-1/2}}{\sigma}\right)^p\int_0^1\frac{|S(nt)|^p}{q(t)}\,dt=\int_0^1\frac{|W_n(nt)|^p}{q(t)}\,dt+o_P(1). \tag{3.35}$$

Putting together (3.23) and (3.35) we get

$$D_n^{(3)}\xrightarrow{\mathcal{D}}\int_0^1\frac{|W(t)|^p}{q(t)}\,dt. \tag{3.36}$$

Using (3.22) again we have

$$\begin{aligned}&n^{-p/2}\int_0^1\frac{\big||S(nt)-R(nt)|^p-|S(nt)|^p\big|}{t^{1+p/2}}\,dt\\&\leq p2^pn^{-p/2}\int_{1/n}^1\frac{|R(nt)|^p}{t^{1+p/2}}\,dt+p2^pn^{-p/2}\int_{1/n}^1\frac{|S(nt)|^{p-1}|R(nt)|}{t^{1+p/2}}\,dt\\&=A_n^{(5)}+A^{(6)}.\end{aligned} \tag{3.37}$$

Lemma 2.3 gives

$$A_n^{(5)}=O_P(1). \tag{3.38}$$

Applying Lemma 2.3 and (3.8) we obtain

$$\begin{aligned}A_n^{(6)}&=O_P(1)n^{(1-p)/2}\int_{1/n}^1\frac{|S(nt)|^{p-1}}{t^{p/2}}\,dt\\&=O_P(1)\left(n^{-1/2}\sup_{0<t<1}\frac{|S(nt)|}{t^{1/2-1/(4(p-1))}}\right)^{p-1}\int_0^1 t^{-3/4}\,dt\\&=O_P(1).\end{aligned} \tag{3.39}$$

Now (3.37)-(3.39) yield

$$D_n^{(4)}=\left(\frac{n^{-1/2}}{\sigma}\right)^p\int_0^1\frac{|S(nt)|^p}{t^{1+p/2}}\,dt+O_P(1), \tag{3.40}$$

and therefore by Lemma 2.6 we conclude

$$\lim_{n\to\infty}P\{(2d\log n)^{-1/2}(D_n^{(4)}-m\log n)\leq y\}=\Phi(y). \tag{3.41}$$

The proof of (1.17) is complete if we can establish the asymptotic independence of $D_n^{(3)}$ and $D_n^{(4)}$. By (3.23) and (3.40) it is enough to show the asymptotic independence of $n^{-p/2}\int_0^1|S(nt)|^p/q(t)\,dt$ and $n^{-p/2}\int_0^1|S(nt)|^p/t^{1+p/2}\,dt$. Using Lemma 2.6 we get

$$n^{-p/2}\int_0^1\frac{|S(nt)|^p}{t^{1+p/2}}\,dt=n^{-p/2}\int_0^{1/\log n}\frac{|S(nt)|^p}{t^{1+p/2}}\,dt+O_P(\log\log n), \tag{3.42}$$

and therefore (3.23) gives

$$D_n^{(4)} = \left(\frac{n^{-1/2}}{\sigma}\right)^p \int_0^{1/\log n} \frac{|S(nt)|^p}{t^{1+p/2}}\,dt + O_P(\log\log n). \tag{3.43}$$

Similarly to (3.17) we have

$$n^{-p/2} \int_{1/\log n}^{1} \frac{|S(nt) - J(nt)|^p}{q(t)}\,dt = o_P(1),$$

which immediately implies

$$D_n^{(3)} = \left(\frac{n^{-1/2}}{\sigma}\right)^p \int_0^1 \frac{|J(nt)|^p}{q(t)}\,dt + o_P(1). \tag{3.44}$$

For each n, $\{S(t), 0 \le t \le \log n\}$ and $\{J(u), 0 \le u \le n\}$ are independent, and therefore the asymptotic independence follows from (3.42) and (3.44). □

Proof of Theorem 1.2. (i) If $I^*(q, c) < \infty$ for some $c > 0$, then we have (3.1) and

$$\lim_{t \to 1} \frac{q(t)}{(1-t)^{1/2}} = \infty. \tag{3.45}$$

We start with the proof of

$$n^{-1/2} \sup_{0<t<1} \frac{|\hat{Z}_n(t) - \hat{S}_n(t)|}{q(t)} = o_P(1). \tag{3.46}$$

By (2.5) it suffices to establish

$$n^{-1/2} \sup_{0<t<1} \frac{|\hat{R}_n(t)|}{q(t)} = o_P(1). \tag{3.47}$$

For all $0 < \varepsilon < \frac{1}{2}$ we have

$$\begin{aligned} n^{-1/2} \sup_{0<t<1} \frac{|\hat{R}_n(t)|}{q(t)} &\le \sup_{0<t<1/(n+1)} \frac{|n^{-1/2}\hat{R}_n(t)|}{q(t)} + \sup_{1/(n+1) \le t \le \varepsilon} \frac{|n^{-1/2}\hat{R}_n(t)|}{q(t)} \\ &\quad + \sup_{\varepsilon \le t \le 1-\varepsilon} \frac{|n^{-1/2}\hat{R}_n(t)|}{q(t)} + \sup_{1-\varepsilon \le t \le n/(n+1)} \frac{|n^{-1/2}\hat{R}_n(t)|}{q(t)} \\ &\quad + \sup_{n/(n+1)<t<1} \frac{|n^{-1/2}\hat{R}_n(t)|}{q(t)} \\ &= L_n^{(1)} + \cdots + L_n^{(5)}. \end{aligned} \tag{3.48}$$

Using (2.7) and Lemma 2.4 we get

$$L_n^{(1)} \le \frac{n^{-1/2}}{n+1} |R(n)| = o_P(1) \tag{3.49}$$

and

$$L_n^{(5)} = O_P(1). \tag{3.50}$$

Applying Lemmas 2.3 and 2.4 we obtain

$$L_n^{(2)} = O_P(1) \sup_{0<t\leq\varepsilon} \frac{t}{q(t)}, \tag{3.51}$$

$$L_n^{(4)} = O_P(1) \sup_{0<t\leq\varepsilon} \frac{t}{q(t)} \tag{3.52}$$

and

$$L_n^{(3)} = o_P(1) \tag{3.53}$$

for all $0<\varepsilon<\frac{1}{2}$. Combining (3.1) and (3.45) with (3.48)-(3.53), we conclude (3.47).

The Brownian bridge construction in Lemma 2.5 yields

$$\begin{aligned} &\sup_{1/(n+1)\leq t\leq n/(n+1)} \frac{|n^{-1/2}\hat{S}_n(t)-\sigma B_n(t)|}{q(t)} \\ &\quad\leq \sup_{1/(n+1)\leq t\leq\varepsilon} \frac{|n^{-1/2}\hat{S}_n(t)-\sigma B_n(t)|}{q(t)} + \sup_{\varepsilon\leq t\leq 1-\varepsilon} \frac{|n^{-1/2}\hat{S}_n(t)-\sigma B_n(t)|}{q(t)} \\ &\qquad + \sup_{1-\varepsilon\leq t\leq n/(n+1)} \frac{|n^{-1/2}\hat{S}_n(t)-\sigma B_n(t)|}{q(t)} \\ &\quad= L_n^{(6)} + L_n^{(7)} + L_n^{(8)}. \end{aligned} \tag{3.54}$$

Choosing $\nu=0$ in (2.23) we get

$$L_n^{(6)} = O_P(1) \sup_{0<t\leq\varepsilon} \frac{t^{1/2}}{q(t)} \tag{3.55}$$

and

$$L_n^{(8)} = O_P(1) \sup_{1-\varepsilon\leq t<1} \frac{(1-t)^{1/2}}{q(t)}. \tag{3.56}$$

By (2.23) we have

$$L_n^{(7)} = o_P(1) \tag{3.57}$$

for all $0<\varepsilon<\frac{1}{2}$. Now (3.1), (3.45) and (3.55)-(3.57) imply (3.54). We note

$$n^{-1/2} \sup_{0<t<1/(n+1)} \frac{|\hat{S}_n(t)|}{q(t)} = n^{-1/2}|S(n)| \sup_{0<t<1/(n+1)} \frac{t}{q(t)} = o_P(1) \tag{3.58}$$

and

$$n^{-1/2} \sup_{n/(n+1)<t<1} \frac{|\hat{S}_n(t)|}{q(t)} = o_P(1). \tag{3.59}$$

The assumption $I^*(q, c) < \infty$ holds, if and only if the random variable $\sup_{0<t<1}|B(t)|/q(t)$ is almost surely finite. Thus by (3.54)–(3.59) we conclude

$$n^{-1/2} \sup_{0<t<1} \frac{|\hat{S}_n(t)|}{q(t)} = \sigma \sup_{0<t<1} \frac{|B_n(t)|}{q(t)} + o_P(1). \tag{3.60}$$

From (3.46) and (3.60) it follows

$$C_n^{(1)} \xrightarrow{\mathcal{D}} \sup_{0<t<1} \frac{|B(t)|}{q(t)}. \tag{3.61}$$

Lemma 2.3 yields

$$n^{-1/2} \sup_{0<t<1} \frac{|\hat{R}_n(t)|}{(t(1-t))^{1/2}} = O_P(1). \tag{3.62}$$

Using Lemma 2.6 we can verify

$$n^{-1/2} \sup_{0<t<(\log n)/n} \frac{|\hat{S}_n(t)|}{(t(1-t))^{1/2}} = O_P((\log\log\log n)^{1/2}), \tag{3.63}$$

$$n^{-1/2} \sup_{1-(\log n)/n \le t<1} \frac{|\hat{S}_n(t)|}{(t(1-t))^{1/2}} = O_P((\log\log\log n)^{1/2}), \tag{3.64}$$

and hence we get

$$n^{-1/2} \frac{1}{\sigma} a(\log n) \max\left\{ \sup_{0<t\le(\log n)/n} \frac{|\hat{Z}_n(t)|}{(t(1-t))^{1/2}}, \sup_{1-(\log n)/n\le t<1} \frac{|\hat{Z}_n(t)|}{(t(1-t))^{1/2}} \right\}$$
$$-(b(\log n)+y) \xrightarrow{P} -\infty. \tag{3.65}$$

Using Lemma 2.4 again we obtain

$$n^{-1/2} \sup_{(\log n)/n \le t \le 1/\log n} \frac{|\hat{R}_n(t)|}{(t(1-t))^{1/2}} = O_P(1/(\log n)^{1/2}) \tag{3.66}$$

and

$$n^{-1/2} \sup_{1-1/\log n \le t \le 1-(\log n)/n} \frac{|\hat{R}_n(t)|}{(t(1-t))^{1/2}} = o_P(1/(\log n)^{1/2}). \tag{3.67}$$

Lemma 2.6 and (3.62) imply

$$a(\log n) \frac{n^{1/2}}{\sigma} \sup_{1/\log n \le t \le 1-1/\log n} \frac{|\hat{S}_n(t)|}{(t(1-t))^{1/2}} - (b(\log n)+y) \xrightarrow{P} -\infty \tag{3.68}$$

and

$$a(\log n) \frac{n^{1/2}}{\sigma} \sup_{1/\log n \le t \le 1-1/\log n} \frac{|\hat{Z}_n(t)|}{(t(1-t))^{1/2}} - (b(\log n)+y) \xrightarrow{P} -\infty. \tag{3.69}$$

Lemma 2.6 and (3.68) yield

$$\lim_{n\to\infty} P\left\{\frac{n^{-1/2}}{\sigma} a(\log n) \max\left(\sup_{(\log n)/n\le t\le 1/\log n} \frac{|\hat{S}_n(t)|}{(t(1-t))^{1/2}}, \sup_{1-1/\log n\le t\le 1-(\log n)/n} \frac{|\hat{S}_n(t)|}{(t(1-t))^{1/2}}\right) \le b(\log n)+y\right\}$$

$$= \lim_{n\to\infty} P\left\{\frac{n^{-1/2}}{\sigma} a(\log n) \sup_{(\log n)/n\le t\le 1-(\log n)/n} \frac{|\hat{S}_n(t)|}{(t(1-t))^{1/2}} \le b(\log n)+y\right\}$$

$$= \lim_{n\to\infty} P\left\{\frac{n^{-1/2}}{\sigma} a(\log n) \sup_{0<t<1} \frac{|\hat{S}_n(t)|}{(t(1-t))^{1/2}} \le b(\log n)+y\right\}. \tag{3.70}$$

Collecting together (3.65)-(3.70) we obtain

$$\lim_{n\to\infty} P\{a(\log n)C_n^{(2)} \le b(\log n)+y\} = \exp(-2\exp(-y)). \tag{3.71}$$

It is clear from the proofs of (3.61) and (3.71) that (1.18) is established if we show the asymptotic independence of $n^{-1/2}\sup_{0<t<1}|\hat{S}_n(t)|/q(t)$ and

$$\xi_n = n^{-1/2}a(\log n)\max\left\{\frac{1}{\sigma} \sup_{(\log n)/n\le t\le 1/\log n} \frac{|\hat{S}_n(t)|}{(t(1-t))^{1/2}}, \frac{1}{\sigma} \sup_{1-1/\log n\le t\le 1-(\log n)/n} \frac{|\hat{S}_n(t)|}{(t(1-t))^{1/2}}\right\} - b(\log n).$$

Using the central limit theorem we get

$$n^{-1/2} \sup_{(\log n)/n\le t\le 1/\log n} \frac{t|S(n)|}{(t(1-t))^{1/2}} = O_P(1/(\log n)^{1/2})$$

$$n^{-1/2} \sup_{1-1/\log n\le t\le 1-(\log n)/n} \frac{(1-t)|S(n)|}{(t(1-t))^{1/2}} = O_P(1/(\log n)^{1/2}),$$

and therefore we have

$$\xi_n = \frac{n^{-1/2}}{\sigma} a(\log n)\max\left\{ \sup_{0<t\le 1/\log n} \frac{|S((n+1)t|}{(t(1-t))^{1/2}}, \sup_{1-1/\log n\le t<1} \frac{|S((n+1)t)-S(n)|}{(t(1-t))^{1/2}}\right\} - b(\log n)+O_P(1). \tag{3.72}$$

Let

$$\hat{J}_n(u) = \begin{cases} 0, & 0\le u\le n/\log n, \\ S(u)-S(n/\log n), & n/\log n\le u\le n-n/\log n, \\ S(n-n/\log n)-S(n/\log n), & n-n/\log n\le u\le n, \end{cases}$$

and

$$U_n(t) = \hat{J}_n((n+1)t) - t\hat{J}_n(n), \quad 0 < t < 1.$$

It is clear that

$$\{U_n(t), 0 < t < 1\}, \qquad \{S((n+1)t), 1 < t < 1/\log n\}$$
$$\text{and} \quad \{S((n+1)t) - S(n), 1 - 1/\log n \le t < 1\} \tag{3.73}$$

are independent. Similarly to (3.18) we can show

$$n^{-1/2} \sup_{0<t<1} \frac{|\hat{S}_n(t)|}{q(t)} = n^{-1/2} \sup_{0<t<1} \frac{|U_n(t)|}{q(t)} + o_P(1). \tag{3.74}$$

The asymptotic independence follows immediately from (3.72)-(3.74).

(ii) First we prove

$$C_n^{(3)} = \left(\frac{n^{-1/2}}{\sigma}\right)^p \int_0^1 \frac{|\hat{S}_n(t)|^p}{q(t)} \, dt + o_p(1). \tag{3.75}$$

Similarly to (3.24)-(3.27) we have

$$n^{-p/2} \int_0^\varepsilon \frac{||\hat{S}_n(t) - \hat{R}_n(t)|^p - |\hat{S}_n(t)|^p|}{q(t)} \, dt = O_P(1) \int_0^\varepsilon \frac{t^{p/2}}{q(t)} \, dt. \tag{3.76}$$

The symmetricity of the processes $\hat{S}_n$ and $\hat{R}_n$ gives

$$n^{-1/2} \int_{1-\varepsilon}^1 \frac{||\hat{S}_n(t) - \hat{R}_n(t)|^p - |\hat{S}_n(t)|^p|}{q(t)} \, dt = O_P(1) \int_{1-\varepsilon}^1 \frac{(1-t)^{p/2}}{q(t)} \, dt, \tag{3.77}$$

and Lemma 2.4 implies

$$n^{-1/2} \int_0^{1-\varepsilon} \frac{||\hat{S}_n(t) - \hat{R}_n(t)|^p - |\hat{S}_n(t)|^p|}{q(t)} \, dt = o_P(1), \tag{3.78}$$

for all $0 < \varepsilon < \frac{1}{2}$. Condition (1.19) and (3.76)-(3.78) yield (3.75). If the approximation with Wiener processes in (3.29)-(3.34) is replaced by the corresponding approximation with Brownian bridges, the symmetricity of $\hat{S}_n$ and B_n gives

$$\left(\frac{n^{-1/2}}{\sigma}\right)^p \int_0^1 \frac{|\hat{S}_n(t)|^p}{q(t)} \, dt = \int_0^1 \frac{|B_n(t)|^p}{q(t)} \, dt + o_P(1). \tag{3.79}$$

By (3.75) and (3.79) we have

$$C_n^{(3)} \xrightarrow{\mathcal{D}} \int_0^1 \frac{|B(t)|^p}{q(t)} \, dt. \tag{3.80}$$

We follow (3.37)-(3.39) and obtain

$$C_n^{(4)} = \left(\frac{n^{-1/2}}{\sigma}\right)^p \int_0^1 \frac{|\hat{S}_n(t)|^p}{(t(1-t))^{1+p/2}} \, dt + O_P(1). \tag{3.81}$$

Thus we conclude

$$\lim_{n\to\infty} P\{(4d \log n)^{-1/2}(C_n^{(4)} - 2m \log n) \leq y\} = \Phi(y). \tag{3.82}$$

The proof of the asymptotic independence is very simple. We note that

$$n^{-p/2} \int_0^1 \frac{|\hat{S}_n(t)|^p}{q(t)} \mathrm{d}t = n^{-p/2} \int_0^1 \frac{|U_n(t)|^p}{q(t)} \mathrm{d}t + o_P(1). \tag{3.83}$$

Using Lemma 2.6 and (3.22) we obtain

$$\begin{aligned} n^{-p/2} \int_0^1 \frac{|\hat{S}_n(t)|}{(t(1-t))^{1+p/2}} \mathrm{d}t &= n^{-p/2} \int_0^{1/\log n} \frac{|S((n+1)t)|^p}{(t(1-t))^{1+p/2}} \mathrm{d}t \\ &\quad + n^{-p/2} \int_{1-1/\log n}^1 \frac{|S(n) - S((n+1)t)|}{(t(1-t))^{1+p/2}} \mathrm{d}t \\ &\quad + O_P(\log \log n), \end{aligned} \tag{3.84}$$

which concludes the proof of (1.20). □

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