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Torsion-Free Groups Having Finite Automorphism Groups. I

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INTRODUCTION

If G is an infinite periodic group, then its automorphism group is also infinite (Baer [*1*]); if G, in addition, is abelian, then more detailed information is available on the cardinal number of Aut (G) (Boyer [*2*]; Walker [*13*]). But in contrast, if G is torsion-free, then Aut (G) may well be a finite group. The simplest example shows this: the infinite cyclic group C_∞, which has only one automorphism other than the identity.

The problem we shall discuss in this paper is the following: for what finite groups A is there a torsion-free group G such that Aut (G) is isomorphic to A?[1] We remark immediately that under these circumstances G is necessarily abelian. For if Aut (G) is finite, then so is its subgroup consisting of the inner automorphisms, which is isomorphic to the factor group of G over its center $Z(G)$. But by a celebrated theorem of Schur,[2] if the center of a group is of finite index, then its derived group G' is finite. And in our case, since G is torsion-free, this means that $G' = 1$ or that G is abelian.

We do not concern ourselves with the apparently hopeless task of finding all the torsion-free abelian groups whose automorphism group is a given finite group. It may suffice here to state that if a finite group A occurs at all, then it will become clear from the examples we shall construct in Part II that even among countable torsion-free abelian groups G of finite rank there are always uncountably many nonisomorphic ones having the given A as their automorphism group.

In fact, much more is known even in the simplest case when $A = C_2$ is cyclic of order 2. Preliminary results by de Groot [7], Hulanicki [*10*], Fuchs [*5*], and Saşiada [*11*] showed successively that for every cardinal

[1] To save circumlocution we shall occasionally say in this case that A "occurs" as an automorphism group.

[2] For a short proof see [*8*].

481-2

number r less than $2^{\aleph_0}$ $2^{2^{\aleph_0}}$, $2^{2^{2^{\aleph_0}}}$ there are torsion-free abelian groups G of rank r with $|\operatorname{Aut}(G)| = 2$, and in 1959 Fuchs [6] stated that there is no restriction whatever on the cardinal number r of the rank of such a group. True, a flaw in Fuchs' argument was revealed by Corner [3], but he at least was able to save the result for all ranks r smaller than the hypothetical first "strongly inaccessible" cardinal number.

Our interest in the present problem arises from the paper by de Vries and de Miranda [12] who investigated what groups of small order (not exceeding 8) occur as the automorphism groups of other groups. Of course, every torsion-free abelian group G has the "inversion" automorphism $\mu = -1 : g\mu = -g$ for all $g \in G$, so that the order of Aut (G), if finite, must be even.[3] De Vries and de Miranda show that of the ten groups of order 2, 4, 6, or 8 seven do occur as automorphism groups of torsion-free groups, and three do not. The latter are the cyclic group C_8 and the dihedral groups D_3 and D_4. Note that among their examples there is a single nonabelian automorphism group, the quaternion group Q_8.

The first part of this paper is devoted to a search for conditions that are necessary for a finite (or for that matter, periodic) group A to occur as the automorphism group of a torsion-free group G. The subsequent second part will deal with the sufficiency of these conditions, that is, with the task of constructing torsion-free groups having a prescribed finite automorphism group.

I

We may perhaps anticipate our final result in the form of a

MAIN THEOREM. *If a finite group A is the automorphism group of a torsion-free group G, then A is a subgroup of a direct product of a finite number of groups of the following six types*[4]*:*

cyclic groups C_2, C_4, C_6 of order 2, 4, 6;
the quaternion group Q_8 of order 8;
the dicyclic group DC_{12} of order 12;
the binary tetrahedral group BT_{24} of order 24.

[3] And it is well known that groups of odd prime order cannot be automorphism groups.

[4] It will be shown in Part II that these six groups and all finite direct products of them do, in fact, occur as automorphism groups. Note that each of these groups has a single element of order 2 and is therefore indecomposable qua automorphism group.

The last three groups can be given conveniently by generators and defining relations:

$$Q_8 = \{\alpha, \beta \parallel \alpha^2 = \beta^2 = (\alpha\beta)^2\};$$
$$DC_{12} = \{\alpha, \beta \parallel \alpha^3 = \beta^2 = (\alpha\beta)^2\};$$
$$BT_{24} = \{\alpha, \beta \parallel \alpha^3 = \beta^3 = (\alpha\beta)^2\}.$$

The theorem indicates that the class of finite automorphism groups of torsion-free abelian groups is rather special; it should be contrasted with the remarkable result of Corner [*4*] that every countable, reduced, torsion-free ring (associative and with unit element) is the endomorphism ring of a torsion-free abelian group.

We begin the proof with a description of the (quite elementary) method by which we derive information on A from the assumption that it is the automorphism group of a torsion-free abelian group G. We denote by $\mathbf{Z}A$ the integral group ring of A. Its elements $\sum_i x_i\alpha_i$, $(x_i \in \mathbf{Z}, \alpha_i \in A)$, induce endomorphisms in G in the obvious way: $g(\sum_i x_i\alpha_i) = \sum_i x_i(g\alpha_i)$. We form the twosided ideal Γ of those $\gamma \in \mathbf{Z}A$ for which this is the zero endomorphism: $g\gamma = 0$ for all $g \in G$. The residue class ring $\mathbf{Z}A/\Gamma$ can now be embedded in End (G), the endomorphism ring of G. We set $\mathbf{Z}A/\Gamma = R(G)$ and call it the automorphism ring of G, that is, the subring of End (G) generated by Aut (G).

Now the units of the ring End (G) are precisely the automorphisms, and they are contained, as monomials $1.\alpha$, in $R(G)$. Following G. Higman [*9*] we call them *trivial* units of $R(G)$. But suppose that we can deduce from intrinsic properties of the finite group A (without specific information on the way the elements of A act on those of G) that no matter what the group G is, the ring $R(G)$ must contain other, nontrivial units. Then our assumption that A is the automorphism group of a suitable G is false and A cannot occur. And if such a nontrivial unit is of infinite order, then A cannot even be a subgroup of a finite (or periodic) automorphism group. To illustrate our method we take the case of the dihedral group D_4 of order 8.

$$D_4 = \{\alpha, \beta \parallel \alpha^4 = \beta^2 = (\alpha\beta)^2 = 1\}.$$

Here the element $\eta = 1 + \alpha(1 + \beta)$ of $R(G)$ turns out to be a unit, because with $\eta' = 1 - \alpha(1 + \beta)$ we have $\eta\eta' = \eta'\eta = 1$. It is easy to show that η cannot be one of the 8 trivial units of $R(G)$.[5]

[5] It may be useful to compare our method with that of de Vries and de Miranda. Apart from the obvious fact that the inversion automorphism must be given by $\alpha^2 = -1$ we have not used any knowledge about the action of the elements of D_4 on those of G. A slight mistake in their paper concerning D_4 is corrected in *Math. Rev.* **22**, 1365/6, #8061.

Alternatively, in their notation, as soon as the statement $\varphi R = P$, $\varphi P = R$ is reached, a contradiction arises to their assumption that $\varphi\delta \neq \delta\varphi$.

The ring End (G) (and hence $R(G)$) has the following two properties of which we shall make repeated use.

P1. End (G) *is itself torsion-free*, that is, integers are not divisors of zero. If for an element $\epsilon \in \text{End}(g)$, $\epsilon \neq 0$, we have $n\epsilon = 0$, i.e., $g(n\epsilon) = 0$ for all $g \in G$, then $(ng)\,\epsilon = 0$, but $\epsilon \neq 0$, hence $ng = 0$ and, as G is torsion-free, $n = 0$.

P2. End (G) *contains no nilpotent elements*, other than 0. If for an element $\epsilon \in \text{End}(G)$, $\epsilon \neq 0$, we have $\epsilon^k = 0$ with $k \geqslant 2$, then $(\epsilon^{k-1})^2 = 0$. So we may assume that $\epsilon^2 = 0$. But then $\eta = 1 + \epsilon$ is a unit of $R(G)$ with $\eta' = 1 - \epsilon$ as its two-sided inverse. This unit η is nontrivial, because it is of infinite order:

$$(1 + \epsilon)^n = 1 + n\epsilon \neq 1 \qquad \text{for} \qquad n \neq 0.$$

The fact that every torsion-free abelian group G has the inversion automorphism $\mu : g\mu = -g$ for all $g \in G$, gives us trivially:

N_0. *The center of A contains an element of order 2.*

The next condition imposes a severe restriction on the orders of the elements of A.

N_1. *All the elements of A have orders dividing 12.* Hence A is of exponent 2, 4, 6, or 12.

Proof. (i) Let $\alpha \in A$ be an element of odd prime power order $k = p^l$. We shall show that $k = 3$. For if $k > 3$, we form the element[6] of $R(G)$:

$$\eta = 1 - \alpha + \alpha^2 - + \cdots + \alpha^{k-3},$$

which we can write unambiguously as

$$\eta = \frac{1 + \alpha^{k-2}}{1 + \alpha}.$$

This η is a unit of $R(G)$. To see this we remark that its inverse η', if it exists, has to be

$$\eta' = \frac{1 + \alpha}{1 + \alpha^{k-2}} = \alpha^2 \frac{1 + \alpha}{1 + \alpha^2},$$

and all we have to do is to write this fraction as a polynomial in α. Using the relation $\alpha^k = 1$ we find explicitly:

[6] When k is a prime number and α a primitive kth root of unity, then η is a unit of the cyclotomic field $\mathbf{Q}(\alpha)$.

for $k \equiv 1 \pmod 4$

$$\eta' = \alpha^2 \frac{1 + \alpha^{k+1}}{1 + \alpha^2} = \alpha^2 - \alpha^4 + - \cdots + \alpha^{k+1};$$

for $k \equiv 3 \pmod 4$

$$\eta' = \alpha^2 \frac{\alpha^k + \alpha}{1 + \alpha^2} = \alpha^3 - \alpha^5 + - \cdots + \alpha^k.$$

We now show that η is a unit of infinite order, hence nontrivial. Let $g(x)$ be the minimal polynomial for which $g(\alpha)$ annihilates G. Then $g(x)$ divides

$$x^{p^l} - 1 = (x^{p^{l-1}} - 1)\, \Phi_{p^l}(x),$$

where the second factor is the cyclotomic polynomial. But $g(x)$ does not divide $x^{p^{l-1}} - 1$, because α is of order p^l, and since $\Phi_{p^l}(x)$ is irreducible, $\Phi_{p^l}(x)$ must divide $g(x)$. Thus we can map α to a primitive kth root of unity ω and extend this mapping to a homomorphism of the (commutative) subring of $R(G)$ generated by α into the complex numbers. Then the image of η is $(1 + \omega^{-2})/(1 + \omega)$. If η were of finite order, then its image would also be, so that the complex number $(1 + \omega^{-2})/(1 + \omega)$ would have absolute value 1. But this implies that $\omega^{-2} = \omega$ or $\bar{\omega}$, and we have a contradiction to our assumption that $k > 3$.

(ii) Let $\alpha \in A$ be an element whose order is a power of 2, say 2^l. We shall show that $l \leqslant 2$. For if $l > 2$, we may assume that $l = 3$, replacing α, if necessary, by $\alpha^{2^{l-3}}$. We now examine the element[7]

$$\eta = 1 + (1 - \alpha^4)(1 + \alpha(1 - \alpha^2)).$$

A short calculation, which we omit, will show that

$$\eta' = 1 + (1 - \alpha^4)(1 - \alpha(1 - \alpha^2))$$

is a two-sided inverse of η. Hence η is a unit of $R(G)$ and is nontrivial, because the mapping $\alpha \to \omega = (1 + i)/\sqrt{2}$ gives $\eta \to 3 + 2\sqrt{2}$, a fundamental unit of $\mathbf{Q}(\omega)$, and shows that η is of infinite order.

The next condition is vacuous when A does not contain elements of order 12.

N_2. *A contains an element of order 2 that is not the sixth power of any element of order 12.*

[7] The choice of this element η is motivated by the construction of a nontrivial unit in the integral group ring of a cyclic group of order 8. For details see Higman [*9*].

Proof. If every element of A of order 2 is the sixth power of an element of order 12, then so is, in particular, the inversion automorphism $\mu = -1$. But if $\alpha^6 = -1$, then

$$\eta = \frac{1+\alpha^5}{1+\alpha} = 1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4 = 1 - \alpha(1+\alpha^2)(1-\alpha)$$

turns out to be a unit whose inverse η' can be written in the form

$$\eta' = \frac{1+\alpha}{1+\alpha^5} = 1 - \alpha^2 - \alpha^3 - \alpha^4 - \alpha^5 = 1 - \alpha^2(1+\alpha^2)(1+\alpha),$$

and again η can be shown in the same way as before to be of infinite order.[8]

N_3. *All elements of A of order 2 are contained in the center $Z(A)$.*

Proof. Let $\alpha, \beta \in A$, $\alpha^2 = 1$. Consider the elements of $R(G)$

$$\epsilon_1 = (1+\alpha)\beta(1-\alpha) \qquad \text{and} \qquad \epsilon_2 = (1-\alpha)\beta(1+\alpha).$$

Here $\epsilon_1^2 = \epsilon_2^2 = 0$, hence $\epsilon_1 = \epsilon_2 = 0$, by P_2. But then

$$\epsilon_1 - \epsilon_2 = 2(\alpha\beta - \beta\alpha) = 0$$

and so $\alpha\beta = \beta\alpha$, by P_1. Therefore $\alpha \in Z(A)$, as required.

In deriving the next two conditions we shall make repeated use of an important lemma on divisors of zero in $R(G)$.

LEMMA. *Let $f(t)$ and $g(t)$ be coprime polynomials with integer coefficients, and*

$$n = a(t)f(t) + b(t)g(t)$$

a representation of their greatest common divisor, where n and the coefficients of $a(t)$ and $b(t)$ are integers. Suppose that $f(\alpha)g(\alpha) = 0$ for some $\alpha \in A$. We define the subgroups H and K of G by

$$H = \{y \in G \mid\mid yf(\alpha) = 0\}, \qquad K = \{z \in G \mid\mid zg(\alpha) = 0\}.$$

Then $H \cap K = 0$, $nG \leqslant H \oplus K$, H and K are characteristic in G, and End (H) *and* End (K) *contain no nilpotent elements other than* 0.

[8] It will appear later, in Part II, that unlike $\alpha^6 + 1 = 0$, the condition $\alpha^{12} - 1 = 0$ does not lead to a contradiction so that A may contain elements of order 12. Our restrictions on $R(G)$ and A are less stringent than those of Higman [9] on a finite group with only trivial units in its integral group ring. In fact, in the homomorphism $\mathbf{Z}A \to R(G)$ a nontrivial unit of the integral group ring may become trivial. See also p. 296.

Proof. For every element $x \in G$ we have

$$nx = x \cdot a(\alpha) f(\alpha) + x \cdot b(\alpha) g(\alpha).$$

Here the first term lies in K, the second in H, and so $nG \leqslant H + K$.

If an element $x \in G$ is annihilated by both $f(\alpha)$ and $g(\alpha)$, then also by $n : nx = 0$, and so $x = 0$. Hence $H \cap K = 0$.

To show that H is characteristic in G, we take an arbitrary element $\beta \in A$. Then $(g(\alpha)\beta f(\alpha))^2 = 0$ and so $g(\alpha)\beta f(\alpha) = 0$, by P_2. For every element $y \in H$ we have

$$ny = y(a(\alpha) f(\alpha) + b(\alpha) g(\alpha)) = y \cdot b(\alpha) g(\alpha).$$

Hence $ny(\beta f(\alpha)) = 0$ or $y(\beta f(\alpha)) = 0$. This shows that $y\beta$ also lies in H, as required. Similarly, K is characteristic in G.

Finally, for every $x \in G$ we have $nx = y + z$, $y \in H$, $z \in K$, and so

$$\begin{aligned} nx \cdot b(\alpha) g(\alpha) &= y \cdot b(\alpha) g(\alpha) + z \cdot b(\alpha) g(\alpha) \\ &= y \cdot b(\alpha) g(\alpha) \\ &= y \cdot b(\alpha) g(\alpha) + y \cdot a(\alpha) f(\alpha) \\ &= ny. \end{aligned}$$

In this way every endomorphism ϵ of H gives rise to an endomorphism $b(\alpha) g(\alpha) \epsilon$ of nG and hence of G, because G is torsion-free. But if ϵ were nonzero and nilpotent on H, then $b(\alpha) g(\alpha) \epsilon$ would also be nilpotent on G and nonzero, because $b(\alpha) g(\alpha)$ annihilates K and acts as multiplication by n on H. This is a contradiction to P_2; therefore End (H), and similarly End (K), contains no nonzero nilpotent elements.

N_4. *The Sylow 3-subgroups of A are (elementary) abelian.*

Proof. If this were not the case, then two noncommuting elements of a Sylow 3-subgroup would generate a subgroup of A of order 27 and exponent 3.

(i) We show, first of all, that if α and β are two elements of a Sylow 3-subgroup and

$$1 + \alpha + \alpha^2 = 1 + \beta + \beta^2 = 1 + (\alpha\beta) + (\alpha\beta)^2 = 0$$

in $R(G)$, then $\alpha = \beta$. Indeed,

$$(\alpha\beta)^2 = (\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1} = \beta^2\alpha^2 = -(1 + \beta) \cdot -(1 + \alpha) = 1 + \alpha + \beta + \beta\alpha;$$

on the other hand,

$$(\alpha\beta)^2 = -(1 + \alpha\beta).$$

So

$$1 + \alpha + \beta + \beta\alpha = -1 - \alpha\beta$$

or

$$\alpha\beta + \beta\alpha = -(1 + \alpha) - (1 + \beta) = \alpha^2 + \beta^2.$$

Hence

$$\alpha^2 + \beta^2 - \alpha\beta - \beta\alpha = (\alpha - \beta)^2 = 0,$$

and so $\alpha = \beta$, by P_2.

(ii) We now assume that α and β are non-commuting elements of a Sylow 3-subgroup. We write $\gamma = [\alpha, \beta]$ for their commutator and apply the preceding lemma to

$$f(t) = 1 + t + t^2, \qquad g(t) = 1 - t, \qquad n = 3, \qquad a(t) = 1, \qquad b(t) = 2 + t.$$

Here $f(\gamma)\,g(\gamma) = 0$ in $R(G)$, hence $3G \leqslant H \oplus K$, where γ is the identity on K, so that by assumption $G \neq K$. We shall show that γ is the identity on H and so derive a contradiction.

In accordance with the lemma, let H_1 and K_1 be the characteristic subgroups of H such that $3H \leqslant H_1 \oplus K_1$ and $H_1(1 + \alpha + \alpha^2) = 0$, $K_1(1 - \alpha) = 0$. Since $\alpha = 1$ on K_1, we also have $\gamma = 1$ on $K_1 \leqslant H$, hence $K_1 = 0$. So $3H \leqslant H_1$, but

$$3H(1 + \alpha + \alpha^2) = 0$$

implies that

$$H(1 + \alpha + \alpha^2) = 0.$$

We now proceed in the same way with β and $\alpha\beta$ and find that on H:

$$1 + \alpha + \alpha^2 = 1 + \beta + \beta^2 = 1 + (\alpha\beta) + (\alpha\beta)^2 = 0.$$

Hence $\alpha = \beta$ on H, by the lemma and what was proved under (i). But then $\gamma = 1$ on H. Together with $H \cap K = 0$ this shows that $H = 0$ or $3G \leqslant K$. But $3G(1 - \gamma) = 0$ implies that $g(1 - \gamma) = 0$. Hence $G = K$ and we have reached a contradiction to the assumption that α and β do not commute.

N_5. *Let α be an arbitrary element of A of order 2. Then G has a characteristic subgroup H_α with the following properties: if φ_α denotes the restriction homomorphism of A into* Aut (H_α), *then*

1. *$\alpha\varphi_\alpha$ is the inversion automorphism on H_α, and*
2. *the order of $A\varphi_\alpha$ divides 24.*

Proof. (i) We begin by applying the lemma to

$$f(t) = 1 + t, \qquad g(t) = 1 - t, \qquad n = 2, \qquad a(t) = b(t) = 1.$$

Here $f(\alpha)\,g(\alpha) = 0$, $2G \leqslant H_1 \oplus K_1$, $H_1 \neq 0$, where α acts as the inversion automorphism on H_1 and as the identity on K_1. Let φ_1 be restriction homomorphism of A into Aut (H_1), i.e., $\lambda\varphi_1 = \lambda_{|H_1}$ for all $\lambda \in A$.

The conditions N_1 and N_4 are clearly inherited by homomorphic images of A, but N_2 and N_3 need not be. We shall now show that in the present circumstances it is still true that every element of $A\varphi_1$ of order 2 lies in the center of Aut (H_1). If this were not so, then we could find a $\beta \in A\varphi_1$, $\beta^2 = 1$, and a $\gamma \in \text{Aut}(H_1)$ such that $[\beta, \gamma] \neq 1$. Now consider the elements

$$\epsilon_1 = (1 + \beta)\gamma(1 - \beta) \qquad \text{and} \qquad \epsilon_2 = (1 - \beta)\gamma(1 + \beta)$$

of the endomorphism ring End (H_1). Both are nilpotent, $\epsilon_1^2 = \epsilon_2^2 = 0$. But if both ϵ_1 and ϵ_2 were zero, then $\epsilon_1 - \epsilon_2 = 2(\beta\gamma - \gamma\beta) = 0$, so that β and γ would commute after all. Hence End (H_1) would contain at least one nonzero nilpotent element. This is a contradiction to the lemma and shows that β lies in the center of Aut (H_1).

(ii) Now if $A\varphi_1$ contains, apart from $\alpha\varphi_1$, another element of order 2, say β (in the center as we have just seen), then we can again apply the lemma with respect to β and "split" $H_1 : 2H_1 \leqslant H_2 \oplus K_2$, where β is the inversion automorphism on H_2 and the identity on K_2. Here H_2 and K_2 are characteristic in H_1, consequently in G. Continuing in this manner we arrive after a finite number of steps at a characteristic subgroup H of G such that the restriction $\alpha\varphi$ of α to H is the only element of order 2 in the image $A\varphi$ of A in Aut (H).

Now a finite 2-group having a single element of order 2 in its center is cyclic or a generalized quaternion group. Bearing N_1 in mind, we can say at this stage that the Sylow 2-subgroups of $A\varphi$ are cyclic of order 2 or 4, or quaternion groups. Hence $A\varphi$ is of order 2^r3^s, $r \leqslant 3$.

(iii) If $s = 0$ or 1, we set $H_\alpha = H$, $\varphi_\alpha = \varphi$ and have satisfied the conditions of N_5. But if $s > 1$, we proceed again to apply the lemma with respect to an element $\delta \in A\varphi$ of order 3, this time with

$$f(t) = 1 + t + t^2, \quad g(t) = 1 - t, \quad n = 3, \quad a(t) = 1, \quad b(t) = 2 + t.$$

Then $3H \leqslant H' \oplus K'$, $H' \neq 0$, where $1 + \delta + \delta^2 = 0$ on H' and δ is the identity on K'. If the restriction of $A\varphi$ to H' contains a further element ζ of order 3 for which $H'(1 + \zeta + \zeta^2) \neq 0$, we continue the process. Eventually we reach a subgroup H_α that is characteristic in H', hence in H and in G, such that for any two distinct elements β, γ of a Sylow 3-subgroup of $A\varphi_\alpha = A_{|H_\alpha}$ we have $1 + \beta + \beta^2 = 1 + \gamma + \gamma^2 = 0$ on H_α. But then the proof of N_4, (i) shows that β and γ are inverses of each other, so that the order of a Sylow 3-subgroup of $A\varphi_\alpha$ is 3 and the order of $A\varphi_\alpha$ divides 24, as required.

We are now ready for the proof of the Main Theorem.

Proof. (i) Suppose that the subgroup of A consisting of the elements of order 2 (and the unit element) is of order 2^k and that the Sylow 3-subgroups of A are of order 3^l. By a repeated application of the process described in the proof of N_5 we arrive at the relationship of the form

$$nG \leqslant H_1 \oplus H_2 \oplus \cdots \oplus H_m ,$$

where each H_i is characteristic in G and the (even) order of $A\varphi_i = A_{|H_i}$ divides 24. Here n divides 2^k3^l and m is less than or equal to 2^{k+l}.

We now define a map φ of A into the direct product of the groups $A\varphi_i$:

$$\alpha\varphi = (\alpha\varphi_1 , \alpha\varphi_2 , \cdots, \alpha\varphi_m) \qquad \text{for all } \alpha \in A.$$

This is clearly a homomorphism. But if $\alpha \in \operatorname{Ker} \varphi$, then $\alpha\varphi_i = \alpha_{|H_i} = 1$. Hence α induces the identity automorphism on $H_1 \oplus H_2 \oplus \cdots \oplus H_m$, hence on nG, and as G is torsion-free, also on G.

Hence $\operatorname{Ker} \varphi = 1$ and we can embed A in the direct product:

$$A \leqslant A\varphi_1 \otimes A\varphi_2 \otimes \cdots \otimes A\varphi_m .$$

(ii) A straightforward inspection of the groups of order dividing 24 that satisfy the relevant conditions N_0, N_1, N_3, and N_4 shows that each $A\varphi_i$ is of one of the following types:

the six groups C_2, C_4, C_6, Q_8, DC_{12}, BT_{24}

mentioned in the statement of the theorem,

or the two groups C_{12} and $DC_{24} = \{\alpha, \beta \parallel \alpha^{12} = 1, \beta^2 = \alpha^6, \beta^{-1}\alpha\beta = \alpha^{-1}\}$. The last two cannot occur as automorphism groups, because they have elements of order 12 and a single element of order 2, and hence violate the the condition N_2. But they can occur as subgroups of automorphism groups, e.g., $C_{12} \leqslant C_4 \otimes C_6$, and our final step is to eliminate these two groups if they occur among the $A\varphi_i$ and to replace them by other homomorphic images of A.

(iii) Suppose that A contains an element α of order 12. Then the group ring $\mathbf{Z}A$ contains a nontrivial unit η of infinite order (see Higman [9]). How can this η, a polynomial in α, become a trivial unit in $R(G)$? Since the group rings of the cyclic groups C_4 and C_6 have no nontrivial units, it is easy to see that this can only happen if the kernel Γ of the homomorphism from $\mathbf{Z}A$ to $R(G)$ contains $(1 - \alpha^6)(1 + \alpha^2)$.

But with $(1 - \alpha^6)(1 + \alpha^2) = 0$ on $R(G)$ we can again apply the lemma to

$$f(t) = 1 - t^6, \quad g(t) = 1 + t^2, \quad n = 2, \quad a(t) = 1, \quad b(t) = 1 - t^2 + t^4,$$
$$f(\alpha)\, g(\alpha) = 0,$$

and find that $2G \leqslant H \oplus K$, where $\alpha^6{}_{|H} = 1$, $\alpha^2{}_{|K} = -1$. If the image of A in Aut (H) or Aut (K) still contains an element of order 12, we can continue the procedure until all the elements of order 12 are eliminated from the groups $A\varphi_i$.

This concludes the proof of the Main Theorem. We may perhaps illustrate the last step of the proof on the example of the group of order 48

$$A\{\alpha, \beta \parallel \alpha^{12} = 1, \beta^4 = 1, \beta^{-1}\alpha\beta = \alpha^{-1}\}.$$

If we use as kernels of restriction homomorphisms the subgroups $K_1 = \{\alpha\}$ and $K_2 = \{\alpha^6\beta^2\}$, then $A\varphi_1 \cong C_4$ and $A\varphi_2 \cong DC_{24}$, $A \leqslant C_4 \otimes DC_{24}$. But with the more judicious choice of $K_3 = \{\alpha^3\}$ and $K_4 = \{\alpha^4, \alpha^2\beta^2\}$, we have $A\varphi_3 \cong Q_8$ and $A\varphi_4 \cong DC_{12}$, hence $A \leqslant Q_8 \otimes DC_{12}$. It will become apparent, in Part II, that A is, in fact, an automorphism group of a torsion-free group G.

In conclusion, we mention that there is a somewhat shorter but more sophisticated approach to the Main Theorem, starting from the group algebra $\mathbf{Q}A$ over the rationals rather than the group ring. This will be the subject of a separate paper [*9a*].

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References

1. Baer, R. Finite extensions of Abelian groups with minimum condition. *Trans. Am. Math. Soc.* **79** (1955), 521-540.

2. Boyer, D. L. Enumeration theorems in infinite abelian groups. *Proc. Am. Math. Soc.* **7** (1956), 567-570.

3. Corner, A. L. S. Ph. D. Thesis, Cambridge 1961.

4. Corner, A. L. S. Every countable reduced torsion-free ring is an endomorphism ring. *Proc. London Math. Soc.* (3), **13** (1963), 687-710.

5. Fuchs, L. On a directly indecomposable abelian group of power greater than continuum. *Acta. Math. Acad. Sci. Hung.* **8** (1957), 543-454.

6. Fuchs, L. The existence of indecomposable abelian groups of arbitrary power. *Acta Math. Acad. Sci. Hung.* **10** (1959), 453-457.

7. de Groot, J. Indecomposable abelian groups. *Ned. Akad. Wetenschap. Proc. Ser. A, Indagationes Math.* **19** (1957), 137-145.

8. Hall, P. "Nilpotent Groups." Lecture Notes, Canadian Math. Congr., Summer Seminar 1957.

9. Higman, G. The units of group rings. *Proc. London Math. Soc.* (2) **46** (1960), 231-248.

9a. Hirsch, K. A. and Zassenhaus, H. Finite automorphism groups of torsion-free groups. *J. London Math. Soc.*, in press.

10. HULANICKI, A. Note on a paper of de Groot. *Ned. Akad. Wetenschap. Proc. Ser. A, Indagationes Math.* **20** (1958), 114.

11. SAŞIADA, E. Construction of directly indecomposable Abelian groups of a power higher than that of the continuum, *Bull. Acad. Polon. Sci., Ser. Sci. Math., Astron. Phys.* **5** (1957), 701-703; *ibid.* **7** (1959), 23-26.

12. DE VRIES, H. AND DE MIRANDA, A. B. Groups with a small number of automorphisms. *Math. Z.* **68** (1958), 450-464.

13. WALKER, E. A. On the orders of the automorphism groups of infinite torsion abelian groups. *J. London Math. Soc.* **35** (1960), 385-388.