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# Translating solitons of mean curvature flow of noncompact spacelike hypersurfaces in Minkowski space☆

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**Abstract**

In this paper, we study the existence, uniqueness and asymptotic behavior of rotationally symmetric translating solitons of the mean curvature flow in Minkowski space. We also study the asymptotic behavior and the strict convexity of general solitons of such flows.


*MSC:* 35J60; 52C44

*Keywords:* Spacelike hypersurface; Elliptic equation; Singular ODE; Asymptotic behavior; Strictly convex solution

## 1. Introduction and main results

Minkowski space $R^{n,1}$ is the linear space $R^{n+1}$ endowed with the Lorentz metric

$$ds^2 = \sum_{i=1}^{n} dx_i^2 - dx_{n+1}^2.$$

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Spacelike hypersurfaces in $R^{n,1}$ are Riemanian $n$-manifolds, having an everywhere lightlike normal field $v$ which we assume to be future directed and thus satisfy the condition $\langle v, v\rangle = -1$. Locally, such surfaces can be expressed as graphs of functions $x_{n+1} = u(x_1, \ldots, x_n) : R^n \longmapsto R$ satisfying the spacelike conditions $|\nabla u(x)| < 1$ for all $x \in R^n$.

If a family of spacelike embeddings $X_t = X(\cdot, t) : R^n \mapsto R^{n,1}$ with corresponding hypersurfaces $M_t = X(R^n, t)$ satisfy the evolution equation

$$\frac{\partial X}{\partial t} = Hv \tag{1.1}$$

on some time interval, we say that the surfaces $\{M_t\}$ are evolved by *mean curvature flow* (MCF). Here $H = div_{M_t} v$ denotes the mean curvature of the hypersurface $M_t$. Let $V(\cdot, t)$ be the graph expression of $M_t$. Then $|\nabla V(\cdot, t)| < 1$ and MCF equation (1.1) is equivalent, up to a diffeomorphism in $R^n$, to the equation

$$\frac{\partial V}{\partial t} = \sqrt{1 - |\nabla V|^2}\, div \left( \frac{\nabla V}{\sqrt{1 - |\nabla V|^2}} \right) \quad \text{in } R^n. \tag{1.2}$$

MCF has been extensively studied in Euclidean space; see [8] and the references therein, while in Minkowski space, MCF was studied in [4,9] for compact hypersurfaces and in [2,3] for noncompact hypersurfaces. The method of MCF was used in [4,9] to constructed spacelike hypersurfaces with prescribed mean curvature, which, as it is well-known, have played important roles in studying Lorentzian manifolds. In particular, maximal hypersurfaces, i.e., the ones with zero mean curvature, were used by Schoen and Yau in the first proof of the famous positive mass theorem [12].

The solutions of MCF (1.1) (or (1.2), equivalently) which move by vertical translation are called *Translating Solitons*. Therefore, a translating soliton of MCF (1.2) is characterized by $V(x, t) = u(x) + t$, where $u : R^n \mapsto R$ is an initial spacelike hypersurface satisfying

$$div \left( \frac{\nabla u(x)}{\sqrt{1 - |\nabla u(x)|^2}} \right) = \frac{1}{\sqrt{1 - |\nabla u(x)|^2}}, \quad \forall x \in R^n. \tag{1.3}$$

The spacelike condition reads as

$$|\nabla u(x)| < 1, \quad \forall x \in R^n. \tag{1.4}$$

Translating solitons can be regarded as a natural way of foliating spacetimes by almost nulllike hypersurfaces. It may be expected that this kind of translating solitons would have applications in general relativity [3]. For this purpose, it is useful to understand their geometric structure sufficiently. In [3], the existence of smooth solutions of (1.3)–(1.4) was proved by a PDE method. However, using ODE techniques we can find strictly convex radially symmetric solutions of (1.3)–(1.4).

**Theorem 1.1.** *There exists exact one solution $r \in C^2[0,\infty)$ to initial value problem*

$$\frac{r''(t)}{1-(r'(t))^2} + \frac{n-1}{t}r'(t) = 1, \quad t \in (0,\infty) \tag{1.5}$$

*and*

$$r(0) = r'(0) = 0 \tag{1.6}$$

*such that $u(x) = r(|x-x_0|) + u(x_0)$ in $R^n$ for any radially symmetric $C^2$ solution $u$ of* (1.3)–(1.4), *where $x_0$ is the vertex of $u$. Moreover, the function $r \in C^\infty[0,\infty)$ satisfies*

$$\frac{t}{\sqrt{n^2+t^2}} \leqslant r'(t) < 1, \quad \forall t \in [0,\infty) \tag{1.7}$$

*and*

$$0 < r''(t) \leqslant 1, \quad \forall t \in [0,\infty). \tag{1.8}$$

*Therefore, all rotationally symmetric spacelike translating solitons of MCF* (1.2) *is smooth, strictly convex, unique up to a translation in $R^{n+1}$, and of linear growth.*

To describe the asymptotic behavior of general solitons as $|x| \to \infty$, we use the tangent cones methods in [1,13] for entire spacelike convex hypersurfaces of constant mean curvature and in [6] for constant Gauss curvature. Define the *blow down* of $F$ at infinity by

$$V_F(x) = \lim_{\rho\to\infty} \frac{F(\rho x)}{\rho}. \tag{1.9}$$

Since $\frac{d}{d\rho}(\frac{F(\rho x)}{\rho} - \frac{F(0)}{\rho}) \geqslant 0$ if $F$ is convex, and $\frac{F(\rho x)}{\rho} - \frac{F(0)}{\rho} \leqslant |x|$ if $F$ is spacelike. $V_F$ is well-defined over $R^n$ and the limit in (1.9) is uniform on any compact set in $R^n$ if $F$ is a convex function satisfying (1.4). Using Theorem 1.1 and the methods in [1,13], we will prove

**Theorem 1.2.** *Suppose that $u$ is a convex solution to* (1.3)–(1.4). *Then the blow-down function $V_u$ is a positive homogeneous degree one convex function satisfying the* 1-*Lipschitz condition*

$$|V_u(x) - V_u(y)| \leqslant |x-y|, \quad \forall x, y \in R^n \tag{1.10}$$

*and the null condition, i.e., for any $x \in R^n$ and any $\delta > 0$, there is $y \in R^n$ such that*

$$|V_u(x) - V_u(y)| = |x - y| = \delta. \tag{1.11}$$

*Furthermore, one has*

$$V_u(y) = \lim_{\rho\to\infty} \frac{u(\rho y)}{\rho} = 1 \quad \textit{uniformly for } y \in \overline{\nabla u(R^n)} \bigcap S^{n-1} \tag{1.12}$$

*and*

$$V_u(x) = \lim_{\rho\to\infty} \frac{u(\rho x)}{\rho} = |x| \quad \textit{for } x \in \overline{\nabla u(R^n)}, \tag{1.13}$$

*where $\overline{\nabla u(R^n)}$ is the smallest closed set containing $\{y : y = \nabla u(x), x \in R^n\}$ in $R^n$.*

A natural question is whether any solution to (1.3)–(1.4) is convex. This question seems very difficult to the author. However, we obtain the following result which is related to this question in some way.

**Theorem 1.3.** *Let $u$ be a convex solution of Eqs.* (1.3)–(1.4). *If the set $\Omega_0 = \{x \in R^n : (u_{ij}(x)) > 0\}$ is nonempty, then $\Omega_0 = R^n$.*

A similar result was obtained for the equation $\Delta u = f(u, \nabla u)$ in [11], for the equation of entire spacelike hypersurfaces of constant mean curvature in [13] and for the mean curvature flow in Euclidean space in [10,7,14].

This paper is organized as follows. In Section 2, we will use ODE theory and a priori estimate techniques to prove Theorem 1.1. In Section 3, we will give the proof of Theorem 1.2. In Section 4, we will prove Theorem 1.3.

## 2. Radially symmetric solutions

We start with some simple facts which will be used throughout this section.

If $u(x) = r(|x-x_0|)+u(x_0)$ and $u \in C^{k,\alpha}(R^n)$ for some $k\geqslant 1, 0\leqslant\alpha\leqslant 1$ with $k+\alpha\geqslant 2$, then $r \in C^{k,\alpha}[0,\infty)$ since $r(t) = u((t,0)+x_0) = u((-t,0)+x_0)$ for all $t\geqslant 0$. Thus $r'(0) = 0$ and Eq. (1.3) is equivalent to

$$\frac{r''(t)}{1-(r'(t))^2} + \frac{n-1}{t} r'(t) = 1, \quad \forall t \in (0,\infty) \tag{2.1}$$

and

$$r(0) = r'(0) = 0, \tag{2.2}$$

the spacelike condition (1.4) is equivalent to

$$0 < r'(t) < 1, \quad \forall t \in (0, \infty) \tag{2.3}$$

and the strict convexity to

$$1 \geqslant r''(t) > 0, \quad \forall t \in [0, \infty). \tag{2.4}$$

Conversely, if $r \in C^2[0, \infty)$ is a solution to (2.1)–(2.2), then it follows from a direct computation that $u(x) = r(|x|) \in C^{1,1}(R^n)$ is a solution to (1.3)–(1.4). By the standard regularity theory of elliptic equations in [5] we see that $r(|x|) \in C^\infty(R^n)$ and thus $r \in C^\infty[0, \infty)$.

**Lemma 2.1.** *If* $r \in C^2[0, \infty)$ *is a solution to* (2.1)–(2.4), *then it satisfies* (1.7).

**Proof.** If $r'(t) < 1 - \delta$ for all $t \in [0, \infty)$ and some $\delta \in (0, 1)$, then $r''(t) \geqslant \frac{\delta}{2}$ for all $t \geqslant t_0$ and for some large $t_0 > 0$ by (2.1). Integrating this inequality over $[t_0, t)$ we obtain

$$1 - \delta > r'(t) \geqslant \frac{\delta}{2}(t - t_0) - r'(t_0)$$

for all $t \geqslant t_0$, a contradiction. Therefore, there is a sequence $t_k \to \infty$ such that $r'(t_k) \to 1$. Using (2.4), we get

$$\lim_{t \to +\infty} r'(t) = 1. \tag{2.5}$$

Note that the inequality on the right-hand sides of (1.7) follows directly from (2.3). We want only to prove

$$r'(t) \geqslant \frac{t}{\sqrt{n^2 + t^2}}, \quad \forall t \geqslant 0. \tag{2.6}$$

On the contrary that (2.6) is false. Then we have a $t_0 > 0$ such that

$$r'(t_0) < \frac{t_0}{\sqrt{n^2 + t_0^2}}.$$

Observing that $r'(0) = 0$ and

$$\lim_{t \to +\infty} \left( r'(t) - \frac{t}{\sqrt{n^2 + t^2}} \right) = 0$$

by (2.5), we see that the function $r'(t) - \frac{t}{\sqrt{n^2+t^2}}$ attains its negative minimum at a point $t_1 > 0$. Hence

$$r''(t_1) = \left( \frac{t_1}{\sqrt{n^2 + t_1^2}} \right)' = n^2(n^2 + t_1^2)^{-\frac{3}{2}}$$

and

$$r'(t_1) < \frac{t_1}{\sqrt{n^2 + t_1^2}}.$$

This, together with (2.1), imply

$$\begin{aligned} 1 &= \frac{r''(t_1)}{1 - (r'(t_1))^2} + \frac{n-1}{t_1} r'(t_1) \\ &< \frac{n^2(n^2 + t_1^2)^{-\frac{3}{2}}}{1 - \frac{t_1^2}{n^2+t_1^2}} + \frac{n-1}{t_1} \cdot \frac{t_1}{\sqrt{n^2 + t_1^2}} = \frac{n}{\sqrt{t_1^2 + n^2}} < 1, \end{aligned}$$

a contradiction! $\square$

**Lemma 2.2.** *There exists a* $r \in C^\infty[0, \infty)$ *to* (2.1)–(2.4).

**Proof.** Since Eq. (2.1) is singular at $t = 0$, we consider the approximation problem

$$\frac{r''(t)}{1 - (r'(t))^2} + \frac{n-1}{t + \varepsilon} r'(t) = 1, \quad \forall t \in (0, \infty) \tag{2.7}$$

$$|r'(t)| < 1, \quad \forall t \in (0, \infty) \tag{2.8}$$

and

$$r(0) = 0, \quad r'(0) = \frac{\varepsilon}{n}. \tag{2.9}$$

Integrating (2.7) over $[0, t)$ we have

$$\frac{1}{2} \left[ \ln \frac{1 + r'(t)}{1 - r'(t)} - \ln \frac{1 + r'(0)}{1 - r'(0)} \right] + (n-1) \int_0^t \frac{r'(s)}{s + \varepsilon} \, ds = t,$$

which implies that for any $R > 0$, there exist a constant $0 < C(R) < 1$ depending on $R$ such that

$$|r'(t)| < 1 - C(R), \quad \forall t \in [0, R).$$

Therefore, by local existence result of ODE, we see that for any $\varepsilon \in (0, 1)$ there is a unique smooth solution to (2.7)–(2.9). Denote this solution by $r_\varepsilon$. Obviously,

$$r_\varepsilon''(0) = \left[1 - \frac{n-1}{\varepsilon} r'(0)\right][1 - (r'(0))^2] = \frac{n^2 - \varepsilon^2}{n^3}. \tag{2.10}$$

This leads us to conclude that

$$r_\varepsilon''(t) \geqslant 0, \quad \forall t \in [0, \infty). \tag{2.11}$$

Otherwise, there is a $t_1 \in (0, \infty)$ such that $r_\varepsilon''(t_1) < 0$. Then we may choose $t_0 > 0$ and $\delta > 0$ such that

$$r_\varepsilon''(t_0) = 0, \quad r_\varepsilon''(t) < 0, \quad \forall t \in (t_0, t_0 + \delta). \tag{2.12}$$

By (2.9) and (2.10), we may further assume

$$r_\varepsilon'(t) > 0, \quad \forall t \in [t_0, t_0 + \delta). \tag{2.13}$$

Hence, $0 < r_\varepsilon'(t) < r_\varepsilon'(t_0)$ for all $t \in (t_0, t_0 + \delta)$. But this, together with (2.7), (2.8), (2.12) and (2.13), implies

$$1 = \frac{n-1}{t_0 + \varepsilon} r_\varepsilon'(t_0) > \frac{n-1}{t + \varepsilon} r_\varepsilon'(t) > \frac{r_\varepsilon''(t)}{1 - (r_\varepsilon'(t))^2} + \frac{n-1}{t + \varepsilon} r_\varepsilon'(t) = 1$$

for all $t \in (t_0, t_0 + \delta)$, a contradiction! This proves (2.11).

It follows from (2.11) and (2.9) that

$$r_\varepsilon'(t) \geqslant \frac{\varepsilon}{n}, \quad \forall t \in (0, \infty). \tag{2.14}$$

Using this, (2.10) and (2.11) again, we claim that

$$r_\varepsilon''(t) > 0, \quad \forall t \in [0, \infty). \tag{2.15}$$

In fact, on the contrary that there is a $t_2 > 0$ such that $r''_\varepsilon(t_2) = 0$. Then the function

$$y(t) := \frac{n-1}{t+\varepsilon} r'_\varepsilon(t) = 1 - \frac{r''_\varepsilon(t)}{1-(r'_\varepsilon(t))^2}$$

attains a maximum at $t_2$. Hence, $y'(t_2) = 0$ and therefore, $r'_\varepsilon(t_2) = 0$, contradicting with (2.14). This proves (2.15).

Now, we use (2.8), (2.9), (2.14), (2.15) and (2.7) to see that

$$\frac{\varepsilon}{n} \leqslant r'_\varepsilon(t) < 1, \quad \forall t \in [0,\infty) \tag{2.16}$$

$$\frac{t\varepsilon}{n} \leqslant r_\varepsilon(t) \leqslant t, \quad \forall t \in [0,\infty) \tag{2.17}$$

$$0 < r''_\varepsilon(t) = \left(1 - \frac{n-1}{t+\varepsilon} r'_\varepsilon(t)\right)(1-(r'_\varepsilon(t))^2) \leqslant 1 - \frac{(n-1)\varepsilon}{n(t+\varepsilon)}, \quad \forall t \in [0,\infty). \tag{2.18}$$

By estimates (2.16)–(2.18) we can choose a subsequence $\varepsilon_k \to 0$ $(k \to \infty)$ and a function $r \in C^{1,\alpha}[0,\infty)$ ($\alpha \in (0,1)$ fixed) such that

$$r_{\varepsilon_k} \to r \quad \text{in } C^{1,\alpha}[0,\infty) \text{ as } k \to \infty. \tag{2.19}$$

Obviously,

$$r(0) = 0 = r'(0), \tag{2.20}$$

$$0 \leqslant r'(t) \leqslant 1 \quad \text{and} \quad 0 \leqslant r''(t) \leqslant 1, \ \forall t \in [0,\infty). \tag{2.21}$$

Furthermore, we can conclude that

$$0 \leqslant r'(t) < 1, \quad \forall t \in [0,\infty) \tag{2.22}$$

Otherwise, there is a $t_3 > 0$ such that $r'(t_3) = 1$ and $0 \leqslant r'(t) < 1$ for all $t \in [0, t_3)$. Integrating (2.7) for $r_{\varepsilon_k}$ over $[\frac{t_3}{2}, t)$ we have

$$\frac{1}{2}\left[\ln \frac{1+r'_{\varepsilon_k}(t)}{1-r'_{\varepsilon_k}(t)} - \ln \frac{1+r'_{\varepsilon_k}(\frac{t_3}{2})}{1-r'_{\varepsilon_k}(\frac{t_3}{2})}\right] + (n-1)\int_{\frac{t_3}{2}}^{t} \frac{r'_{\varepsilon_k}(s)}{s+\varepsilon_k}\, ds = \frac{t_3}{2}, \quad \forall t \in \left(\frac{t_3}{2}, t_3\right).$$

Letting $k \to \infty$ and $t \to t_3^-$ then, we obtain

$$+\infty - \ln \frac{1+r'(\frac{t_3}{2})}{1-r'(\frac{t_3}{2})} + 2(n-1)\int_{\frac{t_3}{2}}^{t_3} \frac{r'(s)}{s}\, ds = \frac{t_3}{2},$$

a contradiction! This shows (2.22). Observing that $r_\varepsilon$ satisfies Eq. (2.7), we use (2.19)–(2.22) to see that $r \in C^2[0,\infty)$ satisfies (2.1) and (2.2), which implies $r \in C^\infty[0,\infty)$ as we have said in the beginning of this section.

Therefore, in order to finish the proof of Lemma 2.2, we want only to prove

$$r'(t) > 0, \quad \forall t \in (0,\infty) \tag{2.23}$$

and

$$r''(t) > 0, \quad \forall t \in [0,\infty). \tag{2.24}$$

In fact, by (2.10) we have $r''(0) = \frac{1}{n}$. Then (2.23) follows from the fact that $r'(0) = 0$ and $r''(t) \geqslant 0$ for all $t > 0$ as in (2.20) and (2.21).

If there were a $t_4 \in (0,\infty)$ such that $r''(t_4) = 0$, then it follows from (2.21) that the function

$$Z(t) := \frac{n-1}{t} r'(t) = 1 - \frac{r''}{1-(r')^2}$$

attains a maximum at the point $t_4$. Thus

$$Z'(t_4) = 0 \quad \text{and therefore} \quad r'(t_4) = 0,$$

contradicting (2.23). This proves (2.24) and thus Lemma 2.2.

**Lemma 2.3.** *If $r_1, r_2 \in C^2[0,\infty)$ are both solutions to initial problem* (2.1)–(2.3), *then $r_1(t) = r_2(t)$ for all $r \geqslant 0$.*

**Proof.** Let $u_i(x) = r_i(|x|)$ $(i = 1, 2)$. As we have seen, $u_i \in C^\infty(R^n)$ are solutions of (1.3)–(1.4). Fix $t > 0$, arbitrarily. We see that both $u_1(x)$ and $u_2(x) + r_1(t) - r_2(t)$ are solutions of the Dirichlet problem of Eqs. (1.3)–(1.4) over the ball $B_t(0)$ with the same boundary value $r_1(t)$. Thus $u_1(x) = u_2(x) + r_1(t) - r_2(t)$ for all $x \in B_t(0)$ by the uniqueness theorem [5, Theorem 10.2]. Taking $x = 0$, we obtain $r_1(t) = r_2(t)$.

**Proof of Theorem 1.1.** Observing the simple facts at the beginning of this section and using Lemmas 2.1–2.3, we immediately obtain Theorem 1.1. □

## 3. Proof of theorem 1.2

In this section, we use the concept of tangent cones at infinity to describe the asymptotic behavior of the solitons as $|x| \to \infty$. This method was used in [1,13] for entire spacelike convex hypersurfaces of constant mean curvature and in [6] for constant Gauss curvature.

Recall that the *blow down* function

$$V_F(x) = \lim_{\rho \to 0\infty} \frac{F(\rho x)}{\rho} \tag{3.1}$$

is well-defined over $R^n$ and the limit is uniform on any compact set in $R^n$ if $F$ is a convex function satisfying (1.4).

**Lemma 3.1.** *If $u$ is a convex function satisfying* (1.4), *then $V_u$ is a positively homogeneous degree one convex function satisfying the* 1-*Lipschitz condition*

$$|V_u(x) - V_u(y)| \leqslant |x - y|, \quad \forall x, y \in R^n, \tag{3.2}$$

*while if $u$ is a convex solution to* (1.3)–(1.4), *then $V_u$ satisfies the null condition*, *i.e.*, *for any $x \in R^n$ and any $\delta > 0$*, *there is $y \in R^n$ such that*

$$|V_u(x) - V_u(y)| = |x - y| = \delta. \tag{3.3}$$

**Proof.** The convexity and the positive homogeneity are obviously from the definition of $V_u$ and the convexity of $u$.

For any $x, y \in R^n$, by (1.4) we have

$$|V_u(x) - V_u(y)| \leqslant \limsup_{\rho \to \infty} \frac{|u(\rho x) - u(\rho y)|}{\rho} \leqslant |x - y|.$$

Hence, it is sufficient to prove the null condition. On the contrary that there would exist an $x \in R^n$, $\delta > 0$ and $\theta > 0$ such that

$$V_u(y) \leqslant V_u(x) + (1 - 2\theta)\delta$$

for all $y \in R^n$ with $|x - y| = \delta$. Observing that the limit in (3.1) is uniform on any compact set, we may choose a $\rho_0 > 0$ so that

$$u_\rho(y) \leqslant V_u(x) + (1 - \theta)\delta \tag{3.4}$$

for all $\rho > \rho_0$ and all $y \in B(x, \delta)$, where we have used the notation

$$B(x, \delta) = \{y \in R^n : |y - x| < \delta\} \quad \text{and} \quad u_\rho(x) = \frac{u(\rho x)}{\rho}.$$

It follows from (1.3)–(1.4) that $u_\rho$ satisfies

$$\left(\delta_{ij} + \frac{(u_\rho)_i(x)(u_\rho)_j(x)}{1 - |\nabla u_\rho(x)|^2}\right)(u_\rho)_{ij} = \rho, \quad \forall x \in R^n \tag{3.5}$$

and

$$|\nabla u_\rho(x)| < 1, \quad \forall x \in R^n. \tag{3.6}$$

Let $r(|x|)$ be the same solution to (1.3)–(1.4) as in Theorem 1.1, where $r$ is the unique solution of (1.5) and (1.6). Then the function

$$W(y) = W(y;\rho) := V_u(x) + \left(\delta - \frac{r(\rho\delta)}{\rho} + \frac{r(\rho|y-x|)}{\rho}\right) - \theta\delta$$

also satisfies the same (3.5)–(3.6) as $u_\rho$ for any $\rho > 0$ and any $x \in R^n$. Note that

$$W(y) = V_u(x) + (1-\theta)\delta, \quad \forall y \in \partial B(x,\delta).$$

We use (3.4) and maximum principle on the domain $B(x,\delta)$ to obtain

$$u_\rho(y) \leqslant W(y;\rho), \quad \forall y \in B(x,\delta).$$

Letting $\rho \to \infty$, we have

$$\begin{aligned} V_u(x) &\leqslant V_u(x) + (\delta - \theta\delta) + \lim_{\rho\to\infty} \frac{r(\rho|y-x|)}{\rho} - \lim_{\rho\to\infty} \frac{r(\rho\delta)}{\rho} \\ &= V_u(x) + (\delta - \theta\delta) + (|y-x| - \delta) \\ &= V_u(x) + |y-x| - \theta\delta. \end{aligned}$$

Here, in order to determine the limit, we have used the estimate

$$\sqrt{n^2+t^2} - n \leqslant r(t) \leqslant t,$$

which follows directly from (1.6) and (1.7) in Theorem 1.1. Taking $y = x$ yields

$$V_u(x) \leqslant V_u(x) - \theta\delta,$$

a contradiction. In this way, we have shown the desired lemma. □

Recall that the tangential mapping of convex function $V_u$ at a point $x_0 \in R^n$ is defined by

$$T_{V_u}(x_0) = \{\alpha \in R^n : V_u(x) \geqslant \alpha \cdot (x - x_0) + V_u(x_0), \ \forall x \in R^n\}.$$

Obviously, it is a closed, convex set and equals to $\nabla V_u(x_0)$ if $V_u$ is differential at $x_0$. The tangent cone of $u$ is defined by

$$T_{V_u}(R^n) = \bigcup_{x \in R^n} T_{V_u}(x).$$

**Lemma 3.2.** *If $u$ is a convex function satisfying* (1.4), *then its tangent cone satisfies*

$$\overline{T_{V_u}(R^n)} = T_{V_u}(0) = \overline{\nabla u(R^n)}.$$

**Proof.** To show $T_{V_u}(0) \subset \overline{\nabla u(R^n)}$, we choose $\xi \in T_{V_u}(0)$. Since $V_u(0) = 0$, $V_u(y) \geqslant \xi \cdot y$ for all $y \in R^n$. Given a $\delta > 0$. Observing that the limit

$$V_u(y) = \lim_{\rho \to 0} \frac{u(\rho y)}{\rho} = \lim_{\rho \to 0} \frac{u(\rho x) - u(0)}{\rho}$$

holds uniformly on any compact set in $R^n$, we see that

$$\phi(y) := \frac{u(\rho_\delta y) - u(0)}{\rho_\delta} - \xi \cdot y + |y|^2 \geqslant \frac{\delta^2}{2}$$

for all $y \in \partial B(0, \delta)$ and some large $\rho_\delta > 1$. But $\phi(0) = 0$, so $\phi$ attains its minimum at a point $x_\delta \in B(0, \delta)$. Thus

$$\nabla \phi(x_\delta) = \nabla u(\rho_\delta x_\delta) - \xi + 2x_\delta = 0.$$

Letting $\delta \to 0$ we get $\xi \in \overline{\nabla u(R^n)}$. Therefore, $T_{V_u}(0) \subset \overline{\nabla u(R^n)}$.

To finish the proof, we follow the arguments in [1, p. 793]. Let $\xi \in T_{V_u}(R^n)$. Then there is an $x \in R^n$ such that

$$V_u(\rho y) \geqslant \xi \cdot (\rho y - x) + V_u(x), \quad \forall y \in R^n, \ \forall \rho > 0.$$

Dividing this inequality by $\rho$, using the homogeneity of $V_u$ and then letting $\rho \to \infty$, we get

$$V_u(y) \geqslant \xi \cdot y, \quad \forall y \in R^n.$$

This means $\xi \in T_{V_u}(0)$. Thus, $\overline{T_{V_u}(R^n)} = T_{V_u}(0)$ Since $T_{V_u}(0)$ is closed.

Now for any $x \in R^n$, the convexity implies

$$u(\rho y) \geqslant \nabla u(x) \cdot (\rho y - x) + u(x), \quad \forall y \in R^n, \ \forall \rho > 0.$$

Dividing this by $\rho$, and letting $\rho \to \infty$, we see that

$$V_u(y) \geqslant \nabla u(x) \cdot y, \quad \forall y \in R^n,$$

which implies $\nabla u(x) \in T_{V_u}(0)$. Since $x$ is arbitrary and $T_{V_u}(0)$ is closed, we conclude that

$$\overline{\nabla u(R^n)} \subset T_{V_u}(0) = \overline{T_{V_u}(R^n)}.$$

This proves the Lemma.

**Proof of Theorem 1.2.** Since we have Lemma 3.1, it is enough to prove (1.12) and (1.13).

Choose $y \in \overline{\nabla u(R^n)}$. By Lemma 3.2, $y \in T_{V_u}(0)$. Because of $V_u(0) = 0$, we have

$$V_u(y) \geqslant y \cdot y = |y|^2.$$

On the other hand, Lemma 3.1 yields

$$V_u(y) \leqslant |V_u(y) - V_u(0)| \leqslant |y|.$$

Thus, we have

$$|y|^2 \leqslant V_u(y) \leqslant |y|, \quad \forall y \in \overline{\nabla u(R^n)}. \tag{3.7}$$

Hence (1.12) follows. Note that

$$\frac{u(\rho x)}{\rho} = |x| \frac{u(\rho|x| \cdot x/|x|)}{\rho|x|}$$

and $\frac{x}{|x|} \in S^{n-1}$ for $x \neq 0$. This, together with (1.12), yields (1.13).

## 4. Proof of Theorem 1.3

On the contrary that there exists a $x_1 \in R^n \backslash \Omega_0$. We will derive a contradiction. We may assume $\Omega_0$ is nonempty and connected. (Otherwise, we replace it by one of its connected components.) Then there exists a short segment $l \subset \Omega_0$ such that $\bar{l} \cap \partial\Omega_0 = \{x_1\}$ and $\varepsilon_1 = dist(l, \partial\Omega) > 0$. Take $x_2 \in l$ such that $B_\varepsilon(x_2) \subset \Omega_0$ for some $\varepsilon \in (0, \varepsilon_1)$. Translating the ball $B_\varepsilon(x_2)$ along the line $l$ we come to a point $\bar{x}$ where the ball and $\partial\Omega_0$ are touched at the first time. It follows that

$$\bar{x} \in R^n \backslash \Omega_0, \quad B_\varepsilon(x_0) \subset \Omega_0 \quad \text{and} \quad \overline{B_\varepsilon(x_0)} \cap \partial\Omega_0 = \{\bar{x}\} \tag{4.1}$$

for some $x_0 \in \Omega_0$. Moreover, the minimum eigenvalue $\lambda(x)$ of the Hessian $(u_{ij}(x))$ satisfies $\lambda(\bar{x}) = 0$. By a coordinate translation and rotation we may arrange that

$$\bar{x} = 0, \quad u(0) = 0, \quad \nabla u(0) = 0 \quad \text{and} \quad u_{11}(0) = \lambda(0) = 0. \tag{4.2}$$

Thus, the origin $0 \in \partial B_\varepsilon(x_0)$ and

$$(u_{ij}(x)) > 0 \quad \text{in } B_\varepsilon(x_0). \tag{4.3}$$

Rewrite Eq. (1.3) as

$$\Delta u = 1 + A(|\nabla u|^2) u_i u_j u_{ij} \quad \text{in } R^n, \tag{4.4}$$

where $A(t) = \frac{1}{t-1}$ is analytic for $t \in (-1, 1)$. Differentiating (4.4) twice with respect to $\frac{\partial}{\partial x_1}$, we have

$$\begin{aligned} \Delta u_{11} = {} & 4A'' u_l u_{l1} u_m u_{m1} u_i u_j u_{ij} + 2A' u_{m1} u_{m1} u_i u_j u_{ij} \\ & + 2A' u_m u_{m11} u_i u_j u_{ij} + 8A' u_m u_{m1} u_{i1} u_j u_{ij} \\ & + 4A' u_m u_{m1} u_i u_j u_{ij1} + 2A u_{i11} u_j u_{ij} \\ & + 2A u_{i1} u_{j1} u_{ij} + 4A u_{i1} u_j u_{ij1} \\ & + A u_i u_j u_{ij11} \quad \text{in } R^n. \end{aligned} \tag{4.5}$$

Since $u$ is analytic in $R^n$, we expand $u_{11}$ at $x = 0$ as a power series to obtain $u_{11}(x) = P_k(x) + R(x)$ for all $x \in \overline{B_\varepsilon(x_0)}$ (one can choose a smaller $\varepsilon$ in advance if necessary), where $P_k(x)$ is the lowest order term, which, by (4.2) and (4.3), is a nonzero homogeneous polynomial of degree $k$, and $R(x)$ is the rest. The convexity of $u$ yields $k \geqslant 2$. It follows from (4.3) that $u_{ii} u_{11} - (u_{i1})^2 > 0$ in $B_\varepsilon(x_0)$. Summing over $i$ we have

$$\Delta u u_{11} > \sum_{j=1}^{n} u_{j1}^2 \geqslant u_{i1}^2 \tag{4.6}$$

for each $i = 1, 2, \ldots, n$.

We claim that each $u_{i1}$ is of order at least $\frac{k}{2}$. Otherwise, we expand $u_{i1}$ at $x = 0$ as a power series so that the lowest order term $h(x)$ must be a nonzero homogeneous polynomial. Choose

$$a = (a_1, a_2, \ldots, a_n) \in B_\varepsilon(x_0) \backslash \{x \in B_\varepsilon(x_0) : h(x) = 0\}$$

so that the segment

$$L = \{ta : t \in (0, 1)\} \subset B_\varepsilon(x_0).$$

Now restricting (4.6) on $L$, multiplying the both sides by $t^{-k}$ and then letting $t \to 0^+$, we see the limit of the left-hand side of (4.6) is a nonzero constant multiplied by $\Delta u(0)$ which equals to 1 by (4.4), but the limit of the right-hand side is positive infinite. This is a contradiction.

Therefore, each $u_{i1}$ is of order at least $\frac{k}{2}$. Hence, $u_{ij1}$, $u_{11i}$ and $u_{11ij}$ are of order at least $\frac{k}{2} - 1$, $k - 1$ and $k - 2$, respectively. Also note that each $u_i$ is of order at least 1 by (4.2). With these facts one can check that the right-hand side of Eq. (4.5) is of order at least of $k$; while the left-hand side, $\Delta u_{11}$, is either of order $k - 2$, or $\Delta P_k = 0$ for all $x \in B_\varepsilon(x_0)$. Since the first case is impossible by comparing the orders of the two sides, we obtain that $P_k$ is a harmonic polynomial in $B_\varepsilon(x_0)$.

We claim that $P_k \geqslant 0$ for all $x \in B_\varepsilon(x_0)$. Otherwise, there exists $a = (a_1, a_2, \dots, a_n) \in B_\varepsilon(x_0)$ such that $P_k(a) < 0$. Then

$$\frac{u_{11}(ta)}{t^k} = P_k(a) + \frac{R(ta)}{t^k}, \quad \forall t \in (0, 1),$$

which implies $\lim_{t \to 0^+} \frac{u_{11}(ta)}{t^k} = P_k(a) < 0$ contradicting the fact that $u_{11} > 0$ in $B_\varepsilon(x_0)$ (see (4.3)).

Now we use the strong maximum principle to see that $P_k > 0$ for all $x \in B_\varepsilon(x_0)$. But $P_k(0) = 0$, and it follows from Hopf's lemma that $\frac{\partial P_k}{\partial \nu}(0) < 0$, where $\nu$ is the unit outward normal to the sphere $\partial B_\varepsilon(x_0)$. This means that the degree of $P_k$ is only one, contradicting the fact $k \geqslant 2$. This contradiction proves the theorem.

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