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Effective actions of a Gauss–Bonnet brane world with brane curvature terms

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Abstract

We consider a warped brane world scenario with two branes, Gauss–Bonnet gravity in the bulk, and brane localised curvature terms. When matter is present on both branes, we investigate the linear equations of motion and distinguish three regimes. At very high energy and for an observer on the positive tension brane, gravity is four-dimensional and coupled to the brane bending mode in a Brans–Dicke fashion. The coupling to matter and brane bending on the negative tension brane is exponentially suppressed. In an intermediate regime, gravity appears to be five dimensional while the brane bending mode remains four-dimensional. At low energy, matter on both branes couple to gravity for an observer on the positive tension brane, with a Brans–Dicke description similar to the 2-brane Randall–Sundrum setup. We also consider the zero mode truncation at low energy and show that the moduli approximation fails to reproduce the low energy action.

1. Introduction

Since the pioneering work of Randall and Sundrum [1,2], brane world models have been studied intensively. In the simplest setup, which provides a potential solution of the hierarchy problem, two branes of tension T_i $(i = \pm)$ are embedded in a 5D bulk AdS spacetime with negative cosmological constant Λ. The action for the system is

$$S_{\mathrm{RS}} = \frac{1}{2\kappa_5^2}\int d^5x\,\sqrt{-g_5}\,[-2\Lambda + \mathcal{R}] + \sum_{i=\pm}\frac{1}{\kappa_5^2}\int_i d^4x\,\sqrt{-\bar{g}^i}\,(-T_i + 2K_i), \tag{1.1}$$

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where $\mathcal{R}$ is the Ricci scalar, κ_5^2 the gravitational constant, and $\bar{g}^i_{\mu\nu}$ denotes the induced metric on the ith brane. We have also included the Gibbons–Hawking boundary term for outgoing normal vectors. The AdS_5 warped solution with 4D Poincaré invariance and Z_2 symmetry about each brane, located at constant z, is

$$ds^2 = e^{-2kz}\eta_{\mu\nu}\,dx^\mu\,dx^\nu + dz^2, \tag{1.2}$$

requiring the well-known fine-tunings

$$\Lambda = -6k^2 \quad (<0), \qquad T_+ = -T_- = 6k \quad (>0). \tag{1.3}$$

In general, when matter is added to the branes, the physics of the RS model cannot be derived from a 4D action since the brane is not decoupled from the bulk, and hence the system of brane equations is not closed [3]. At *low energies* $E \ll |T_i|$, the situation is different and the 4D low energy effective action corresponding to (1.1) has been thoroughly studied. In this limit, the degrees of freedom are the two brane positions and the 4D graviton zero mode [4,5]. In the Einstein frame, one of the two moduli, the dilaton, decouples leaving only one physical modulus, the radion. As stressed in [6], the resulting effective action is non-perturbative and hence can describe the physics of strong gravity systems such as black holes on the brane [7].

Our aim is to derive a similar 4D low energy effective action when Gauss–Bonnet (GB) gravity rather than Einstein gravity acts in the bulk. This particular higher derivative combination is the only one which gives equations of motion depending on the metric and its first two derivatives:

$$S_{\mathrm{GB}} = \frac{1}{2\kappa_5^2}\int d^5x\,\sqrt{-g_5}\big[-2\Lambda + \mathcal{R} + \alpha\big(\mathcal{R}^2 - 4\mathcal{R}_{ab}\mathcal{R}^{ab} + \mathcal{R}_{abcd}\mathcal{R}^{abcd}\big)\big] + \sum_{i=\pm}\frac{1}{\kappa_5^2}\int_i d^4x\,\sqrt{-\bar{g}^i}\big(-T_i + 2\mathcal{L}^i_{\mathrm{boundary}}\big), \tag{1.4}$$

where the boundary term is given in [8]. The coupling constant α has mass dimension -2, and when interpreted as the string slope in a derivative expansion, $\alpha > 0$. Action (1.4) has a solution of the same form as (1.2), but now with corrections[2] linear in α [11]

$$\Lambda = -6k^2\big(1 - 2\alpha k^2\big), \qquad T_+ = -T_- = 6k\left(1 - \frac{4}{3}\alpha k^2\right). \tag{1.5}$$

Static brane worlds with Gauss–Bonnet gravity have been intensively studied [11] while time-dependent solutions have also been considered in [9,12,13]. The addition of a bulk scalar field has been investigated in [8,14,16]. However, the effective brane gravity in a system consisting of *two* Minkowski branes and Gauss–Bonnet gravity in the bulk has not yet been studied: it is the aim of this Letter.

As opposed to (1.1), the action (1.4) is not a suitable starting point to derive a low energy effective action when GB gravity acts in the bulk. One reason is that in contrast with the RS model, the AdS solution (1.5) is unstable: the spin 2 fluctuations contain a tachyonic mode which is localised around the negative tension brane [17]. This instability is a generic problem of any GB system containing a negative tension brane. Clearly in order for the effective action to make any sense, this mode must be 'removed'. Here we follow the procedure analysed in [17] and add induced gravity terms to the brane so that the 5D action we consider is

$$S_{\mathrm{totalGB}} = S_{\mathrm{GB}} + S_{\mathrm{ind}}, \tag{1.6}$$

where

$$S_{\mathrm{ind}} = \sum_{i=\pm}\frac{\beta_i}{2\kappa_5^2}\int_i d^4x\,\sqrt{-\bar{g}^i}\,\bar{\mathcal{R}}_i, \tag{1.7}$$

[2] Note that due to an improper brane delta function regularization, the corresponding relations given in [9,10] have an incorrect coefficient.

where $\bar{\mathcal{R}}_i$ is the Ricci scalar constructed from the 4D induced metric on each brane $\bar{g}^i_{\mu\nu}$. The required constraints on β_i have been discussed in [17] (see also (2.21)). Note that warped brane worlds with brane curvature terms have been studied before, for instance, in [18,19].

The outline of the Letter is the following. First, we recall the linear equations of motion for GB brane worlds with two branes and induced gravity on each brane. We analyse the high energy regime from the point of view of an observer on the positive tension brane. We find that the coupling to matter on the negative tension brane is exponentially suppressed. Gravity becomes 4D with a Brans–Dicke coupling to the brane bending mode. At intermediate energy, gravity becomes 5D while the brane bending mode retains its 4D character. Finally, at low energy, we find that the effective gravity and brane bending equations are equivalent to the field equations obtained from an effective action involving only one scalar field, i.e., the radion. We then consider the same brane world model from the point of view of the moduli approximation and show that the resulting action obtained after integration over the fifth dimension differs from the low energy action derived from the linear equations of motion.

2. Low energy action and linear equations of motion

2.1. Propagator

Following [4,5], we first give the equations of motion for perturbations about the background solution given in (1.2) and (1.5). Starting from a general gauge for the metric with the two branes located at constant unperturbed positions $\xi_0^\pm$, we then impose the GN gauge $h_{\mu 5} = h_{55} = 0$, so that the perturbed metric takes the form

$$ds^2 = \left(a^2(z)\eta_{\mu\nu} + h_{\mu\nu}\right) + dz^2, \tag{2.1}$$

where

$$a(z) = e^{-kz}. \tag{2.2}$$

In addition, we furthermore impose the transverse-traceless gauge condition

$$h \equiv \eta^{\mu\nu} h_{\mu\nu} = 0 = \partial_\mu h^{\mu\nu}, \tag{2.3}$$

so that the branes are no longer straight but located at perturbed positions

$$z^\pm(x) = \xi_0^\pm + \xi^\pm(x). \tag{2.4}$$

Note that throughout the following, 4D indices are raised with the flat metric $\eta_{\mu\nu}$. Furthermore, it will be useful to introduce

$$\gamma_{\mu\nu} = a^{-2}(z) h_{\mu\nu}. \tag{2.5}$$

The perturbed bulk Einstein equations now take the form [14]

$$\left(1 - 4\alpha k^2\right)\left(\partial_z^2 - 4k\partial_z + a^{-2}\Box^{(4)}\right)\gamma_{\mu\nu} = 0, \tag{2.6}$$

where the GB term acts as an overall multiplicative constant. Note that the quadratic expansion of the GB term around a *flat background* vanishes [15], therefore not modifying the propagator. For an AdS_5 background, however, the quadratic contribution is non-zero though it preserves the linearized bulk equations of motion. Thus, as long as $4\alpha k^2 \neq 1$, the solution of (2.6) is just as in the RS model: in momentum space, where $\Box^{(4)}\gamma_{\mu\nu} = -p^2\gamma_{\mu\nu}$, it is given by

$$\gamma_{\mu\nu}(p,z) = -\frac{(ky)^2}{p^2}\big(A_{\mu\nu}(p)J_2(y) + B_{\mu\nu}(p)Y_2(y)\big). \tag{2.7}$$

Here

$$y = \frac{\sqrt{-p^2}}{ka(z)} \tag{2.8}$$

is the conformal variable rescaled by $\sqrt{-p^2}$ and J_2, Y_2 are the Bessel functions of the first and second kind. The p-dependent functions $A_{\mu\nu}$ and $B_{\mu\nu}$ are determined by the boundary conditions for the gravitational perturbation which, in this gauge, are given by [14]

$$\partial_z\gamma_{\mu\nu}(p,z)\big|_\pm - p^2\ell_\pm a_\pm^{-1}\gamma_{\mu\nu}(p,z)\big|_\pm = \mp\kappa_5^2 a_\pm^{-2}\Sigma_{\mu\nu}^\pm(p). \tag{2.9}$$

Here

$$a_\pm = a\big(\xi_0^\pm\big) \tag{2.10}$$

are the scale factors at the unperturbed brane positions, and the length scales $\ell_\pm$ are given by

$$\ell_\pm = \frac{1}{ka_\pm}\left(\frac{\pm\beta_\pm k + 8\alpha k^2}{2(1-4\alpha k^2)}\right). \tag{2.11}$$

These scales, which will play an important role later, vanish in the RS limit but more generally can be either positive or negative. Note that in (2.9) we have added matter with stress-energy tensor $T_{\mu\nu}^\pm$ to each brane so that the source term is

$$\Sigma_{\mu\nu}^\pm = \frac{1}{1-4\alpha k^2}\left[\left(T_{\mu\nu}^\pm - \frac{1}{3}T^\pm\eta_{\mu\nu}\right) \mp 2\kappa_5^{-2}w_\pm\partial_\mu\partial_\nu\xi^\pm\right], \tag{2.12}$$

where we have defined

$$w_\pm = \big(1 \pm \beta_\pm k + 4\alpha k^2\big). \tag{2.13}$$

The stress-energy tensors are defined with respect to the induced metrics:

$$T_{\mu\nu}^\pm \equiv -\frac{2}{\sqrt{-\bar{g}_\pm}}\frac{\delta\mathcal{L}_{\text{matter}}^\pm}{\delta\bar{g}_\pm^{\mu\nu}}. \tag{2.14}$$

Finally, the relative signs in (2.9) arise from the change of orientation on the second brane compared to the first brane, and these equations generalise those of [4] to GB gravity. From (2.9) and $\gamma = 0$, it follows that $\Sigma^\pm = 0$ and hence

$$\Box^{(4)}\xi^\pm = \mp\frac{\kappa_5^2}{6w_\pm}T^\pm. \tag{2.15}$$

On substituting the solution (2.7) into (2.9), the boundary conditions become

$$ky_\pm\big\{A_{\mu\nu}(p)\tilde{J}_\pm(p) + B_{\mu\nu}(p)\tilde{Y}_\pm(p)\big\} = \mp\kappa_5^2\Sigma_{\mu\nu}^\pm(p), \tag{2.16}$$

where from (2.8)

$$y_\pm = y_\pm(p) = \frac{\sqrt{-p^2}}{ka_\pm} \tag{2.17}$$

and

$$\tilde{J}_\pm(p) \equiv J_1(y_\pm) + (k\ell_\pm)a_\pm y_\pm J_2(y_\pm), \tag{2.18}$$

$$\tilde{Y}_\pm(p) \equiv Y_1(y_\pm) + (k\ell_\pm)a_\pm y_\pm Y_2(y_\pm). \tag{2.19}$$

Let us consider the homogeneous solutions of (2.16) corresponding to $\Sigma^\pm_{\mu\nu} = 0$. A first solution of (2.16) is when $y_\pm = 0$ so that $p^2 = 0$—the zero mode corresponding to the massless graviton. The other solutions are obtained when the relevant determinant of (2.16) vanishes:

$$\mathrm{Det}(p) \equiv \tilde{J}_-(p)\tilde{Y}_+(p) - \tilde{J}_+(p)\tilde{Y}_-(p) = 0. \tag{2.20}$$

As discussed in [17], for $\alpha \neq 0$ and $\beta_\pm = 0$, Eq. (2.20) has solutions when y is imaginary, and these tachyonic modes with $p^2 > 0$ are non-perturbative in α. However, for non-zero induced gravity terms $\beta_\pm$ they can be prevented provided [17]

$$\ell_+\ell_- < 0. \tag{2.21}$$

For real $y_\pm$, Eq. (2.20) yields the Kaluza–Klein tower.

We now assume that (2.21) holds and solve the linear equation in the presence of matter on both branes. From the boundary conditions (2.16) we find (away from the locus $\mathrm{Det}(p) = 0$ which corresponds to a discrete spectrum in p^2)

$$A_{\mu\nu}(p) = \frac{\kappa_5^2}{k}\frac{1}{\mathrm{Det}(p)}\left(\frac{\Sigma^+_{\mu\nu}(p)\tilde{Y}_-(p)}{y_+} + \frac{\Sigma^-_{\mu\nu}(p)\tilde{Y}_+(p)}{y_-}\right), \tag{2.22}$$

$$B_{\mu\nu}(p) = -\frac{\kappa_5^2}{k}\frac{1}{\mathrm{Det}(p)}\left(\frac{\Sigma^+_{\mu\nu}(p)\tilde{J}_-(p)}{y_+} + \frac{\Sigma^-_{\mu\nu}(p)\tilde{J}_+(p)}{y_-}\right). \tag{2.23}$$

Thus, from (2.7), the general solution for $\gamma_{\mu\nu}$ is

$$h_{\mu\nu}(x,z) = a^2(z)\gamma_{\mu\nu}(x,z) = \int d^4x' \left(\Delta^+(x,x',z)\Sigma^+_{\mu\nu}(x') + \Delta^-(x,x',z)\Sigma^-_{\mu\nu}(x')\right), \tag{2.24}$$

where the propagators are given by

$$\begin{aligned}\Delta^\pm(x,x',z) &\equiv \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x'-x)}\Delta^\pm(p,z) \\ &= \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x'-x)} \frac{\kappa_5^2 a_\pm}{\sqrt{-p^2}}\left(\frac{\tilde{Y}_\mp(p)J_2(y) - \tilde{J}_\mp(p)Y_2(y)}{\mathrm{Det}(p)}\right)\end{aligned} \tag{2.25}$$

with $y = y(p,z)$ given in (2.8).

Finally, from (2.24), the perturbed metric on each brane can be calculated. For the positive (respectively, negative) tension brane, we transform to a GN coordinate system $\tilde{x}^a = x^a - \xi^a$, giving a straight brane located at $\tilde{z} = \xi_0^\pm$, as well as $\tilde{h}_{\mu 5} = \tilde{h}_{55} = 0$. After a 4D gauge transformation [4,5], the perturbed metric on each brane is then $\tilde{h}_{\mu\nu}(x, \tilde{z} = \xi_0^\pm)$ with

$$\begin{aligned}\tilde{h}_{\mu\nu}(x,\xi_0^{\pm}) &= h_{\mu\nu} - 2ka_{\pm}^2\eta_{\mu\nu}\xi^{\pm}\\ &= \frac{1}{1-4\alpha k^2}\left\{\int d^4x'\left[\Delta^+\left(x,x',\xi_0^{\pm}\right)\left(T_{\mu\nu}^+ - \frac{1}{3}\eta_{\mu\nu}T^+\right)(x')\right.\right.\\ &\qquad \left.\left.+\,\Delta^-\left(x,x',\xi_0^{\pm}\right)\left(T_{\mu\nu}^- - \frac{1}{3}\eta_{\mu\nu}T^-\right)(x')\right]\right\} \pm \frac{k\kappa_5^2 a_{\pm}^2}{3w_{\pm}}\frac{1}{\Box^{(4)}}T^{\pm}\eta_{\mu\nu}.\end{aligned} \tag{2.26}$$

This expression together with (2.25) and (2.15) captures the physics of the Gauss–Bonnet brane world models with induced gravity on the branes. Notice that the perturbation $\tilde{h}_{\mu\nu}(x,z)$ depends on the sources on both branes. In particular, the brane positions play an important role in the dynamics of the system. At low energy, we will show that there is only one effective scalar degree of freedom. Before considering the low energy action reproducing the linear equations of motion, let us concentrate on the high energy regime.

2.2. *High energy limit*

At high energy, the effect of the induced brane terms is highly relevant. In particular, we find that at very high energy gravity propagates in 4D while its behaviour is 5D in an intermediate range.

Consider first the positive tension brane and set $T_{\mu\nu}^- = \xi^- = 0$. In order to evaluate the propagators on the positive tension brane it is convenient to work in Euclidean space and define $q = -i\sqrt{-p^2}$ with q real. Notice that this also corresponds to space-like momenta $p^2 > 0$ as relevant when computing the static potential between point sources. In the high energy limit $|y_{\pm}| = |q|/(ka_{\pm}) \gg 1$, we obtain the propagator

$$\Delta^+\left(q,\xi_0^+\right) \approx \frac{\kappa_5^2 a_+}{q}\left(\frac{1}{q\ell_+ + 1}\right), \tag{2.27}$$

from which two different energy regimes appear.

- At large momenta or small distances, $q^{-1} \ll |\ell_+|$ the propagator $\propto q^{-2}$ leading to

$$\begin{aligned}\frac{1}{a_+^2}\tilde{h}_{\mu\nu}\left(q,\xi_0^{\pm}\right) &= \frac{1}{q^2}\frac{2k\kappa_5^2}{\beta_+k + 8\alpha k^2}\left[T_{\mu\nu}^+ - \frac{1}{2}\eta_{\mu\nu}T^+ + \frac{1-4\alpha k^2}{6w_+}\eta_{\mu\nu}T^+\right]\\ &\equiv \frac{1}{q^2}\frac{2\kappa_4^2}{\Phi_0}\left[T_{\mu\nu}^+ - \frac{1}{2}\eta_{\mu\nu}T^+ + \frac{1}{2(3+2\omega(\Phi_0))}\eta_{\mu\nu}T^+\right].\end{aligned} \tag{2.28}$$

  We consider $\frac{1}{a_+^2}\tilde{h}_{\mu\nu}$ as the gravitational perturbation associated to a Minkowski background $\eta_{\mu\nu}$. In this limit, the interaction with matter mediated by gravity is a *four-dimensional* tensor–scalar theory which is given in a Brans–Dicke parametrisation [20] by a background Brans–Dicke parameter

$$\omega(\Phi_0) = \frac{3}{2}\frac{\beta_+k + 8\alpha k^2}{1-4\alpha k^2}, \tag{2.29}$$

  where the background Brans–Dicke field is

$$\frac{\Phi_0}{\kappa_4^2} = \frac{\beta_+k + 8\alpha k^2}{k\kappa_5^2}, \tag{2.30}$$

  and its fluctuation

$$\frac{\delta\Phi}{\Phi_0} = -2k\frac{1-4\alpha k^2}{\beta_+k + 8\alpha k^2}\xi^+. \tag{2.31}$$

  It coincides with the results of [14] and the Minkowski limit in [13]. This should be contrasted to the RS model in which $\ell_+ = 0$ and where gravity is always five-dimensional at short distance.

- If $k|\ell_+|a_+ \ll 1$, there is an intermediate high-energy regime in which $\frac{1}{ka_+} \gg q^{-1} \gg |\ell_+|$. In this case the propagator $\propto q^{-1}$ leading to an effective gravity given by

$$\tilde{h}_{\mu\nu}(q,\xi_0^+) = \frac{\kappa_5^2 a_+}{q(1-4\alpha k^2)}\left(T^+_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T^+\right) - \frac{\kappa_5^2 k a_+^2}{3w_+ q^2}T^+\eta_{\mu\nu}. \tag{2.32}$$

  The $1/q$ momentum dependence associated with the $1/3$ trace factor instead of $1/2$ means that there is 5D propagation of a combination of a 4D-tensor and a 4D-scalar mode. The term $1/q^2$ term corresponds to the 4D propagation of a 4D-scalar mode. Again note that in the RS model one is always in this regime at high energy.

So far we have not taken into account the presence of a second brane. In fact, one finds

$$\Delta^-\left(\xi_0^+\right) \approx 2\kappa_5^2\sqrt{a_+a_-}\exp\left\{\frac{q}{k}\left(\frac{1}{a_+} - \frac{1}{a_-}\right)\right\}\left[\frac{1}{q}\frac{1}{(1+\ell_+q)}\frac{1}{(1-\ell_-q)}\right]. \tag{2.33}$$

Notice that the propagator from the negative tension brane to the positive tension brane is exponentially suppressed, i.e., no gravitational effect is transmitted from one brane to another at high energy. Hence, at high energy, the two brane system behaves like a single brane system with no influence from the second brane.

In the following section we show that in the low energy limit, gravity is always 4D.

2.3. Low energy limit

Here we are interested in determining the dynamics and the number of degrees of freedom in the low energy limit, $y_\pm \ll 1$. In that limit, Eqs. (2.22) and (2.23) reduce to

$$A_{\mu\nu} \approx -\frac{2k\kappa_5^2}{p^2}a_-^2\frac{w_-\Sigma^+_{\mu\nu} + w_+\Sigma^-_{\mu\nu}}{w_+ - \frac{a_-^2}{a_+^2}w_-}, \tag{2.34}$$

$$B_{\mu\nu} \approx \left(1-4\alpha k^2\right)\frac{\pi\kappa_5^2}{2k}\frac{\Sigma^+_{\mu\nu} + \frac{a_-^2}{a_+^2}\Sigma^-_{\mu\nu}}{w_+ - \frac{a_-^2}{a_+^2}w_-}, \tag{2.35}$$

while we have

$$\Box^{(4)}\gamma_{\mu\nu} \approx -\frac{4k^2}{\pi}B_{\mu\nu} \tag{2.36}$$

$$\approx -\frac{1}{a_+^2w_+ - a_-^2w_-}\Big[2\kappa_4^2\left(a_+^2T^+_{\mu\nu} + a_-^2T^-_{\mu\nu}\right) - 4ka_+^2w_+\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box^{(4)}\right)\xi^+ + 4ka_-^2w_-\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box^{(4)}\right)\xi^-\Big] \tag{2.37}$$

where $\kappa_4^2 = k\kappa_5^2$ and we have used (2.15).

These equations have the structure of the equations of motion from a low energy effective action involving tensor gravity $\gamma_{\mu\nu}(x)$ and two scalar fields $\xi^\pm(x)$. They can be reproduced by a quadratic action, expanding

$$\frac{1}{2\kappa_4^2}\int d^4x\sqrt{-g}\left(\left[F_+(\xi^+) - F_-(\xi^-)\right]\mathcal{R} - B_+(\xi^+)(\partial\xi^+)^2 - B_-(\xi^-)(\partial\xi^-)^2\right) + S^+_{\text{matter}}\left(A_+(\xi^+)g_{\mu\nu}\right) + S^-_{\text{matter}}\left(A_-(\xi^-)g_{\mu\nu}\right) \tag{2.38}$$

to second order around $g_{\mu\nu} = \eta_{\mu\nu}$, $\xi^{\pm} = 0$. Here matter on each brane is minimally coupled to the indicated metric. We find that one can identify

$$F_{\pm} = \pm w_{\pm} a^2 \big(\xi_0^{\pm} + \xi^{\pm}\big) \tag{2.39}$$

and the sigma model coefficients

$$B_{\pm} = \mp 6k^2 w_{\pm} a^2 \big(\xi_0^{\pm} + \xi^{\pm}\big). \tag{2.40}$$

The coupling functions to matter are given by

$$A_{\pm} = a^2 \big(\xi_0^{\pm} + \xi^{\pm}\big) \tag{2.41}$$

implying that matter couples to the induced metric on each brane. Notice that in the $\alpha \to 0$ and $\beta_{\pm} \to 0$ limits, one obtains the scalar–tensor theory corresponding to the Randall–Sundrum case.

A quick glance at the action that we have just derived seems to indicate that there are two scalar degrees of freedom while there is only one effective scalar degree of freedom in the R–S case. To determine the structure of the effective action, it is convenient to go to the Einstein frame where the Planck mass is fixed. In the following we will assume that

$$w_+ a_+^2 > w_- a_-^2 \tag{2.42}$$

guaranteeing that the squared effective Planck mass is positive in the brane frame (and thus in all frames so that the graviton is not a ghost). The corresponding Einstein frame action is the quadratic expansion, around $g_{\mu\nu} = (F_+^0 - F_-^0)\eta_{\mu\nu}$ and $\xi^{\pm} = 0$, of

$$S_{\mathrm{EF}} = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g} \big(\mathcal{R} - \sigma_{ij} \partial\xi^i \partial\xi^j\big) + S_{\mathrm{mat}}^{+}\left(\frac{A_+}{F_+ - F_-} g_{\mu\nu}\right) + S_{\mathrm{mat}}^{-}\left(\frac{A_-}{F_+ - F_-} g_{\mu\nu}\right) \tag{2.43}$$

with

$$\sigma_{ij} = \begin{pmatrix} \frac{3}{2}\big(\frac{F_+'}{F_+ - F_-}\big)^2 + \frac{B_+}{F_+ - F_-} & -\frac{3}{2}\frac{F_+' F_-'}{(F_+ - F_-)^2} \\ -\frac{3}{2}\frac{F_+' F_-'}{(F_+ - F_-)^2} & \frac{3}{2}\big(\frac{F_-'}{F_+ - F_-}\big)^2 + \frac{B_-}{F_+ - F_-} \end{pmatrix}, \tag{2.44}$$

and $i, j = 1, 2 = +, -$. This sigma model matrix simplifies drastically in our case and takes the form

$$\sigma_{ij} = \frac{6k^2 F_+ F_-}{(F_+ - F_-)^2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{2.45}$$

It is easy to see that this matrix has a zero eigenvalue leading to the presence of only one physical scalar degree of freedom, the radion $r = R + \xi^- - \xi^+$, where $R = \xi_0^- - \xi_0^+$ is the unperturbed interbrane distance. Therefore, the action is the quadratic expansion of

$$S_{\mathrm{EF}} = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g} \left(\mathcal{R} - \frac{6k^2 w_- a^2(r)}{w_+ \big(1 - \frac{w_-}{w_+} a^2(r)\big)^2} (\partial r)^2\right) + S_{\mathrm{mat}}^{+}\left(\frac{g_{\mu\nu}}{w_+ - a^2(r) w_-}\right) + S_{\mathrm{mat}}^{-}\left(\frac{g_{\mu\nu}}{a^{-2}(r) w_+ - w_-}\right). \tag{2.46}$$

Requiring that the radion is not a ghost implies that

$$w_+ w_- > 0. \tag{2.47}$$

When the graviton and the radion are not ghosts, the low energy effective action provides useful information on the Gauss–Bonnet brane world at low energy.

Let us assume that $|w_-| \leqslant |w_+|$ and define

$$\sqrt{\frac{w_-}{w_+}}\, e^{-kr} = \tanh\rho. \tag{2.48}$$

The effective action becomes now (the quadratic expansion of)

$$S_{\mathrm{EF}} = \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-g}\,\big(\mathcal{R} - 6(\partial\rho)^2\big) + S^+_{\mathrm{mat}}\left(\frac{\cosh^2\rho}{w_+} g_{\mu\nu}\right) + S^-_{\mathrm{mat}}\left(\frac{\sinh^2\rho}{w_-} g_{\mu\nu}\right). \tag{2.49}$$

Notice that the only difference with the RS effective action resides in the prefactors $w_\pm$ in the coupling of the radion to matter. When these prefactors are equal to unity, the effective action is the RS one as derived within the moduli space approximation. We will compare the effective action obtained from the linear equations of motion and the moduli space approximation in the following section.

The coupling to gravity has to be such that the presence of a massless degree of freedom does not modify gravity. To carry out this analysis, it is convenient to use another form of the action. The action can be put in the Brans–Dicke form using the metric on each brane as the gravitational field. For the positive tension brane matter, the action becomes (the quadratic expansion of)

$$S^+_{\mathrm{BD}} = \frac{w_+}{2k\kappa_5^2}\int d^4x\,\sqrt{-g}\left(\Psi\mathcal{R} - \frac{\omega_+(\Psi)}{\Psi}(\partial\Psi)^2\right) + S^+_{\mathrm{mat}}(g_{\mu\nu}) + S^-_{\mathrm{mat}}\big(A^{\mathrm{BD}}_-(\Psi) g_{\mu\nu}\big) \tag{2.50}$$

where the Brans–Dicke field is

$$\Psi = 1 - \frac{w_-}{w_+} e^{-2kr} \tag{2.51}$$

with a Brans–Dicke parameter

$$\omega_+(\Psi) = \frac{3}{2}\frac{\Psi}{1-\Psi} \tag{2.52}$$

and a coupling to matter of the second brane

$$A^{\mathrm{BD}}_-(\Psi) = \frac{w_+}{w_-}(1-\Psi). \tag{2.53}$$

Notice that the Brans–Dicke parameter can be arbitrarily large when the branes are far apart. Hence ordinary matter can be located on the positive tension brane. This coincides with the usual R–S result.

Similarly for the negative tension brane this is the second order expansion of

$$S^-_{\mathrm{BD}} = \frac{w_-}{2k\kappa_5^2}\int d^4x\,\sqrt{-g}\left(\Phi\mathcal{R} - \frac{\omega_-(\Phi)}{\Phi}(\partial\Phi)^2\right) + S^+_{\mathrm{mat}}\big(A^{\mathrm{BD}}_+(\Phi) g_{\mu\nu}\big) + S^-_{\mathrm{mat}}(g_{\mu\nu}) \tag{2.54}$$

with a Brans–Dicke parameter

$$\omega_-(\Phi) = -\frac{3}{2}\frac{\Phi}{1+\Phi} \tag{2.55}$$

and a coupling to matter

$$A^{\mathrm{BD}}_+(\Phi) = \frac{w_-}{w_+}(1+\Phi), \tag{2.56}$$

where the Brans–Dicke field is

$$\Phi = e^{2kr}\,\frac{w_+}{w_-} - 1. \tag{2.57}$$

Notice that the Brans–Dicke parameter is here negative for large brane distances, ruling out the possibility of having ordinary matter on the second brane.

2.4. The projective approach

The previous action can be retrieved using the projective approach [3,6], in which the Einstein equations on both branes are written in terms of the matter energy–momentum tensors and the projected Weyl tensor $E_{\mu\nu}$. Eliminating the projected Weyl tensor between the two brane equations leads to the effective Einstein equation on either brane. Here we will concentrate on the case $\alpha = 0$ for simplicity. The projective approach for Gauss–Bonnet branes has been first considered in [21]. This general case is beyond the scope of the present Letter. At low energy one can neglect the quadratic terms in the matter content of the branes. The Einstein equation on the first brane reads

$$w_+ G_{\mu\nu}\big(\bar{g}^+_{\mu\nu}\big) = k\kappa_5^2 T^+_{\mu\nu} - E_{\mu\nu}, \tag{2.58}$$

where we have indicated the dependence on the induced metric explicitly, and the contribution in β_+ comes from the brane curvature term. Similarly, on the second brane,

$$w_- G_{\mu\nu}\big(\bar{g}^-_{\mu\nu}\big) = k\kappa_5^2 T^-_{\mu\nu} - \frac{E_{\mu\nu}}{\Omega^4}, \tag{2.59}$$

where $\Omega = a_-/a_+$ corresponds to the radion field. Using $\bar{g}^-_{\mu\nu} = \Omega^2 \bar{g}^+_{\mu\nu}$ to lowest order in a derivative expansion, one can eliminate $E_{\mu\nu}$ and obtain the Einstein equation

$$\begin{aligned} G_{\mu\nu}(\bar{g}^+_{\mu\nu}) = {} & \frac{\kappa_4^2}{\Psi}\left(T^+_{\mu\nu} + \frac{w_+}{w_-}(1-\Psi)T^-_{\mu\nu}\right) \\ & + \frac{\omega(\Psi)}{\Psi^2}\left(D_\mu \Psi D_\nu \Psi - \frac{1}{2}(D\Psi)^2 \bar{g}^+_{\mu\nu}\right) + \frac{1}{\Psi}\big(D_\mu D_\nu \Psi - D^2\Psi \bar{g}^+_{\mu\nu}\big), \end{aligned} \tag{2.60}$$

which coincides with the Einstein equations deduced from the effective action obtained in the previous section. Hence, the projective approach leads to the same results as the linear equations of motion.

2.5. The failure of the moduli space approximation

In the RS case, it has been shown that the effective action can also be deduced using the moduli space approximation. In this section, we examine the validity of the moduli space approximation for Gauss–Bonnet brane worlds, i.e., keeping only the massless degrees of freedom represented here by the 4D metric $g_{\mu\nu}$ plus two real 4D scalar fields giving the brane positions in the fifth dimension. Within this approximation, one obtains the Einstein frame effective action

$$S = \frac{1}{2k\kappa_5^2}\int d^4x\,\sqrt{-g}\big[\mathcal{R} - \gamma_{\sigma\sigma}(\partial\sigma)^2 - \gamma_{\rho\rho}(\partial\rho)^2 - 2\gamma_{\rho\sigma}(\partial\rho)(\partial\sigma)\big] \tag{2.61}$$

with the normalization matrix

$$\begin{aligned} \gamma_{\sigma\sigma} &= \frac{96\alpha k^2 + 6\beta_+ k}{1 + 4\alpha k^2 + \beta_+ k}\cosh^2(\rho) - \frac{96\alpha k^2 - 6\beta_- k}{1 + 4\alpha k^2 - \beta_- k}\sinh^2(\rho), \\ \gamma_{\rho\rho} &= \frac{6 - 72\alpha k^2}{1 + 4\alpha k^2 - \beta_- k}\cosh^2(\rho) - \frac{6 - 72\alpha k^2}{1 + 4\alpha k^2 + \beta_+ k}\sinh^2(\rho), \\ \gamma_{\rho\sigma} &= \frac{k(\beta_+ + \beta_-)(6 - 72\alpha k^2)}{(1 + 4\alpha k^2 + \beta_+ k)(1 + 4\alpha k^2 - \beta_- k)}\sinh(\rho)\cosh(\rho). \end{aligned} \tag{2.62}$$

As can be easily seen the sigma model matrix is of rank two, leading to the existence of two massless degrees of freedom in the scalar sector. This contradicts the linear equations and therefore invalidates the moduli approximation in the Gauss–Bonnet case. The failure of the moduli space approximation here, and the non-equivalence with the projective approach deserves further study. In particular, its link with either the presence of higher derivative terms or the necessity of extending the moduli ansatz needs to be investigated. This is left for future work.

3. Conclusion

We have analysed brane worlds with a bulk Gauss–Bonnet term and induced brane gravity terms. We have studied the high energy and low energy limits. In particular, we have shown that the low energy effective action involves only one field, the radion, and differs from the RS case. The difference with the RS case arises in the coupling of the radion to matter and the value of the effective Planck mass.

We have also noted that the moduli approximation fails for Gauss–Bonnet brane worlds. Indeed, it fails to reproduce the linear equations of motion and involves a spurious scalar degree of freedom. This means that dimensional reduction does not commute with taking the equations of motions from the action; the correct procedure consists in first taking the higher-dimensional equations of motion and then dimensionally reducing them. Similar cases of non-commutativity have been described in [22] where it is specifically due to the Gauss–Bonnet term, or in [23] where it has been shown more generally that it can arise from symmetries of the equations of motion which are not symmetries of the action. In this context, a better understanding of the link between the moduli approximation and the projective approach deserves to be further investigated and is left for future work.

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