JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 60, 47–57 (1977)

A Higher-Order Hopf Bifurcation Formula and Its Application to Fitzhugh's Nerve Conduction Equations*

IN-DING HSÜ

Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260

Submitted by C. L. Dolph

1. INTRODUCTION AND HOPF THEOREM

This work is a continuation of the recent research of Hsü and Kazarinoff [3] on the Hopf bifurcation formula and it's application to the Fitzhugh system. In [3], Hsü and Kazarinoff derived a criterion for the stability of bifurcating periodic solutions of the n-dimensional autonomous differential system (1.1) below. Their criterion applies if a certain quantity μ_2 is not zero. We obtain a new criterion in this paper to determine the direction and stability of periodic solutions of (1.1) if that quantity is zero. We also apply this criterion to Fitzhugh's system. We prove that this system has a family of unstable periodic solutions even if the parameters satisfy the condition $1 + \rho b^2 - 2b = 0$.

The general n-dimensional autonomous differential system we study is

$$\dot{x} = A(\mu)\, x + \hat{F}(x, \mu), \tag{1.1}$$

where $x = \operatorname{col}(x_1, x_2, \ldots, x_n)$, $\hat{F} = \operatorname{col}(\hat{F}_1, \ldots, \hat{F}_n)$, $\hat{F}(0, \mu) = 0$, and $\hat{F}_x(0, \mu) = 0$. Suppose the $\hat{F}_i$ are real analytic functions on $G \times (-c, c)$, where G is an open connected domain in R^n, $c > 0$, and A is a real $n \times n$ analytic matrix defined on $(-c, c)$ with exactly two purely imaginary eigenvalues at $\mu = 0$ whose continuous extensions $\alpha(\mu)$, $\bar{\alpha}(\mu)$ satisfy the conditions:

$$\alpha(0) = -\bar{\alpha}(0), \qquad \operatorname{Re}(\alpha'(0)) \neq 0,$$

with $\operatorname{Im}(\alpha(0)) = w_0 > 0$. Under this assumption, the Hopf theorem is

THEOREM (E. Hopf). *There exists an $\epsilon_0 > 0$ such that for each ϵ on an interval $(-\epsilon_0, \epsilon_0)$ there exists a periodic solution $p(t, \epsilon)$ with period $T(\epsilon)$ of (2.1), where the parameter ϵ is related to μ by a functional relation $\mu = \mu(\epsilon)$ such that $\mu(0) = 0$, $p(t, 0) = 0$, and $p(t, \epsilon) \neq 0$ for all sufficiently small $\epsilon \neq 0$. Moreover, $\mu(\epsilon)$, $p(t, \epsilon)$ and $T(\epsilon)$ are analytic at $\epsilon = 0$, and $T(0) = 2\pi/w_0$. These periodic solutions exist for one of three cases: either only for $\mu > 0$, or only for $\mu < 0$, or only for $\mu = 0$. Furthermore, for each $L > T(0)$ there exist $a > 0$, $b > 0$ such that if $|\mu| < b$, then there is no nonconstant periodic solution with period less than L, besides the bifurcating periodic solutions $p(t, \epsilon)$ with $\epsilon > 0$, that lies entirely in $\{x: |x| < a\}$.*

* This research was partially supported by NIH Grant GM 23156-0.

ISSN 0022-247X

2. General Formula for Direction of Bifurcation

Since $\hat{F}(x, \mu)$ is an analytic vector function of x and μ, we may rewrite (1.1) using Taylor's theorem as:

$$\dot{x} = A(\mu)\, x + Q_\mu(x, x) + K_\mu(x, x, x) + H_\mu(x, x, x, x) + M_\mu(x, x, x, x, x) + \cdots, \tag{2.1}$$

where the vector functions

$$Q_\mu(x, x),\ K_\mu(x, x, x),\ H_\mu(x, x, x, x),\ M_\mu(x, x, x, x, x),$$

are analytic on μ and depend linearly on each argument vector. By the Hopf theorem, the periodic solutions $p(t, \epsilon)$ of (2.1), the period $T(\epsilon)$, and the parameter $\mu(\epsilon)$ are analytic functions of ϵ at $\epsilon = 0$:

$$T(\epsilon) = T_0(1 + \tau_1\epsilon + \tau_2\epsilon^2 + \cdots), \tag{2.2}$$

$$\mu(\epsilon) = \mu_1\epsilon + \mu_2\epsilon^2 + \mu_3\epsilon^3 + \cdots, \tag{2.3}$$

$$p(t, \epsilon) = \epsilon(p_0(t) + p_1(t)\,\epsilon + p_2(t)\,\epsilon^2 + \cdots), \tag{2.4}$$

where the $p_i(t)$ are $T(\epsilon)$-periodic functions. The sign of the first nonzero μ_i plays an important role for the stability of the periodic solutions $p(t, \epsilon)$. From the proof of the Hopf theorem, it follows that if $\mu(\epsilon) \not\equiv 0$, then the first nonzero coefficients in $\mu(\epsilon)$ and $T(\epsilon)$ are of even order. In particular, $\mu_1 = \tau_1 = 0$. Now, suppose $\mu_2 = 0$; then $\mu_3 = 0$. In order to find a formula for μ_4, we introduce a new time variable s by

$$t = s(1 + \tau_2\epsilon^2 + \tau_3\epsilon^2 + \cdots). \tag{2.5}$$

We write $\epsilon y(s, \epsilon)$ for $p(t, \epsilon)$ and $y^i(s)$ for $p_i(t)$. Then both $y(s, \epsilon)$ and $y^i(s)$ are T_0-periodic functions. Substituting (2.5) and (2.4) into (2.1), we have

$$\begin{aligned}(1 - \tau_2\epsilon^2 - \tau_3\epsilon^3 - (\tau_4 - \tau_2^2)\,\epsilon^4 - \cdots)\,(\dot{y}^0(s) + \epsilon\dot{y}^1(s) + \epsilon^2\dot{y}^2(s) + \cdots) \\ = A(\mu)\, y + \epsilon Q_\mu(y, y) + \epsilon^2 K_\mu(y, y, y) \\ + \epsilon^3 H_\mu(y, y, y, y) + \epsilon^4 M_\mu(y, y, y, y, y) + \cdots.\end{aligned} \tag{2.6}$$

Since $A(\mu)$, Q_μ, K_μ, H_μ, and M_μ are analytic in μ, we may write

$$A(\mu) = A(0) + \mu A'(0) + (\mu^2/2)\, A''(0) + \cdots, \tag{2.7}$$

$$Q_\mu = Q_0 + \mu Q_0' + (\mu^2/2)\, Q_0'' + \cdots, \tag{2.8}$$

$$K_\mu = K_0 + \mu K_0' + (\mu^2/2)\, K_0'' + \cdots, \tag{2.9}$$

$$H_\mu = H_0 + \mu H_0' + (\mu^2/2)\, H_0'' + \cdots, \tag{2.10}$$

$$M_\mu = M_0 + \mu M_0' + (\mu^2/2)\, M_0'' + \cdots. \tag{2.11}$$

We substitute (2.7)–(2.11) and (2.3) into (2.6) with $\mu_2 = 0$ and we compare the coefficients of ϵ^i $(i = 0, 1, 2,..)$. We obtain

$$\dot{y}^0 = A(0)\, y^0 \tag{2.12}$$

$$\dot{y}^1 = A(0)\, y^1 + Q_0(y^0, y^0) \tag{2.13}$$

$$-\tau^0 \dot{y}^0 + \dot{y}^2 = A(0)\, y^2 + 2Q_0(y^0, y^1) + K_0(y^0, y^0, y^0) \tag{2.14}$$

$$\begin{aligned} &-\tau_3 \dot{y}^0 - \tau_2 \dot{y}^1 + \dot{y}^3 \\ &\quad = A(0)\, y^3 + Q_0(y^1, y^1) + 2Q_0(y^0, y^2) + 3K_0(y^0, y^0, y^1) \\ &\qquad + H_0(y^0, y^0, y^0, y^0) \end{aligned} \tag{2.15}$$

$$\begin{aligned} &-(\tau_4 - \tau_2^2)\, \dot{y}^0 - \tau_2 \dot{y}^2 - \tau_3 \dot{y}^1 + \dot{y}^4 \\ &\quad = A(0)\, y^4 + \mu_4 A'(0)\, y^0 + 2Q_0(y^0, y^3) + 2Q_0(y^1, y^2) \\ &\qquad + 3K_0(y^0, y^0, y^2) \\ &\qquad + 3K_0(y^0, y^1, y^1) + 4H_0(y^0, y^0, y^0, y^1) + M_0(y^0, y^0, y^0, y^0, y^0). \end{aligned} \tag{2.16}$$

By using (2.12)–(2.16), we are able to determine μ_4 as well as τ_2, τ_3, τ_4 and y^0, y^1, y^2, y^3, y^4. However, we do need the following lemma, which was introduced in Hopf's paper.

LEMMA 1. *Suppose $g(t)$ is a T_0-periodic function. Then the system*

$$\dot{Z} = A(0)\, Z + g(t) \tag{2.17}$$

has a periodic solution with period T_0 if and only if

$$\int_0^{T_0} g(t) \cdot Z^*(t)\, dt = 0 \tag{2.18}$$

for all T_0-periodic solutions $Z^*(t)$ of the adjoint differential equation

$$\dot{Z}^* = -A^t(0)\, Z^*, \tag{2.19}$$

where $A^t(0)$ is the transposed matrix of $A(0)$.

By our assumptions, there are two linearly independent T_0-periodic solutions $Z_1{}^*$ and $Z_2{}^*$ of (2.19). We may choose $Z_1{}^*$ and $Z_2{}^*$ so that

$$\int_0^{T_0} y^0(s) \cdot Z_1{}^*(s)\, ds = \int_0^{T_0} \dot{y}^0(s) \cdot Z_2{}^*(s)\, ds = 1 \tag{2.20}$$

and

$$\int_0^{T_0} y^0(s) \cdot Z_2{}^*(s)\, ds = \int_0^{T_0} \dot{y}^0(s) \cdot Z_1{}^*(s)\, ds = 0 \tag{2.21}$$

where $\cdot$ is the inner product and $y^0(t)$ is the T_0-periodic solution of (2.12).

By applying Lemma 1 and Hopf's scheme for deriving the formula for $\mu_2 \operatorname{Re}(\alpha'(0))$ to the system (2.12)–(2.16), we easily obtain

THEOREM 1. *Under the hypotheses of the Hopf theorem, if* $\mu_2 = 0$, *we have*

$$\begin{aligned}\mu_4 \operatorname{Re}(\alpha'(0)) = -\int_0^{T_0} [&2Q_0(y^0, y^3) + 2Q_0(y^1, y^2) + 3K_0(y^0, y^0, y^2) \\ &+ 3K_0(y^0, y^1, y^1) + 4H_0(y^0, y^0, y^0, y^1) \\ &+ M_0(y^0, y^0, y^0, y^0, y^0) + \tau_3 A(0)\, y^1 + \tau_2 A(0)\, y^2] \cdot Z_1^*(s)\, ds,\end{aligned} \tag{2.22}$$

where $y^0(s)$ *and* $y^1(s)$ *are the* T_0*-periodic solutions of* (2.12) *and* (2.13), *respectively,* $y^2(s)$ *and* $y^3(s)$ *are* T_0*-periodic solutions of*

$$\dot{y}^2 = A(0)\, y^2 + 2Q_0(y^1, y^1) + K_0(y^0, y^0, y^0) + \tau_2 A(0)\, y^0 \tag{2.23}$$

and

$$\begin{aligned}\dot{y}^3 = T(0)\, y^3 &+ Q_0(y^1, y^1) + 2Q_0(y^0, y^2) + 3K_0(y^0, y^0, y^1) \\ &+ H_0(y^0, y^0, y^0, y^0) + \tau_2 A(0)\, y^1 + \tau_2 Q_0(y^0, y^0) + \tau_3 A(0)\, y^0,\end{aligned} \tag{2.24}$$

respectively, and

$$\tau_2 = -\int_0^{T_0} [2Q_0(y^0, y^1) + K(y^0, y^0, y^2)] \cdot Z_2^*(s)\, ds \tag{2.25}$$

$$\begin{aligned}\tau_3 = -\int_0^{T_0} [&Q_0(y^1, y^1) + 2Q_0(y^0, y^2) + 3K_0(y^0, y^0, y^1) + H_0(y^0, y^0, y^0, y^1) \\ &+ \tau_2 A(0)\, y^1] \cdot Z_2^*(s)\, ds.\end{aligned} \tag{2.26}$$

To facilitate the computations that arise in applying Theorem 1, we use the method in Hsü and Kazarinoff [3]. We use the system

$$\dot{y} = L(\mu)\, y + F(y, \mu) \tag{2.27}$$

instead of the system (1.1), where $L(\mu) = P^{-1}A(\mu)\, P$, $y = P^{-1}x$, $F(y, \mu) = P^{-1}\hat{F}(Py, \mu)$, and

$$P^{-1}A(0)\, P = \begin{pmatrix} 0 & w_0 & 0 \\ -w_0 & 0 & 0 \\ 0 & 0 & D \end{pmatrix}, \tag{2.28}$$

where $w_0 = \operatorname{Im} \alpha(0) > 0$; see [3, Lemma 2.1] for the form of D.

By using the system (2.27), we can easily find $y^0(s)$ and $Z_i^*(s)$ such that (2.20) and (2.21) hold:

$$y^0(s) = (\cos(w_0 , s), -\sin(w_0 s), 0,..., 0)^t, \tag{2.29}$$

$$z_1^*(s) = (1/T_0)(\cos(w_0 s), -\sin(w_0 s), 0,..., 0)^t, \tag{2.30}$$

and

$$z_2^*(s) = -(1/(T_0 w_0))(\sin(w_0 s), \cos(w_0 s), 0,..., 0)^t. \tag{2.31}$$

By using (2.29)–(2.31) (see [3]), we are able to write down the solutions $y^1(s)$ of (2.13) precisely. Similarly, we obtain $y^2(s)$ and $y^3(s)$, and thus we compute $\mu_4 \operatorname{Re}(\alpha'(0))$ by (2.22). We shall not give the details in general. However, we illustrate the computations in Section 4 relative to Fitzhugh's system.

Remark. From the proof of the Hopf theorem, it is easy to see that the "analytic" assumption for $\hat{F}(x, \mu)$ can be reduced to C^{k+1} if we are only interested in the existence of C^k-periodic solutions; see [5] also. However, in order to obtain the formula (2.22), we do need to assume that $\hat{F}(x, \mu)$ is a C^6-function.

3. Stability of Bifurcating Periodic Solutions

The characteristic exponents of a periodic solution $p(t, \epsilon)$ are well known to be the eigenvalues of the eigenvalue problem

$$\begin{gathered} \dot{V}(t) + \lambda V(t) = L(t, \epsilon)\, V(t), \\ V(0) = V(T(\epsilon)), \end{gathered} \tag{3.1}$$

where $V(t)$ has the same period $T(\epsilon)$ as the solution $p(t, \epsilon)$ and the function

$$L(t, \epsilon) = A(\mu(\epsilon)) + \hat{F}_x(p(t, \epsilon), \mu(\epsilon)) \tag{3.2}$$

is periodic in t with period $T(\epsilon)$ and is analytic in t and ϵ at $\epsilon = 0$. The characterstic exponents depend continuously upon ϵ and are determined only $\operatorname{mod}(2\pi i/T(\epsilon))$. As $\epsilon \to 0$, the exponents tend $\operatorname{mod}(2\pi i/T_0)$ to the exponents of the stationary solution of (1.1) with $\mu = 0$. According to our assumptions, there are exactly two exponents that tend to the imaginary axis as $\epsilon \to 0$. One of them is the one which vanishes identically, the other $\beta = \beta(\epsilon)$ must be real and analytic at $\epsilon = 0$ and $\beta(0) = 0$. We may write

$$\beta = \beta_1 \epsilon + \beta_2 \epsilon^2 + \beta_3 \epsilon^3 + \beta_4 \epsilon^4 + \cdots . \tag{3.3}$$

By the conclusions of the Hopf theorem, $\beta_1 = 0$ and

$$\beta_2 = -2\mu_2 \operatorname{Re}(\alpha'(0)). \tag{3.4}$$

We may use the same scheme without any difficulty to derive

$$\beta_3 = -2\mu_3 \operatorname{Re}(\alpha'(0)) \qquad \text{if} \qquad \mu_2 = 0. \tag{3.5}$$

Hence

$$\beta_4 = -2\mu_4 \operatorname{Re}(\alpha'(0)) \tag{3.6}$$

if $\mu_2 = 0$ (since $\mu_3 = 0$). Thus we have the following analog to [3, Theorem 3.1]:

THEOREM 2. *Under the assumptions of Hopf's theorem, if $A(0)$ has exactly two pure imaginary eigenvalues and the other $n - 2$ have negative real parts and if $\mu_2 = 0$ and $\mu_4 \operatorname{Re}(\alpha'(0)) > 0$, then a bifurcating periodic solution whose existence is asserted by Hopf's theorem is asymptotically orbitally stable with asymptotic phase; however, if $\mu_4 \operatorname{Re}(\alpha'(0)) < 0$ or if any one of the other $n - 2$ eigenvalues has a positive real part, then the bifurcating periodic solutions are orbitally unstable.*

4. PERIODIC SOLUTIONS OF THE FITZHUGH EQUATIONS

In this section we apply the results of Sections 2 and 3 to the system of Fitzhugh which was recently studied by Hsü and Kazarinoff [3] and Troy [4]. The Fitzhugh system is

$$\begin{aligned} \dot{x}_1 &= \alpha + x_2 + x_1 - \tfrac{1}{3}x_1^3, \\ \dot{x}_2 &= \rho(a - x_1 - bx_2), \end{aligned} \tag{4.1}$$

where $\alpha \in (-\infty, \infty)$, $0 < b < 1$, $a \in (-\infty, \infty)$, and $\rho \in (0, 1)$. Hsü and Kazarinoff [3] and Troy [4] have proved that if $b \in (0, \frac{1}{2}]$, $\rho \in (0, 1)$, and $a \in (-\infty, \infty)$, there exist α_1 and α_2 such that the Fitzhugh system (4.1) has an asymptotical, periodic solution for each $\alpha \in (\alpha_1, \alpha_1 + \epsilon)$ or $\alpha \in (\alpha_2 - \epsilon, \alpha_2)$ for some $\epsilon > 0$. However, if $b \in (\frac{1}{2}, 1)$, $\rho \in (0, 1)$, $a \in (-\infty, \infty)$, and $1 + \rho b^2 - 2b = 0$, they were unable to determine the direction of bifurcation and stability of bifurcating periodic solutions. We determine them under these conditions.

By Hsü and Kazarinoff [3], the system (4.1) can be rewritten as

$$\dot{y} = L(\mu)\, y + \begin{pmatrix} -2x_1(\alpha)\, y_1^2 - \frac{4}{3}\, y_1^3 \\ \dfrac{-2\rho b}{w_0}\, x_1(\alpha)\, y_1^2 - \dfrac{4\rho b}{3w_0} y_1^3 \end{pmatrix}, \tag{4.2}$$

where $\alpha = \alpha_j + \mu$ $(j = 1, 2)$, $(x_1(\alpha), x_2(\alpha))$ is the unique steady state $(x_1(\alpha_1) = -(1 - \rho b)^{1/2}$, $x_1(\alpha_2) = (1 - \rho b)^{1/2})$, $L(\mu) = P^{-1}A(\mu)\, P$ with

$$A(\mu) = \begin{pmatrix} 1 - x_1^2(\alpha) & 1 \\ -\rho & -\rho b \end{pmatrix} \tag{4.3}$$

which is the linearized matrix of (4.1), and

$$P = \begin{pmatrix} 2 & 0 \\ -2\rho b & 2w_0 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ \dfrac{\rho b}{2w_0} & \dfrac{1}{2w_0} \end{pmatrix}. \tag{4.4}$$

Here $\pm w_0 i = \pm(\rho - \rho^2 b^2)^{1/2}\, i$ are the purely imaginary eigenvalues of $A(0)$. Writing $F(y, \mu)$ for the nonlinear part of (4.3), we have

$$\begin{aligned} F^1_{11} &= \frac{\partial^2 F^1(0, 0)}{\partial {y_1}^2} = -4x_1(\alpha_j), \\ F^2_{11} &= \frac{\partial^2 F^2(0, 0)}{\partial {y_1}^2} = \frac{-4\rho b}{w_0}\, x_1(\alpha_j), \\ F^1_{111} &= \frac{\partial^3 F^1(0, 0)}{\partial {y_1}^3} = -8, \\ F^2_{111} &= \frac{\partial^3 F^2(0, 0)}{\partial {y_1}^3} = -\frac{8\rho b}{w_0} \end{aligned} \tag{4.5}$$

and all other F^l_{ij} and F^l_{ijk} and any higher derivatives are zero. From [3, Lemma 2.2], we see directly that

$$\begin{aligned} {y_1}^1(s) = \frac{1}{6w_0}\{&F^1_{11}[\sin(w_0 s) + 2\sin(w_0 s)\cos(w_0 s)] \\ &+ F^2_{11}[1 + 2\cos(w_0 s) + \sin^2(w_0 s)]\}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} {y_2}^1(s) = \frac{1}{6w_0}\{&F^1_{11}[-2 + \cos(w_0 s) + \cos^2(w_0 s)] \\ &+ F^2_{11}[-2\sin(w_0 s) + 2\sin(w_0 s)\cos(w_0 s)]\}. \end{aligned} \tag{4.7}$$

By (2,25), we have

$$\begin{aligned} \tau_2 &= \frac{1}{T_0 w_0}\int_0^T [F^1_{11}{y_1}^1(s)\cos(w_0 s)\sin(w_0 s) + \tfrac{1}{6}F^1_{111}\cos^2(w_0 s)\sin(w_0 s) \\ &\quad + F^2_{11}{y_1}^1(s)\cos^2(w_0 s) + \tfrac{1}{6}F^2_{111}\cos^4(w_0 s)]\, ds \\ &= \frac{1}{24{w_0}^2}(F^1_{11})^2 + \frac{5}{48w^2}(F^2_{11})^2 + \frac{1}{16w}F^2_{111}. \end{aligned}$$

For the system (4.3), the differential equation (2.23) becomes

$$\begin{aligned} \begin{pmatrix} \dot{y}_1{}^2 \\ \dot{y}_2{}^2 \end{pmatrix} &= \begin{pmatrix} 0 & w_0 \\ -w_0 & 0 \end{pmatrix}\begin{pmatrix} {y_1}^2 \\ {y_2}^2 \end{pmatrix} \\ &\quad + \begin{pmatrix} F^1_{11}{y_1}^1(s)\cos(w_0 s) + \tfrac{1}{6}F^1_{111}\cos^3(w_0 s) - \tau_2 w_0\sin(w_0 s) \\ F^2_{11}{y_1}^1(s)\cos(w_0 s) + \tfrac{1}{6}F^2_{111}\cos^3(w_0 s) - \tau_2 w_0\cos(w_0 s) \end{pmatrix}. \end{aligned} \tag{4.9}$$

It is not difficult to find the unique T_0-periodic solution $y^2(s)$ of (4.9) which satisfies the initial conditions

$$\dot{y}^2(0) \cdot e = 0 \qquad \text{and} \qquad \dot{y}^2(0) \cdot e = 0, \tag{4.10}$$

where $e = (0, -1/2w_0)$ was introduced in [3]:

$$\begin{aligned} y_1^2(s) &= -\tau_2 \cos(w_0 s) \\ &\quad + \frac{(F_{11}^1)^2}{6w_0^2}\left[\frac{1}{3} + \frac{1}{3}\cos(w_0 s) - \frac{2}{3}\cos^2(w_0 s) + \frac{3}{4}\sin^2(w_0 s)\cos(w_0 s)\right] \\ &\quad + \frac{(F_{11}^2)^2}{6w_0^2}\left[\frac{4}{3} + \frac{7}{3}\cos(w_0 s) - \frac{2}{3}\cos^2(w_0 s) - \frac{1}{8}\sin^2(w_0 s)\cos(w_0 s)\right] \\ &\quad + \frac{F_{11}^1 F_{11}^2}{6w_0^2}[\sin(w_0 s)\cos(w_0 s) + 2\sin(w_0 s) - \sin(w_0 s)\cos^2(w_0 s)] \\ &\quad + \frac{F_{111}^2}{48w_0}[\sin^2(w_0 s)\cos(w_0 s) + 8\cos(w_0 s)] \end{aligned} \tag{4.11}$$

$$\begin{aligned} y_2^2(s) &= \tau_2 \sin(w_0 s) + \frac{F_{11}^1 F_{11}^2}{6w_0^2}[-1 + \cos(w_0 s) + \sin^2(w_0 s)\cos(w_0 s)] \\ &\quad + \frac{(F_{11}^1)^2}{6w_0^2}\left[-\frac{5}{6}\sin(w_0 s) + \frac{1}{3}\sin(w_0 s)\cos(w_0 s) + \frac{1}{4}\sin(w_0 s)\cos^2(w_0 s)\right] \\ &\quad + \frac{(F_{11}^2)^2}{6w_0^2}\left[-\frac{3}{4}\sin(w_0 s) + \frac{4}{3}\sin(w_0 s)\cos(w_0 s) - \frac{3}{8}\sin(w_0 s)\cos^2(w_0 s)\right] \\ &\quad + \frac{F_{111}^2}{6w_0}\left[-\frac{3}{4}\sin(w_0 s) + \frac{3}{8}\sin(w_0 s)\cos^2(w_0 s)\right]. \end{aligned} \tag{4.12}$$

Now, we are ready to compute τ_3 by (2.26). We obtain:

$$\begin{aligned} \tau_3 = \frac{1}{T_0 w_0} \Bigg\{ &\int_0^{T_0} \left[\frac{1}{2} F_{11}^1 (y_1^1(s))^2 + F_{11}^1 y_1^2(s)\cos(w_0 s)\right. \\ &+ \left.\frac{1}{2} F_{111}^1 y_1^1(s)\cos^2(w_0 s) + \tau_2 w_0 y_2^1(s)\right]\sin(w_0 s)\, ds \\ &+ \int_0^{T_0} \left[\frac{1}{2} F_{11}^2 (y_1^1(s))^2 + F_{11}^2 y_1^2(s)\cos(w_0 s)\right. \\ &+ \left.\frac{1}{2} F_{111}^2 y_1^1(s)\cos^2(w_0 s) - \tau_2 w_0 y_{11}(s)\right]\cos w_0 s\, ds \Bigg\} \\ = \frac{1}{72 w_0^3} &[5(F_{11}^2)^3 + 2(F_{11}^1)^2 F_{11}^2 + 3w_0 F_{111}^2 F_{11}^2]. \end{aligned} \tag{4.13}$$

With τ_2 and τ_3 determined, the differential equation (2.24) is well defined. It becomes

$$\begin{pmatrix} \dot{y}_2{}^3 \\ \dot{y}_1{}^3 \end{pmatrix} = \begin{pmatrix} w_0 y_2{}^3 + \frac{1}{2}F^1_{11}(y_1{}^1(s))^2 + F^1_{11} y_1{}^2(s)\cos(w_0 s) \\ -w_0 y_1{}^3 + \frac{1}{2}F^2_{11}(y_1{}^1(s))^2 + F^2_{11} y_1{}^2(s)\cos(w_0 s) \end{pmatrix}$$
$$+ \begin{pmatrix} \frac{1}{2}F^1_{111} y_1{}^1(s)\cos^2(w_0 s) - \tau_3 w_0 \sin(w_0 s) \\ \frac{1}{2}F^2_{111} y_1{}^1(s)\cos^2(w_0 s) - \tau_3 w_0 \cos(w_0 s) \end{pmatrix} \tag{4.14}$$
$$+ \begin{pmatrix} + \tau_2 w_0 y_2{}^1(s) + \frac{1}{2}\tau_2 F^1_{11}\cos^2(w_0 s) \\ - \tau_2 w_0 y_1{}^1(s) + \frac{1}{2}\tau_2 F^2_{11}\cos^2(w_0 s) \end{pmatrix}$$

We can find a unique T_0-periodic solution $y^3(s)$ of (4.14) which satisfies the initial conditions

$$y^3(0) \cdot e = 0 \quad \text{and} \quad \dot{y}^3(0) \cdot e = 0; \tag{4.15}$$

namely,

$$\begin{aligned} y_1{}^3(s) &= \cos(w_0 s)\,[1] + \sin(w_0 s)\,[2], \\ y_2{}^3(s) &= -\sin(w_0 s)\,[1] + \cos(w_0 s)\,[2], \end{aligned} \tag{4.16}$$

where

$$\begin{aligned}
[1] = \frac{1}{72 w_0{}^3} \Big\{ &(F^1_{11})^3 \Big[2\sin(w_0 s) - \sin(2w_0 s) + 4\sin^3(w_0 s) \\
&- \frac{3}{8}\sin(4w_0 s) - \frac{13}{5}\sin^5(w_0 s) \Big] \\
&+ F^1_{11}(F^2_{11})^2 \Big[26\sin(w_0 s) + \sin(2w_0 s) - \frac{93}{6}\sin^3(w_0 s) \\
&- \frac{3}{16}\sin(4w_0 s) - \frac{3}{2}\sin^5(w_0 s) \Big] \\
&+ (F^1_{11})^2 F^2_{11} \Big[\frac{35}{6} - \frac{22}{3}\cos^3(w_0 s) - 5\cos^4(w_0 s) + 3\cos^5(w_0 s) - \sin^4(w_0 s) \Big] \\
&+ (F^2_{11})^3 \Big[-\frac{1019}{60} + \frac{3}{2}\cos(w_0 s) - 2\sin^2(w_0 s) \\
&+ \frac{19}{3}\cos^3(w_0 s) - \sin^4(w_0 s) + \frac{1}{5}\cos^5(w_0 s) \Big] \Big\} \\
&+ \frac{F^2_{111}}{48 w_0{}^2} \Big\{ F^1_{11} \Big[-\frac{13}{3}\sin^3(w_0 s) + \frac{7}{5}\sin^5(w_0 s) + \frac{1}{8}\sin(4w_0 s) + 6\sin(w_0 s) \Big] \\
&+ F^2_{11} \Big[-\frac{35}{6} + 5\cos^3(w_0 s) - \cos(w_0 s) + 2\cos^4(w_0 s) - \cos^5(w_0 s) \Big] \Big\}
\end{aligned}$$

It is not difficult to find the unique T_0-periodic solution $y^2(s)$ of (4.9) which satisfies the initial conditions

$$\dot{y}^2(0) \cdot e = 0 \qquad \text{and} \qquad \dot{y}^2(0) \cdot e = 0, \tag{4.10}$$

where $e = (0, -1/2w_0)$ was introduced in [3]:

$$\begin{aligned} y_1^2(s) &= -\tau_2 \cos(w_0 s) \\ &+ \frac{(F_{11}^1)^2}{6w_0^2}\left[\frac{1}{3} + \frac{1}{3}\cos(w_0 s) - \frac{2}{3}\cos^2(w_0 s) + \frac{3}{4}\sin^2(w_0 s)\cos(w_0 s)\right] \\ &+ \frac{(F_{11}^2)^2}{6w_0^2}\left[\frac{4}{3} + \frac{7}{3}\cos(w_0 s) - \frac{2}{3}\cos^2(w_0 s) - \frac{1}{8}\sin^2(w_0 s)\cos(w_0 s)\right] \\ &+ \frac{F_{11}^1 F_{11}^2}{6w_0^2}[\sin(w_0 s)\cos(w_0 s) + 2\sin(w_0 s) - \sin(w_0 s)\cos^2(w_0 s)] \\ &+ \frac{F_{111}^2}{48w_0}[\sin^2(w_0 s)\cos(w_0 s) + 8\cos(w_0 s)] \end{aligned} \tag{4.11}$$

$$\begin{aligned} y_2^2(s) &= \tau_2 \sin(w_0 s) + \frac{F_{11}^1 F_{11}^2}{6w_0^2}[-1 + \cos(w_0 s) + \sin^2(w_0 s)\cos(w_0 s)] \\ &+ \frac{(F_{11}^1)^2}{6w_0^2}\left[-\frac{5}{6}\sin(w_0 s) + \frac{1}{3}\sin(w_0 s)\cos(w_0 s) + \frac{1}{4}\sin(w_0 s)\cos^2(w_0 s)\right] \\ &+ \frac{(F_{11}^2)^2}{6w_0^2}\left[-\frac{3}{4}\sin(w_0 s) + \frac{4}{3}\sin(w_0 s)\cos(w_0 s) - \frac{3}{8}\sin(w_0 s)\cos^2(w_0 s)\right] \\ &+ \frac{F_{111}^2}{6w_0}\left[-\frac{3}{4}\sin(w_0 s) + \frac{3}{8}\sin(w_0 s)\cos^2(w_0 s)\right]. \end{aligned} \tag{4.12}$$

Now, we are ready to compute τ_3 by (2.26). We obtain:

$$\begin{aligned} \tau_3 = \frac{1}{T_0 w_0}\Bigg\{&\int_0^{T_0}\left[\frac{1}{2}F_{11}^1(y_1^1(s))^2 + F_{11}^1 y_1^2(s)\cos(w_0 s)\right. \\ &\left. + \frac{1}{2}F_{111}^1 y_1^1(s)\cos^2(w_0 s) + \tau_2 w_0 y_2^1(s)\right]\sin(w_0 s)\,ds \\ &+ \int_0^{T_0}\left[\frac{1}{2}F_{11}^2(y_1^1(s))^2 + F_{11}^2 y_1^2(s)\cos(w_0 s)\right. \\ &\left. + \frac{1}{2}F_{111}^2 y_1^1(s)\cos^2(w_0 s) - \tau_2 w_0 y_{11}(s)\right]\cos w_0 s\,ds\Bigg\} \\ = \frac{1}{72w_0^3}&[5(F_{11}^2)^3 + 2(F_{11}^1)^2 F_{11}^2 + 3w_0 F_{111}^2 F_{11}^2]. \end{aligned} \tag{4.13}$$

Substituting the quantities of (4.5) into (4.17), we find

$$\mu_4 \operatorname{Re}(\alpha'(0)) < 0. \tag{4.18}$$

By using Theorem 2, we obtain

THEOREM 3. *For any given* $b \in (\frac{1}{2}, 1)$, $\rho \in (0, 1)$, *and* $a \in (0, \infty)$, *there exist* α_1 *and* α_2 *and two positive numbers* ϵ_1 *and* ϵ_2 *such that:* (1) *If* $1 + \rho b^2 - 2b > 0$, *then the Fitzhugh system* (4.1) *has a periodic solution for each* $\alpha \in (\alpha_1, \alpha_1 + \epsilon_1)$ *and each* $\alpha \in (\alpha_2 - \epsilon_2, \alpha_2)$, *and these bifurcating periodic solutions are asymptotically, orbitally stable with asymptotic phase.* (2) *If* $1 + \rho b^2 - 2b \leqslant 0$, *then* (4.1) *has a periodic solution for each* $\alpha \in (\alpha_1 - \alpha_1, \epsilon_1)$ *and each* $\alpha \in (\alpha_2, \alpha_2 + \epsilon_2)$, *and these periodic solutions are orbitally unstable. The* α_i *and* ϵ_i *depend on* a, b, ρ, *and* $\alpha_1 < \alpha_2$.

REFERENCES

1. R. FITZHUGH, Impulses and physiological states in theoretical models of membrane, *Biophys. J.* 1 (1961), 455–466.
2. E. HOPF, Abzweigning einer periodischen losung von einer stationaren losung eines differential systems, *Ber. Verh. Sachs. Akad. Wiss. Leipsig. Math. Nat.* **94** (1942), 3–22.
3. I. D. HSÜ AND N. D. KAZARINOFF, An applicable Hopf bifurcation formula and instability of small periodic solutions of the Field–Noyes model, *J. Math. Anal. Appl.*, **55** (1976), 61–89.
4. W. C. TROY, Bifurcation phenomena in Fitzhugh's nerve conduction equations, *J. Math. Anal. Appl.* **54** (1976), 678–690.
5. Y. H. WAN, "A Remark on Hopf Bifurcation Formula," preprint.