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A multifacility location problem on median spaces

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Abstract

This paper is concerned with the problem of locating n new facilities in the median space when there are k facilities already located. The objective is to minimize the weighted sum of distances. Necessary and sufficient conditions are established. Based on these results a polynomial algorithm is presented. The algorithm requires the solution of a sequence of minimum-cut problems. The complexity of this algorithm for median graphs and networks and for finite median spaces with $|V|$ points is $O(|V|^3 + |V|\psi(n))$, where $\psi(n)$ is the complexity of the applied maximum-flow algorithm. For a simple rectilinear polygon P with N edges and equipped with the rectilinear distance the analogical algorithm requires $O(N + k(\log N + \log k + \psi(n)))$ time and $O(N + k\psi(n))$ time in the case of the vertex-restricted multifacility location problem.

1. Introduction

In this paper we discuss the multifacility location problem on median spaces. This is a well-known problem in location theory and its solving in different classes of metric spaces is a topic that has received considerable attention. Recall that the classical *multifacility location problem* is as follows.

Let (S, r) be a metric space and let Y be a set of fixed (old) facilities which are located at points $y_1, \ldots, y_k \in S$. The problem is to determine the location of n variable points $x_1, \ldots, x_n$ such that the following sum is minimized

$$(P) \qquad \bar{F}(x_1, \ldots, x_n) = \sum_{j=1}^{k} \sum_{i=1}^{n} w_{ij} r(x_i, y_j) + (1/2) \sum_{j=1}^{n} \sum_{i=1}^{n} v_{ij} r(x_i, x_j),$$

where w_{ij} and v_{ij} are nonnegative weights and $v_{ij} = v_{ji}$ for all i and j.

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The multifacility location problem involving Euclidean distances can be solved using an extension of the well-known Weiszfeld algorithm; consult [29, 41]. However, unlike the Weiszfeld algorithm for single facility location problem the iterations obtained by this procedure converge only under some additional hypothesis; see [41, 43]. Several authors [29, 38] use the generalized Weiszfeld algorithm for solving the multifacility location problem with L_p-distances; see also [37] where optimality conditions has been given for problem (P) involving any norm.

The rectilinear multifacility location problem is decomposable into one-dimensional problems, which can be viewed as instances of a multifacility problem on a line. Variations of this one-dimensional problem have been investigated by a number of researchers; see [13, 18, 21, 20, 29, 32, 36, 42, 49, 54, 55]. The majority of these approaches give algorithms of the same complexity $O(dkn^3)$ for space R^d with L_1-metric.

Now consider the multifacility location problem on discrete (finite) metric spaces and networks. The problem on general graphs and networks was shown to be NP-hard [36, 48]. As well as for other optimization problems involving network distances this problem is solved only in the case where spaces are trees or tree-like. For trees all approaches used for solving the multifacility problem on a line may be applied; consult [20, 21, 36, 48]. For a review of location problems on networks consult [48]. This survey deals with the following problem: "What special network structure, more general than a tree structure, can be exploited to facilitate the analysis of network location problems?".

In this paper we study the multifacility location problem on median metric spaces. As with the one-dimensional problem [18, 42, 49] our approach determines an optimal solution by solving a sequence of minimum cut problems, each on a network containing at most $n + 2$ vertices. We reduce the problem on a general median space to a similar problem on a median graph. Using the general approach we obtain an $O(|V|^3 + |V|n^3)$-time algorithm for solving the multifacility problem on a median graph with $|V|$ vertices. The same complexity algorithm is derived for this problem on a median network with $|V|$ vertices and on a median space with $|V|$ points. Applying some results of computational geometry we present an $O(N + k(\log N + \log k + n^3))$-time algorithm for this problem on a simple rectilinear polygon P with N edges and endowed with rectilinear distance. We use also the fact that P is a median space. We conclude the paper with a discussion of computational problems which appear for solving multifacility and other metric problems on cubical polyhedrons and other multidimensional median spaces. The obtained results generalize the author's results from [16]. Note also that the single facility location problem on median networks and discrete median spaces has been considered in papers [4, 5, 15, 47]; consult also [9, 10] for application of this problem in group choice and cluster analysis. An algorithm for solving the single facility location problem on a simple rectilinear polygon is presented in [17].

The rest of the paper is organized as follows. In Section 2 we give a problem formulation. Section 3 introduces some terminology and preliminary results on median spaces and median graphs. Section 4 presents the optimality conditions for multifacility location on median graphs. In this section we also reduce the problem on general median spaces to a similar problem on median graphs. Section 5 proves the results that will be needed later in the proof of the correctness of our algorithm. Section 6 presents an algorithm for solving the multifacility location problem on median graphs and finite median spaces. Section 6.2 deals with the case of simple rectilinear polygons. Section 7 concludes.

2. Problem formulation

Now we present a slightly modified version of a multifacility location problem; see also [37].

Let $Y = \{y_1, \ldots, y_k\}$ be a set of fixed facilities of a space (S, r) and let $X = \{x_1, \ldots, x_n\}$ denote the location of new facilities. Put $J = \{1, \ldots, k\}$ and $I = \{1, \ldots, n\}$. Assume also a graph Γ is given with vertex set $I \cup J$ and edge set $E_\Gamma = E' \cup E''$, where E' denotes the set of edges connecting new facilities with fixed facilities and E'' denotes the set of edges connecting two new facilities. The problem can be written as follows [37]:

$$(\mathscr{P}) \qquad \bar{F}(X) = \sum_{(i,j)\in E'} w_{ij} r(x_i, y_j) + (1/2) \sum_{(i,j)\in E''} v_{ij} r(x_i, x_j),$$

where w_{ij} and v_{ij} are strictly positive weights associated with the corresponding edges and $v_{ij} = v_{ji}$.

Note that $(\mathscr{P})$ is a refined version of the problem (P). This allow us to use the more efficient algorithms for minimum cut (maximum flow) problems. So, in all sections, except the section with algorithms, we consider the standard version of a multifacility location problem, i.e. problem (P).

3. Median spaces

Let (S, r) be a metric space. The *interval* xy between two points $x, y \in S$ is defined by

$$xy = \{z \in S\colon r(x,z) + r(z,y) = r(x,y)\}.$$

A metric space (S, r) is called *median* if for any triple x, y, z of points intersection $xy \cap yz \cap zx$ is a singleton, that is, there is a unique "median" point $m(x,y,z)$ between each pair of x, y, z [3, 8, 33, 40, 45, 51]. The median spaces represent a common generalization of different mathematical structures such as median semilattices (including distributive lattices) [3, 8, 45], median algebras [8, 33, 45], median graphs (including trees and hypercubes) [8, 40], $L_1(\mu)$-spaces [51, 52], acyclic cubical complexes [7], median complexes and median cubical polyhedrons [51, 52] and some classes of convexity structures [50, 51]. Between all these types of mathematical objects, close relationships has been established. For classical results on median spaces the reader is referred to [8, 33, 40, 51]. We present only those properties of median spaces and median graphs which are needed in the sequel.

Recall that a set M of a space (S, r) is *convex* if for any points $x, y \in M$ and $z \in S$ the equality $r(x, z) + r(z, y) = r(x, y)$ implies that $z \in M$. A subset H of S is a *half-space* provided both H and $S \backslash H$ are convex. Following [51], the convexity on a space (S, r) is a *S_2-convexity* if any two distinct points may be separated by complementary half-spaces, i.e. if x and y are distinct points then there is a half-space H of S with $x \in H$ and $y \in S \backslash H$.

A set M of a median space (S, r) is *median stable* [51] provided $m(x, y, z) \in M$ for any triple $x, y, z \in M$. The intersection of median stable sets is a median stable set too. So, for any $N \subset S$ we can define the smallest median stable set containing N. It is known that for any finite set $N \subset S$ this set is finite too; see [51, p. 130, Lemma 6.20(2)].

Recall that a subset M of a space (S, r) is a *gated set* [27, 30] if for every point x outside M there exists a (necessarily unique) point $x_M \in M$ such that $x_M \in xy$ for all $y \in M$. The point x_M is called the *gate* for x in M. Any gated set of a metric space is convex [30]. The converse holds for median spaces.

Lemma 1. *Let (S, r) be a median space. Then*
(1) *any closed convex set $M \subset S$ is gated;*
(2) *the convexity in S is an S_2-convexity;*
(3) *(Helly property) any finite collection of pairwise intersecting convex sets has a common point.*

For a proof of these properties of median spaces see for example [51].

Now we present some properties of median graphs. The graph $G = (V, E)$ is *median* if its shortest-path metric d defines a median space (V, d). In other words, the graph G is median if for any triple x, y, z there exists a unique vertex $m(x, y, z)$ which is simultaneously on shortest path from x to y, y to z and z to x [40]. The interval between vertices x and y of a graph G we will denote by $I(x, y)$.

Let (S, r) be a finite median space. An interval xy is called an edge if $xy = \{x, y\}$ and points x and y are distinct; the edges then form the *graph* of the space (S, r).

Lemma 2 (Avann [3] and Bandelt and Hedlíková [5]). *The underlying graph G of a finite median space (S, r) is a median graph and $z \in xy$ if and only if $z \in I(x, y)$.*

For vertices x, y of a graph G we put

$$W(x, y) = \{z \in V\colon x \in I(y, z)\}, \quad W(y, x) = \{z \in V\colon y \in I(x, z)\}.$$

An *isometric embedding* of a graph $G = (V, E)$ in a graph $G' = (V', E')$ is a map from V to V' which preserves distances. The *hypercube Q_d* of dimension d (the d-cube, for short) has (0, 1)-vertices of length d as vertices, two vertices being joined if they differ in exactly one coordinate.

Lemma 3. *Let $G = (V, E)$ be a median graph. Then*
(1) *G is bipartite* [40];
(2) *for any two adjacent vertices x and y the sets $W(x,y)$ and $W(y,x)$ are gated complementary half-spaces* [8,40];
(3) *G is isometrically embeddable in a hypercube* [40].

Now we present one of the existing isometric embeddings of a median graph G in a hypercube, the one more suitable for us. For other such embeddings see [2, 40, 47]. Note also that in [2] an $O(|V||E|)$-time algorithm for this problem was developed.

Following [26], we define a binary relation θ on the set E of edges of a median graph G. Let $e_1 = (x_1, y_1)$ and $e_2 = (x_2, y_2)$ be edges of G. We say that $e_1 \theta e_2$ if and only if $x_2 \in W(x_1, y_1)$ and $y_2 \in W(y_1, x_1)$. It is clear that θ is reflexive. By convexity of the sets $W(x_1, y_1)$ and $W(x_2, y_2)$ we conclude that $W(x_1, y_1) = W(x_2, y_2)$ and $W(y_1, x_1) = W(y_2, x_2)$, i.e. θ is also symmetric and transitive. Hence, θ is an equivalence relation [26]. Let $E_1, \ldots, E_m$ be the set of equivalence classes, i.e. edge set of a graph G partitions into m sets of equivalent edges. Each equivalence class E_i defines a pair of complementary half-spaces W_i^1 and W_i^2, where $W_i^1 = W(x,y)$ and $W_i^2 = W(y,x)$ for each edge $(x,y) \in E_i$.

Let G be a graph of a finite median space (S, r). Denote by $r(e)$ the length of the edge $e = (x,y)$ of a space S. By Lemma 2 we conclude that $r(e) = r(e')$ for all pairs e, e' of equivalent edges. Let r_i be the length of any edge from the equivalence class E_i. For all pairs x, y of vertices of a graph G define the numbers $t^i(x,y) = 0$ if $x, y \in W_i^1$ or $x, y \in W_i^2$ and $t^i(x,y) = 1$ otherwise.

Lemma 4 (Soltan and Chepoi [47]). *Let (S, r) be a finite median space and G be a graph of S. Then for all vertices $x, y \in S$ we have*

$$d(x,y) = \sum_{i=1}^{m} t^i(x,y), \quad r(x,y) = \sum_{i=1}^{m} r_i t^i(x,y).$$

For any vertex x of a median graph G define $\beta(x) = (r^1, \ldots, r^m)$, where $r^i = 0$, if $x \in W_i^1$ and $r^i = 1$ if $x \in W_i^2$.

Lemma 5 (Soltan and Chepoi [47]). *The map β is an isometric embedding of a median graph in the m-cube Q_m.*

4. Problem reduction and optimality conditions

For a set Y of fixed facilities we denote by V some finite median stable set of (S, r) containing Y. Then (V, r) is a finite median space. Let $G = (V, E)$ be the underlying median graph of this space. Denote by (P_s) and (P_g) the multifacility location problems with the set of fixed facilities Y in space (V, r) and graph G, respectively. Explicitly we formulate the second problem only:

$$(P_g) \qquad F(X) = \sum_{(i,j)\in E_1} w_{ij} r(x_i, y_j) + (1/2) \sum_{(i,j)\in E_2} v_{ij} r(x_i, x_j).$$

In a similar way we can define the problems $(\mathscr{P}_s)$ and $(\mathscr{P}_g)$.

Our immediate goal is to show that optimal solutions of problems (P_s) and (P_g) coincide and that any optimal location for (P_s) is optimal for initial problem (P) too. Also we formulate the optimality conditions for (P_g).

4.1. Equivalence classes and q-local networks

Let $X = \{x_1, \dots, x_n\}$ be any solution, not necessarily optimum, to problem (P_g). For $q = 1, \dots, m$ if we put

$$X_q^1 = \{i\colon x_i \in W_q^1\}, \qquad X_q^2 = \{i\colon x_i \in W_q^2\},$$

then we can write the general expression for the "contribution" of the equivalence class E_q as

$$\begin{aligned} C_q(X_q^1, X_q^2) &= \sum_{i\in X_q^1} \sum_{y_j \in W_q^2} w_{ij} + \sum_{i\in X_q^2} \sum_{y_j\in W_q^1} w_{ij} + \sum_{i\in X_q^1} \sum_{j\in X_q^2} v_{ij} \\ &= \sum_{i\in X_q^1} w_i(W_q^2) + \sum_{i\in X_q^2} w_i(W_q^1) + \sum_{i\in X_q^1} \sum_{j\in X_q^2} v_{ij}. \end{aligned}$$

(For a subset $M \subset V$ by $w_i(M) = \sum_{y_j\in M} w_{ij}$ we denote the i-weight of M.)

Lemma 6. (1) *For the problem* (P_s) *we have*

$$\bar{F}(X) = \sum_{q=1}^{m} C_q(X_q^1, X_q^2) r_q.$$

(2) *In particular, for problem* (P_g) *we obtain*

$$F(X) = \sum_{q=1}^{m} C_q(X_q^1, X_q^2).$$

Proof. By Lemma 4 for any vertices $u, u' \in V$ it is true that

$$r(u, u') = \sum_{q=1}^{m} t^q(u, u') r_q.$$

Therefore

$$\begin{aligned} \bar{F}(X) &= \sum_{i=1}^{n} \sum_{j=1}^{k} w_{ij} \left[\sum_{q=1}^{m} t^q(x_i, y_j) r_q \right] + (1/2) \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} \left[\sum_{q=1}^{m} t^q(x_i, x_j) r_q \right] \\ &= \sum_{q=1}^{m} \left[\sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} t^q(x_i, y_j) + (1/2) \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} t^q(x_i, x_j) \right] r_q = \sum_{q=1}^{m} F_q(X) r_q. \end{aligned}$$

Since $t^q(u,u') = 0$ if vertices u and u' belong to one and the same half-spaces W_q^1 or W_q^2 and $t^q(u,u') = 1$ otherwise, we can rewrite $F_q(X)$ as follows:

$$F_q(X) = \sum_{x_i \in W_q^1} w_i(W_q^2) + \sum_{x_i \in W_q^2} w_i(W_q^1) + \sum_{x_i \in W_q^1} \sum_{x_i \in W_q^2} v_{ij}$$

$$= \sum_{i \in X_q^1} w_i(W_q^2) + \sum_{i \in X_q^2} w_i(W_q^1) + \sum_{i \in X_q^1} \sum_{i \in X_q^2} v_{ij} = C_q(X_q^1, X_q^2).$$

This concludes the proof. □

For each equivalence class E_q, $q = 1, \ldots, m$, consider a network N_q having vertices $s, 1, \ldots, n, t$, where each vertex i corresponds to new facility i for $i = 1, \ldots, n$. Define undirected arcs (s,i) with capacities $c(s,i) = w_i(W_q^1)$ for $i = 1, \ldots, n$, undirected arcs (t,i) with capacities $c(t,i) = w_i(W_q^2)$ for $i = 1, \ldots, n$, and undirected arcs (i,j) with capacities $c(i,j) = v_{ij}$ for all distinct i and j. Following [42] we will call this network N_q the *q-local network*. For problem $(\mathscr{P}_g)$ the q-local network is defined in a similar way, except the fact that arcs (i,j) are defined only if (i,j) is an edge of a graph Γ.

If we partition the vertices $\{1, \ldots, n\}$ of the q-local network N_q into two sets Z_q^1 and Z_q^2, then the sets $Z_q^1 \cup \{s\}$ and $Z_q^2 \cup \{t\}$ define a *cut* in N_q. The arcs in this cut are those with one end in $Z_q^1 \cup \{s\}$ and the other end in $Z_q^2 \cup \{t\}$. We will denote the cut by (Z_q^1, Z_q^2). The *capacity* of the cut (Z_q^1, Z_q^2) is defined to be the sum of the capacities of all arcs in the cut and is denoted by $C_q(Z_q^1, Z_q^2)$. We will refer to the cut having minimum capacity as the *minimum cut*.

Remark 1. The capacity of any cut (Z_q^1, Z_q^2) in the q-local network is given by formula for $C_q(X_q^1, X_q^2)$ with $X_q^1 = Z_q^1$ and $X_q^2 = Z_q^2$.

Remark 2. Defining sets X_q^1 and X_q^2 that minimize $C_q(X_q^1, X_q^2)$ is equivalent to finding a minimum cut in the q-local network N_q.

Remark 3. If all fixed facilities are contained in one of the complementary half-spaces W_q^1 and W_q^2 then in the network N_q vertices s and t are disconnected and the minimum cut in N_q is the cut with capacity zero.

We conclude with the following definition. Let $X = \{x_1, \ldots, x_n\}$ be any solution, not necessarily optimum to problem (P). Solution X is *q-optimal* if the cut (X_q^1, X_q^2) defined by X is a minimum cut of a q-local network.

4.2. Optimality conditions

We begin with an auxiliary result. Let E_q be an equivalence class of a median graph G. Let also $Z = \{z_1, \ldots, z_p\}$ be some subset of a half-space W_q^1. Denote by v_i the gate for vertex z_i in the other half-space W_q^2. Put $d(Z) = \max\{d(z_i, v_i)\colon i = 1, \ldots, p\}$.

Lemma 7. *There exist vertices $u_i \in I(z_i, v_i)$ adjacent to z_i such that $d(u_i, u_j) \leqslant d(z_i, z_j)$ for all $i, j \in \{1, \dots, p\}$.*

Proof. Assume the contrary and choose the smallest subset Z at the minimal distance $d(Z)$ from W_q^2 for which our assumptions fail. Pick the vertex z_t and let u_t be the vertex adjacent to z_t in the interval $I(z_t, v_t)$. Suppose that the edge (z_t, u_t) belongs to the equivalence class E_s and let $z_t \in W_s^1$ and $u_t \in W_s^2$. Since $u_t \in I(z_t, v_t)$ and for any vertex $w \in W_q^2$ we have $I(z_t, v_t) \subset I(z_t, w)$ then we obtain that $W_q^2 \subset W_s^2$.

First we consider the case when all vertices $z_1, \dots, z_p$ belong to the half-space W_s^1. Denote by v_i' the gate for z_i in W_s^2. In particular $v_t' = u_t$. If $d(Z) = 1$ then $q = s$ and $v_i = v_i'$ for $i = 1, \dots, p$. In this case vertices z_i and v_i are adjacent. From the fact that W_q^1 and W_q^2 are gated sets we conclude that $d(v_i, v_j) = d(z_i, z_j)$ for all $i, j \in \{1, \dots, p\}$. So assume that $d(Z) > 1$. Then

$$\max\{d(z_i, v_i'): i = 1, \dots, p\} < d(Z).$$

From the choice of set Z and equivalence class E_q we conclude that there exist vertices $u_i \in I(z_i, v_i')$ with the desired property. Since $I(z_i, v_i') \subset I(z_i, v_i)$ we get a contradiction with our choice.

Now assume that $z_1, \dots, z_t \in W_s^1$ and $z_{t+1}, \dots, z_p \in W_s^2$. As in the first case denote by v_i' the gate of vertex z_i in the half-space W_s^2. By our assumption we deduce that there exist vertices $u_i \in I(z_i, v_i')$ adjacent to z_i, such that $d(u_i, u_j) \leqslant d(z_i, z_j)$ for all $i, j \in \{1, \dots, t\}$. By the same argument there exist vertices $u_{t+i} \in I(z_{t+i}, v_{t+i})$ adjacent to z_{t+i} such that $d(u_{t+i}, u_{t+j}) \leqslant d(z_{t+i}, z_{t+j})$ for all $i, j \in \{1, \dots, p-t\}$. Now we will show that for any $i \in \{1, \dots, t\}$ and for any $j \in \{1, \dots, p-t\}$ we have $d(u_i, u_{t+j}) \leqslant d(z_i, z_{t+j})$. Since $u_i \in I(z_i, v_i')$ and $v_i' \in I(z_i, z_{t+j})$ then $d(z_i, z_{t+j}) = d(u_i, z_{t+j}) + 1$. On the other hand, since z_{t+j} and u_{t+j} are adjacent then $d(u_i, u_{t+j}) \leqslant d(u_i, z_{t+j}) + 1$. From these two expressions we immediately conclude that $d(u_i, u_{t+j}) \leqslant d(z_i, z_{t+j})$. So our initial assumption lead us to the contradiction. □

Let $X = \{x_1, \dots, x_n\}$ be any solution, not necessarily optimum, to problem (P_g). Pick any equivalence class E_q of a graph G. For any $i \in X_q^1$ denote by v_i the gate for x_i in the half-space W_q^2. In a similar way for any $i \in X_q^2$ let v_i be the gate for vertex x_i in W_q^1. Now consider any cut (R_q^1, R_q^2) of the q-local network N_q. Define a new solution $Z = \{z_1, \dots, z_n\}$ using the following rules:

(A1) for any $i \in X_q^1 \cap R_q^2$ let z_i be the vertex adjacent to x_i in the interval $I(x_i, v_i)$ such that $d(z_i, z_j) \leqslant d(x_i, x_j)$ for all $i, j \in X_q^1 \cap R_q^2$;

(B1) for any $i \in X_q^2 \cap R_q^1$ let z_i be the vertex adjacent to x_i in the interval $I(x_i, v_i)$ such that $d(z_i, z_j) \leqslant d(x_i, x_j)$ for all $i, j \in X_q^2 \cap R_q^1$;

(C1) $z_i = x_i$ for any $i \in (X_q^1 \cap R_q^1) \cup (X_q^2 \cap R_q^2)$.

Existence of such a solution Z follows from Lemma 7.

Lemma 8. $F(Z) - F(X) \leqslant C_q(R_q^1, R_q^2) - C_q(X_q^1, X_q^2)$.

Proof. For any fixed facility y_j denote $\delta_i(y_j) = d(z_i, y_j) - d(x_i, y_j)$. Put also $\delta_{ij} = d(z_i, z_j) - d(x_i, x_j)$, $i, j \in \{1, \ldots, n\}$. Substituting this in the parts of the required inequality yields

$$F(Z) - F(X) = F_1 + F_2, \quad C_q(R_q^1, R_q^2) - C_q(X_q^1, X_q^2) = L_1 + L_2,$$

where

$$F_1 = \sum_{i=1}^{n} \sum_{j=1}^{k} w_{ij}\delta_i(y_j), \quad F_2 = (1/2) \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij}\delta_{ij},$$

$$L_1 = \sum_{i \in R_q^1 \cap X_q^2} [w_i(W_q^2) - w_i(W_q^1)] + \sum_{i \in R_q^2 \cap X_q^1} [w_i(W_q^1) - w_i(W_q^2)],$$

$$L_2 = \sum_{i \in R_q^1} \sum_{j \in R_q^2} v_{ij} - \sum_{i \in X_q^1} \sum_{j \in X_q^2} v_{ij}.$$

Observe that $\delta_i(y_j) = 0$ if $i \in (R_q^1 \cap X_q^1) \cup (R_q^2 \cap X_q^2)$. Now suppose that $i \in (R_q^1 \cap X_q^2) \cup (R_q^2 \cap X_q^1)$. In this case $\delta_i(y_j) = 1$ if $y_j \in W(x_i, z_i)$ and $\delta_i(y_j) = -1$ if $y_j \in W(z_i, x_i)$. Therefore we obtain the following:

$$F_1 = \sum_{i \in R_q^1 \cap X_q^2} [w_i(W(x_i, z_i)) - w_i(W(z_i, x_i))] + \sum_{i \in R_q^2 \cap X_q^1} [w_i(W(x_i, z_i)) - w_i(W(z_i, x_i))]$$

and

$$\begin{aligned} F_1 - L_1 = & \sum_{i \in R_q^1 \cap X_q^2} [w_i(W(x_i, z_i)) - w_i(W_q^2)] \\ & + \sum_{i \in R_q^1 \cap X_q^2} [w_i(W_q^1) - w_i(W(z_i, x_i))] \\ & + \sum_{i \in R_q^2 \cap X_q^1} [w_i(W(x_i, z_i)) - w_i(W_q^1)] \\ & + \sum_{i \in R_q^2 \cap X_q^1} [w_i(W_q^2) - w_i(W(z_i, x_i))]. \end{aligned}$$

From the definition of Z we get that

$$W_q^1 \subseteq W(z_i, x_i) \quad \text{and} \quad W(x_i, z_i) \subseteq W_q^2 \qquad \text{if } i \in R_q^1 \cap X_q^2,$$

and

$$W(x_i, z_i) \subseteq W_q^1 \quad \text{and} \quad W_q^2 \subseteq W(z_i, x_i) \qquad \text{if } i \in R_q^2 \cap X_q^1.$$

So, as all four sums in the last expression for $F_1 - L_1$ are nonpositive, we now come to the desired inequality $F_1 - L_1 \leqslant 0$.

For a proof of a second inequality $F_2 - L_2 \leqslant 0$ first we consider the coefficients δ_{ij}, $i, j \in \{1, \ldots, n\}$. Observe that $\delta_{ij} = 0$ if $i, j \in (R_q^1 \cap X_q^1) \cup (R_q^2 \cap X_q^2)$. Note also that $\delta_{ij} \leqslant 0$ if $i, j \in R_q^1 \cap X_q^2$ or $i, j \in R_q^2 \cap X_q^1$. This is a consequence of Lemma 7. Now assume that $i \in R_q^1 \cap X_q^2$ and $j \in R_q^2 \cap X_q^1$. From the same Lemma 7 and since W_q^1 and

W_q^2 are gated sets we conclude that $\delta_{ij} = 0$ if edges (x_i, z_i) and (x_j, z_j) belong to E_q and $\delta_{ij} = -2$ otherwise. Observe also that $\delta_{ij} = -1$ in any case when vertices x_i and x_j lie in different half-spaces W_q^1, W_q^2 and only one of these vertices is changed in Z. Therefore the coefficients δ_{ij} may be positive only when x_i and x_j lie in the same half-space W_q^1 and W_q^2 and only one of these vertices is changed in the new solution Z.

Now notice that F_2 may be represented as a sum of distinct terms, each of the type

$$\sum_{i \in R_q^a \cap X_q^b} \sum_{j \in R_q^c \cap X_q^d} v_{ij} \delta_{ij},$$

where $a, b, c, d \in \{1, 2\}$. By invoking the properties of δ_{ij} we obtain that all terms, for which $a + b \equiv c + d \equiv 0 (\mathrm{mod}\, 2)$, are equal to zero. From the same arguments we conclude that all terms, for which $a = c$ and $b = d$ or $a = d = 1$ and $b = c = 2$, are nonpositive. Therefore,

$$\begin{aligned} F_2 \leqslant & \sum_{i \in R_q^1 \cap X_q^2} \left[\sum_{j \in R_q^1 \cap X_q^1} v_{ij} \delta_{ij} + \sum_{j \in R_q^2 \cap X_q^2} v_{ij} \delta_{ij} \right] \\ & + \sum_{i \in R_q^2 \cap X_q^1} \left[\sum_{j \in R_q^1 \cap X_q^1} v_{ij} \delta_{ij} + \sum_{j \in R_q^2 \cap X_q^2} v_{ij} \delta_{ij} \right] \\ = & \sum_{i \in R_q^1 \cap X_q^2} \left[- \sum_{j \in R_q^1 \cap X_q^1} v_{ij} + \sum_{j \in R_q^2 \cap X_q^2} \{ v_{ij} \colon x_j \in W(x_i, z_i) \} \right. \\ & \left. - \sum_{j \in R_q^2 \cap X_q^2} \{ v_{ij} \colon x_j \in W(z_i, x_i) \} \right] \\ & + \sum_{i \in R_q^2 \cap X_q^1} \left[- \sum_{j \in R_q^2 \cap X_q^2} v_{ij} + \sum_{j \in R_q^1 \cap X_q^1} \{ v_{ij} \colon x_j \in W(x_i, z_i) \} \right. \\ & \left. - \sum_{j \in R_q^1 \cap X_q^1} \{ v_{ij} \colon x_j \in W(z_i, x_i) \} \right]. \end{aligned}$$

Now consider L_2. Observe that each of the expressions $\sum_{i \in R_q^1} \sum_{j \in R_q^2} v_{ij}$ and $\sum_{i \in X_q^1} \sum_{j \in X_q^2} v_{ij}$ may be represented as a sum of four terms, each of the type

$$\sum_{i \in R_q^1 \cap X_q^a} \sum_{j \in R_q^2 \cap X_q^b} v_{ij}, \qquad \sum_{i \in R_q^a \cap X_q^1} \sum_{j \in R_q^b \cap X_q^2} v_{ij} \qquad (a, b \in \{1, 2\}).$$

So we arrive at

$$L_2 = \sum_{i \in R_q^1 \cap X_q^2} \left[\sum_{j \in R_q^2 \cap X_q^2} v_{ij} - \sum_{j \in R_q^1 \cap X_q^1} v_{ij} \right] + \sum_{i \in R_q^2 \cap X_q^1} \left[\sum_{j \in R_q^1 \cap X_q^1} v_{ij} - \sum_{j \in R_q^2 \cap X_q^2} v_{ij} \right].$$

Comparing the expressions obtained for F_2 and L_2 we have established that each of the terms for F_2 is less than or equal to the respective term of L_2, i.e. $F_2 \leqslant L_2$. Hence $F_1 + F_2 \leqslant L_1 + L_2$ and

$$F(Z) - F(X) \leqslant C_q(R_q^1, R_q^2) - C_q(X_q^1, X_q^2),$$

as desired. □

Using this result we formulate the optimality conditions for the problem (P_g). Also we present a reduction of problems (P) and (P_s) to (P_g).

Let $Y = \{y_1, \ldots, y_k\}$ be the set of fixed facilities of a given median space (S, r). Recall that by V we denote some finite median stable set of S containing Y, and by $G = (V, E)$ we denote the median graph associated with space (V, r).

Theorem 1. (1) *$X = \{x_1, \ldots, x_n\}$ is an optimal solution to (P_g) if and only if for each $q = 1, \ldots, m$ the q-local network N_q has (X_q^1, X_q^2) as a minimum cut, i.e. X is q-optimal;*

(2) *X is optimum to (P_s) if and only if X is optimum to (P_g);*

(3) *if X is optimum to (P_s) then X is optimum to (P).*

Proof. (1) Assume that for some q the cut generated by X is not minimal in N_q. Consider a minimum cut (R_q^1, R_q^2) of the network N_q. According to Lemma 8 there exists a solution Z such that

$$F(Z) - F(X) \leqslant C_q(R_q^1, R_q^2) - C_q(X_q^1, X_q^2) < 0.$$

Hence X is not optimum to (P_g).

Now suppose that X is q-optimal for all $q = 1, \ldots, m$. By Lemma 6

$$F(X) = \sum_{q=1}^{m} C_q(X_q^1, X_q^2).$$

Therefore X_q^1 and X_q^2 minimize the qth term in the sum for $F(X)$. Since each term is minimized, then the sum of terms is also minimum.

(2) The fact that any optimum to (P_g) is an optimum to (P_s) is an immediate consequence of the first part of this theorem and Lemma 6(1). On the other hand, if X is an optimal solution for (P_g) and Z is a solution, not necessarily optimum, for (P_s) then $C_q(X_q^1, X_q^2) \leqslant C_q(Z_q^1, Z_q^2)$ for $q = 1, \ldots, m$. Since

$$\bar{F}(Z) = \sum_{i=1}^{m} C_q(Z_q^1, Z_q^2) r_q$$

then Z is an optimum for (P_s) only if $C_q(Z_q^1, Z_q^2) = C_q(X_q^1, X_q^2)$, $q = 1, \ldots, m$. Hence any optimum for (P_s) is an optimum for (P_g) too.

(3) Let X^* be an optimal solution to the multifacility location problem on space (S, r). Denote by V^* some finite median stable set of S, that contains sets X^* and V. Clearly X^* is an optimum for our problem and on median space (V^*, r). Let $E_1^*, \ldots, E_{m^*}^*$ be the equivalence classes of edges of the space (V^*, r). Since V is a median stable subset of (V^*, r) then (V, r) is an isometric subspace of (V^*, r). So, any equivalence class E_q of (V, r) is contained in some equivalence class of the space (V^*, r). Let us suppose, for example, that $E_1 \subseteq E_1^*, \ldots, E_m \subseteq E_m^*$. Note also that the complementary half-spaces W_q^1 and W_q^2 defined by E_q are contained in the corresponding half-spaces defined by E_q^*. Finally remark that for any equivalence class $E_{q+1}^*, \ldots, E_{m^*}^*$ the whole set Y of fixed facilities is contained in one of the half-spaces defined by this class.

Consider any optimal solution X for problem (P_s) on space (V, r). By the previous result we conclude that (X_q^1, X_q^2) is a minimum cut in the q-local network N_q for $q = 1, \dots, m$. Since N_q is also a q-local network for equivalence class E_q^*, $q = 1, \dots, m$, and $C_q(X_q^1, X_q^2) = 0$ for $q = m+1, \dots, m^*$, then we obtain that X is an optimum for our problem on the space (V^*, r). Hence $\bar{F}(X) = \bar{F}(X^*)$ and so X is an optimum to problem (P). □

The argument proving part (1) of Theorem 1 may be interpreted in the following way. Let $\tilde{G}^n$ be the direct product of n copies of a graph G. Vertices of $\tilde{G}^n$ are the n-tuples $(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are the vertices of G. Two vertices $(x_1, \dots, x_n)$ and $(x'_1, \dots, x'_n)$ are adjacent provides

$$\max\{d(x_i, x'_i)\colon i = 1, \dots, n\} = 1.$$

Then from Theorem 1(1) we conclude that in the graph $\tilde{G}^n$ any local minimum of the function $F(x_1, \dots, x_n)$ is a global minimum; compare with the analogical results for tree networks [20, 32].

5. On a way to an algorithm

Our previous results reduce the initial problem (P) to a similar problem (P_g) on a median graph $G = (V, E)$. While Theorem 1 gives necessary and sufficient conditions for a solution to be optimum, it does not suggest an algorithm for solving (P_g). The results of this section provide the basis for such an algorithm.

Let $E_1, \dots, E_m$ be the equivalence classes of edges of a median graph $G = (V, E)$. Following [16, 51], two classes E_i and E_j are *compatible* if one of the complementary half-spaces W_i^1, W_i^2 is contained in one of the complementary half-spaces W_j^1, W_j^2; they are called *incompatible* otherwise.

For equivalence classes $E_1, \dots, E_p$ consider a family of minimum cuts $(R_1^1, R_1^2), \dots, (R_p^1, R_p^1)$ of their local networks. This system of cuts is called a *compatible system* (a *c-system* for short) if for each pair of compatible classes E_i and E_j from the inclusion $W_i^a \subset W_j^b$ it follows that $R_i^a \subseteq R_j^b$, where $a, b \in \{1, 2\}$. (In addition, we have also that $W_i^{3-a} \supset W_j^{3-b}$ and $R_i^{3-a} \supset R_j^{3-b}$.)

Theorem 2. *Let $\mathscr{R} = \{(R_1^1, R_1^2), \dots, (R_{q-1}^1, R_{q-1}^2)\}$ be a c-system of minimum cuts and let E_q be a new equivalence class of G. Then there exists a minimum cut (R_q^1, R_q^2) of a q-local network such that $\mathscr{R} \cup (R_q^1, R_q^2)$ is a c-system too.*

Proof. We first show that if two classes E_i and E_j are compatible with E_q and $W_i^a \subset W_q^1$, $W_j^b \subset W_q^2$ then $R_i^a \cap R_j^b = \emptyset$ $(a, b \in \{1, 2\})$. To prove this assume for example that $a = b = 1$. Then $W_q^1 \subset W_j^2$ and $W_q^2 \subset W_i^1$ and therefore classes E_i and E_j are compatible. Since $\mathscr{R}$ is a c-system then we have $R_i^1 \subseteq R_j^2$ and $R_j^1 \subseteq R_i^2$, i.e. $R_i^1 \cap R_j^1 = \emptyset$.

Now assume that the class E_q is compatible with classes $E_1, \ldots, E_r$. Let us suppose that

$$W_i^1 \subset W_q^1, \qquad W_i^2 \supset W_q^2, \qquad i = 1, \ldots, t$$

and

$$W_j^1 \supset W_q^1, \quad W_j^2 \subset W_q^2, \quad j = t+1, \ldots, r.$$

Put $R_0^1 = \bigcup_{i=1}^t R_i^1$ and $R_0^2 = \bigcup_{j=t+1}^r R_j^2$. From the previous remark it follows that $R_i^1 \cap R_j^2 = \emptyset$ for all $i \in \{1, \ldots, t\}$ and $j \in \{t+1, \ldots, r\}$. So $R_0^1 \cap R_0^2 = \emptyset$.

In the q-local network N_q consider all cuts (Z_q^1, Z_q^2) with the property $R_0^1 \subseteq Z_q^1$ and $R_0^2 \subseteq Z_q^2$. Among these cuts choose the cut (X_q^1, X_q^2) with minimal capacity. For a proof of the theorem, we must show that (X_q^1, X_q^2) is a minimum cut in the q-local network N_q. Assume the contrary and let (R_q^1, R_q^2) be a minimum cut in N_q.

For each vertex $p \in \{1, \ldots, n\}$ of a q-local network put

$$V_p = \bigcap_{i=1}^{q-1} \{W_i^a : p \in R_i^a\}.$$

Next we shall show that each of these sets $V_1, \ldots, V_n$ is nonempty. By Helly's Theorem for median spaces, the set V_p is nonempty if the intersection of each two half-spaces W_i^a and W_j^b with $p \in R_i^a \cap R_j^b$ is nonempty. Suppose that the intersection of two such half-spaces is empty. Let, for example, $p \in R_i^1 \cap R_j^2$, whereas $W_i^1 \cap W_j^2 = \emptyset$. Then $W_i^1 \subset W_j^1$ and $W_i^2 \supset W_j^2$. Hence, classes E_i and E_j are compatible. As $\mathcal{R}$ is a c-system, then we conclude that $R_i^1 \subseteq R_j^1$ and $R_i^2 \supseteq R_j^2$ and so $R_i^1 \cap R_j^2 = \emptyset$. This contradicts the assumption that $p \in R_i^1 \cap R_j^2$. So, all the sets $V_1, \ldots, V_n$ are nonempty.

For any $p = 1, \ldots, n$, in the assumption that $p \in X_q^a$, choose in the set V_p a vertex x_p at minimal distance from the half-space W_q^{3-a}. Put $X = \{x_1, \ldots, x_n\}$. Observe that

$$\{p: x_p \in W_q^1\} = X_q^1, \qquad \{p: x_p \in W_q^2\} = X_q^2.$$

Moreover, for each equivalence class E_i, $i = 1, \ldots, q-1$, the set X defines the cut (R_i^1, R_i^2) in the i-local network. For minimum cut (R_q^1, R_q^2) and solution X defines a new solution $Z = \{z_1, \ldots, z_n\}$ by rules (A1), (B1) and (C1). Take

$$I^+ = (X_q^1 \cap R_q^2) \cup (X_q^2 \cap R_q^1),$$

$$X^+ = \{x_p : p \in I^+\}, \qquad Z^+ = \{z_p : p \in I^+\}.$$

It is easy to see that for any $p \in I^+$ the new vertex z_p does not belong to V_p. Therefore for each $p \in I^+$ there exist an $i \in \{1, \ldots, q-1\}$ such that vertices x_p and z_p are separated by half-spaces W_i^1 and W_i^2. Hence $(z_p, x_p) \in E_i$. We claim that classes E_i and E_q are compatible. To show this it suffices to observe that

$$W_i^1 = W(x_p, z_p) \subset W_q^1, \qquad W_q^2 \subset W(z_p, y_p) = W_i^2$$

if $p \in X_q^1 \cap R_q^2$ and

$$W_i^1 = W(x_p, z_p) \subset W_q^2, \qquad W_q^1 \subset W(z_p, y_p) = W_i^2$$

if $p \in X_q^2 \cap R_q^1$.

So, for each $p \in I^+$ the edge (x_p, z_p) belongs to an equivalence class compatible with E_q. For each $i \in \{1, \ldots, r\}$ let us denote by X_i^+ the set of all vertices $x_p \in X^+$ such that $(x_p, z_p) \in E_i$. Also let

$$I_i^+ = \{p \in I^+ : x_p \in X_i^+\}, \quad Z_i^+ = \{z_p : x_p \in X_i^+\}.$$

From the above we conclude that each of the families $\{I_i^+\}$, $\{X_i^+\}$ and $\{Z_i^+\}$ defines a partition for I^+, X^+ and Z^+, respectively.

As we already mentioned, for each nonempty set X_i^+ the class E_i is compatible with E_q. Since $E_i \neq E_q$ then all the vertices of Z_i^+ are obtained by moving along equivalent edges the corresponding vertices of X_i^+. Hence the sets X_i^+ and Z_i^+ are contained in the same half-space W_q^1 or W_q^2. Therefore each set I_i^+ is contained in one of the sets $X_q^1 \cap R_q^2$ or $X_q^2 \cap R_q^1$. Assume for example, that nonempty sets $I_{i_1}^+, \ldots, I_{i_p}^+$ are contained in $X_q^1 \cap R_q^2$ and the remaining sets $I_{i_{p+1}}^+, \ldots, I_{i_f}^+$ are contained in $X_q^2 \cap R_q^1$. Hence

$$\bigcup_{j=1}^{p} I_{i_j}^+ \subseteq X_q^1 \cap R_q^2, \qquad \bigcup_{j=p+1}^{f} I_{i_j}^+ \subseteq X_q^2 \cap R_q^1.$$

On the other hand, from the definition of the solution X we obtain that $I_{i_j}^+ \subseteq R_{i_j}^1$, $j = 1, \ldots, p$ and $I_{i_j}^+ \subseteq R_{i_j}^2, j = p+1, \ldots, f$. So

$$\bigcup_{j=1}^{p} I_{i_j}^+ \subseteq R_0^1, \qquad \bigcup_{j=p+1}^{f} I_{i_j}^+ \subseteq R_0^2.$$

For each class E_i compatible with E_q define

$$Z_i^1 = \{j : z_j \in W_i^1\}, \qquad Z_i^2 = \{j : z_j \in W_i^2\},$$

i.e. (Z_i^1, Z_i^2) is the cut defined by Z in the network N_i. From the above we obtain the following:
(1) $Z_i^1 = R_i^1, Z_i^2 = R_i^2$ if $i \notin \{i_1, \ldots, i_f\}$;
(2) $Z_i^1 = R_i^1 \setminus I_i^+$, $Z_i^2 = R_i^2 \cup I_i^+$ if $i \in \{i_1, \ldots, i_p\}$;
(3) $Z_i^1 = R_i^1 \cup I_i^+$, $Z_i^2 = R_i^2 \setminus I_i^+$ if $i \in \{i_{p+1}, \ldots, i_f\}$.
Let us now define the following sequence of solutions $T_0, T_1, \ldots, T_r$:

put $T_0 = Z$ and let $T_i = (T_{i-1} \setminus Z_i^+) \cup X_i^+$ for $i = 1, \ldots, r$.

Since $\{Z_i^+\}$ and $\{X_i^+\}$ represent partitions of the sets Z^+ and X^+ and $X \setminus X^+ = Z \setminus Z^+ = X \cap Z$ we obtain that $T_r = X$. Pick any $1 \leqslant i \leqslant r$. Then each of the solutions $T_0, T_1, \ldots, T_{i-1}$ defines in the i-local network one and the same cut (Z_i^1, Z_i^2), while each of the solutions $T_i, T_{i+1}, \ldots, T_r$ defines in this network the cut (R_i^1, R_i^2). This follows from the fact that for any $j \in \{1, \ldots, r\}, j \neq i$, the sets Z_j^+ and

X_j^+ are contained in the same half-space W_i^1 or W_i^2. Hence the solutions $T_i, T_{i+1}, \ldots, T_r$ are i-optimal. From the relation between the cuts (Z_i^1, Z_i^2) and (R_i^1, R_i^2) it follows that each solution T_i is defined from T_{i-1} by using the rules (A1), (B1) and (C1). As (R_i^1, R_i^2), $i = 1, \ldots, r$ are minimum cuts and by Lemma 8 we have established that

$$F(T_1) - F(Z) \leqslant C_1(R_1^1, R_1^2) - C_1(Z_1^1, Z_1^2) \leqslant 0,$$

$$\vdots$$

$$F(T_i) - F(T_{i-1}) \leqslant C_i(R_i^1, R_i^2) - C_i(Z_i^1, Z_i^2) \leqslant 0,$$

$$\vdots$$

$$F(X) - F(T_{r-1}) \leqslant C_r(R_r^1, R_r^2) - C_r(Z_r^1, Z_r^2) \leqslant 0.$$

These inequalities involve $F(X) - F(Z) \leqslant 0$. Recall however that Z is obtained from X by using the rules (A1), (B1) and (C1). By Lemma 8 we have

$$F(Z) - F(X) \leqslant C_q(R_q^1, R_q^2) - C_q(X_q^1, X_q^2) < 0.$$

Hence the assumption that (X_q^1, X_q^2) is not a minimum cut of the network N_q leads us to a contradiction.

So, assume that (X_q^1, X_q^2) is a minimum cut. Since $R_0^1 \subseteq X_q^1$ and $R_0^2 \subseteq X_q^2$, then for any class E_i compatible with E_q we have $R_i^1 \subseteq X_q^1$ for $i = 1, \ldots, t$ and $R_i^2 \subseteq X_q^2$ for $i = t+1, \ldots, r$. Hence the extended system of cuts $\mathscr{R} \cup (X_q^1, X_q^2)$ is also a c-system. □

Let $X = \{x_1, \ldots, x_n\}$ be an i-optimal solution for $i = 1, \ldots, q-1$. Observe that $(X_1^1, X_1^2), \ldots, (X_{q-1}^1, X_{q-1}^2)$ define a c-system of minimal cuts. Let (X_q^1, X_q^2) be the cut of the q-local network defined by X. According to Theorem 2 there exists a minimum cut (R_q^1, R_q^2) of the network N_q which together with $(X_1^1, X_1^2), \ldots, (X_{q-1}^1, X_{q-1}^2)$ form a compatible system of cuts too. Now define a new solution $Z = \{z_1, \ldots, z_n\}$ using the following rules:

(A2) z_i is the gate for vertex x_i in the half-space W_q^2 for $i \in X_q^1 \cap R_q^2$;
(B2) z_i is the gate for vertex x_i in the half-space W_q^1 for $i \in X_q^2 \cap R_q^1$;
(C2) $z_i = x_i$ for $i \in (X_q^1 \cap R_q^1) \cup (X_q^2 \cap R_q^2)$.

Theorem 3. *Let Z be the solution obtained from an i-optimal solution X, $i = 1, \ldots, q-1$, using rules* (A2), (B2) *and* (C2). *Then*
(1) *$F(Z) \leqslant F(X)$ and $F(Z) < F(X)$ if X is not q-optimal;*
(2) *solution Z is i-optimal for $i = 1, \ldots, q-1, q$. In fact, the cuts defined by Z coincide with $(X_1^1, X_1^2), \ldots, (X_{q-1}^1, X_{q-1}^2)$, (R_q^1, R_q^2).*

Proof. (1) We proceed by induction on $d(X) = \max\{d(x_i, z_i)\colon\ i = 1, \ldots, n\}$. If $d(X) = 1$ then the solution Z is obtained from X by rules (A1), (B1) and (C1). By

Lemma 8 we conclude that

$$F(Z) - F(X) \leqslant C_q(R_q^1, R_q^2) - C_q(X_q^1, X_q^2).$$

So, assume that $d(X) \geqslant 2$. Let $X' = \{x'_1, \dots, x'_n\}$ be the solution obtained from X by using the rules (A1), (B1) and (C1). Then we have

$$F(X') - F(X) \leqslant C_q(R_q^1, R_q^2) - C_q(X_q^1, X_q^2) \leqslant 0.$$

We claim that the solution X' is p-optimal for all $p \in \{1, \dots, q-1\}$. To show this it is sufficient to consider the case when some edge (x_i, x'_i) belongs to an equivalence class E_p, where $p \in \{1, \dots, q-1\}$. Observe that each vertex z_j is the gate for both vertices x_j and x'_j in the half-space W_q^1 if $j \in X_q^2 \cap R_q^1$ or in the half-space W_q^2 if $j \in X_q^1 \cap R_q^2$. Note also that each edge (x_j, x'_j) belongs to an equivalence class compatible with E_q. In particular classes E_p and E_q are compatible. From the compatibility of the cuts (X_p^1, X_p^2) and (R_q^1, R_q^2) we deduce that the solution X' is p-optimal. Therefore, by moving the vertices from x_i to x'_i we do not affect the p-optimality for $p = 1, \dots, q-1$.

Further, since $d(x'_i, z_i) = d(x_i, z_i) - 1$ for any $i \in (X_q^1 \cap R_q^2) \cup (X_q^2 \cap R_q^1)$ then $d(X') = d(X) - 1$. By the induction hypothesis we conclude that $F(Z) \leqslant F(X')$. If $C_q(R_q^1, R_q^2) < C_q(X_q^1, X_q^2)$ then $F(X') < F(X)$ and so $F(Z) < F(X)$.

(2) We first note that Z defines in the q-local network the cut (R_q^1, R_q^2). Since $(X_1^1, X_1^2), \dots, (X_{q-1}^1, X_{q-1}^2)$ are minimum cuts then for a proof of part (2) it suffices to show that the solution Z defines these cuts in the corresponding local networks. In other words, we must prove that for any $i = 1, \dots, n$ both the vertices x_i and z_i belong to one and the same half-space of the pair W_p^1, W_p^2 for $p = 1, \dots, q-1$.

First consider the case when classes E_p and E_q are compatible. Without loss of generality assume that $W_p^1 \subset W_q^1$ and $W_p^2 \supset W_q^2$. Since the cuts (R_q^1, R_q^2) and (X_q^1, X_q^2) are compatible we conclude that $X_p^1 \subseteq R_q^1$ and $X_p^2 \supseteq R_q^2$. Moreover, $X_p^1 \subseteq X_q^1$ and $X_p^2 \supseteq X_q^2$ as cuts generated by the solution X. By definition of the new solution Z we infer that $z_i = x_i$ as only $i \in (X_q^1 \cap R_q^1) \cup (X_q^2 \cap R_q^2)$. So assume that $i \in (X_q^1 \cap R_q^2) \cup (X_q^2 \cap R_q^1)$. If $i \in X_q^1 \cap R_q^2$ then z_i is the gate for x_i in W_q^2. As $X_p^1 \subseteq R_q^1$ we obtain that $x_i \notin X_p^1$, whence $x_i \in W_p^2$. Since $z_i \in W_q^2 \subset W_p^2$ then both the vertices x_i and z_i belong to W_p^2. Now suppose that $i \in X_q^2 \cap R_q^1$, i.e. z_i is a gate for x_i in the half-space W_q^1. Then $x_i \in W_q^2 \subset W_p^2$. If we assume that $z_i \in W_p^1$ then from inclusion $W_p^1 \subset W_q^1$ we infer that z_i is the gate for x_i in the half-space W_p^1. But then $W_p^1 = W(z_i, v_i) = W_q^2$, where by v_i we denote the vertex adjacent to x_i of the interval $I(z_i, x_i)$. Hence the classes E_p and E_q must coincide, which is impossible.

Finally consider the case when classes E_p and E_q are incompatible, i.e. for all $a, b \in \{1, 2\}$ we have $W_p^a \cap W_q^b \neq \emptyset$. Assume that for some index $i \in X_q^1 \cap R_q^2$ the vertices x_i and z_i are separated by half-spaces W_p^1 and W_p^2. Let, for example, $x_i \in W_p^1$ and $z_i \in W_p^2$. As z_i is the gate for x_i in W_q^2 then for any vertex $x \in W_p^1 \cap W_q^2$ we have $z_i \in I(x, x_i)$. This however contradicts the convexity of the set W_p^1. This completes the proof of the theorem. □

Let $X = \{x_1, \ldots, x_n\}$ be an i-optimal solution for all $i = 1, \ldots, q-1$. By Theorem 2 there exists a minimum cut (R_q^1, R_q^2) in the q-local network, which together with $(X_1^1, X_1^2), \ldots, (X_{q-1}^1, X_{q-1}^2)$ form a c-system of cuts. The cut (R_q^1, R_q^2) may be found in the following way. First, find all equivalence classes $E_{i_1}, \ldots, E_{i_r}$ compatible with E_q. Let us assume that

$$W_i^1 \subset W_q^1, \quad W_i^2 \supset W_q^2 \qquad \text{for } i \in \{i_1, \ldots, i_t\},$$

$$W_i^1 \supset W_q^1, \quad W_i^2 \subset W_q^2 \qquad \text{for } i \in \{i_{t+1}, \ldots, i_r\}.$$

Put

$$R_0^1 = \bigcup_{j=1}^{t} X_{i_j}^1, \qquad R_0^2 = \bigcup_{j=t+1}^{r} X_{i_j}^2, \qquad R = \{1, \ldots, n\} \setminus (R_0^1 \cup R_0^2).$$

Now consider the network having vertex set $R \cup \{s, t\}$. For each $i \in R$ define undirected arcs (s, i) and (t, i) with capacities

$$c(s,i) = w_i(W_q^1) + \sum_{j \in R_0^1} v_{ij},$$

$$c(t,i) = w_i(W_q^2) + \sum_{j \in R_0^2} v_{ij}.$$

For all $i, j \in R$, $i \neq j$, define undirected arcs (i, j) with capacities $c(i, j) = v_{ij}$.

Remark 4. If (R_+^1, R_+^2) is a minimum cut in this network, then $(R_+^1 \cup R_0^1, R_+^2 \cup R_0^2)$ is a minimum cut in the q-local network, i.e. $R_q^1 = R_+^1 \cup R_0^1$ and $R_q^2 = R_+^2 \cup R_0^2$.

6. Algorithms

In this section we present the algorithms for solving the multifacility location problem on median graphs and networks, finite median spaces and simple rectilinear polygons.

6.1. *Median spaces and median networks*

The results of the previous section lead to the following algorithm for solving problems (P_g) and (P_s). We assume that a median graph $G = (V, E)$ is given in standard *adjacency list representation*. Then applying a simplified version of Dijkstra's shortest paths algorithm to each vertex $v \in V$, we compute the distance matrix $D(G)$ of graph G in a total $O(|V||E|)$ time and $O(|V|^2)$ space. Once $D(G)$ is available, then using the algorithm from [2] in $O(|V||E|)$ time we find the equivalence classes $E_1, \ldots, E_m$ of the graph G. As we already mentioned, there exists an isometric embedding of the graph G into a hypercube Q_m. That is, each vertex x can be augmented with a 0–1-address $\beta(x)$ such that the Hamming distance $d_H(\beta(x), \beta(y))$ equals $d(x, y)$ for all $x, y \in V$. By Lemma 5 the ith bit r^i of $\beta(x)$ is set to 0 if $d(x, v_i) < d(x, u_i)$ and 1 otherwise; (v_i, u_i) is some edge of class E_i, $1 \leqslant i \leqslant m$. Let $IE(G)$ be the 0–1 matrix that stores at the i-line the address for the ith vertex of the graph G.

input: a median graph G and the distance matrix $D(G)$
isometric embedding matrix $IE(G)$
initialize start with some solution $X = \{x_1, \dots, x_n\}$

- (1) for $q = 1$ through m put

$$R := \{1, \dots, n\}, \qquad R_0^1 := \emptyset, \quad R_0^2 := \emptyset.$$

- (2) for $i = 1$ through q do
  begin
- (3) using the matrix $IE(G)$ determine if the classes E_i and E_q are compatible. If E_i and E_q are compatible go to the following step, otherwise go to the next equivalence class.
- (4) find

$$X_i^1 = \{j\colon r^i(x_j) = 0\},$$

$$X_i^2 = \{j\colon r^i(x_j) = 1\}.$$

- (5) assign $R_0^1 := R_0^1 \cup X_i^a$ if $W_i^a \subset W_q^1$ or assign $R_0^2 := R_0^2 \cup X_i^a$ if $W_i^a \subset W_q^2$, where $a \in \{1, 2\}$.
- (6) set $R := R \setminus \{R_0^1 \cup R_0^2\}$. Construct the network N_q with the vertex set $R \cup \{s, t\}$ and the following arc capacities:

$$c(s, p) = w_p(W_q^1) + \sum_{i \in R_0^1} v_{pi},$$

$$c(t, p) = w_p(W_q^2) + \sum_{i \in R_0^2} v_{pi},$$

  and $c(p, p') = v_{pp'}$ for all $p, p' \in R$, $p \neq p'$.
- (7) find a minimum cut (R_q^1, R_q^2) in the network N_q.
- (8) find

$$X_q^1 = \{j\colon r^q(x_j) = 0\},\ X_q^2 = \{j\colon r^q(x_j) = 1\}.$$

- (9) for each $i \in X_q^1 \cap R_q^2$ using the matrix $D(G)$ find the gate v_i for vertex x_i in the half-space W_q^2.
- (10) for each $i \in X_q^2 \cap R_q^1$ using the matrix $D(G)$ find the gate v_i for vertex x_i in the half-space W_q^1.
- (11) set $x_i =: v_i$ for each $i \in (X_q^1 \cap R_q^2) \cup (X_q^2 \cap R_q^1)$.
  end

output: the optimum facility location $X = \{x_1, \dots, x_n\}$.

Theorem 4. *The standard multifacility location problems* (P_g) *and* (P_s) *can be solved in* $O(|V|^3 + |V|n^3)$. *The problems* $(\mathscr{P}_g)$ *and* $(\mathscr{P}_s)$ *with the graph* $\Gamma = (I \cup J, E' \cup E'')$ *can be solved in* $O(|V|^3 + |V||E'| + |V|(n + |E''|)n \log n)$ *time.*

Proof. The correctness proof of the algorithm is based on Theorems 1 and 3. By Theorem 3 and Remark 4 the solution $X = \{x_1, \dots, x_n\}$ defined on step (11) is i-optimal for each $1 \leqslant i \leqslant q$. Hence, if $q = m$ then from Theorem 1 we conclude that X is an optimal solution for problems (P_g) and (P_s).

We now come to the analysis of the running time. First observe that $m \leqslant |V| - 1$. Execution of steps (2)–(5) (for fixed classes E_q and E_i) require $O(|V|)$ time. These steps must be repeated $m(m-1)/2$ times and so the total complexity of these steps are $O(m^2|V|)$ operations. The next step of the algorithm is to find the minimum cut in the network defined on step (6). For problems (P_g) and (P_s) this is done in $O(n^3)$ time using the Dinic and Karzanov algorithm [25,34,44]. For problems $(\mathscr{P}_g)$ and $(\mathscr{P}_s)$ the interconnections between facilities are given by the graph $\Gamma = (I \cup J, E' \cup E'')$. In this case the network N_q contains at most $(2n + |E_2|)$ edges, whose capacities can be computed in $O(|E'| + |E''|)$ time. So we need $O(m(|E'| + |E''|))$ overall time to compute the local networks N_q, $q = 1, \ldots, m$. Therefore in the case when all N_q are sparse networks then the Sleator and Tarjan algorithm find the minimum cut in time $O((n + |E''|)n \log n)$ [44]. Since the steps (6) and (7) must be repeated m times, then the total complexity of these steps is $O(mn^3)$ or $O(m(n + |E''|)n \log n)$ operations. Further, finding of the gate v_i for vertex x_i in one of the half-spaces W_q^1 or W_q^2 takes $O(|V|)$ operations. Hence steps (9) and (10) require $O(n|V|)$ operations and the overall time complexity of these steps is bounded by $O(mn|V|)$. Since $m \leqslant |V| - 1$

$$mn|V| \leqslant |V|n \leqslant \max(|V|n^3, |V|^3).$$

Summarizing the above we obtain an $O(|V|^3 + |V|n^3)$ bound for our algorithm in the case of problems (P_g) and (P_s) and an $O(|V|^3 + |V||E'| + |V|(n + |E''|)n \log n)$ bound for problems $(\mathscr{P}_g)$ and $(\mathscr{P}_s)$.

Finally we consider the case when the finite median space $S = (V, r)$ is given by distance matrix $R(S)$. Then the underlying graph $G = (V, E)$ and equivalence classes of this space can be constructed in $O(|V|^3)$ time by using the following algorithm. The algorithm is based on a characterization of median graphs from [40,8].

input a median space $S = (V, r)$ and the distance matrix $R(S)$
initialize set $E := \emptyset$

- (1) while $\max\{r(x,y) \in R(S)\} > 0$ do begin
- (2) choose a point x with at least one nonzero element in the corresponding line of matrix $R(S)$.
- (3) find a point y for which $r(x,y) = \min\{r(x,z)\colon r(x,z) \neq \emptyset\}$.
- (4) set $E(x,y) := \{(x,y)\}$.
- (5) find the sets

$$W(x,y) = \{z\colon r(x,z) < r(y,z)\}, \qquad W(y,x) = \{z\colon r(y,z) < r(x,z)\}.$$

- (6) for any points $u \in W(x,y)$ and $v \in W(y,x)$ put $r(u,v) := r(u,v) - r(x,y)$.
- (7) find all new pairs of points (u,v) such that $r(u,v) = 0$ and set $E(x,y) := E(x,y) \cup (u,v)$.
- (8) set $E := E \cup E(x,y)$.

end

output: the median graph G of the space (S,r) and the equivalence classes $E_1, \ldots, E_m$ of G.

On step (3) of this algorithm we find an edge (x, y) of the graph G. Further we compute the sets $W(x, y)$ and $W(y, x)$. Using these complementary half-spaces we find the equivalence class $E(x, y)$ generated by the edge (x, y). On step (6) we contract the space S into a new median space. This operation allows us to detect correctly a new edge of G. Note that in the new median space distances between points from the same half-space $W(x, y)$ or $W(y, x)$ remain invariant. On the other hand, the distances between points from distinct half-spaces are decreased by the length of the edge (x, y). The complexity of this algorithm is $O(m|V|^2)$. Since $m \leqslant |V| - 1$ then we obtain that the graph $G = (V, E)$ and the equivalence classes of G can be constructed in $O(|V|^3)$ time. This concludes the proof of the theorem. □

Remark 5. As to the complexity of the algorithm for problems (P) and $(\mathscr{P})$ on a median space (S, r), its mainly depends on the complexity of the procedure for finding some median stable set V, containing the fixed facilities, and on the cardinals of this set V.

Our approach to the multifacility location problem on finite median spaces may be applied for solving the similar problem on median network. A *network* N consists of a finite set V of vertices and a set of links joining certain pairs of vertices. Each link uv between two vertices u and v has a positive length $r(u, v)$ and consists of a continuum of points. The network N can be regarded as a metric space where the distance $r(x, y)$ of two points x and y is the length of a shortest route from x to y. It is assumed (as in [4, 32]) that every link constitutes a shortest route between its endvertices. According to [4], the network N is called *median* if for every triple u, v, w of vertices intersection $uv \cap vw \cap wu$ is a singleton (recall that uv is the interval between vertices u and v). The underlying graph of a network N consists of the vertex set V of N and the edge set of all linked pairs of vertices. The underlying graph of a median network is a median graph [4].

Consider now a median network N and let all fixed facilities $y_1, \ldots, y_k$ be located only at the vertices of the network. Location of the new facilities $x_1, \ldots, x_n$ is allowed at any point of the network. The multifacility location problem on a median network N is to find n new facilities such that the function $F(x_1, \ldots, x_n)$ is minimized. A well-known result with respect to a multifacility location problem on a network is that there exists an optimal solution with $x_1, \ldots, x_n \in V$; see for example [48]. We may therefore restrict our search for an optimum to solutions of this type. Since V endowed with distance $r(u, v)$ is a median space, such a solution may be found using the above algorithms.

As we already noticed, the best algorithm for an isometric embedding of a median graph into a hypercube requires $O(|V||E|)$ operations [2]. Also remark that any median graph contains at most $O(|V|\log|V|)$ edges. On the other hand, in our algorithm we use $O(m^2|V|)$ bite operations for computing all compatible pairs of edges. We raise the question whether $O(|V||E|)$ time suffices for solving this problem. Below we mention some difficulties in this direction. We also analyze the work of the algorithm in cases when the graph is a path, a tree or a hypercube.

Let $\mathscr{H}$ be the family of all half-spaces of a median graph G. Denote by H the covering graph of the partially ordered set $(\mathscr{H}, \subset)$. Remark that our algorithm may be represented as a sweeping procedure on the poset $(\mathscr{H}, \subset)$. On each step of this procedure for a given equivalence class E_q we must find only the half-spaces which are covered in H by half-spaces W_q^1 and W_q^2. Therefore such a sweeping will be more successful in the case when the chains in the graph H are longer and each half-space covers only fixed number of the half-spaces.

For example, if G is a path P then H consists of two disjoint chains. In this case each half-space of P covers only one other half-space. Hence beginning with one end of the path P at the step q of our algorithm we compute all new facilities which must be located at the vertex q in the optimal solution.

Now assume that G is a hypercube Q_d. Then the covering graph H consists of two disjoint antichains, each of them with $d = \log|V|$ elements. In this case we must solve d independent cut problems, one for each coordinate of the hypercube. Let $(R_1^1, R_1^2), \ldots, (R_d^1, R_d^2)$ be the obtained minimum cuts. Then the new facility i will be located at the vertex with coordinates $(\alpha_1, \ldots, \alpha_d)$, where $\alpha_q = 0$ if $i \in R_q^1$ and $\alpha_q = 1$ if $i \in R_q^2$. Unlike the path P, for hypercubes the efficiency of the algorithm is based on the fact that in Q_d any two half-spaces are incompatible.

A quite different approach may be used for trees. Assume that the median graph G is a tree T. For trees any two equivalence classes of edges are compatible. This property may be used in the algorithm, besides that the covering graph H of a tree in general has a more complex structure. (For example, if T is a star then H is a complete bipartite graph minus a complete matching.) In this case we can preprocess the tree in $O(|V|)$ time and obtain an ordering $v_1, v_2, \ldots, v_N$ of the vertices, such that v_i is a leaf of a subtree induced by the vertices $v_i, \ldots, v_N$ $(N = |V|)$. Using this ordering at the step q of the algorithm we will find all new facilities which must be located at the vertex v_i. This example lead us to the conclusion that the third requirement to a successful sweeping is the condition that for each equivalence class there is a fixed number of incompatible classes of edges.

Unfortunately, these conditions need not be fulfilled by a class of median graphs. In this respect an interesting class is formed by simplex graphs [10, 51]. Let F be an arbitrary graph. The collection of all cliques in F is denoted by $\mathscr{C}(F)$. Define $C_1, C_2 \in \mathscr{C}(F)$ to form an edge provided their symmetric difference consists of at most one point. According to [10, 51], the resulting graph $\mathscr{C}(F)$ is called the *simplex graph* of F. As was shown in [10] $\mathscr{C}(F)$ is a median graph. For example, if F is a cycle C_N with N vertices then $\mathscr{C}(C_N)$ has $2N + 1$ vertices. The graph $\mathscr{C}(C_N)$ contains N equivalence classes; each class is compatible with $N - 3$ other classes. In the covering graph H the degree of any vertex is $N - 3$ and any maximal chain has length two. Therefore H contains $O(N^2)$ edges and $O(N^2)$ directions.

6.2. Simple rectilinear polygons

Let P be a *simple rectilinear polygon* (i.e. a simple polygon having all edges axis-parallel) with N edges. A *rectilinear path* π is a polygonal chain consisting of axis-parallel segments lying inside P. The length of the path π is defined as the sum of the length of the segments π consists of. For any two points u and v in P, the *rectilinear distance* between u and v, denoted as $r(u,v)$, is defined as the length of the minimum length rectilinear path connecting u and v. We will regard the polygon P with the distance $r(u,v)$ as a metric space (P,r). Denote by (P_P) the multifacility location problem on (P,r). An important particular case is the multifacility location problem with all fixed facilities located only at the vertices of P. We will denote this problem by (P_{PV}). Below we will prove that for any simple rectilinear polygon P the space (P,r) is median. Using this property and some results from computational geometry we present a sweeping version of our algorithm for solving problems (P_P) and (P_{PV}).

The rectilinear version of the multifacility location problem, like all other distance problems on rectilinear polygons, is motivated by applications in areas such as wire layout, circuit design, plant and facility layout, urban transportation, and robot motion (see [1, 12, 19, 22, 23, 24, 31, 39, 53] for distance problems on rectilinear polygons and polyhedrons).

An axis-parallel segment is called a *cut segment* of a polygon P if it connects two edges of P and lies entirely inside P. Note that any edge or any cut of P is a convex subset of (P,r).

The following auxiliary property is a well-known property of metric spaces, see [12].

Lemma 9. *If x,y,z,v are points of a metric space (S,r) such that $v \in xy$ and $z \in xv$ then $v \in zy$.*

The fact that (P,r) is a median space can be derived from the van de Vel general matching theorem for median convex structures [50, 51]. We present a direct proof of this result; see also [17].

Lemma 10. *(P,r) is a median space.*

Proof. We proceed by induction on the number of vertices N of P. The statement is evident for rectangles, i.e. for $N = 4$. Now assume that $N > 4$ and let c be the cut segment of P with one endpoint at the concave vertex of P. Then c cuts P into two simple rectilinear polygons P' and P'' with at most $N - 1$ vertices each. By induction hypothesis P' and P'' are median spaces. The segment c is convex in each of these spaces. By Lemma 1(1) c is a gated set in P' and P''. Note also that P' and P'' are convex subsets of P.

Let u, v, w be arbitrary points of P. Assume without loss of generality that $u \in P'$ and $v, w \in P''$. Denote by u_c the gate of u in c. Consider any point $p \in P''$. Any shortest path from u to p intersect the cut c in some point u'. As $u_c \in uu'$ and $u' \in up$ then by Lemma 9 we obtain that $u_c \in up$. Hence u_c is the gate for u in the subpolygon P''. Let z be the median of u_c, v and w. Since $u_c \in uv \cap uw$ then z is a median of points u, v, w too. Now assume that z^+ is another median for points u, v and w. As P'' is convex and $v, w \in P''$ then $z^+ \in P''$. On the other hand, since $u_c \in uz^+$ then by Lemma 9 we conclude that

$z^+ \in u_c v \cap u_c w$. Therefore the triple u_c, v, w admits in P'' two median points z and z^+, in contradiction with our induction assumption. □

Consider all horizontal and all vertical cuts which pass through fixed facilities or vertices of a polygon P. These cuts together with the edges of P generate a rectilinear grid. Denote by V the vertices (intersection points) of this grid. Obviously, all vertices of P and all fixed facilities are contained in V.

Lemma 11. *V is a median stable set.*

Proof. First observe that the repeated application of the above operation with respect to V as the set of fixed facilities give the same grid V. Therefore for a proof of our assertion it is enough to consider any three points $u, v, w \in V$ which are vertices of P or fixed facilities. We proceed by induction on the number N of vertices of a polygon P. Choose a cut c with the endpoint in the concave vertex of P and which divides P into a rectangle P' and subpolygon P''. (Such a cut always exists.) Assume that c is a vertical cut. Suppose without loss of generality that c separates points u and v, w. First consider the case when $u \in P'$ and $v, w \in P''$. Let u' be the gate for u in P''. Remark that u' is the intersection of c with the horizontal cut which passes through u. The median of the triple u, v, w coincide with the median of points u', v, w. Consider the grid of P'' generated by vertices of P'' and the set of fixed facilities $Y \setminus \{u\} \cup \{u'\}$. Since the horizontal cuts which pass through points u and u' coincide then we obtain a subgrid of a grid for P. By induction assumption the median of the points u', v, w is a grid point of V. Now assume that $u \in P''$ and $v, w \in P'$. Then the horizontal cuts which pass through v and w divide P'' into three subpolygons P_1'', P_2'' and P_3''. If $u \in P_1'' \cup P_3''$ then the median $m(u, v, w)$ coincide with one of the intersections of horizontal and vertical cuts which pass through v and w. So, assume that $u \in P_2''$. Let v' and w' be the gates for v and w in the subpolygon P''. By induction assumption the median of points u, v', w'' is a grid point. Therefore $m(u, v', w')$ is the intersection of the segments $v'w''$ with some cut c_+ of P''. Observe that c_+ is a part of a cut c_+^*, which belongs to the grid of P. It remains to note that $m(u, v, w)$ is the intersection of c_+^* with one of the vertical cuts which passes through v and w, i.e. $m(u, v, w) \in V$. □

Let (V, r) be the median space generated by the stable set V. Observe that any pair of complementary half-spaces of (V, r) may be represented as intersections of V with the subpolygons defined by some cut of P, which passes through the point from V. In the algorithm presented below we will avoid the construction of the set V. Although the space (V, r) contains $O(N + k)$ pairs of complementary half-spaces we will solve only $O(k)$ minimum cuts problems on local networks. The algorithm is based on the Chazelle algorithm for computing all vertex-edge visible pairs of edges of a simple polygon [14] and on optimal point location methods [28, 35].

By the first algorithm we obtain a decomposition $\mathscr{D}_0^{\mathrm{h}}$ of the polygon P into rectangles, using only horizontal cuts which pass through the vertices of P. Now we have to compute which rectangles of the decomposition $\mathscr{D}_0^{\mathrm{h}}$ contain each of the fixed facilities. Using one of the optimal point location methods [28, 35] this can be done in time $O(k \log N)$ with a structure that uses $O(N)$ storage. Observe that the induced subdivision $\mathscr{D}_0^{\mathrm{h}}$ is monotone and, hence, the point location structure can be built in linear time. At the following step we sort by y-coordinate all fixed facilities from each rectangle. Having these sorted lists we obtain a decomposition of each rectangle from $\mathscr{D}_0^{\mathrm{h}}$, using horizontal cuts which pass through fixed facilities. (Some of these cuts may be edges of P.) As a result we derive a new decomposition $\mathscr{D}^{\mathrm{h}}$ of P into $O(N + k)$ rectangles, which is a refinement of the decomposition $\mathscr{D}_0^{\mathrm{h}}$. The dual graph of this decomposition is a tree $\mathscr{T}^{\mathrm{h}}$: the vertices of a tree are the rectangles of $\mathscr{D}^{\mathrm{h}}$ and two vertices in $\mathscr{T}^{\mathrm{h}}$ are adjacent if and only if the corresponding rectangles in the decomposition are bounded by the common cut. Assign to each edge of the subdivision $\mathscr{D}^{\mathrm{h}}$ the fixed facilities which lie on this edge. In a similar way we define the decompositions $\mathscr{D}_0^{\mathrm{v}}$ and $\mathscr{D}^{\mathrm{v}}$ of P into rectangles, using only vertical cuts. Let $\mathscr{T}^{\mathrm{v}}$ be the dual graph of $\mathscr{D}^{\mathrm{v}}$. The decompositions $\mathscr{D}^{\mathrm{h}}$ and $\mathscr{D}^{\mathrm{v}}$ and their graphs $\mathscr{T}^{\mathrm{h}}$ and $\mathscr{T}^{\mathrm{v}}$ may be constructed in time $O(N + k(\log k + \log n))$. If all fixed facilities are vertices of P then $\mathscr{D}^{\mathrm{h}} = \mathscr{D}_0^{\mathrm{h}}$ and $\mathscr{D}^{\mathrm{v}} = \mathscr{D}_0^{\mathrm{v}}$. In this case we avoid the application of point location methods and sorting of fixed facilities. So, we require only $O(N + k)$ time.

In what follows, suppose that the rectangles $R_1^{\mathrm{h}}, \dots, R_p^{\mathrm{h}}$ of the decomposition $\mathscr{D}^{\mathrm{h}}$ are numbered in such a way that any R_i^{h} is a leaf in the subtree with vertices $R_i^{\mathrm{h}}, R_{i+1}^{\mathrm{h}}, \dots, R_p^{\mathrm{h}}$ of a tree $\mathscr{T}^{\mathrm{h}}$. Such an ordering may be obtained in linear time with respect to a number of rectangles. Having done this, we obtain the list $e_1^{\mathrm{h}}, \dots, e_{p*}^{\mathrm{h}}$ of the horizontal edges of the rectangles from $\mathscr{D}^{\mathrm{h}}$. In a similar way by preprocessing the tree $\mathscr{T}^{\mathrm{v}}$ we obtain an analogical ordering $R_1^{\mathrm{v}}, \dots, R_t^{\mathrm{v}}$ of the rectangles from $\mathscr{D}^{\mathrm{v}}$ and the list $e_1^{\mathrm{v}}, \dots, e_{t*}^{\mathrm{v}}$ of their vertical edges.

Any edge e_i^{h} divide the polygon P into two subpolygons P_i^1 and P_i^2. Denote by W_i^1 and W_i^2 the intersection of these subpolygons with the median stable set V. Then we obtain two compatible pairs $\Pi_{i1}^{\mathrm{h}} = (W_i^1, V \setminus W_i^1)$ and $\Pi_{i2}^{\mathrm{h}} = (V \setminus W_i^2, W_i^2)$ of complementary half-spaces of the space (S, r). Observe that all horizontal edges of the subdivision $\mathscr{D}^{\mathrm{h}}$ contained in one of the subpolygons P_i^1 or P_i^2, say all such edges from P_i^1, have the indices smaller than i. In P_i^1 there exists an edge e_j^{h} such that the pairs of complementary half-spaces Π_{i1}^{h} and Π_{j2}^{h} coincide. This remark allows us to consider in future only the pairs of the type Π_{i1}^{h}. In a similar way, for any vertical edge e_i^{v} we define the pairs of complementary half-spaces Π_{i1}^{v} and Π_{i2}^{v}. Note that the pairs of complementary half-spaces defined by two edges of our decompositions $\mathscr{D}^{\mathrm{h}}$ and $\mathscr{D}^{\mathrm{v}}$ are incompatible only if these edges have a nonempty intersection. In particular, we obtain that all parallel edges define compatible pairs of half-spaces. Two parallel edges are called *equivalent* if the subpolygons defined by them contain the same sets of fixed facilities. Remark that any two equivalent edges define one and the same local network.

Now we are ready to present the final steps of the algorithm. First by sweeping the segments $e_1^{\mathrm{h}}, \dots, e_{p*}^{\mathrm{h}}$ from left to right we compute the y-coordinates of all new facilities. For each $q \in \{1, \dots, p^*\}$ let R be the set of new facilities already located at the previous $q-1$ steps on the segments $e_1^{\mathrm{h}}, \ldots, e_{q-1}^{\mathrm{h}}$. Then $R_0 = R_0^1 \cup R_0^2$, where R_0^1 is the set of new facilities located in the subpolygon P_q^1 and R_0^2 is the set of new facilities located in P_q^2. Put $R_q = R \backslash R_0$. If the edge e_q^{h} is equivalent to some previously considered edge then set $q = q + 1$. Otherwise, for pair of complementary half-spaces Π_{q1}^{h} construct the q-local network N_q. This network has $R_q \cup \{s, t\}$ as vertex set. As in the case of median graphs each $j \in R_0^1$ is treated as a fixed facility from W_q^1 and each $j \in R_0^2$ is treated as a fixed facility from $V \setminus W_q^1$. Let (R_q^1, R_q^2) be the minimum cut of the q-local network N_q. Then locate all new facilities from R_q^1 on the edge e_q^{h}. After p^* steps we find an optimal location of all new facilities with respect to horizontal edges of the subdivision $\mathscr{D}^{\mathrm{h}}$.

Observe that among the edges which pass through the vertices of the polygon P only at most k of them may be nonequivalent to some of already considered edges. Hence we must solve at most $2k$ minimum cut problems in the local networks (k of such problems must be solved with respect to horizontal edges which pass through the fixed facilities). So, the total complexity of this sweeping procedure is $O(k\psi(n))$, where $\psi(n)$ is the complexity of the applied maximum-flow algorithm.

After this step of the algorithm all the new facilities are located on maximum $2k$ horizontal segments. For each of these segments we find the rectangles of the decomposition $\mathscr{D}^{\mathrm{v}}$ that contain their endpoints. Observe that if the endpoints of the segment e_q^{h} belong to the rectangles R_i^{v} and R_j^{v} then e_q^{h} intersects all the rectangles and their vertical edges from the path between R_i^{v} and R_j^{v} of the tree $\mathscr{T}^{\mathrm{v}}$. Assume that $i < j$. Then temporarily locate all new facilities whose y-coordinate coincide with y-coordinate of e_q^{h} at the endpoint of e_q^{h} which lies in R_i^{v}. Using the optimal point location methods [28, 35] this can be done in $O(k \log N)$ time. If all fixed facilities are vertices of P then any horizontal edge has a vertex P as an endpoint. Therefore the endpoints of $\mathscr{D}^{\mathrm{h}}$ may be located in $O(N)$ time by using the vertex-edge visibility map [14].

By sweeping the vertical edges $e_1^{\mathrm{v}}, \ldots, e_{t^*}^{\mathrm{v}}$ of the subdivision $\mathscr{D}^{\mathrm{v}}$ we will move each new facility i along horizontal edge until $i \in R_q^1$ for the minimum cut (R_q^1, R_q^2) of the network N_q for e_q^{v}. Then definitively locate each new facility $i \in R_q^1$ on the intersection of e_q^{v} and the horizontal edge that contain i. Any other new facility $j \in R_q^2$ is temporarily located at the intersection of the horizontal edge containing j and another vertical edge of the rectangle from $\mathscr{D}^{\mathrm{v}}$ which contains the segment e_q^{v}.

The network N_q for vertical edge e_q^{v} is defined in the following way. Let R_q be the set of new facilities temporarily located on segment e_q^{v}. Note that R_q consists of all new facilities j such that $j \in R_{q'}^1$ for any edge $e_{q'}^{\mathrm{v}}$, $q' < q$, and the horizontal segment containing j must intersect the segment e_q^{v}. The set R_q may be defined also $R_q = R \backslash R_0$, where R_0 consists of all definitively located new facilities and of all temporarily located new facilities for which the horizontal edge does not intersect e_q^{v}. The network N_q has $R_q \cup \{s, t\}$ as vertex set and arcs are defined as in the algorithm for median graphs.

Now we will prove that after the sweeping of vertical edges we obtain an optimal location of all new facilities. First of all, remark that any pair of complementary half-spaces from (V,r) is equivalent to a pair of half-spaces defined by some of the edges from $\mathscr{D}^{\mathrm{h}}$ or $\mathscr{D}^{\mathrm{v}}$. By Theorem 3 we conclude that after each of two sweeping we obtain the solution which is optimal with respect to all edges of the given direction. On the other hand, moving the new facilities horizontally we do not affect the optimality of the solution with respect to any horizontal edge. So, the obtained solution is optimal with respect to all vertical and horizontal edges. From the above remark and Theorem 1 we conclude that the obtained solution is optimal.

Summarizing the results of this section, we have the following theorem.

Theorem 5. *The multifacility location problem in a simple rectilinear polygon P with N vertices can be solved in time $O(N + k(\log N + \log k + \psi(n)))$, where $\psi(n)$ is the complexity of the applied maximum-flow algorithm. The vertex restricted problem can be solved in $O(N + k\psi(n))$ time.*

Remark 6. The results of Theorem 5 remain also true for simple polygons endowed with the following rectilinear-type distance. Let P be a simple polygon and let $x = (x^1, x^2)$ and $y = (y^1, y^2)$ be arbitrary points of P. If the segment $[x, y]$ is contained in P then put $d(x,y) = |x^1 - y^1| + |x^2 - y^2|$, otherwise define $d(x,y)$ as for rectilinear polygons, replacing only rectilinear paths by arbitrary paths inside P. Using Van de Vel results [50, 51] we obtain that P is a median space, so we can apply our results.

7. Conclusions and open problems

We have given an $O(|V|^3 + |V|\psi(n))$ algorithm for solving the multifacility location problem on median graphs and networks and on finite median spaces (recall that $\psi(n)$ is the complexity of the applied maximum-flow algorithm). This algorithm may be applied to any median space, it is necessary only to derive a procedure for finding the median stable set, containing all fixed facilities. In the case of a simple rectilinear polygon P with N edges such a set is easy to describe. Using this property we present an $O(N + k(\log N + \log k + \psi(n)))$ algorithm based on a sweeping the vertical cuts which passes through the fixed facilities and vertices of P. When all fixed facilities are vertices of P this algorithm runs in $O(N + k\psi(n))$ time.

The more efficient algorithms for rectilinear polygons are explained by the fact that, as for trees, the sweeping of the covering graph H of the poset $(\mathscr{H}, \subset)$ is reduced to a sweeping of two chains from H of pairwise compatible pairs of complementary half-spaces. There are some other classes of "multidimensional" median spaces which posses the similar property. These are graphs of acyclic cubical complexes [7], closely related with chordal graphs, and their polyhedrons.

It seems very probable that the median polyhedrons (and more generally, the cubical polyhedrons) will be interesting from the viewpoint of computational geometry. Recall that a *cubical complex* [51] is a graph G together with the collection $\mathscr{C}$ of graphic cubes of G, such that each edge of G is in $\mathscr{C}$ and the intersection of two cubes in $\mathscr{C}$, if nonempty, is in $\mathscr{C}$. A *cubical polyhedron* is a geometric realization of a cubical complex. A *median polyhedron* is a cubical polyhedron whose graph G is median; for more information consult [51, Ch. II].

Besides the multifacility location problem considered in our paper, there are some other problems of computational geometry which may be considered for median and cubical polyhedrons. One of them is the problem of finding the shortest rectilinear path inside a given axis-parallel polyhedron. This problem is thoroughly studied in the case of rectilinear polygons; see [1, 19, 22, 23, 24, 39, 53]. For multidimensional spaces, shortest path problems are considerable harder; see [19, 23] for particular results. We hope that cubical and median polyhedrons are the other class of polyhedrons for which this problem may be efficiently solved. The main reasons are the following. As in the case of other problems, first preprocess the d-dimensional polyhedron P to obtain a decomposition of P into axis-parallel boxes (hyperrectangles). Each cut of the decomposition is recursively decomposed into boxes of smaller dimension, etc. Such a subdivision may be represented as a hierarchical tree. For given queries points first we find the boxes, containing these points. In the obtained data structure of cuts and boxes we compute the lowest common ancestor of these boxes. The obtained cut separates the queries points in the minimum dimensional face of the obtained subdivision that contains these points. If P is a median polyhedron then we must find the gates of the queries points on this cut. Having the shortest paths between the queries points and their gates and the shortest path between the gates then gluing these paths we obtain the desired shortest path.

Another open problem is to generalize the obtained results to other classes of metric spaces and graphs. There are some generalizations of median graphs and median spaces. All these classes of graphs are contained in the class of weakly median graphs. A decomposition theorem of weakly median graphs into simple pieces by using the operations of Cartesian multiplication and gated amalgamation was given in [6]. Using this result an isometric embedding of weakly median graphs into L_1-spaces was obtained. These results may be used in order to decompose the initial multifacility problem into similar problems on smaller graphs.

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