J. Symbolic Computation (2001) **32**, 597–617
doi:10.1006/jsco.2001.0485
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Computing Homomorphisms Between Holonomic D-modules

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Let $K \subseteq \mathbb{C}$ be a subfield of the complex numbers, and let D be the ring of K-linear differential operators on $R = K[x_1, \dots, x_n]$. If M and N are holonomic left D-modules we present an algorithm that computes explicit generators for the finite dimensional vector space $\mathrm{Hom}_D(M, N)$. This enables us to answer algorithmically whether two given holonomic modules are isomorphic. More generally, our algorithm can be used to get explicit generators for $\mathrm{Ext}^i_D(M, N)$ for any i in the sense of Yoneda.

1. Introduction

Let $D = K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$ denote the nth Weyl algebra over a computable subfield $K \subset \mathbb{C}$, and let $\mathrm{Hom}_D(M, N)$ denote the set of left D-module maps between two left D-modules M and N. Then $\mathrm{Hom}_D(M, N)$ is a K-vector space and can also be regarded as the solutions of M inside N in the following way: given a presentation $M \simeq D^{r_0}/D \cdot \{L_1, \dots, L_{r_1}\}$, let S denote the system of vector-valued linear partial differential equations,

$$S = \{L_1 \bullet f = \cdots = L_{r_1} \bullet f = 0\},$$

and let $\mathrm{Sol}(S; N)$ denote the N-valued solutions $f \in N^{r_0}$ to S. Then the homomorphism space $\mathrm{Hom}_D(D^{r_0}/D \cdot \{L_1, \dots, L_{r_1}\}, N)$ is isomorphic to the solution space $\mathrm{Sol}(S; N)$ where a homomorphism φ in $\mathrm{Hom}_D(D^{r_0}/D \cdot \{L_1, \dots, L_{r_1}\}, N)$ corresponds to the solution $[\varphi(e_1), \dots, \varphi(e_{r_0})]^T \in N^{r_0}$ of S, while a solution $f = [f_1, \dots, f_{r_0}]^T \in N^{r_0}$ of S corresponds to the homomorphism which sends e_i to f_i.

If M and N are holonomic (which may be determined algorithmically; see, for example, Saito *et al.*, 1999), then the set $\mathrm{Hom}_D(M, N)$ as well as the higher derived functors $\mathrm{Ext}^i_D(M, N)$ are finite-dimensional K-vector spaces. In this paper, we give algorithms that compute explicit bases for $\mathrm{Hom}_D(M, N)$ and $\mathrm{Ext}^i_D(M, N)$ in this situation. Our algorithms are a refinement of algorithms given in Oaku *et al.* (2001), which were designed to compute the dimensions of $\mathrm{Hom}_D(M, N)$ and $\mathrm{Ext}^i_D(M, N)$ over K. Algebraically, the problem of computing a basis of homomorphisms is easy to describe. Namely, since a map of left D-modules from M to N is uniquely determined by the images of a set of generators of M, we must determine which sets of elements of N constitute legal choices for the images of a fixed set of generators of M under a homomorphism. Since $\mathrm{Hom}_D(M, N)$ lacks any D-module structure in general and is just a K-vector space, this is not a straightforward computation.

In recent years, one of the fundamental advances in computational D-modules has been the development of algorithms by Oaku (1997), Oaku and Takayama (2001) to compute

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the derived restriction modules $\mathrm{Tor}_i^D(D/\{x_1, \ldots, x_d\} \cdot D, M)$ and derived integration modules $\mathrm{Tor}_i^D(D/\{\partial_1, \ldots, \partial_d\} \cdot D, M)$ of a holonomic D-module M to a linear subspace $x_1 = \cdots = x_d = 0$. These algorithms have been the basis for local cohomology and de Rham cohomology algorithms (Oaku and Takayama, 1999; Walther, 1999) and have been extended to algorithms for derived restriction and integration of complexes with holonomic cohomology in Walther (2000).

The algorithm of Oaku *et al.* (2001) to compute the dimensions of $\mathrm{Hom}_D(M, N)$ and $\mathrm{Ext}_D^i(M, N)$ is also based on restriction by using isomorphisms of Kashiwara (1978) and Björk (1979). These isomorphisms are,

$$\mathrm{Ext}_D^i(M, N) \cong \mathrm{Tor}_{n-i}^D(\mathrm{Ext}_D^n(M, D), N), \tag{1}$$

which turns an Ext computation for holonomic M into a Tor computation, and

$$\mathrm{Tor}_j^D(M', N) \simeq \mathrm{Tor}_j^{D_{2n}}(D_{2n}/\{x_i - y_i, \partial_i + \delta_i\}_{i=1}^n \cdot D_{2n}, \tau(M') \boxtimes N), \tag{2}$$

which turns any Tor computation into a twisted restriction computation in twice as many variables (an explanation of the notation used above can be found in Section 4).

In this paper, we will obtain an algorithm for computing an explicit basis of $\mathrm{Ext}_D^i(M, N)$ by analyzing the isomorphisms (1) and (2) and making them compatible with the restriction algorithm. In Section 2, we present a proof of isomorphism (1) adapted from Björk (1979). In Section 3, we give an algorithm for computing $\mathrm{Hom}_D(M, N)$ in the case $N = K[x_1, \ldots, x_n]$, which is used to compute polynomial solutions of a system S. In Section 4, we give our main result, which is an algorithm to compute $\mathrm{Hom}_D(M, N)$ and $\mathrm{Ext}_D^i(M, N)$ for general holonomic modules M, N. In Section 5, we give an algorithm to determine whether M and N are isomorphic and if so to find an isomorphism. Finally, the algorithms described in this paper have been implemented in the (The D-module package, 2000) of the computer algebra system *Macaulay 2 (1999)*.

1.1. Notation

Throughout we shall denote the ring of polynomials $K[x_1, \ldots, x_n]$ by $K[\boldsymbol{x}]$, the ring of polynomials $K[\partial_1, \ldots, \partial_n]$ by $K[\boldsymbol{\partial}]$, and the ring $K[\boldsymbol{x}]\langle\boldsymbol{\partial}\rangle$ of K-linear differential operators on $K[\boldsymbol{x}]$ by D.

Let us also explain the notation we will use to write maps of left or right D-modules. As usual, maps between finitely generated modules will be represented by matrices, but some attention has to be given to the order in which elements are multiplied due to the non-commutativity of D. Let us denote the identity matrix of size r by id_r, and similarly the identity map on the module M by id_M.

Let A be an $r \times s$ matrix $A = [a_{ij}]$ with entries in D. We get a map of free left D-modules,

$$D^r \xrightarrow{\cdot A} D^s \quad : \quad [\ell_1, \ldots, \ell_r] \mapsto [\ell_1, \ldots, \ell_r] \cdot A,$$

where D^r and D^s are regarded as modules of row vectors, and the map is matrix multiplication. Under this convention, the composition of maps $D^r \xrightarrow{\cdot A} D^s$ and $D^s \xrightarrow{\cdot B} D^t$ is the map $D^r \xrightarrow{\cdot AB} D^t$ where AB is usual matrix multiplication.

In general, suppose M and N are left D-modules with presentations D^r/M_0 and D^s/N_0. A induces a left D-module map $(D^r/M_0) \xrightarrow{\cdot A} (D^s/N_0)$ from M to N precisely when $L \cdot A \in N_0$ for all row vectors $L \in M_0$. This condition need only be checked for a generating set of M_0. Conversely, any map of left D-modules between M and N can be represented by some matrix A in the manner above.

Now let us discuss maps of right D-modules. The $r \times s$ matrix A also defines a map of right D-modules in the opposite direction,

$$(D^s)^T \xrightarrow{A\cdot} (D^r)^T \quad : \quad [\ell'_1, \ldots, \ell'_s]^T \mapsto A \cdot [\ell'_1, \ldots, \ell'_s]^T,$$

where the superscript-T means to regard the free modules $(D^s)^T$ and $(D^r)^T$ as consisting of column vectors. $(D^s)^T$ may be regarded as the dual module $\mathrm{Hom}_D(D^s, D)$. The map $(D^s)^T \xrightarrow{A\cdot} (D^r)^T$ is equivalent to the map obtained by applying $\mathrm{Hom}_D(-, D)$ to $D^r \xrightarrow{\cdot A} D^s$. We will suppress the superscript-T when the context is clear. A induces a right D-module map between right D-modules $N' = (D^s)^T/N'_0$ and $M' = (D^r)^T/M'_0$ whenever $A \cdot L \in M'_0$ for all column vectors $L \in N'_0$. We denote the map by $(D^s)^T/N'_0 \xrightarrow{A\cdot} (D^r)^T/M'_0$.

1.2. Left–right correspondence

The category of left D-modules is equivalent to the category of right D-modules, and for convenience, we will sometimes prefer to work in one category rather than the other—for instance, we will phrase all algorithms in terms of left D-modules. In the Weyl algebra, the correspondence is given by the algebra involution

$$D \xrightarrow{\tau} D \quad : \quad x^\alpha \partial^\beta \mapsto (-\partial)^\beta x^\alpha.$$

The map τ is called the standard transposition or adjoint operator. Given a left D-module D^r/M_0, the corresponding right D-module is

$$\tau\left(\frac{D^r}{M_0}\right) := \frac{D^r}{\tau(M_0)}, \qquad \tau(M_0) = \{\tau(L) | L \in M_0\}.$$

Similarly, given a homomorphism of left D-modules $\phi : D^r/M_0 \longrightarrow D^s/N_0$ defined by right multiplication by the $r \times s$ matrix $A = [a_{ij}]$, the corresponding homomorphism of right D-modules $\tau(\phi) : D^r/\tau(M_0) \longrightarrow D^s/\tau(N_0)$ is defined by right multiplication by the $s \times r$ matrix $\tau(A) := [\tau(a_{ij})]^T$. The map τ is used similarly to go from right to left D-modules. For more details, see Oaku *et al.* (2001).

1.3. Restriction and integration

Our algorithm for $\mathrm{Hom}_D(M, N)$ relies on the algorithms in Oaku and Takayama (2001) for derived restriction and integration of D-modules. We give here a brief summary of their algorithm. Let $X = K^{n+d}$ and $Y = K^n$ with coordinates $(x_1, \ldots, x_n, t_1, \ldots, t_d)$ and $(x_1, \ldots, x_n)$. Put $D_X = K\langle \boldsymbol{x}, \boldsymbol{t}, \boldsymbol{\partial_x}, \boldsymbol{\partial_t} \rangle$ and $D_Y = K\langle \boldsymbol{x}, \boldsymbol{\partial_x} \rangle$, and let M be a left D_X-module.

Definition. The ith restriction of M to Y (with respect to the inclusion $\iota : Y \to X$ where $\iota(\boldsymbol{x}) = (\boldsymbol{x}, 0)$) is equal to $\mathrm{Tor}_i^{D_X}(\Lambda_Y, M)$ as a left D_Y-module, where Λ_Y is the right D_X-module $D_X/\{t_1, \ldots, t_d\} \cdot D_X$.

Definition. The ith integration of M along Y (with respect to the projection $\pi : X \to Y$ where $\pi(\boldsymbol{x}, \boldsymbol{t}) = \boldsymbol{x}$) is equal to $\mathrm{Tor}_i^{D_X}(\Omega_Y, M)$ as a left D_Y-module, where Ω_Y is the right D_X-module $D_X/\{\partial_{t_1}, \ldots, \partial_{t_d}\} \cdot D_X$.

In principle, restriction and integration to Y can thus be computed as the homology of the Koszul complexes $\mathcal{K}^\bullet(M; t_1, \ldots, t_d)$ and $\mathcal{K}^\bullet(M; \partial_{t_1}, \ldots, \partial_{t_d})$, or as the homology of the complexes $\Lambda_Y \otimes_{D_X} P^\bullet$ and $\Omega_Y \otimes_{D_X} P^\bullet$ where $P^\bullet$ is a projective resolution of M. The problem is that the maps in $\mathcal{K}^\bullet(M; t_1, \ldots, t_d)$ and $\mathcal{K}^\bullet(M; \partial_{t_1}, \ldots, \partial_{t_d})$ are not maps of left D_X-modules while the modules in $\Lambda_Y \otimes_{D_X} P^\bullet$ and $\Omega_Y \otimes_{D_X} P^\bullet$ are no longer left D_X-modules. All of these complexes are indeed complexes of left D_Y-modules but the modules are no longer finitely generated as D_Y-modules.

Oaku and Takayama's algorithm identifies quasi-isomorphic complexes which do consist of finitely generated D_Y-modules. Let us describe the main ideas.

Definition. The V_Y-filtration $F_Y^\bullet(-)$ of a shifted free module $D^r[\mathfrak{m}]$ with respect to Y is $F_Y^i(D_X^r[\mathfrak{m}]) = \operatorname{Span}_K\{\boldsymbol{x}^\mu \boldsymbol{\partial}_{\boldsymbol{x}}^\nu \boldsymbol{t}^\alpha \boldsymbol{\partial}_{\boldsymbol{t}}^\beta \vec{e}_l : \mu, \nu, \alpha, \beta \in \mathbb{N}^n, |\beta| - |\alpha| \leq i + \mathfrak{m}_l\}$.

Definition. The $\widetilde{V}_Y$-filtration $\widetilde{F}_Y^\bullet(-)$ of a shifted free module $D^r[\mathfrak{m}]$ with respect to Y is $\widetilde{F}_Y^i(D^r[\mathfrak{m}]) = \operatorname{Span}_K\{\boldsymbol{x}^\mu \boldsymbol{\partial}_{\boldsymbol{x}}^\nu \boldsymbol{t}^\alpha \boldsymbol{\partial}_{\boldsymbol{t}}^\beta \vec{e}_l : \mu, \nu, \alpha, \beta \in \mathbb{N}^n, |\alpha| - |\beta| \leq i + \mathfrak{m}_l\}$.

The V_Y-filtration and $\widetilde{V}_Y$-filtration induce filtrations on submodules and quotients in the usual manner, and there are associated graded objects $\operatorname{gr}_Y^\bullet(-)$ and $\tilde{\operatorname{gr}}_Y^\bullet(-)$ respectively.

Definition. The b-function of M for restriction (respectively integration) to Y is the monic polynomial $b(s) \in K[s]$ of least degree, if any, which satisfies $b(\theta)\operatorname{gr}_Y^0(M) = 0$ where $\theta = t_1\partial_{t_1} + \cdots + t_d\partial_{t_d}$ (respectively $b(\widetilde{\theta})\tilde{\operatorname{gr}}_Y^0(M) = 0$ where $\widetilde{\theta} = -\theta - d$).

Definition. A free resolution $P^\bullet : \cdots \longrightarrow D_X^{r_{j+1}}[\mathfrak{m}_{j+1}] \xrightarrow{\psi_{j+1}} D_X^{r_j}[\mathfrak{m}_j] \longrightarrow \cdots$ of M is said to be V_Y-strict (respectively $\widetilde{V}_Y$-strict) if

$$\psi_{j+1}(\Gamma^i(D_X^{r_{j+1}}[\mathfrak{m}_{j+1}])) \subset \Gamma_Y^i(D_X^{r_j}[\mathfrak{m}_j]))$$

for all i and j where Γ is the V_Y-filtration F_Y (respectively the $\widetilde{V}_Y$-filtration $\widetilde{F}_Y$), and if the graded complex associated to Γ

$$\operatorname{gr}_\Gamma(P^\bullet) : \cdots \longrightarrow \operatorname{gr}_\Gamma(D_X^{r_{j+1}}[\mathfrak{m}_{j+1}]) \xrightarrow{\operatorname{gr}_\Gamma(\psi_{j+1})} \operatorname{gr}_\Gamma(D_X^{r_j}[\mathfrak{m}_j]) \longrightarrow \cdots$$

is in fact a resolution.

With these definitions, Oaku and Takayama prove the following theorem about restriction and integration. They also provide algorithms based on Gröbner bases for b-functions and strict resolutions, which makes the theorem algorithmic.

Theorem 1.1. *Let M be holonomic and let $P^\bullet$ be a V_Y-strict (respectively $\widetilde{V}_Y$-strict) resolution of M with $\mathfrak{m}_0 = 0$. Then the b-function of M for restriction (respectively integration) to Y is nonzero, and if k is its maximum integer root, then the ith restriction (respectively integration) of M is equal to the ith cohomology of*

$$\Gamma^k(\Pi \otimes_{D_X} P^\bullet) : \cdots \longrightarrow \Gamma^k(\Pi \otimes_{D_X} D_X^{r_{j+1}}[\mathfrak{m}_{j+1}]) \xrightarrow{\psi_{j+1}} \Gamma_Y^k(\Pi \otimes_{D_X} D_X^{r_j}[\mathfrak{m}_j]) \longrightarrow \cdots$$

where $\Gamma = F_Y$ and $\Pi = \Lambda_Y$ (respectively $\Gamma = \widetilde{F}_Y$ and $\Pi = \Omega_Y$).

2. Basic Isomorphism

The following identification, taken with its proof from Björk (1979), is our main theoretical tool to explicitly compute homomorphisms of holonomic D-modules.

Theorem 2.1. (Björk, 1979) *Let M and N be holonomic left D-modules. Then*

$$\mathrm{Ext}^i_D(M,N) \cong \mathrm{Tor}^D_{n-i}(\mathrm{Ext}^n_D(M,D),N). \tag{3}$$

Proof. Since it will be useful to us later, we give the main steps of the proof here. The interesting bit of the construction is the transformation of a Hom into a tensor product. Let $X^\bullet$ be a free resolution of M,

$$X^\bullet : 0 \to D^{r_{-a}} \xrightarrow{\cdot M_{-a+1}} \cdots \to D^{r_{-1}} \xrightarrow{\cdot M_0} D^{r_0} \to M \to 0.$$

We may assume it is of finite length by virtue of Hilbert's syzygy theorem—namely, Schreyer's proof and method carries over to D (see, for example, Cox *et al.*, 1998). The dual of $X^\bullet$ is the complex of right D-modules,

$$\mathrm{Hom}_D(X^\bullet, D) : 0 \leftarrow \underbrace{(D^{r_{-a}})^T}_{\text{degree } a} \xleftarrow{M_{-a+1}\cdot} \cdots \leftarrow (D^{r_{-1}})^T \xleftarrow{M_0\cdot} \underbrace{(D^{r_0})^T}_{\text{degree } 0} \leftarrow 0.$$

Since $\mathrm{Hom}_D(D^r, D) \otimes_D N \simeq \mathrm{Hom}_D(D^r, N)$, we see that $\mathrm{Hom}_D(X^\bullet, D) \otimes_D N \simeq \mathrm{Hom}_D(X^\bullet, N)$, whose cohomology groups are by definition $\mathrm{Ext}^i_D(M,N)$. Now replace N by a free resolution $Y^\bullet$ of finite length,

$$Y^\bullet : 0 \to D^{s_{-b}} \xrightarrow{\cdot N_{-b+1}} \cdots \to D^{s_{-1}} \xrightarrow{\cdot N_0} D^{s_0} \to N \to 0. \tag{4}$$

We get the double complex $\mathrm{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet$,

$$\begin{array}{ccccccccc}
 & 0 & & & & 0 & & 0 & \\
 & \uparrow & & & & \uparrow & & \uparrow & \\
0 \leftarrow & (D^{r_{-a}})^T \otimes_D D^{s_0} & \xleftarrow{(M_{-a+1}\cdot)\otimes \mathrm{id}_{s_0}} & \cdots & \leftarrow & (D^{r_{-1}})^T \otimes_D D^{s_0} & \xleftarrow{(M_0\cdot)\otimes \mathrm{id}_{s_0}} & (D^{r_0})^T \otimes_D D^{s_0} & \leftarrow 0 \\
 & \uparrow {\scriptstyle (-\mathrm{id}_{r_{-a}})^a \otimes (\cdot N_0)} & & & & {\scriptstyle -\mathrm{id}_{r_{-1}} \otimes (\cdot N_0)} \uparrow & & {\scriptstyle \mathrm{id}_{r_0} \otimes (\cdot N_0)} \uparrow & \\
0 \leftarrow & (D^{r_{-a}})^T \otimes_D D^{s_{-1}} & \xleftarrow{(M_{-a+1}\cdot)\otimes \mathrm{id}_{s_{-1}}} & \cdots & \leftarrow & (D^{r_{-1}})^T \otimes_D D^{s_{-1}} & \xleftarrow{(M_0\cdot)\otimes \mathrm{id}_{s_{-1}}} & (D^{r_0})^T \otimes_D D^{s_{-1}} & \leftarrow 0 \\
 & \uparrow & & & & \uparrow & & \uparrow & \\
 & \vdots & & & & \vdots & & \vdots & \\
 & \uparrow {\scriptstyle (-\mathrm{id}_{r_{-a}})^a \otimes (\cdot N_{-b+1})} & & & & {\scriptstyle -\mathrm{id}_{r_{-1}} \otimes (\cdot N_{-b+1})} \uparrow & & {\scriptstyle \mathrm{id}_{r_0} \otimes (\cdot N_{-b+1})} \uparrow & \\
0 \leftarrow & (D^{r_{-a}})^T \otimes_D D^{s_{-b}} & \xleftarrow{(M_{-a+1}\cdot)\otimes \mathrm{id}_{s_{-b}}} & \cdots & \leftarrow & (D^{r_{-1}})^T \otimes_D D^{s_{-b}} & \xleftarrow{(M_0\cdot)\otimes \mathrm{id}_{s_{-b}}} & (D^{r_0})^T \otimes_D D^{s_{-b}} & \leftarrow 0 \\
 & \uparrow & & & & \uparrow & & \uparrow & \\
 & 0 & & & & 0 & & 0 &
\end{array} \tag{5}$$

Since the columns of the double complex are exact except for at positions in the top row, it follows that the cohomology of the total complex equals the cohomology of the complex induced on the table of E_1 terms (vertical cohomologies),

$$0 \leftarrow \underbrace{\operatorname{Hom}_D(D^{r_{-a}}, N)}_{\text{degree } a} \xleftarrow{\operatorname{Hom}_D((M_{-a+1}\cdot), N)} \cdots \xleftarrow{\operatorname{Hom}_D((M_0\cdot), N)} \underbrace{\operatorname{Hom}_D(D^{r_0}, N)}_{\text{degree } 0} \leftarrow 0 \qquad (6)$$

which are the $\operatorname{Ext}^i_D(M, N)$.

On the other hand, since M is holonomic, the complex $\operatorname{Hom}_D(X^\bullet, D)$ is exact except in degree n, where its cohomology is by definition $\operatorname{Ext}^n_D(M, D)$. Hence the rows of the double complex are also exact except at positions in the column containing terms $(D^{r_{-n}} \otimes_D (-))$. It follows that the cohomology of the total complex also equals the cohomology of the complex induced on the other table of E_1 terms (horizontal cohomologies), which in this case is

$$0 \to \operatorname{Ext}^n_D(M, D) \otimes_D D^{s_{-b}} \to \cdots \xrightarrow{\operatorname{id}_{\operatorname{Ext}^n_D(M,D)} \otimes (\cdot N_0)} \operatorname{Ext}^n_D(M, D) \otimes_D D^{s_0} \to 0. \qquad (7)$$

By definition, the complex (7) has cohomology groups $\operatorname{Tor}^D_j(\operatorname{Ext}^n_D(M, D), N)$, which establishes the identification. □

Our goal is to compute an explicit basis of cohomology classes of the complex (6). In particular, the cohomology in degree 0 corresponds explicitly to $\operatorname{Hom}_D(M, N)$ because any map $\psi \in \operatorname{Hom}_D(D^{r_0}, N)$ which is in the degree 0 kernel, i.e. in

$$H^0(\underbrace{\operatorname{Hom}_D(D^{r_{-1}}, N)}_{\text{degree } 1} \xleftarrow{\operatorname{Hom}_D((M_0\cdot), N)} \underbrace{\operatorname{Hom}_D(D^{r_0}, N)}_{\text{degree } 0} \leftarrow 0), \qquad (8)$$

factors through $M \simeq D^{r_0}/M_0$, hence defines a homomorphism $\overline{\psi} : M \to N$. The reason why it is hard to compute these cohomology classes is that the modules $\operatorname{Hom}_D(D^{r_i}, N)$ in the complex (6) are left D-modules while the maps $\operatorname{Hom}_D((M_i\cdot), N)$ are not maps of left D-modules. In the next few sections, we will explain how the ingredients of the proof of Theorem 2.1 can be combined with the restriction algorithm to compute the desired representatives of cohomology classes.

3. Polynomial Solutions

In this section, we give an algorithm to compute $\operatorname{Hom}_D(M, K[\boldsymbol{x}])$ for holonomic M. This vector space is more efficiently computed by Gröbner deformations as described in Oaku *et al.* (2001), but we wish to discuss this special case in order to introduce the general methodology.

For $N = K[\boldsymbol{x}]$, the isomorphism (3) of Theorem 2.1 specializes to

$$\operatorname{Ext}^i_D(M, K[\boldsymbol{x}]) \simeq \operatorname{Tor}^D_{n-i}(\operatorname{Ext}^n_D(M, D), K[\boldsymbol{x}]). \qquad (9)$$

In this case, the proof of Theorem 2.1 also leads directly to an algorithm. As a D-module, the polynomial ring has the presentation $K[\boldsymbol{x}] \simeq D/D \cdot \{\partial_1, \ldots, \partial_n\}$ and can be resolved by the Koszul complex,

$$\mathcal{K}^\bullet : 0 \to \underbrace{D}_{\text{degree } n} \xrightarrow{\cdot[(-1)^{n-1}\partial_n, \ldots, \partial_1]} D^n \to \cdots \to D^n \xrightarrow{\cdot\begin{bmatrix}\partial_1 \\ \vdots \\ \partial_n\end{bmatrix}} \underbrace{D}_{\text{degree } 0} \to 0.$$

The complex (7) whose cohomology computes $\mathrm{Tor}^D_{n-i}(\mathrm{Ext}^n_D(M,D),K[\boldsymbol{x}])$ then specializes to $\mathrm{Ext}^n_D(M,D)\otimes_D \mathcal{K}^\bullet$ and is equivalently the derived integration complex of Ext^n_D (M,D) in the category of right D-modules. The integration algorithm of Oaku and Takayama (2001) can now be applied to obtain a basis of explicit cohomology classes in $H^n(\mathrm{Ext}^n_D\ (M,D)\otimes_D \mathcal{K}^\bullet)\simeq \mathrm{Tor}^D_n(\mathrm{Ext}^n_D(M,D),K[\boldsymbol{x}])$. These classes can then be transferred via the double complex (5) to cohomology classes in the complex (8), where they represent homomorphisms in $\mathrm{Hom}_D(M,K[\boldsymbol{x}])$. The method and details are probably best illustrated through an example.

Example 3.1. Consider the Gelfand–Kapranov–Zelevinsky hypergeometric system $M_A(\beta)$ associated to the matrix $A=\{1,2\}$ and parameter vector $\beta=\{5\}$. Thus, $n=2$ and $M_A(\beta)$ is the D-module associated to the equations,

$$u=\theta_1+2\theta_2-5, \qquad v=\partial_1^2-\partial_2.$$

Here, θ_i stands for the operator $x_i\partial_i$.

A free resolution for $M_A(\beta)$ is

$$X^\bullet : 0 \to D^1 \xrightarrow{\cdot[-v\ \ u+2]} D^2 \xrightarrow{\cdot\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]} D^1 \to 0$$

while a resolution for $K[x_1,x_2]$ is the Koszul complex,

$$\mathcal{K}^\bullet : 0 \to D \xrightarrow{\cdot[\partial_1,\partial_2]} D^2 \xrightarrow{\cdot\left[\begin{smallmatrix}\partial_2\\ -\partial_1\end{smallmatrix}\right]} D \to 0.$$

The augmented double complex $\mathrm{Hom}_D(X^\bullet,D)\otimes_D\mathcal{K}^\bullet$ is

$$\begin{array}{ccccccc}
 & & K[x_1,x_2] & \xleftarrow{[-v\ \ u+2]\bullet} & K[x_1,x_2]^2 & \xleftarrow{\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]\bullet} & K[x_1,x_2] \\
 & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{Ext}^2_D(M_A(\beta),D) & \longleftarrow & D^1 & \xleftarrow{[-v\ \ u+2]\cdot} & D^2 & \xleftarrow{\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]\cdot} & D^1 \\
\uparrow{\scriptstyle \cdot\left[\begin{smallmatrix}\partial_2\\ -\partial_1\end{smallmatrix}\right]} & & \uparrow{\scriptstyle \cdot\left[\begin{smallmatrix}\partial_2\\ -\partial_1\end{smallmatrix}\right]} & & \uparrow{\scriptstyle \cdot\left[\begin{smallmatrix}\partial_2 & 0\\ 0 & \partial_2\\ -\partial_1 & 0\\ 0 & -\partial_1\end{smallmatrix}\right]} & & \uparrow{\scriptstyle \cdot\left[\begin{smallmatrix}\partial_2\\ -\partial_1\end{smallmatrix}\right]} \\
\mathrm{Ext}^2_D(M_A(\beta),D)^2 & \longleftarrow & D^2 & \xleftarrow{\left[\begin{smallmatrix}-v & u+2 & 0 & 0\\ 0 & 0 & -v & u+2\end{smallmatrix}\right]\cdot} & D^4 & \xleftarrow{\left[\begin{smallmatrix}u & 0\\ v & 0\\ 0 & u\\ 0 & v\end{smallmatrix}\right]\cdot} & D^2 \\
\uparrow{\scriptstyle \cdot[\partial_1\ \ \partial_2]} & & \uparrow{\scriptstyle \cdot[\partial_1\ \ \partial_2]} & & \uparrow{\scriptstyle \cdot\left[\begin{smallmatrix}\partial_1 & 0 & \partial_2 & 0\\ 0 & \partial_1 & 0 & \partial_2\end{smallmatrix}\right]} & & \uparrow{\scriptstyle \cdot[\partial_1\ \ \partial_2]} \\
\mathrm{Ext}^2_D(M_A(\beta),D) & \longleftarrow & \boxed{D^1} & \xleftarrow{[-v\ \ u+2]\cdot} & D^2 & \xleftarrow{\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]\cdot} & D^1
\end{array}$$

Here, we interpret an element of a module in the above diagram as a column vector for purposes of the horizontal maps and as a row vector for purposes of the vertical maps. The induced complex at the left-hand wall is the derived integration to the origin of $\mathrm{Ext}^2_D(M_A(\beta),D)$ in the category of right D-modules. Applying the integration algorithm, we find that the cohomology at the module $\boxed{D^1}$ in the bottom left-hand corner is one-dimensional and spanned by the residue class of

$$L_{1,0}=-(2x_1^5x_2-40x_1^3x_2^2+120x_1x_2^3)\partial_1-(x_1^6-30x_1^4x_2+180x_1^2x_2^2-120x_2^3).$$

We lift this class to a cohomology class of the complex induced in the top row via a "transfer" sequence in the total complex given schematically by

$$
\begin{array}{ccccc}
 & & D^2 & \xleftarrow{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\cdot} & D^1 \ni L_{1,2} \\
 & & \Big\uparrow {\scriptstyle \cdot\left[\begin{smallmatrix} \partial_2 & 0 \\ 0 & \partial_2 \\ -\partial_1 & 0 \\ 0 & -\partial_1 \end{smallmatrix}\right]} & & \\
D^2 & \xleftarrow{\left[\begin{smallmatrix} -v & u+2 & 0 & 0 \\ 0 & 0 & -v & u+2 \end{smallmatrix}\right]\cdot} & D^4 \ni L_{1,1} & & \\
\Big\uparrow {\scriptstyle \cdot[\partial_1 \ \partial_2]} & & & & \\
D^1 \ni L_{1,0} & & & &
\end{array}
$$

In other words, $L_{1,1}$ is obtained by taking the image of $L_{1,0}$ under the vertical map and then a pre-image under the horizontal map, and similarly for $L_{1,2}$. We find that,

$$
\begin{aligned}
L_{1,1} &= \begin{bmatrix} 2x_1^5x_2 - 40x_1^3x_2^2 + 120x_1x_2^3 \\ -(x_1^5 - 20x_1^3x_2 + 60x_1x_2^2) \\ -(x_1^6 - 20x_1^4x_2 + 60x_1^2x_2^2) \\ (x_1^5 - 20x_1^3x_2 + 60x_1x_2^2)\partial_1 + (10x_1^4 - 120x_1^2x_2 + 120x_2^2) \end{bmatrix}, \\
L_{1,2} &= \left[\ x_1^5 - 20x_1^3x_2 + 60x_1x_2^2\ \right].
\end{aligned}
$$

(The elements $L_{1,0}$, $L_{1,1}$ and $L_{1,2}$ are, as opposed to the cohomology classes of $L_{1,0}$ in $\mathrm{Ext}_D^2(M_A(\beta), D)$ and of $L_{2,1}$ in $K[x_1, x_2]$, not unique.)

The space of polynomial solutions is spanned by the residue class of $L_{1,2}$ in $K[x_1, x_2]$, which is $x_1^5 - 20x_1^3x_2 + 60x_1x_2^2$.

Remark. The transfer sequence above is used to show that Tor is a balanced functor in Weibel (1994). A generalization of the transfer sequence is also used to compute the cup product structure for de Rham cohomology of the complement of an affine variety in Walther (2001).

From a practical standpoint, the method outlined above is not quite the final story. The detail we have left out is how the integration algorithm of Oaku and Takayama (1999) actually computes the cohomology classes of a Koszul complex such as $\mathrm{Ext}_D^n(M, D) \otimes_D \mathcal{K}^\bullet$. Their algorithm does not compute these classes directly. Rather, their method (phrased in terms of right D-modules) is to first compute a $\tilde{V}$-strict resolution $Z^\bullet$ of $\mathrm{Ext}_D^n(M, D)$ (more details about the $\tilde{V}$-filtration can be found in Walther, 2001). Then they give a technique to compute explicitly the cohomology classes of $Z^\bullet \otimes_D K[\boldsymbol{x}]$. This complex is quasi-isomorphic to $\mathrm{Ext}_D^n(M, D) \otimes_D \mathcal{K}^\bullet$, and cohomology classes can be transferred to $\mathrm{Ext}_D^n(M, D) \otimes_D \mathcal{K}^\bullet$ by setting up another double complex $Z^\bullet \otimes_D \mathcal{K}^\bullet$. Thus, our method as described to compute polynomial solutions would require two transfers via two double complexes.

Given the true nature of the integration algorithm, the two transfers can be collapsed into a single step. Namely, we start with $\mathrm{Hom}_D(X^\bullet, D)$,

$$
\mathrm{Hom}_D(X^\bullet, D) : 0 \leftarrow \cdots \xleftarrow{M_{-n}\cdot} \underbrace{(D^{r_{-n}})^T}_{\text{degree } n} \xleftarrow{M_{-n+1}\cdot} \cdots \xleftarrow{M_0\cdot} \underbrace{(D^{r_0})^T}_{\text{degree } 0} \leftarrow 0
$$

which is exact except in cohomological degree n because M is holonomic. We are interested in explicit cohomology classes for $H^0(\mathrm{Hom}_D(X^\bullet, D) \otimes_D \mathcal{K}[\boldsymbol{x}])$. To obtain them, we replace $\mathrm{Hom}_D(X^\bullet, D)$ with a quasi-isomorphic $\widetilde{V}$-adapted resolution $E^\bullet$ along with an explicit quasi-isomorphism $\pi_\bullet$ from $E^\bullet$ to $\mathrm{Hom}_D(X^\bullet, D)$. That is, we make a map π_n from a free module $(D^{s_{-n}})^T$ onto some choice of generators of $\ker(M_{-n}\cdot)$, take the preimage P of $\mathrm{im}(M_{-n+1}\cdot)$ under π_n, and compute a $\widetilde{V}$-adapted resolution $E^\bullet$ of $D^{s_{-n}}/P$. Schematically,

$$\begin{array}{ccccccccc}
0 \leftarrow \dfrac{(D^{s_{-n}})^T}{P} & \longleftarrow & (D^{s_{-n}})^T & \xleftarrow{N_{-n+1}\cdot} & (D^{s_{-n+1}})^T \cdots & \xleftarrow{N_0\cdot} & (D^{s_0})^T \leftarrow (D^{s_1})^T & \leftarrow \cdots \\
 & & \downarrow \pi_n & & \downarrow & & \downarrow & \\
0 \leftarrow \cdots \leftarrow (D^{r_{-n-1}})^T & \xleftarrow{M_{-n}\cdot} & \underbrace{(D^{r_{-n}})^T}_{\text{degree } n} & \xleftarrow{M_{-n+1}\cdot} & (D^{r_{-n+1}})^T \cdots & \xleftarrow{M_0\cdot} & \underbrace{(D^{r_0})^T}_{\text{degree } 0} & \leftarrow 0
\end{array}$$

Using the integration algorithm, the cohomology classes of the top row can now be computed. In order to transfer them to $\mathrm{Hom}_D(X^\bullet, D) \otimes_D \mathcal{K}[\boldsymbol{x}]$, a chain map lifting π_n is computed and utilized as suggested by the dashed arrows.

4. Holonomic Solutions

In this section, we give an algorithm to compute a basis of $\mathrm{Hom}_D(M, N)$ for holonomic left D-modules M and N. We will use the following notation. As before, D will denote the ring of differential operators in the variables $x_1, \ldots, x_n$ with derivations $\partial_1, \ldots, \partial_n$. Occasionally we will write D_x for D. In a similar fashion, D_y will stand for the ring of differential operators in the variables $y_1, \ldots, y_n$ with derivations $\delta_1, \ldots, \delta_n$.

If X is a D_x-module and Y a D_y-module then we denote by $X \boxtimes Y$ the external product of X and Y over K. It equals the tensor product of X and Y over the field K, equipped with its natural structure as a module over $D_{2n} = D_x \boxtimes D_y$, the ring of differential operators in $x_1, \ldots, x_n, y_1, \ldots, y_n$ with derivations $\{\partial_i, \delta_j\}_{1 \le i,j \le n}$. In addition, let η denote the algebra isomorphism,

$$\eta : D_{2n} \longrightarrow D_{2n} \qquad \left\{ \begin{array}{ll} x_i \mapsto \frac{1}{2}x_i - \delta_i, & \partial_i \mapsto \frac{1}{2}y_i + \partial_i, \\ y_i \mapsto -\frac{1}{2}x_i - \delta_i, & \delta_i \mapsto \frac{1}{2}y_i - \partial_i \end{array} \right\}_{i=1}^{n},$$

and let Δ and Λ denote the right D_{2n}-modules,

$$\Delta := \frac{D_{2n}}{\{x_i - y_i, \partial_i + \delta_i : 1 \le i \le n\} \cdot D_{2n}}, \qquad \Lambda := \frac{D_{2n}}{\boldsymbol{x}D_{2n} + \boldsymbol{y}D_{2n}} = \eta(\Delta).$$

An algorithm to compute the dimensions of $\mathrm{Ext}^i_D(M, N)$ was given in Oaku *et al.* (2001) based upon the K-isomorphisms (1) and (2):

$$\mathrm{Ext}^i_D(M, N) \cong \mathrm{Tor}^D_{n-i}(\mathrm{Ext}^n_D(M, D), N),$$
$$\mathrm{Tor}^D_{n-i}(M', N) \cong \mathrm{Tor}^{D_{2n}}_{n-i}(D_{2n}/\{x_i - y_i, \partial_i + \delta_i\}_{i=1}^n \cdot D_{2n}, \tau(M') \boxtimes N).$$

Combining these isomorphisms where $M' = \mathrm{Ext}^n_D(M, D)$ produces

$$\mathrm{Ext}^i_D(M, N) \simeq \mathrm{Tor}^{D_{2n}}_{n-i}(D_{2n}/\{x_i - y_i, \partial_i + \delta_i\}_{i=1}^n \cdot D_{2n}, \tau(\mathrm{Ext}^n_D(M, D)) \boxtimes N). \qquad (10)$$

In order to compute $\mathrm{Hom}_D(M, N)$ explicitly, we will trace the isomorphism (10). We explain how to do this step by step in the following algorithm. The motivation behind the algorithm is discussed in the proof.

ALGORITHM 4.1. (HOLONOMIC SOLUTIONS BY DUALITY)

INPUT: Presentations $M = D^{r_0}/M_0$ and $N = D^{s_0}/N_0$ of holonomic left D-modules.
OUTPUT: A basis for $\mathrm{Hom}_D(M, N)$.

(1) Compute finite free resolutions $X^\bullet$ and $Y^\bullet$ of M and N,

$$X^\bullet : 0 \to \underbrace{D^{r_{-a}}}_{\text{degree } -a} \xrightarrow{\cdot M_{-a+1}} \cdots \to D^{r_{-1}} \xrightarrow{\cdot M_0} \underbrace{D^{r_0}}_{\text{degree } 0} \to M \to 0.$$

$$Y^\bullet : 0 \to \underbrace{D^{s_{-b}}}_{\text{degree } -b} \xrightarrow{\cdot N_{-b+1}} \cdots \to D^{s_{-1}} \xrightarrow{\cdot N_0} \underbrace{D^{s_0}}_{\text{degree } 0} \to N \to 0.$$

Also, dualize $X^\bullet$ and apply the standard transposition to obtain

$$\tau(\mathrm{Hom}_D(X^\bullet, D)) : 0 \leftarrow \underbrace{D^{r_{-a}}}_{\text{degree } a} \xleftarrow{\cdot\tau(M_{-a+1})} \cdots \leftarrow D^{r_{-1}} \xleftarrow{\cdot\tau(M_0)} \underbrace{D^{r_0}}_{\text{degree } 0} \leftarrow 0.$$

(2) Form the double complex $\tau(\mathrm{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet$ of left D_{2n}-modules and its total complex

$$Z^\bullet : 0 \leftarrow \underbrace{{D_{2n}}^{t_a}}_{\text{degree } a} \leftarrow \cdots \leftarrow \underbrace{{D_{2n}}^{t_0}}_{\text{degree } 0} \leftarrow \cdots \leftarrow {D_{2n}}^{t_{-b}} \leftarrow 0$$

where

$${D_{2n}}^{t_k} = \bigoplus_{i-j=k} D^{r_{-i}} \boxtimes D^{s_{-j}}.$$

Let the part of $Z^\bullet$ in cohomological degree n be denoted

$${D_{2n}}^{t_{n+1}} \xleftarrow{\cdot T_n} {D_{2n}}^{t_n} \xleftarrow{\cdot T_{n-1}} {D_{2n}}^{t_{n-1}}.$$

(3) Compute a surjection $\pi_n : {D_{2n}}^{u_n} \twoheadrightarrow \ker(\cdot\eta(T_n))$, and find the preimage $P := \pi_n^{-1}$ $(\mathrm{im}(\cdot\eta(T_{n-1})))$.

(4) Compute the derived restriction module $H^0((\Lambda \otimes^L_{D_{2n}} {D_{2n}}^{u_n}/P)[n])$ using the restriction algorithm of Oaku and Takayama (2001). In particular, this algorithm produces,

(i) A V-strict free resolution $E^\bullet$ of $D_{2n}^{u_n}/P$ of length $n+1$,

$$E^\bullet : 0 \leftarrow \underbrace{{D_{2n}}^{u_n}}_{\text{degree } n} \leftarrow {D_{2n}}^{u_{n-1}} \leftarrow \cdots \leftarrow {D_{2n}}^{u_1} \leftarrow \underbrace{{D_{2n}}^{u_0}}_{\text{degree } 0} \leftarrow {D_{2n}}^{u_{-1}}.$$

(ii) Elements $\{g_1, \ldots, g_k\} \subset {D_{2n}}^{u_0}$ whose images in $\Lambda \otimes_{D_{2n}} E^\bullet$ form a basis for

$$H^0\left(\left(\Lambda \otimes^L_{D_{2n}} \frac{{D_{2n}}^{u_n}}{P}\right)[n]\right)$$
$$\simeq H^0(\Lambda \otimes_{D_{2n}} E^\bullet) \simeq \frac{\ker(\Lambda \otimes_{D_{2n}} {D_{2n}}^{u_1} \leftarrow \Lambda \otimes_{D_{2n}} {D_{2n}}^{u_0})}{\mathrm{im}(\Lambda \otimes_{D_{2n}} {D_{2n}}^{u_0} \leftarrow \Lambda \otimes_{D_{2n}} {D_{2n}}^{u_{-1}})}$$

(5) Lift π_n to a chain map $\pi_\bullet : E^\bullet \to \eta(Z^\bullet)$ with $\pi_i : D_{2n}^{u_i} \to D_{2n}^{t_i}$.

(6) Compute the image of each g_i under the composition of chain maps,

$$\begin{array}{ccccc}
E^\bullet & & \Delta \otimes_{D_{2n}} Z^\bullet & \xrightarrow{\simeq} & \mathrm{Tot}^\bullet(\mathrm{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet) \\
\Big\downarrow \pi^\bullet & & \Big\uparrow & & \Big\downarrow p_1 \\
\eta(Z^\bullet) & \xrightarrow{\eta^{-1}} & Z^\bullet & & \mathrm{Hom}_D(X^\bullet, N)
\end{array}$$

Here p_1 is the projection onto $\mathrm{Hom}_D(X^\bullet, D) \otimes Y^0$ followed by factorization through N_0. These are all chain maps of complexes of vector spaces. Step by step, we do the following. Evaluate $\{L_1 = \eta^{-1}(\pi_0(g_1)), \ldots, L_k = \eta^{-1}(\pi_0(g_k))\}$, and write each L_i in terms of the decomposition,

$$L_i = \oplus_j L_{i,j} \in \bigoplus_j D^{r-j} \boxtimes D^{s-j} \qquad \left(= {D_{2n}}^{t_0}\right).$$

Now re-express $L_{i,0}$ modulo $\{x_i - y_i, \partial_i + \delta_i : 1 \le i \le n\} \cdot D_{2n} \otimes_{D_{2n}} (D^{r_0} \boxtimes D^{s_0})$ so that x_i and ∂_j do not appear in any component. Using the identification $D^{r_0} \boxtimes D^{s_0} \simeq D_{2n}^{s_0} e_1 \oplus \cdots \oplus D_{2n}^{s_0} e_{r_0}$, where $\{e_i\}$ forms the canonical D-basis for D^{r_0}, we then get an expression

$$L_{i,0} = \ell_{i,1} e_1 + \cdots + \ell_{i,r_0} e_{r_0} \in (D_y)^{s_0} e_1 \oplus \cdots \oplus (D_y)^{s_0} e_{r_0}.$$

Let $\{\bar{\ell}_{i,1}, \ldots, \bar{\ell}_{i,r_0}\}$ be the images in $(D^{s_0}/N_0) \simeq N$. Finally, set $\phi_i \in \mathrm{Hom}_D(M, N)$ to be the map induced by

$$\{e_1 \mapsto \bar{\ell}_{i,1}, e_2 \mapsto \bar{\ell}_{i,2}, \ldots, e_{r_0} \mapsto \bar{\ell}_{i,r_0}\}.$$

(7) Return $\{\phi_1, \ldots, \phi_k\}$, a basis for $\mathrm{Hom}_D(M, N)$.

Proof. The main idea behind the algorithm is to adapt the proof of Theorem 2.1. In that proof, we saw that $\mathrm{Tot}^\bullet(\mathrm{Hom}(X^\bullet, D) \otimes_D Y^\bullet) \xrightarrow{p_1} \mathrm{Hom}_D(X^\bullet, N)$ is a quasi-isomorphism. Thus it suffices to compute explicit generating classes for

$$H^0(\mathrm{Tot}^\bullet(\mathrm{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet)) \xrightarrow{\simeq} H^0(\mathrm{Hom}_D(X^\bullet, N)) \simeq \mathrm{Hom}_D(M, N).$$

Here, the double complex $\mathrm{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet$ is in some sense easier to digest because it consists entirely of free D-modules. However, it too only carries the structure of a complex of (infinite dimensional) K-vector spaces, making its cohomology no easier to compute than the cohomology of $\mathrm{Hom}_D(X^\bullet, N)$.

Instead we consider the double complex $\tau(\mathrm{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet$ of Step 2, whose total complex $T^\bullet$ does carry the structure of a complex of left D_{2n}-modules. Moreover, we claim that as a double complex of vector spaces, $\mathrm{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet$ can be naturally identified with the double complex,

$$\Delta \otimes_{D_{2n}} (\tau(\mathrm{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet),$$

the "restriction to the diagonal". To make the identification, first note that the natural map

$$D_y \longrightarrow \frac{D_{2n}}{\{x_i - y_i, \partial_i + \delta_i : 1 \le i \le n\} \cdot D_{2n}} = \Delta$$

is an isomorphism of left D_y-modules. Let $\{e_1, \ldots, e_r\}$ denote the canonical basis of a free module D^r. Then an arbitrary element of $\Delta \otimes_{D_{2n}} ({D_x}^r \boxtimes {D_y}^s)$ can be expressed uniquely as $\sum_k e_k \boxtimes m_k$, where $m_k \in {D_y}^s$. Similarly, an element of $D^r \otimes_D D^s$ can be expressed uniquely as $\sum_k e_k \otimes m_k$ where $m_k \in D^s$. Hence we get an isomorphic identification as D_n-modules of $\Delta \otimes_{D_{2n}} ({D_x}^r \boxtimes {D_y}^s)$ and $D^r \otimes_D D^s$. In particular, this shows that the modules appearing in the double complexes are the same.

It remains to show that the maps in the double complexes can also be identified. An arbitrary vertical map of $\Delta \otimes_{D_{2n}} (\tau(\mathrm{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet)$ acts on an arbitrary element $\sum_k 1 \otimes e_k \boxtimes m_k$ according to,

$$\begin{array}{ccc} \Delta \otimes_{D_{2n}} ({D_x}^{r_i} \boxtimes {D_y}^{s_j}) & & \sum_i (-1)^i e_k \boxtimes (\cdot N_j)(m_k) \\ \Big\uparrow {\scriptstyle \mathrm{id}_\Delta \otimes (-\mathrm{id}_{r_i})^i \boxtimes (\cdot N_j)} & & \Big\uparrow \\ \Delta \otimes_{D_{2n}} ({D_x}^{r_i} \boxtimes {D_y}^{s_{j+1}}) & & \sum_k 1 \otimes e_k \boxtimes m_k \end{array}$$

This is exactly the way the corresponding vertical map in $\mathrm{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet$ works on the corresponding element:

$$\begin{array}{ccc} {D_x}^{r_i} \otimes_D {D_y}^{s_j} & & \sum_k (-1)^i e_k \otimes (\cdot N_j)(m_k) \\ \Big\uparrow {\scriptstyle (-\mathrm{id}_{r_i})^i \otimes (\cdot N_j)} & & \Big\uparrow \\ {D_x}^{r_i} \otimes_D {D_y}^{s_{j+1}} & & \sum_k e_k \otimes m_k \end{array}$$

Likewise, an arbitrary horizontal map of $\Delta \otimes_{D_{2n}} (\tau(\mathrm{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet)$ acts on an arbitrary element according to,

$$\begin{array}{ccc} \Delta \otimes_{D_{2n}} ({D_x}^{r_{i+1}} \boxtimes {D_y}^{s_j}) & \xrightarrow{\mathrm{id}_\Delta \otimes (\cdot \tau(M_i)) \boxtimes 1} & \Delta \otimes_{D_{2n}} ({D_x}^{r_i} \boxtimes {D_y}^{s_j}) \\ \sum_k 1 \otimes e_k \boxtimes m_k & \longrightarrow & \sum_k 1 \otimes (\cdot \tau(M_i))(e_k) \boxtimes m_k. \end{array}$$

Here, we would like to re-express the image $\sum_k 1 \otimes (\cdot \tau(M_i))(e_k) \boxtimes m_k$ in the form $\sum_k 1 \otimes e_k \boxtimes n_k$. To help us, note the following computation in $\Delta \otimes_{D_{2n}} ({D_x}^r \boxtimes {D_y}^s)$:
$(1 \otimes x^\alpha \partial^\beta e_i \boxtimes m) = 1 \otimes \partial^\beta e_i \boxtimes y^\alpha m = 1 \otimes e_i \boxtimes (-\delta)^\beta y^\alpha m = 1 \otimes e_i \boxtimes \tau(y^\alpha \delta^\beta) m$.
Using it, we get that

$$\begin{aligned} \sum_k 1 \otimes (\cdot \tau(M_i))(e_k) \boxtimes m_k &= \sum_k \sum_j 1 \otimes \tau(M_i)_{jk} e_j \boxtimes m_k \\ &= \sum_k \sum_j 1 \otimes e_j \boxtimes \tau(\tau(M_i)_{jk}) m_k \\ &= \sum_k \sum_j 1 \otimes e_j \boxtimes (M_i)_{jk} m_k. \end{aligned}$$

This is exactly the way the corresponding horizontal map in $\mathrm{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet$ works on an arbitrary element:

$$\begin{array}{ccc} D^{r_{i+1}} \otimes_D D^{s_j} & \xrightarrow{(M_i \cdot) \otimes \mathrm{id}_{s_j}} & D^{r_i} \otimes_D D^{s_j} \\ \sum_k e_k \otimes m_k & \longrightarrow & \sum_k \sum_j e_j \otimes (M_i)_{jk} m_k. \end{array}$$

Thus, we have given an explicit identification of $\Delta \otimes_D (\tau(\mathrm{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet)$ and $\mathrm{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet$.

The task now becomes to compute explicit cohomology classes which are a basis for $H^0(\Delta \otimes_{D_{2n}} Z^\bullet)$. To do this, we note that $Z^\bullet$ is exact except in cohomological degree n, where its cohomology is $\tau(\mathrm{Ext}^n_D(M, D)) \boxtimes N$. This follows because $\tau(\mathrm{Hom}_D(X^\bullet, D))$ is exact by holonomicity except in degree n, where its cohomology is $\tau(\mathrm{Ext}^n_D(M, D))$, and $Y^\bullet$ is exact except in degree 0, where its cohomology is N. Thus, the complex $\Delta \otimes_{D_{2n}} Z^\bullet$ is essentially a restriction complex: after applying the algebra isomorphism η, we get an honest restriction complex $\Lambda \otimes \eta(Z^\bullet)$ for the restriction of $\eta(\tau(\mathrm{Ext}^n_D(M, D)) \boxtimes N)$ to the origin (the restriction complex of a left D_{2n}-module M' is by definition $\Lambda \otimes^L_{D_{2n}} M'$).

We can therefore compute the cohomology groups of $\Lambda \otimes_{D_{2n}} \eta(Z^\bullet)$ by applying the restriction algorithm. Since we are after explicit representatives for the cohomology classes, we need to use a presentation of $\eta(\tau(\mathrm{Ext}^n_D(M, D)) \boxtimes N)$ which is compatible with $\eta(Z^\bullet)$. This is the content of Step 3. Once equipped with a compatible presentation, we apply the restriction algorithm to it, which is the content of Step 4. This step produces explicit cohomology classes of $\Lambda \otimes_{D_{2n}} E^\bullet$, where $E^\bullet$ is a V-strict resolution of $\eta(\tau(\mathrm{Ext}^n_D(M, D)) \boxtimes N)$. To then get explicit cohomology classes of $\Lambda \otimes_{D_{2n}} \eta(Z^\bullet)$, we construct a chain map between $E^\bullet$ and $\eta(Z^\bullet)$, which is the content of Step 5. The cohomology classes can now be transported to $\Lambda \otimes_{D_{2n}} \eta(Z^\bullet)$ using the chain map, then to $\Delta \otimes_{D_{2n}} Z^\bullet$ using η^{-1}, then to $\mathrm{Tot}^\bullet(\mathrm{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet)$ using the natural identification described earlier, and finally to the complex $\mathrm{Hom}_D(X^\bullet, N)$ using the natural augmentation map. These steps are all grouped together in Step 6. This completes the proof of the algorithm. □

EXAMPLE 4.2. Let $M = D/D \cdot (\partial - 1)$ and $N = D/D \cdot (\partial - 1)^2$, where D is the first Weyl algebra. Then for Step 1, we have the resolutions,

$$X^\bullet : 0 \to D^1 \xrightarrow{\cdot(\partial-1)} D^1 \to 0 \qquad Y^\bullet : 0 \to D^1 \xrightarrow{\cdot(\partial-1)^2} D^1 \to 0.$$

For Step 2, we form the complex $Z^\bullet = \mathrm{Tot}(\tau(\mathrm{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet)$,

$$Z^\bullet : 0 \leftarrow \underbrace{{D_2}^1}_{\text{degree } 1} \xleftarrow{\cdot\begin{bmatrix}(\partial_x+1)\\(\partial_y-1)^2\end{bmatrix}} \underbrace{{D_2}^2}_{\text{degree } 0} \xleftarrow{\cdot[(\partial_y-1)^2, -(\partial_x+1)]} \underbrace{{D_2}^1}_{\text{degree } -1} \leftarrow 0.$$

For Steps 3–5, we get the output,

$$\begin{array}{ccccccc}
\eta(Z^\bullet) : 0 \leftarrow & {D_2}^1 & \xleftarrow{\cdot\begin{bmatrix}\frac{1}{2}y+\partial_x+1\\(\frac{1}{2}y-\partial_x-1)^2\end{bmatrix}} & {D_2}^2 & \xleftarrow{\cdot[(\frac{1}{2}y-\partial_x-1)^2, -\frac{1}{2}y-\partial_x-1]} & {D_2}^1 & \leftarrow 0 \\
& \uparrow {\scriptstyle \pi_1 = \cdot[1]} & & \uparrow {\scriptstyle \pi_0 = \cdot\begin{bmatrix}1 & 0\\ \frac{3}{2}y-\partial_x-1 & 1\end{bmatrix}} & & & \\
E^\bullet : 0 \leftarrow & {D_2}^1[0] & \xleftarrow{\cdot\begin{bmatrix}\frac{1}{2}y+\partial_x+1\\y^2\end{bmatrix}} & {D_2}^2[-1,2] & \xleftarrow{\cdot[y^2, -\frac{1}{2}y-\partial_x-1]} & {D_2}^1[1] & \leftarrow 0.
\end{array}$$

The complex $E^\bullet$ is a V-strict resolution of the cohomology of $\eta(Z^\bullet)$ at degree 1, and the restriction b-function is $b(s) = (s + 1)(s + 2)$. Hence $\Lambda \otimes_D E^\bullet$ is quasi-isomorphic to its sub-complex $F^{-1}(\Lambda \otimes_D E^\bullet)$

$$0 \leftarrow 0 \xleftarrow{\cdot\begin{bmatrix}\frac{1}{2}y+\partial_x+1\\ y^2\end{bmatrix}} \mathrm{Span}_K \left\{ \begin{matrix} 0 \oplus \overline{1} \\ 0 \oplus \overline{\partial_x} \\ 0 \oplus \overline{\partial_y} \end{matrix} \right\} \xleftarrow{\cdot[y^2, -\frac{1}{2}y-\partial_x-1]} \mathrm{Span}_K\{\overline{1}\} \leftarrow 0.$$

Hence the cohomology $H^0(\Lambda \otimes_D E^\bullet)$ is spanned by $\{0\oplus\overline{1}, 0\oplus\overline{\partial_y}\}$. Applying π_0, $H^0(\Lambda\otimes_D \eta(Z^\bullet))$ is spanned by the images of $\{(\frac{3}{2}y-\partial_x-1)\oplus 1, \partial_y(\frac{3}{2}y-\partial_x-1)\oplus\partial_y\}$. Next applying η^{-1}, $H^0(\Delta\otimes_D Z^\bullet)$ is spanned by the images of $\{L_1 = (\partial_x+2\partial_y-1)\oplus 1, L_2 = -\frac{1}{2}(x\partial_x + 2y\partial_y + y\partial_x + 2x\partial_y - x - y) \oplus -\frac{1}{2}(x+y)\}$. Modulo the right ideal generated by $\{x-y, \partial_x+\partial_y\}$, we can re-express these cohomology classes by $\{(\partial_y-1)\oplus 1, (y\partial_y-y-1)\oplus -y\}$. Applying p_1 we get $\{L_{1,0} = \partial_y - 1, L_{2,0} = y\partial_y - y - 1\}$, which corresponds to a basis of $\mathrm{Hom}_D(M,N)$ given by,

$$\phi_1 : \frac{D}{D\cdot(\partial-1)} \xrightarrow{\cdot[\partial-1]} \frac{D}{D\cdot(\partial-1)^2}$$

$$\phi_2 : \frac{D}{D\cdot(\partial-1)} \xrightarrow{\cdot[(x\partial-x-1]} \frac{D}{D\cdot(\partial-1)^2}.$$

Example 4.3. Consider the Weyl algebra $D = K\langle x, y, \partial_x, \partial_y\rangle$ and the modules $M = D/D\cdot\{x^2-y^3, 3y^2\partial_x - 2x\,\partial_y\}$ and $N = D/D\cdot\{x,y\}$. Let X stand for the vector field $x\,\partial_x/2 + y\,\partial_y/3$. As a D-module, M is isomorphic to $D[s]/D[s]\cdot\{x^2-y^3, 3y^2\partial_x - 2x\,\partial_y, X - s\}$, which can be interpreted as the module $D[s]\bullet F^s/D[s]\cdot F\bullet F^s$ where $F = x^2 - y^3$ and $D[s]$ operates on F^s by "the power rule" (see, for example, Björk, 1979, Chapter 1). The module M has an s-action defined by $s\cdot\overline{P} = \overline{P\cdot X}$.

The module N can be interpreted as the D-module of δ-functions supported at the point $(0,0)$, which happens to coincide with the singular locus of x^2-y^3. If $f \in \mathrm{Hom}_D(M,N)$, then one can define $s\cdot f$ by $(s\cdot f)(\overline{P}) = f(\overline{PX})$. This gives $\mathrm{Hom}_D(M,N)$ a $K[s]$-structure.

Since $\mathrm{Hom}_D(M,N)$ is a finite dimensional vector space, it has has a minimal polynomial $b^2(s)$. A theorem of Kashiwara asserts that the set of roots of this minimal polynomial, together with -1, coincides with the set of roots of the Bernstein–Sato polynomial of x^2-y^3. (This does not take into account multiplicities, although some statements can be made about that too—for more details, see Yano, 1978, Section 9.)

With *Macaulay 2* one computes $\mathrm{Hom}_D(M,N)$ to be a two-dimensional vector space, spanned by $f : f(\overline{P}) = \overline{P}$ and $g : g(\overline{P}) = \overline{P\partial_y}$. Let us investigate the minimal polynomial $b^2(s)$. Since

$$(s\cdot f)(\overline{P}) = f(\overline{PX}) = \overline{P\cdot(x\,\partial_x/2 + y\,\partial_y/3)} = -\frac{5}{6}\overline{P}$$

we conclude that $(s+5/6)$ kills f. Similarly one calculates that $(s+7/6)$ kills g. Hence the Bernstein–Sato polynomial of x^2-y^3 has roots $-5/6, -1, -7/6$. (It turns out that in this case each root has multiplicity one, which one could also deduce from the statements in Yano, 1978.)

Remark. Algorithm 4.1 for the computation of $\mathrm{Hom}_D(M,N)$ can also be modified to compute explicitly the higher derived functors $\mathrm{Ext}^i_D(M,N)$ for holonomic M and N.

A useful way to represent $\mathrm{Ext}^i_D(M,N)$ is as the ith Yoneda Ext group, which consists of equivalence classes of exact sequences,

$$\xi : 0 \to N \longrightarrow Q \longrightarrow X^{-i+2} \longrightarrow \cdots \longrightarrow X^0 \longrightarrow M \longrightarrow 0,$$

for any list of (not necessarily free) D-modules $Q, X^{-i+2}, \ldots, X^0$. Two exact sequences ξ and ξ' are considered equivalent when there is a chain map of the form,

$$\begin{array}{ccccccccccc}
\xi : 0 \longrightarrow & N & \longrightarrow & Q & \longrightarrow & X^{-i+2} \longrightarrow \cdots \longrightarrow & X^0 & \longrightarrow & M & \longrightarrow 0 \\
 & \downarrow{\scriptstyle \mathrm{id}_N} & & \downarrow & & \downarrow \quad \cdots & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_M} & \\
\xi' : 0 \longrightarrow & N & \longrightarrow & Q' & \longrightarrow & X'^{-i+2} \longrightarrow \cdots \longrightarrow & X'^0 & \longrightarrow & M & \longrightarrow 0.
\end{array}$$

In our modified algorithm we follow the same steps as in Algorithm 4.1, except that in Step 4 we compute $H^{-n+i}(\Lambda \otimes^L_{D_{2n}} (D_{2n}{}^{u_n}/P))$ instead of $H^{-n}(\Lambda \otimes^L_{D_{2n}} (D_{2n}{}^{u_n}/P))$. The output is a basis $\{\varphi_1, \ldots, \varphi_k\}$ of the finite-dimensional K-vector space $H^i(\mathrm{Hom}_D(X^\bullet, N))$, where $X^\bullet$ is a free resolution of M,

$$X^\bullet : 0 \to \underbrace{D^{r_{-a}}}_{\text{degree } -a} \xrightarrow{\cdot M_{-a+1}} \cdots \to D^{r_{-1}} \xrightarrow{\cdot M_0} \underbrace{D^{r_0}}_{\text{degree } 0} \to M \to 0.$$

To obtain the ith Yoneda Ext group from our output for $\mathrm{Ext}^i_D(M,N)$, we follow the presentation of Weibel (1994, Section 3.4) and associate to a cohomology class $\varphi \in H^i(\mathrm{Hom}_D(X^\bullet, N))$ the exact sequence,

$$\xi(\varphi) : 0 \longrightarrow N \longrightarrow Q \longrightarrow D^{r_{-i+2}} \longrightarrow \cdots \longrightarrow D^{r_0} \to M \to 0.$$

Here, Q is the cokernel of $(\cdot M_{-i+1}, \varphi) : D^{r_{-i}} \longrightarrow D^{r_{-i+1}} \oplus N$, and the maps are all the natural ones. The reader may verify that N is indeed a submodule of Q. Notice that the only difference between any $\xi(\varphi)$ and $\xi(\varphi')$ are their corresponding Q's and the maps to and from them.

Example 4.4. Let $D = K\langle x, \partial\rangle$ be the first Weyl algebra and $M = D/D{\cdot}\partial$, $N = D/D{\cdot}x$. Then for Step 1 of Algorithm 4.1, we have the resolutions,

$$X^\bullet : 0 \to D^1 \xrightarrow{\cdot\partial} D^1 \to 0, \qquad Y^\bullet : 0 \to D^1 \xrightarrow{\cdot x} D^1 \to 0.$$

For Step 2, we form the complex $Z^\bullet = \mathrm{Tot}(\tau(\mathrm{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet)$,

$$Z^\bullet : 0 \leftarrow \underbrace{D_2{}^1}_{\text{degree } 1} \xleftarrow{\cdot\begin{bmatrix}\partial_x \\ y\end{bmatrix}} D_2{}^2 \xleftarrow{\cdot[y, -\partial_x]} D_2{}^1 \leftarrow 0.$$

For Steps 3–6, we find that $H^1(\Delta \otimes_{D_2} Z^\bullet)$ is spanned by $\{1\}$, and projecting by p_1, $\mathrm{Ext}^1_D(M,N) \simeq H^1(\mathrm{Hom}_D(X^\bullet, D/D \cdot x)$ is spanned by the natural projection $\varphi : D \to (D/D \cdot x)$. For $\kappa \in K$, the cohomology classes $\kappa\varphi$ correspond to the extensions on the bottom row of the following diagram,

$$\begin{array}{ccccccc}
0 \to & D & \xrightarrow{\cdot \partial} & D & \longrightarrow & \dfrac{D}{D \cdot \partial} & \to 0 \\
& \downarrow{\scriptstyle \cdot \kappa} & & \downarrow{\scriptstyle \cdot [0,1]} & & \downarrow{\scriptstyle \mathrm{id}_{D/D\cdot\partial}} & \\
0 \to & \dfrac{D}{D \cdot x} & \xrightarrow{\cdot [1,0]} & \dfrac{D \cdot \vec{e_1} \oplus D \cdot \vec{e_2}}{D \cdot x\vec{e_1} + D \cdot (\kappa \vec{e_1} + \partial \vec{e_2})} & \xrightarrow{\cdot \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} & \dfrac{D}{D \cdot \partial} & \to 0.
\end{array}$$

When $\kappa \neq 0$, the module $Q(\kappa) = (D \cdot \vec{e_1} \oplus D \cdot \vec{e_2})/(D \cdot x\vec{e_1} + D \cdot (\kappa\vec{e_1} + \partial\vec{e_2}))$ is generated by $\vec{e_2}$ and is always isomorphic to $D/D \cdot x\partial$. When $\kappa = 0$, the module is no longer generated by $\vec{e_2}$ and is not isomorphic to $D/D \cdot x\partial$.

In fact, the module $(D \cdot \vec{e_1} \oplus D \cdot \vec{e_2})/(D \cdot x\vec{e_1} + D \cdot (\kappa\vec{e_1} + \partial\vec{e_2}))$ is always generated by the residue class of $\vec{e_1} + \vec{e_2}$ and has the cyclic presentation $D/D \cdot \{\partial^2 x + \kappa x\partial, x^2\partial\}$ with respect to this generator. Using this presentation, the extensions take the form,

$$0 \to \frac{D}{D \cdot x} \xrightarrow{\cdot[-x\partial]} \frac{D}{D \cdot \{\partial^2 x + \kappa x \partial, x^2 \partial\}} \xrightarrow{\cdot[x\partial + 1]} \frac{D}{D \cdot \partial} \to 0.$$

One can picture $Q(\kappa)$ as the $K[x]$-module $K[x] + x^{-1}K[x^{-1}]$ with the twisted multiplication rule $x \cdot (x^{-1}) = \kappa$ which is a direct sum if $\kappa = 0$.

5. Isomorphism Classes of *D*-modules

In this section, we give an algorithm to determine whether two holonomic D-modules M and N are isomorphic, and if so to produce an explicit isomorphism. Here, $\mathrm{End}_D(M)$ denotes the space of endomorphisms of a D-module M, where an endomorphism is a D-linear map from M to M. Similarly, $\mathrm{Iso}_D(M)$ denotes the units of the ring $\mathrm{End}_D(M)$.

If the holonomic modules M and N are isomorphic, then $\mathrm{Hom}_D(M, N) \simeq \mathrm{End}_D(M)$ is a finite-dimensional K-algebra. In the theory of finite dimensional K-algebras, the Jacobson radical J is the intersection of all maximal left ideals of E, and it has the property that the quotient E/J is a semi-simple K-algebra. By the Wedderburn–Artin theorem, a semi-simple algebra is isomorphic to a product of matrix rings over division algebras, and hence by taking the algebraic closure, we find that $E/J \otimes_K \overline{K}$ is isomorphic to a product of matrix rings over the field $\overline{K}$,

$$(E/\operatorname{Jac}(E)) \otimes_K \bar{K} \cong \prod_{i=1}^{d} \mathrm{End}_{\bar{K}}(\bar{K}^{d_i}). \tag{11}$$

One consequence of this decomposition is that the non-units of $E/J \otimes_K \overline{K}$ form a determinantal hypersurface. In particular, the units of $E/J \otimes_K \overline{K}$ form a Zariski open dense set, and hence the units of E/J also form a Zariski open dense set. Moreover, units and non-units respect the Jacobson radical in the sense that if j is in the Jacobson radical of E and if u is a unit of E then $u + j$ is also a unit, and similarly, if n is not a unit of E then $n + j$ is not a unit. We can thus conclude the following lemma.

Lemma 5.1. *Let M be a holonomic D-module. Then the space of D-linear isomorphisms $\mathrm{Iso}_D(M)$ from M to itself is open and dense in $\mathrm{End}_D(M)$ under the Zariski topology.*

The lemma says that if holonomic M and N are isomorphic then most maps from M to N are isomorphisms. We now give an algorithm to determine whether M and N are isomorphic based on Algorithm 4.1 and Lemma 5.1.

Algorithm 5.1. (Is M Isomorphic to N?)

Input: Presentations $M \simeq D^{m_M}/D \cdot \{P_1, \ldots, P_a\}$ and $N \simeq D^{m_N}/D \cdot \{Q_1, \ldots, Q_b\}$ of left holonomic D-modules.
Output: "No" if $M \not\simeq N$; and "Yes" together with an isomorphism $\iota : M \to N$ if $M \simeq N$.

(1) Compute bases $\{s_1, \ldots, s_\sigma\}$ and $\{t_1, \ldots, t_\tau\}$ for the vector spaces $V = \mathrm{Hom}_D(M, N)$ and $W = \mathrm{Hom}_D(N, M)$ using Algorithm 4.1, where s_i and t_j are respectively $m_M \times m_M$ and $m_N \times m_N$ matrices with entries in D representing homomorphisms by right multiplication. Recall that we view D^{m_M} and D^{m_N} as consisting of row vectors. If $\sigma \neq \tau$, return "No" and exit.
(2) With new indeterminates $\boldsymbol{\mu} = \{\mu_i\}_1^\tau$ and $\boldsymbol{\nu} = \{\nu\}_1^\tau$ form the "generic homomorphisms" $\sum_i \mu_i s_i \in \mathrm{Hom}_D(M, N)$ and $\sum_j \nu_j t_j \in \mathrm{Hom}_D(N, M)$. The compositions $\sum_{i,j} \mu_i \nu_j s_i \cdot t_j : M \to N \to M$ and $\sum_{i,j} \mu_i \nu_j t_j \cdot s_i : N \to M \to N$ are respectively $m_M \times m_M$ and $m_N \times m_N$-matrices over $D[\boldsymbol{\mu}, \boldsymbol{\nu}]$.
(3) Reduce the rows of the matrix $\sum_{i,j} \mu_i \nu_j s_i \cdot t_j - \mathrm{id}_{m_M}$ modulo a Gröbner basis for $D \cdot \{P_1, \ldots, P_a\} \subset D^{m_M}$. Force this reduction to be zero by setting the coefficients (which are inhomogeneous bilinear polynomials in μ_i, ν_j) of every standard monomial in every entry to be zero. Collect these relations in the ideal $I_M \subset K[\boldsymbol{\mu}, \boldsymbol{\nu}]$.
(4) Similarly, reduce the rows of the matrix $\sum_{i,j} \mu_i \nu_j t_j \cdot s_i - \mathrm{id}_{m_N}$ modulo a Gröbner basis for $D \cdot \{Q_1, \ldots, Q_b\} \subset D^{m_N}$. Force this reduction to be zero by setting the coefficients of every standard monomial in every entry to be zero, and collect these relations in the ideal $I_N \subset K[\boldsymbol{\mu}, \boldsymbol{\nu}]$.
(5) If $I(V, W) = I_M + I_N \subseteq K[\boldsymbol{\mu}, \boldsymbol{\nu}]$ contains a unit, return "No" and exit.
(6) Otherwise compute an isomorphism $\sum_{i=1}^\tau k_i s_i$ in $\mathrm{Hom}_D(M, N)$ by finding the first τ coordinates of any point in the zero locus of $I(V, W)$. We can do this by inductively finding $k_i \in K$ for each i from 1 to τ such that $I(V, W) + (\mu_1 - k_1, \ldots, \mu_i - k_i)$ is a proper ideal. For each i, this can be accomplished by trying different numbers for k_i until a suitable choice is found.
(7) Return "Yes" and the isomorphism $(\sum_{i=1}^\tau k_i s_i) : M \to N$.

Remark. Algorithm 5.1 can also be modified to detect whether M is a direct summand of N. Namely M is a direct summand of N if and only if the ideal I_M of Step 3 is not the unit ideal. Similarly N is a direct summand of M if and only if the ideal I_N of Step 4 is not the unit ideal.

Remark. Algorithm 5.1 can be further modified to compute an ideal in $K[\boldsymbol{\nu}]$ defining the closed set of non-isomorphisms, $\mathrm{End}_D(M) \setminus \mathrm{Iso}_D(M)$. Namely, we first perform Steps 1 through 4 with $M = N$ to obtain the ideal $I(V, V) \subset K[\boldsymbol{\mu}, \boldsymbol{\nu}]$. Then we regard each of the ζ generators of $I(V, V)$ as a linear inhomogeneous equation in the variables μ_i with coefficients involving ν_j as parameters, and collect all these equations in a single matrix equation $A \cdot \mu = b$, $A \in K[\boldsymbol{\nu}]^{\zeta \times \tau}$. An ideal defining the non-isomorphisms is generated by all $\tau \times \tau$ minors of A. We leave the proof of this fact as an exercise.

PROOF. (OF THE CORRECTNESS OF ALGORITHM 5.1) Reducing $\sum_{i,j} \mu_i\nu_j s_i \cdot t_j - \mathrm{id}_{m_M}$ modulo $D \cdot \{P_1, \ldots, P_a\}$ in Step 3 leads to a generic remainder which depends on the parameters μ_i, ν_j. Since a Gröbner basis of $D \cdot \{P_1, \ldots, P_a\}$ is parameter-free, this generic remainder has the property that its specialization to a fixed choice of parameters $\mu_i = a_i, \nu_j = b_j$ gives the remainder of $\sum_{i,j} a_i b_j s_i \cdot t_j - \mathrm{id}_{m_M}$ modulo $D \cdot \{P_1, \ldots, P_a\}$. Thus setting the remainder to zero in Step 3 corresponds to deriving conditions on the parameters μ_i, ν_j which makes the endomorphism given by $\sum_{i,j} \mu_i\nu_j s_i \cdot t_j$ equal to the identity on M. This is possible if and only if M is a direct summand of N. The analogous statement holds for reduction of $\sum_{i,j} \mu_i\nu_j t_j \cdot s_i - \mathrm{id}_{m_N}$ modulo $D \cdot \{Q_1, \ldots, Q_b\}$ and setting its resulting remainder to zero. Since setting a remainder to zero is equivalent to the vanishing of the coefficients of its standard monomials, we collect these vanishing conditions in the ideal $I(V, W)$ of $K[\boldsymbol{\mu}, \boldsymbol{\nu}]$.

Now a linear combination $\sum_i a_i s_i : M \to N$ is an isomorphism with inverse $\sum b_j t_j : N \to M$ if and only if the composition $\sum_{i,j} a_i b_j s_i \cdot t_j$ is congruent to id_{m_M} modulo $D \cdot \{P_1, \ldots, P_a\}$ and the opposite composition $\sum_{i,j} a_i b_j t_j \cdot s_i$ is congruent to id_{m_N} modulo $D \cdot \{Q_1, \ldots, Q_b\}$. Thus the common zeroes $(a_1, \ldots, a_\tau, b_1, \ldots, b_\tau)$ of $I(V, W)$ correspond to isomorphisms $\sum_i a_i s_i$ and their inverses $\sum_j b_j t_j$. In particular, if $I(V, W)$ is the entire ring, which we detect by searching for 1 in a Gröbner basis of $I(V, W)$, then there are no isomorphisms.

On the other hand if $I(V, W)$ is proper, then M and N are isomorphic and we obtain an explicit isomorphism from finding any common solution of $I(V, W)$. By Lemma 5.1, the invertible homomorphisms from M to N are Zariski dense in the vector space $\mathrm{Hom}_D(M, N)$. Hence, a common solution can be explicitly found by by intersecting the zero locus of $I(V, W)$ with $\tau = \dim_K(\mathrm{Hom}_D(M, N) \cong \mathrm{End}_D(M))$ hyperplanes $\{\mu_i = k_i\}$. Since $\mathrm{Iso}_D(M)$ is dense in $\mathrm{End}_D(M)$, if $I(V, W) + \langle \mu_1 - k_1, \ldots, \mu_{i-1} - k_{i-1} \rangle$ is proper, then there are only finitely many k_i for which the sum $I(V, W) + \langle \mu_1 - k_1, \ldots, \mu_i - k_i \rangle$ is the unit ideal. Thus the k_i can be found by trial and error. □

REMARK. Once we have specialized the μ_i in a common solution of $I(V, W)$, then the ν_j are determined because of the bilinear nature of the relations (which give linear relations for the ν_j once all μ_i are chosen). This also means that if there is any solution, then the μ_i are rational functions in the ν_j and vice versa. In particular, if $\phi \in \mathrm{Hom}_D(M, N)$ is defined over the field K then ϕ^{-1} is defined over K as well.

We now give some simple examples. Unfortunately, due to the complexity of the Hom_D (M, N) algorithm, our current implementation in Macaulay 2 cannot compute much more complicated examples.

EXAMPLE 5.2. Let $n = 1$ and $M = N = D/D \cdot \partial^2$. One checks that $V = W = \mathrm{Hom}_D(M, N)$ is generated by the four morphisms $s_1 = \cdot(\partial)$, $s_2 = \cdot(x\partial)$, $s_3 = \cdot(1)$, and $s_4 = \cdot(x^2\partial - x)$. We obtain the generic morphism

$$\begin{aligned}\sum_{i=1}^{4}\sum_{j=1}^{4} \mu_i\nu_j t_j \cdot s_i - 1 = {} & (\mu_3\nu_3 - \mu_1\nu_4 - 1) \\ & + (-\mu_4\nu_3 - \mu_2\nu_4 - \mu_3\nu_4)x \\ & + (\mu_3\nu_1 + \mu_1\nu_2 + \mu_1\nu_3)\partial\end{aligned}$$

$$+ (-\mu_4\nu_1 + \mu_2\nu_2 + \mu_3\nu_2 + \mu_2\nu_3 + \mu_1\nu_4)x\partial$$
$$+ (\mu_4\nu_3 + \mu_2\nu_4 + \mu_3\nu_4)x^2\partial$$

plus nine other terms which are in $D \cdot \partial^2$ independently of the parameters.

Hence in order for $\sum_{i=1}^4 \mu_i s_i$ to be an isomorphism, the μ_i need to be part of a solution to the ideal

$$\begin{aligned} I(V, W) = K[\boldsymbol{\mu}, \boldsymbol{\nu}]\{&\mu_3\nu_3 - \mu_1\nu_4 - 1, \\ &-\mu_4\nu_3 - \mu_2\nu_4 - \mu_3\nu_4, \\ &\mu_3\nu_1 + \mu_1\nu_2 + \mu_1\nu_3, \\ &-\mu_4\nu_1 + \mu_2\nu_2 + \mu_3\nu_2 + \mu_2\nu_3 + \mu_1\nu_4, \\ &\mu_4\nu_3 + \mu_2\nu_4 + \mu_3\nu_4\}. \end{aligned}$$

This ideal is not the unit ideal and has degree 8. Hence there are isomorphisms between M and N. Pick "at random" $\mu_1 = 1$, $\mu_2 = 2$, and $\mu_3 = 0$. Then the ideal $I(V, W) + (\mu_1 - 1, \mu_2 - 2, \mu_3 - 0)$ equals the ideal $(\mu_1 - 1, \mu_2 - 2, \mu_3, \nu_4 + 1, \nu_2 + \nu_3, \nu_1 + \frac{1}{2}\nu_3, \mu_4\nu_3 - 2)$. We see that we have to avoid $\mu_4 = 0$ but otherwise have complete choice.

Example 5.3. Let $n = 1$, $M = D/D \cdot \partial^2$, and $N = D/D \cdot \partial$. One checks that $V = \mathrm{Hom}_D(N, M)$ is generated by $t_1 = \cdot(\partial)$ and $t_2 = \cdot(x\partial - 1)$ while $W = \mathrm{Hom}_D(M, N)$ is generated by $s_1 = \cdot(1)$ and $s_2 = \cdot(x)$. The sum $\sum \mu_i\nu_j s_i \cdot t_j$ takes the form $\mu_2\nu_2 x^2\partial + (\mu_1\nu_2 + \mu_2\nu_1)x\partial + \mu_1\nu_1\partial - (\mu_1\nu_2 + \mu_2\nu_2)$. Modulo $D \cdot \partial$ we want this to be 1, so we get the relation

$$K[\boldsymbol{\mu}, \boldsymbol{\nu}] \cdot (\mu_2\nu_1 - \mu_1\nu_2 - 1) = I_N.$$

Since this equation has solutions, M can be realized as a summand of N. On the other hand, the sum $\sum \mu_i\nu_j t_j \cdot s_i$ takes the form $\mu_1\nu_1\partial + (\mu_1\nu_2 + \mu_2\nu_1)x\partial - \mu_1\nu_2 - \mu_2\nu_2 x + \mu_2\nu_2 x^2\partial$. Modulo $D \cdot \partial^2$ we want this to be 1, so we obtain

$$K[\boldsymbol{\mu}, \boldsymbol{\nu}] \cdot \{-\mu_1\nu_2 - 1, \mu_1\nu_1 - 0, \mu_1\nu_2 + \mu_2\nu_1 - 0, \mu_2\nu_2 - 0\} = I_M.$$

$I_M + I_N$ is the unit ideal, and hence M and N are not isomorphic.

Example 5.4. Let $n = 2$ and $D = K\langle x, y, \partial_x, \partial_y\rangle$. Consider the ideals $I_i = D \cdot \{x\partial_x + y\partial_y + 3i, (x^2 + 2xy)\partial_x - (y^2 + 2xy)\partial_y\}$ with associated modules $M_i = D/I_i$.

For $i = 0$ and 1, one computes with *Macaulay 2* that $\mathrm{Hom}_D(M_0, M_1)$ is one-dimensional, generated by (right multiplication by) $s = x^2y + y^2x$, while $\mathrm{Hom}_D(M_1, M_0)$ is generated by $t_1 = \partial_x^2\partial_y - \partial_y^2\partial_x$, $t_2 = y\partial_x\partial_y^3 - y\partial_y^4 + 3\partial_x\partial_y^2 - 3\partial_y^3$, $t_3 = 2y\partial_x^2\partial_y^2 - 2y\partial_x\partial_y^3 + 3\partial_x^2\partial_y - 3\partial_x\partial_y^2$ and $t_4 = y\partial_x^3\partial_y - y\partial_x^2\partial_y^2$.

Then $\mu\nu_1 s \cdot t_1 + \mu\nu_2 s \cdot t_2 + \mu\nu_3 s \cdot t_3 + \mu\nu_4 s \cdot t_4$ reduces modulo I_0 to zero, and similarly $\mu\nu_1 t_1 \cdot s + \mu\nu_2 t_2 \cdot s + \mu\nu_3 t_3 \cdot s + \mu\nu_4 t_4 \cdot s$ is in $D[\mu, \boldsymbol{\nu}] \cdot I_1$. Hence the only D-morphism $M_0 \to M_1 \to M_0$ or $M_1 \to M_0 \to M_1$ is the zero morphism.

In general, it is known that M_i and M_{i+1} are isomorphic for i integral and positive. This follows from the fact that the Bernstein–Sato polynomial of $f = x^2y + y^2x$ is $(s+1)(s+3/4)(s+1)(s+5/4)$ and so $I_i = \mathrm{ann}(f^i)$ and M_i is isomorphic to $K[x, y, f^{-1}]$ for i integral and positive (see Kashiwara, 1978). Similarly, by applying the Fourier transform, M_i and M_{i-1} are isomorphic if i is integral and not positive.

Remark. Methods of computational algebraic geometry can also be used to get structural information about $\mathrm{End}_D(M)$, namely the invariants d_i in the decomposition (11).

One proceeds to compute the de Rham cohomology groups of $\mathrm{Iso}_D(M) \subset \mathrm{End}_D(M) = \mathbb{C}^\tau = \mathrm{Spec}(\mathbb{C}[\boldsymbol{\nu}])$ using the algorithm in Walther (2000). This algorithm allows us to pretend that K is algebraically closed since although it can be used on input defined over any computable subfield of the complex numbers, it always computes $\dim_{\mathbb{C}}(H^\bullet_{dR}(\mathbb{C}^n \setminus Y, \mathbb{C}))$. The obtained output equals the cohomology of the units of $E/\operatorname{Jac}(E) \otimes_K \mathbb{C}$ since this space is homotopy equivalent to the units of $E \otimes_K \mathbb{C}$. Finally, we note that the cohomology of the units $Gl(n, \mathbb{C})$ of a matrix algebra is well known, behaves well under products, and hence can be used to determine the d_i.

To be explicit, in Example 5.2, the non-isomorphisms are defined by the vanishing of the polynomial $\nu_2^2\nu_3^2 + 2\nu_2\nu_3^3 + \nu_3^4 + 2\nu_1\nu_2\nu_3\nu_4 + 2\nu_1\nu_3^2\nu_4 + \nu_1^2\nu_4^2$ in $(\bar{K})^4$. With *Macaulay 2* one obtains that the de Rham cohomology of $\mathrm{Iso}_D(M, N)$ is one-dimensional in degrees 0, 1, 3 and 4 and zero otherwise. From Weyl (1939, Theorems 7.11.A and 8.16.B) one concludes that in the decomposition (11) d equals 1 and $d_1 = 2$.

Let us conclude by mentioning a well-known application of the endomorphism ring $E = \mathrm{End}_D(M)$ towards decompositions of M (see, for example, Lam, 1991 for these basic facts). Namely, there is a bijective correspondence between (1) the decompositions of M into a direct sum of submodules and (2) the decompositions of the identity element $1 = e_1 + \cdots + e_s$ of E into pairwise orthogonal idempotents. The correspondence is obtained by taking a set of orthogonal idempotents $\{e_1, \ldots, e_s\}$ and producing the decomposition $M = e_1 \cdot M \oplus \cdots \oplus e_s \cdot M$. Moreover by the Krull–Schmidt–Azumaya theorem, a D-module M has a (unique up to re-ordering) decomposition into a direct sum of indecomposable submodules (meaning that they cannot be further decomposed into a direct sum of nonzero submodules). Thus, an algorithm which produces a full set of orthogonal idempotents for the K-algebra $\mathrm{End}_D(M)$ combined with Algorithm 4.1 gives a method to decompose holonomic D-modules into indecomposables. We remark that algorithms for computing idempotents as well as for computing many other properties of finite-dimensional K-algebras is a field of active research (see, for example, work of Friedl and Ronyai, 1985 and of Eberly, 1991). Many algorithms have been developed although there are restrictions on the field K.

In Example 5.2, we have the two idempotents $x\partial$ and $1 - x\partial$ which correspond to the decomposition $D/D \cdot \partial^2 \cong (D/D \cdot \partial)^2$ where the two factors are the modules in M generated by $x\partial$ and $1 - x\partial$ respectively.

Acknowledgements

We deeply thank Nobuki Takayama, who initiated this project and who suggested methods to approach these problems. We also thank Bernd Sturmfels for many helpful comments and encouragement, and Mike Artin, Mark Davis, and Wayne Eberly for conversations about non-commutative ring theory.

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Received 18 August 2000
Accepted 7 May 2001