Topology Vol. 8, pp. 47–57. Pergamon Press, 1969. Printed in Great Britain

PARALLEL DYNAMICAL SYSTEMS†

LAWRENCE MARKUS

(*Received* 22 *May* 1968)

§1. TOPOLOGICAL PROPERTIES OF UNSTABLE FLOWS

A dynamical system

$$\mathscr{S}) \quad dx^i/dt = f^i(x^1, \ldots, x^n)$$

on a differentiable n-manifold M^n is a differentiable tangent or contravariant vector field for which the corresponding flow

$$(t, x_0) \to \varphi(t, x_0) : R^1 \times M^n \to M^n$$

is defined for all times $t \in R^1$. Here M^n is a differentiable (that is, of class C^∞) manifold (connected separable metrizable space—without boundary) and $\varphi(t, x_0)$ is the orbit or solution initiating at $x_0 \in M^n$ at $t_0 = 0$. Two dynamical systems in M^n

$$\mathscr{S}_1) \quad \dot{x} = f_1(x) \qquad \text{and} \qquad \mathscr{S}_2) \quad \dot{x} = f_2(x)$$

are defined to be *topologically* (*or differentiably*) *equivalent* in case there exists a homeomorphism (or diffeomorphism) Ψ of M^n onto itself carrying each sensed (but not time-parametrized) solution curve of $\mathscr{S}_1$ onto a sensed solution curve of $\mathscr{S}_2$, and vice versa for Ψ^{-1}.

We shall be primarily concerned with dynamical systems in the real number space R^n. A dynamical system $\mathscr{S}$ in R^n is called *parallel* in case $\mathscr{S}$ is topologically equivalent to the flow along parallel straight lines,

$$\mathscr{P}) \quad \dot{x}^1 = 1, \qquad \dot{x}^2 = 0, \ldots, \qquad \dot{x}^n = 0.$$

Note that $\mathscr{P}$ lies in the same n-dimensional space R^n as does $\mathscr{S}$, and not in some higher dimensional space (or even Hilbert space) as in certain earlier studies [7].

For the 2-dimensional phase plane R^2 Poincaré [9] showed that a dynamical system $\mathscr{S}$ without critical (equilibrium) points has solution curves each of which is a line homeomorph tending towards infinity in R^2 as $|t| \to \infty$. In particular, $\mathscr{S}$ is then (Lagrange) unstable because each solution curve is unbounded in R^2 as $t \to \infty$ and also as $t \to -\infty$. Moreover, $\mathscr{S}$ is completely unstable since each point $x_0 \in R^2$ is wandering (precise definitions below and in [8]).

† Research partially supported by NONR 3776(00).

Clearly a parallel differential system in R^n is completely unstable, but the converse is false as is demonstrated by the elementary example in R^2

$$\dot{x}^1 = x^1(x^1 - 2), \qquad \dot{x}^2 = x^1 - 1,$$

which has the two separatrices $x^1 = 0$ and $x^1 = 2$. It has been established [5] that a completely unstable dynamical system $\mathcal{S}$ in R^2 is parallel if and only if $\mathcal{S}$ has no separatrices. In this paper we give a suitable definition of separatrices in higher dimensions and prove that the analogous assertion holds in R^3 but is false for R^4. We also obtain certain related results for dynamical systems without separatrices on other differentiable manifolds M^n.

We recall [8] that a dynamical system $\mathcal{S}$ in a differentiable manifold M^n is *unstable* in case no compact subset $K \subset M^n$ contains an entire half-orbit $\{\varphi(t, x_0) \mid \text{for } t \geq 0 \text{ or for } t \leq 0\}$. Thus each solution curve of $\mathcal{S}$ is a one-to-one continuous image of R^1 (although it need not be a topological line in M^n), and $\mathcal{S}$ has no critical points or periodic orbits. If all α and ω *limit sets* of $\mathcal{S}$ are empty, that is,

$$\bigcap_{\tau<0} \overline{\bigcup_{t\leq\tau} \varphi(t, x_0)} = \varnothing \qquad \text{and} \qquad \bigcap_{\tau>0} \overline{\bigcup_{t\geq\tau} \varphi(t, x_0)} = \varnothing,$$

then $\mathcal{S}$ is unstable in M^n. Of course, this implies that M^n is non-compact.

Also recall that a dynamical system $\mathcal{S}$ in M^n is *completely unstable* in case each point $x_0 \in M^n$ is *wandering*; that is, there exists a neighborhood U of x_0 whose orbit $\varphi(t, U)$ fails to meet U for all sufficiently large $|t|$. A completely unstable flow $\mathcal{S}$ has empty α and ω limit sets; hence, each solution curve is a topological line tending to infinity (leaving any prescribed compact set $K \subset M^n$) as $|t| \to \infty$, and so $\mathcal{S}$ is unstable.

Nemitsky [7] defined the dynamical system $\mathcal{S}$ in R^n to have a *saddle at infinity* in case: there exists a convergent sequence of points x_i in R^n, and two unboundedly increasing sequences of times $0 < \tau_i < t_i$ such that $\varphi(t_i, x_i)$ converges to some point of R^n yet $\varphi(\tau_i, x_i)$ diverges to infinity. We use the same condition to define a saddle at infinity for $\mathcal{S}$ in M^n. Elementary continuity arguments show [8] that an unstable dynamical system or flow $\mathcal{S}$ in M^n with no saddle at infinity is completely unstable.

Unstable flows are qualitatively simpler than flows with bounded orbits. In R^3 there exist fantastically complicated differential systems without critical points, but with each solution curve bounded [12]. We reject these terrible complexities by considering only unstable flows in this paper.

Nevertheless, even completely unstable flows can be very intricate. For instance, in the plane R^2 there exist a noncountable number of different topological types of completely unstable flows [4 and 5]. Since the definition of an unstable dynamical system $\mathcal{S}$ involves the time parameter t, as well as the topology of M^n, we first note the invariance of this property under topological equivalence.

THEOREM 1. *In a differentiable manifold M^n let*

$$\mathcal{S}_1) \quad \dot{x} = f_1(x) \qquad \text{and} \qquad \mathcal{S}_2) \quad \dot{x} = f_2(x)$$

be topologically equivalent dynamical systems. If $\mathcal{S}_1$ is unstable, or completely unstable, or unstable with no saddle at infinity, then $\mathcal{S}_2$ has the corresponding property.

Proof. Let Ψ be a homeomorphism of M^n onto itself carrying the sensed solution curve family of $\mathscr{S}_1$ onto that of $\mathscr{S}_2$, that is,

$$\Psi\varphi_1(t, x_0) = \varphi_2(s, \Psi x_0),$$

where $\varphi_1(t, x_0)$ is the solution of $\mathscr{S}_1$ initiating at x_0 when time $t = 0$, and $\varphi_2(s, \Psi x_0)$ is the solution of $\mathscr{S}_2$ initiating at Ψx_0 at time $s = 0$.

Assume $\mathscr{S}_1$ is unstable so each solution curve $\varphi_1(t, x_0)$ is a one-to-one continuous image of R^1, and the same holds for the solution $\varphi_2(s, \Psi x_0)$ of $\mathscr{S}_2$. Thus, for each fixed $x_0 \in M^n$, there exists a function

$$s(t, x_0): R^1 \times M^n \to R^1$$

specified by the unique values determined in

$$(t, x_0) \to \varphi_1(t, x_0) \to \Psi\varphi_1(t, x_0) = \varphi_2(s, \Psi x_0).$$

Since the time duration between the endpoints of a solution segment of a dynamical system varies continuously with the two endpoints, $s(t, x_0)$ is continuous on $R^1 \times M^n$. Also, for each fixed x_0, the map $t \to s(t, x_0)$ is an orientation preserving homeomorphism of R^1 onto itself. If a compact subset $K \subset M^n$ contained an entire half-solution of $\mathscr{S}_2$, say $\varphi_2(s, \Psi x_0)$ for $s \geq 0$, then this would imply that $\varphi_1(t, x_0)$ lies in $\Psi^{-1}K$ for all sufficiently large t. Since $\mathscr{S}_1$ is unstable, $\varphi_1(t, x_0)$ is not entirely in the compact set $\Psi^{-1}K$ for all large t, and hence the contradiction proves that $\mathscr{S}_2$ is also unstable.

Next assume that $\mathscr{S}_1$ is completely unstable, so that $\mathscr{S}_1$ and hence $\mathscr{S}_2$ are unstable. Take an initial point $\Psi x_0 \in M^n$ and consider the flow of $\mathscr{S}_1$ initiating from a compact ball B_1 centered at x_0. We take B_1 so small that there exists $t_1 > 0$ such that

$$\varphi_1(t, B_1) \bigcap B_1 = \varnothing \qquad \text{for} \quad |t| > t_1.$$

Then $B_2 = \Psi B_1$ contains Ψx_0 in its interior and has an orbit under $\mathscr{S}_2$ such that

$$\varphi_2(s, B_2) \bigcap B_2 = \varnothing \qquad \text{for all} \quad |s| > s_1 = \max_{x_0 \in B_1} s(t_1, x_0).$$

Thus $\mathscr{S}_2$ is also completely unstable.

Finally assume that $\mathscr{S}_1$ is unstable with no saddle at infinity. Then $\mathscr{S}_1$ and hence $\mathscr{S}_2$ are completely unstable flows. Suppose $\mathscr{S}_2$ has a saddle at infinity; that is, there exists convergent sequences in M^n, $\Psi x_i \to \Psi\bar{x}$ (or $x_i \to \bar{x}$) and $\varphi_2(s_i, \Psi x_i) \to \Psi y$ for some increasing positive time sequence $s_i \to \infty$, with intermediate times $\sigma_i \to \infty$ at which $\varphi_2(\sigma_i, \Psi x_i)$ tends to infinity in M^n. Since $\mathscr{S}_1$ is completely unstable, each point of M^n is a wandering point of $\mathscr{S}_1$ and so it is easy to see that $\bar{x}$ and y do not lie on the same solution curve of $\mathscr{S}_1$. But then $\Psi\bar{x}$ and Ψy lie on distinct solutions of the topologically equivalent system $\mathscr{S}_2$.

Also $\varphi_1(t_i, x_i) \to y$ for the times $t_i > 0$ corresponding to s_i. Clearly the sequence t_i must be unbounded, for otherwise $\bar{x}$ and y would lie on the same solution curve of $\mathscr{S}_1$. Now select subsequences, still denoted t_i and x_i, so that t_i is monotonic increasing to infinity. Let τ_i correspond to the intermediate instant σ_i; that is, $\Psi\varphi_1(\tau_i, x_i) = \varphi_2(\sigma_i, \Psi x_i)$, and so $\varphi_1(\tau_i, x_i)$ diverges to infinity (eventually lies outside any prescribed compact subset of M^n).

Thus τ_i are unbounded. This construction contradicts the assumption that $\mathcal{S}_1$ has no saddle at infinity. Therefore, $\mathcal{S}_2$ has no saddle at infinity. Q.E.D.

§2. ORBIT SPACES OF UNSTABLE FLOWS

Consider a dynamical system

$$\mathcal{S}) \quad \dot{x} = f(x)$$

in a differentiable manifold M^n ($n \geq 2$ always), with orbits or solutions given by the flow $\varphi(t, x_0)$ in C^∞ in $R^1 \times M^n$. The orbit space $M^n/\mathcal{S}$ is the quotient or identification space formed from M^n by the equivalence relation specified by the solution curves of $\mathcal{S}$. That is, two points x_1 and x_2 of M^n are related by $\mathcal{S}$ in case they lie on the same solution curve of $\mathcal{S}$. The orbit space $M^n/\mathcal{S}$ bears the usual quotient topology, defined as the strongest topology for which the projection map

$$p: M^n \to M^n/\mathcal{S}$$

is continuous. Since the orbit $\bigcup_{t \in R^1} \varphi(t, U)$ of an open set $U \subset M^n$ is open in M^n, we note that the saturation set (sat U) of U by $\mathcal{S}$ is open, and hence p is an open map onto $M^n/\mathcal{S}$.

The orbit space $M^n/\mathcal{S}$ is connected and has a countable base. Yet $M^n/\mathcal{S}$ may not be a Hausdorff separated (or T_2) space; as in the example of the non-parallel unstable flow in R^2 described in the first section of this paper. The separation properties of $M^n/\mathcal{S}$ will be basic for our theory of parallel dynamical systems.

Definition. Consider the unstable dynamical system

$$\mathcal{S}) \quad \dot{x} = f(x) \quad \text{in a differentiable manifold } M^n.$$

Two solution curves S_1 and S_2 of $\mathcal{S}$ are inseparable in case any two neighborhoods of S_1 and S_2 in $M^n/\mathcal{S}$ meet; that is, S_1 and S_2 cannot be separated by open sets in $M^n/\mathcal{S}$. A solution curve S of $\mathcal{S}$ is a separatrix in case S lies in the closure of the set of inseparable elements of $M^n/\mathcal{S}$.

Remark. Inseparable solutions of $\mathcal{S}$ occur in pairs and each of them is a separatrix. Thus an element $S \in M^n/\mathcal{S}$ is a separatrix if and only if S is the limit of a sequence of solutions each of which is inseparable (from some other solution). This definition of separatrix solution reduces to that introduced earlier by Markus [5] for the special case $M^n = R^2$, and agrees with the vague usage customary in engineering oscillation analysis.

THEOREM 2. *Consider a completely unstable dynamical system in a differentiable manifold* M^n

$$\mathcal{S}) \quad \dot{x} = f(x).$$

Then the orbit space $M^n/\mathcal{S}$ satisfies the separation axiom T_1. Further, the following three assertions are equivalent:

1) *$\mathcal{S}$ has no saddle at infinity.*
2) *$\mathcal{S}$ has no separatrices.*
3) *$M^n/\mathcal{S}$ is a Hausdorff T_2 space.*

Proof. Let S_1 and S_2 be two different elements of $M^n/\mathscr{S}$; that is, S_1 and S_2 are two solution curves of $\mathscr{S}$. Take a small $(n-1)$-planar disc B_1 orthogonal (in some convenient Riemann metric on M^n) to S_1 at some point P_1, so $\mathscr{S}$ is transverse to B_1. Since S_2 has empty α and ω limit sets, S_2 can meet B_1 at only a finite number of points. Then, upon further restricting the diameter of B_1, we can assume that S_2 fails to meet B_1. Take the neighborhood N_1 of S_1 to be the saturation of B_1. Then S_2 does not lie in the projection $pN_1 \subset M^n/\mathscr{S}$. Hence $M^n/\mathscr{S}$ is a T_1 space.

Assume that $\mathscr{S}$ has no saddle at infinity and we show that $\mathscr{S}$ has no separatrices. Suppose that there were two inseparable solutions S_1 and S_2 of $\mathscr{S}$. Take $(n-1)$-planar discs B_1 and B_2, orthogonal to S_1 at $P_1 \in S_1$ and to S_2 at $P_2 \in S_2$ respectively, so that $\mathscr{S}$ is transverse to both B_1 and B_2. Also require that S_2 fails to meet $N_1 = \text{sat } B_1$ and that S_1 fails to meet $N_2 = \text{sat } B_2$. Since S_1 and S_2 are inseparable, there exists a solution $\varphi(t, x_1)$ on $-\infty < t < \infty$ that lies in $N_1 \bigcap N_2$.

Take an arc $\varphi(t, x_1)$ on $0 \le t \le t_1$ with $\varphi(0, x_1) \in B_1$ and $\varphi(t_1, x_1) \in B_2$ (or vice versa). Repeat the construction for smaller transversal discs at P_1 and P_2 to get a sequence of solution arcs

$$\varphi(t, x_k) \quad \text{on} \quad 0 \le t \le t_k$$

with $\varphi(0, x_k) \in B_1$ and $\varphi(t_k, x_k) \in B_2$ (or vice versa) and with

$$\lim_{k\to\infty} \varphi(0, x_k) = P_1, \qquad \lim_{k\to\infty} \varphi(t_k, x_k) = P_2.$$

If the sequence $\{t_k\}$ were bounded, then a subsequence (still denoted t_k) would converge to a finite limit $t_k \to \bar{t}$. But this would imply that P_2 lies on the solution curve S_1, which is impossible since S_1 and S_2 are different solutions of $\mathscr{S}$. Hence we can assume that $\{t_k\}$ increases monotonically towards infinity.

Some subsequence of points $\varphi(\tau_{k_j}, x_{k_j})$, with $0 < \tau_{k_j} < t_{k_j}$ diverges to infinity in M^n since S_1 approaches infinity with both its ends. Thus the subsequence of arcs $\varphi(t, x_{k_j})$ on $0 \le t \le t_{k_j}$ defines a saddle at infinity for the dynamical system $\mathscr{S}$. This contradicts our assumption and so we conclude that $\mathscr{S}$ has no separatrices.

On the other hand, assume that $\mathscr{S}$ has no separatrices in M^n. Suppose there were a saddle at infinity; that is, a sequence of solution arcs

$$\varphi(t, x_k) \quad \text{on} \quad 0 \le t \le t_k$$

with $\varphi(0, x_k) \to P_1$, $\varphi(t_k, x_k) \to P_2$, $t_k \nearrow \infty$ and $\varphi(\tau_k, x_k)$ diverging to infinity in M^n for an increasing unbounded time sequence $\{\tau_k\}$. It is easy to see that the solution curve S_2 through P_2 must coincide with the solution curve S_1 through P_1; for otherwise S_1 and S_2 would be inseparable in the space $M^n/\mathscr{S}$.

If P_2 followed P_1 along the solution S_1, then an easy continuity argument shows that P_2 is non-wandering. In the other case where P_2 precedes or equals P_1, P_1 is seen to be non-wandering. But $\mathscr{S}$ is completely unstable and every point of M^n is wandering. This contradiction proves that $\mathscr{S}$ has no saddle at infinity.

The assertion that $\mathcal{S}$ has no separatrices is equivalent to the condition that $M^n/\mathcal{S}$ is a T_2 space. This equivalence is immediate if we note that the set of inseparable solutions of $\mathcal{S}$ is empty if and only if its closure, the set of separatrices, is empty. Q.E.D.

COROLLARY. *Consider a dynamical system in* M^n

$$\mathcal{S}) \quad \dot{x} = f(x).$$

Then $\mathcal{S}$ *is completely unstable without separatrices if and only if: for each compact set* $K \subset M^n$ *there exists a time* T *such that the orbit* $\varphi(t, K)$ *fails to meet* K *for* $|t| > T$.

Proof. (cf. [3]).

Assume that $\mathcal{S}$ is completely unstable without separatrices in the differentiable manifold M^n. Let $K \subset M^n$ be a compact set. If $\varphi(t, K)$ meets K for arbitrarily large $|t|$, then there exists two convergent sequences of points $x_i \to P_1$ and $\varphi(t_i, x_i) \to P_2$ in K, for $|t_i|$ unbounded. By re-labeling a subsequence of the points, if necessary, we can assume that $t_i > 0$ are monotonically increasing to infinity. Since the orbit through P_1 tends to infinity in M^n, our construction yields a saddle at infinity for the dynamical system $\mathcal{S}$. This contradiction proves that $\varphi(t, K) \bigcap K = \varnothing$ for all sufficiently large $|t|$.

Conversely, assume that the dynamical system $\mathcal{S}$ has the property that $\varphi(t, K) \bigcap K = \varnothing$ for each compact set $K \subset M^n$, and for $|t|$ then suitably large. If we take K to be the compact closure of a ball neighborhood of a chosen point $P \in M^n$, then we see that P is wandering. Hence $\mathcal{S}$ is completely unstable.

Now suppose that $\mathcal{S}$ has a saddle at infinity in M^n. Then there exist unboundedly increasing sequences of times $0 < \tau_i < t_i$ such that $x_i \to P_1$ and $\varphi(t_i, x_i) \to P_2$, whereas $\varphi(\tau_i, x_i)$ diverges to infinity in M^n. In this case let K_1 be the union of two compact ball neighborhoods of P_1 and P_2. Then, for all large integers i, both x_i and $\varphi(t_i, x_i)$ lie in K_1. Therefore $\varphi(t_i, K_1)$ meets K_1 for all large $t_i \to \infty$. This contradicts the hypothesis and shows that $\mathcal{S}$ has no saddle at infinity. Hence $\mathcal{S}$ has no separatrices. Q.E.D.

For a completely unstable dynamical system $\mathcal{S}$ in a differentiable manifold M^n we shall use local transversals for $\mathcal{S}$ to define local coordinate charts in $M^n/\mathcal{S}$. A *transversal-section* Γ for $\mathcal{S}$ in M^n is defined as a differentiable $(n-1)$- hypersurface in M^n that is transverse to $\mathcal{S}$ (the normal component of the vector field $\mathcal{S}$ on Γ nowhere vanishes), and no solution curve of $\mathcal{S}$ meets Γ in more than one point. We also require that Γ is a topological embedding of an $(n-1)$-manifold in M^n, that is, Γ is a submanifold (without boundary) such that the induced topology coincides with the manifold topology on Γ. Of course, not every solution curve of $\mathcal{S}$ need meet Γ.

We define an *unseparated differentiable r-manifold* to be a topological space (with no separation properties demanded) M with an atlas or covering by local coordinate charts (open sets of M with given homeomorphisms onto open sets of R^r) satisfying the usual requirement that any two overlapping local charts of M are C^∞ differentiably inter-related. The atlas generates a maximal atlas that specifies the differentiable structure on M. The concepts of *differentiable map*, *diffeomorphism*, *differentiable product manifold*, *and differentiable fiber bundle and fibration* [10] are all defined for unseparated differentiable manifolds in the obvious way. We note that a differentiable manifold (without the qualifier "unseparated") is an unseparated differentiable manifold that is also a connected paracompact Hausdorff space with a countable base.

THEOREM 3. *Consider a completely unstable dynamical system in a differentiable manifold* M^n

$$\mathscr{S}) \quad \dot{x} = f(x).$$

Then the orbit space $M^n/\mathscr{S}$ *bears a unique unseparated differentiable structure fulfilling the condition:*

for each transversal-section Γ *of* $\mathscr{S}$ *the projection map* p *restricted to* Γ

$$p|\Gamma : \Gamma \to p\Gamma \subset M^n/\mathscr{S}$$

is a diffeomorphism.

Furthermore, $\{M^n, M^n/\mathscr{S}, R^1, p\}$ *is an unseparated differentiable fiber bundle with total or bundle space* M^n, *base space* $M^n/\mathscr{S}$, *fiber* R^1, *and projection map* p.

Proof. Let S be a solution curve of $\mathscr{S}$ and we shall proceed to define a local chart for a neighborhood U_1 of S in $M^n/\mathscr{S}$. Let B_1 be a transversal-section for $\mathscr{S}$ consisting of a (relatively open) $(n-1)$-disc on a hyperplane orthogonal to S at a point P. Since S approaches infinity at both ends and since P is wandering, the existence of such a transversal-section B_1 is assured. Let $U_1 = pB_1 = p(\text{sat } B_1)$. Since B_1 is a transversal-section for $\mathscr{S}$, the map

$$p|B_1 : B_1 \to U_1$$

defines a homeomorphism of B_1 onto U_1. We use the differentiable structure on B_1 (as a submanifold of M^n) to introduce local coordinates on U_1 so that $p|B_1$ is a diffeomorphism.

Let B_2 be another such transversal-section such that (sat B_2) meets (sat B_1). Then

$$B_1 \cap (\text{sat } B_1) \cap (\text{sat } B_2) = B_{12}$$

and

$$B_2 \cap (\text{sat } B_1) \cap (\text{sat } B_2) = B_{21}$$

are open submanifolds of B_1 and B_2, respectively. Moreover, the differentiable curve family of solutions of $\mathscr{S}$ defines a diffeomorphism of B_{12} onto B_{21}. Thus there exists a diffeomorphism of $pB_{12} = pB_1 \cap pB_2 = pB_{21}$ onto itself, corresponding to the inter-relation of the coordinatizations of pB_{12} within $U_1 = pB_1$ and within $U_2 = pB_2$. Therefore, the atlas of such local charts in such sets $U = pB$, for all transversal-section $(n-1)$-discs B, defines a differentiable structure on $M^n/\mathscr{S}$. This is the only differentiable structure on $M^n/\mathscr{S}$ such that the maps

$$p|B : B \to pB$$

are diffeomorphisms.

Moreover, for each differentiable transversal-section $\Gamma \subset M^n$, the map

$$p|\Gamma : \Gamma \to p\Gamma$$

is a homeomorphism of $\Gamma \subset M^n$ onto $p\Gamma \subset M^n/\mathscr{S}$. But $p|\Gamma$ is also a diffeomorphism since Γ

can be coordinatized locally by means of appropriate tangential $(n-1)$-discs. Thus we have specified $M^n/\mathscr{S}$ as an unseparated differentiable manifold, as required in the theorem.

The fiber bundle structure of $\{M^n, M^n/\mathscr{S}, R^1, p\}$ requires specific diffeomorphisms from $p^{-1}U_\alpha$ onto $R^1 \times U_\alpha$, where $\{U_\alpha\}$ is some open covering of $M^n/\mathscr{S}$. For each element $S_\alpha \in M^n/\mathscr{S}$ take an open neighborhood U_α obtained from a transversal-section disc B_α through S_α, that is,

$$pB_\alpha = p(\text{sat } B_\alpha) = U_\alpha .$$

Then

$$p^{-1}U_\alpha = \text{sat } B_\alpha \subset M^n$$

and we define a diffeomorphism of (sat B_α) with $R^1 \times U_\alpha$. Use the flow $\varphi(t, x_0)$ of $\mathscr{S}$ in M^n and the disc B_α, which is diffeomorphic to U_α under the projection map, to coordinatize each point $Q \in \text{sat } B_\alpha$ by $(t_Q, x_Q) \in R^1 \times B_\alpha$. That is, each point $Q \in \text{sat } B_\alpha$ has a unique representation as

$$Q = \varphi(t_Q, x_Q)$$

for $t_Q \in R^1$ and an initial point $x_Q \in B_\alpha$.

The inverse image of each element of U_α in $p^{-1}U_\alpha \approx R^1 \times U_\alpha$ is a single solution curve of $\mathscr{S}$ in sat B_α, or a line (t, x_Q) in $R^1 \times B_\alpha$. Thus $\{M^n, M^n/\mathscr{S}, R^1, p\}$ is an unseparated differentiable fiber bundle, as required. Q.E.D.

Remark. The group of the bundle $\{M^n, M^n/\mathscr{S}, R^1, p\}$ is the group of all orientation preserving diffeomorphisms of R^1 onto itself. The infinite dimensionality of this group is of no particular significance, but the non-Hausdorff character of the base $M^n/\mathscr{S}$ causes grave difficulties. In particular, the standard results on cross-sections and homotopy lifting are not valid in the generality arising here.

§3. UNSTABLE FLOWS WITHOUT SEPARATRICES

Consider a dynamical system in a differentiable manifold M^n

$$\mathscr{S}) \quad \dot{x} = f(x)$$

with orbits or solutions given by the flow $\varphi(t, x_0)$ in C^∞ in $R^1 \times M^n$. The orbit space $M^n/\mathscr{S}$ is a connected topological space with a countable base. If $\mathscr{S}$ is completely unstable, then $\{M^n, M^n/\mathscr{S}, R^1, p\}$ is an unseparated differentiable fiber bundle, according to Theorem 3. Furthermore, if $\mathscr{S}$ has no separatrices, then $M^n/\mathscr{S}$ is a (paracompact, Hausdorff) differentiable manifold and $\{M^n, M^n/\mathscr{S}, R^1, p\}$ is a differentiable fiber bundle in the usual sense, according to Theorem 2.

In the case where $\mathscr{S}$ is completely unstable with no separatrices in M^n, the projection map

$$p : M^n \to M^n/\mathscr{S}$$

defines a fibration (or Hurewicz fiber space) with the homotopy lifting property, see [10]. In this case the usual exact homotopy sequence applies

$$\cdots \to \pi_m(R^1) \to \pi_m(M^n) \to \pi_m(M^n/\mathscr{S}) \to \pi_{m-1}(R^1) \to \pi_{m-1}(M^n) \to \pi_{m-1}(M^n/\mathscr{S}) \to \cdots$$
$$\cdots \to \pi_0(R^1) \to \pi_0(M^n) \to \pi_0(M^n/\mathscr{S}),$$

where appropriate base points P_1 in a fiber S_1, $P_1 \in M^n$, and $p(P_1) = S_1$ are fixed. Furthermore, the usual differential fiber bundle techniques are valid, as are indicated in the following remark concerning cross-sections.

Remark. If a dynamical system in R^n, $n \neq 5$,

$$\mathscr{S}) \quad \dot{x} = f(x)$$

is topologically equivalent to a parallel system

$$\mathscr{P}) \qquad \dot{x}^1 = 1, \quad \dot{x}^2 = 0, \ldots, \qquad \dot{x}^n = 0,$$

then $\mathscr{S}$ *is differentiably equivalent to* $\mathscr{P}$.

To prove this assertion we note that $\mathscr{P}$ is completely unstable without separatrices, and so $\mathscr{S}$ is also. Then $\{R^n, R^n/\mathscr{S}, R^1, p\}$ is a differentiable fiber bundle with the differentiable manifold $R^n/\mathscr{S}$ as base. The homotopy exact sequence, with $\pi_m(R^n) = 0$ and $\pi_m(R^1) = 0$, shows that $\pi_m(R^n/\mathscr{S}) = 0$ for $m = 1, 2, 3, \ldots$. Hence $R^n/\mathscr{S}$ is a contractible space. Therefore, there exists a differentiable cross-section

$$\Gamma : R^n/\mathscr{S} \to R^n$$

that assigns to each solution curve S of $\mathscr{S}$ a point $P \in S$. In other words, Γ is a transversal-section for $\mathscr{S}$ in R^n such that every solution curve of $\mathscr{S}$ meets Γ in just one point. Then, using the flow defined by $\mathscr{S}$ and initiating at points on Γ (on the image of Γ in R^n), we define a diffeomorphism of R^n onto $R^1 \times \Gamma$ with the solution curves of $\mathscr{S}$ mapping onto the lines corresponding to the factor R^1.

But $\mathscr{S}$ in R^n is topologically equivalent to $\mathscr{P}$, and hence the fiber bundle $\{R^n, R^n/\mathscr{S}, R^1, p\}$ is bundle-homeomorphic to the topological product $R^1 \times R^{n-1}$. Since any two cross-sections of the fiber bundle $\{R^n, R^n/\mathscr{S}, R^1, p\}$ are homeomorphic, we find that Γ is homeomorphic to R^{n-1}. But then, at least for $n - 1 \neq 4$, it is known [6, 11] that Γ is diffeomorphic to R^{n-1}. In this case there is a diffeomorphism of R^n onto $R^1 \times R^{n-1}$ carrying the solution curves of $\mathscr{S}$ onto straight lines. Therefore $\mathscr{S}$ is differentiably equivalent to $\mathscr{P}$.

The arguments used above also prove the next theorem, cf. [1].

THEOREM 4. *Consider a completely unstable dynamical system in a contractible differentiable manifold* M^n

$$\mathscr{S}) \quad \dot{x} = f(x).$$

If $\mathscr{S}$ *has no separatrices, then* $M^n/\mathscr{S}$ *is a (paracompact, Hausdorff) differentiable* $(n-1)$-*manifold that is a contractible space. Furthermore,* $\mathscr{S}$ *in* M^n *is differentiably equivalent to the flow along the lines in the product manifold* $R^1 \times M^n/\mathscr{S}$.

COROLLARY. *Consider an unstable dynamical system in a contractible differentiable manifold* M^n

$$\mathscr{S}) \quad \dot{x} = f(x).$$

If $\mathscr{S}$ *has no saddle at infinity, then* $M^n/\mathscr{S}$ *is a contractible differentiable manifold, and* $\mathscr{S}$ *is differentiably equivalent to the flow along the lines in the product manifold* $R^1 \times M^n/\mathscr{S}$.

Proof. Since $\mathscr{S}$ has no saddle at infinity, $\mathscr{S}$ is completely unstable and $\mathscr{S}$ has no separatrices, according to Theorem 2 and the remarks in Section 1. Hence the Theorem 4 applies. Q.E.D.

We now take the contractible manifold M^n to be the number space R^n to study the case when $\mathscr{S}$ is parallel.

THEOREM 5. *Consider a completely unstable dynamical system in R^n for $n = 2$ or 3,*

$$\mathscr{S}) \quad \dot{x} = f(x).$$

Assume that $\mathscr{S}$ has no separatrices. Then $\mathscr{S}$ is differentiably equivalent to the flow $\mathscr{P}$ along parallel straight lines.

Proof. The differentiable fiber bundle $\{R^n, R^n/\mathscr{S}, R^1, p\}$ has the contractible base space $R^n/\mathscr{S}$, and so it is differentiably bundle-isomorphic with the product $R^1 \times R^n/\mathscr{S}$. For $n = 2$ or 3 the contractible $(n - 1)$-manifold $R^n/\mathscr{S}$ is diffeomorphic to R^1 or R^2, by the known classification of 1 and 2 dimensional manifolds. Thus $\mathscr{S}$ in R_2 or R_3 is differentiably equivalent to the flow along the lines in the product $R^1 \times R^1$ or $R^1 \times R^2$, and $\mathscr{S}$ is parallel. Q.E.D.

If we further assume that $R^n/\mathscr{S}$ is simply-connected at infinity [6, 11] and $n \geq 6$, then we can assert that $R^n/\mathscr{S}$ admits a unique differentiable structure and is diffeomorphic with R^{n-1}. In this case $\mathscr{S}$ in R^n is a parallel dynamical system. However, it does not seem easy to relate the usual theory of dynamical systems $\mathscr{S}$ in R^n to the hypothesis that $R^n/\mathscr{S}$ is simply-connected at infinity (except in certain trivial cases).

We close by presenting an example of a completely unstable dynamical system $\mathscr{S}$ in R^4 without separatrices, and yet $\mathscr{S}$ fails to be parallel.

Example. Let W^3 be the (open) differentiable 3-manifold of *H.* Whitehead [6, 13]. Here W^3 is contractible but W^3 is not homeomorphic to R^3. Consider the product manifold $R^1 \times W^3$ that is known to be combinatorially and hence differentiably isomorphic to R^4, see [2, 6]. The images of the lines $R^1 \times w_0$, for points $w_0 \in W^3$, define a differentiable regular curve family in $R^1 \times W^3 = R^4$. Upon sensing or orienting this regular curve family by unit tangent vectors, we obtain a completely unstable dynamical system $\mathscr{S}$ in R^4. Since $R^4/\mathscr{S}$ is the Hausdorff manifold W^3, $\mathscr{S}$ has no separatrices. But $\mathscr{S}$ is not topologically parallel in R^4, since the parallel system $\mathscr{P}$ yields an orbit space $R^4/\mathscr{P}$ that is homeomorphic to R^4 rather than W^3.

REFERENCES

1. A. ANTOSIEWICZ and J. DUGUNDJI: Parallelizable flows and Lyapounov's second method, *Ann. Math.* 73 (1961), 543–555.
2. J. GLIMM: Two Cartesian products which are Euclidean spaces, *Bull. Soc. math. Fr.* **88** (1960), 131–135.
3. F. HAHN: Recursion of set trajectories in a transformation group, *Proc. Am. math. Soc.* **11** (1960), 527–532.
4. W. KAPLAN: Regular curve families filling the plane I and II, *Duke math. J.* 7 (1940), 154–185; **8** (1941), 11–46.
5. L. MARKUS: Global structure of ordinary differential equations in the plane, *Trans. Am. math. Soc.* **76** (1954), 127–148.
6. J. MUNKRES: Obstructions to the smoothing of piecewise-differentiable homeomorphisms, *Ann. Math.* **72** (1960), 521–554.

7. V. NEMITSKY: Uber vollständig unstabile dynamische Systeme, *Annali Mat. pura appl.* **14** (1936), 275–286.
8. V. NEMITSKY and V. STEPANOV: *Qualitative Theory of Differential Equations*, Princeton, 1961.
9. H. POINCARÉ: Sur les courbes definées par les équations différentielles, *J. Math. pures appl.* **4** (1886), 151–217.
10. E. SPANIER: *Algebraic Topology*, McGraw-Hill, 1966.
11. J. STALLINGS: The piecewise linear structure of Euclidean space, *Proc. Camb. phil. Soc. math. phys. Sci.* **58** (1962), 481–488.
12. H. TERASAKA: On quasi-translations in E^n, *Proc. Japan Acad.* **30** (1954), 80–84.
13. J. H. C. WHITEHEAD: A certain open manifold whose group is unity, *Q. Jl Math.* **6** (1935), 268–279.
14. E. C. ZEEMAN: Polyhedral n-manifolds I and II, *Topology of 3-manifolds and related Topics*, Prentice-Hall (1962), 57–64 and 65–70.

University of Minnesota