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Iteration Processes for Nonlinear Mappings in Convex Metric Spaces

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1. INTRODUCTION

Takahashi [1] introduced a notion of convex metric spaces and studied the fixed point theory for nonexpansive mappings in such setting. For further investigations in this setting the reader may consult [2–4] and references therein.

For the convex metric spaces Kirk [5] and Goebel and Kirk [6] use the term "hyperbolic type space." They studied the iteration processes for nonexpansive mappings in the abstract framework and generalize and unify some known results in [7, 8].

In this paper, we shall deal with Ishikawa's iteration scheme to construct fixed points of quasi-contractive, generalized quasi-contractive, and quasi-nonexpansive mappings in convex metric spaces. Our results generalize and unify the corresponding results in [10–17].

2. CONVERGENCE OF ITERATES OF QUASI-CONTRACTIVE AND GENERALIZED QUASI-CONTRACTIVE MAPS

DEFINITION 2.1. [1]. Let (X, d) be a metric space and $I = [0, 1]$. A mapping $W: X \times X \times I \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leqslant \lambda d(u, x) + (1-\lambda)\, d(u, y). \tag{2.1}$$

X together with a convex structure W is called a convex metric space. A nonempty subset K of X is said to be convex if $W(x, y, \lambda) \in K$ for all $(x, y, \lambda) \in K \times K \times I$.

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Obviously, all normed linear spaces and their convex subsets are convex metric spaces. But there are many examples of convex metric spaces which are not embedded in any normed linear spaces (See [1]).

THEOREM 2.1. *Let K be a nonempty closed convex subset of a complete* convex metric space X and let $T: K \to K$ *be a quasi-contractive mapping* [9], *i.e., there exists a constant $q \in [0, 1)$ such that for all $x, y \in K$,*

$$d(Tx, Ty) \leqslant q \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \tag{2.2}$$

Suppose that $\{x_n\}$ is Ishikawa type iterative scheme defined by

$$\begin{aligned} x_0 \in K, \qquad x_{n+1} &= W(Ty_n, x_n, \alpha_n) \\ y_n = W(Tx_n, x_n, \beta_n), \qquad n &= 0, 1, 2, ..., \end{aligned} \tag{2.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy that $0 \leqslant \alpha_n, \beta_n \leqslant 1$ and $\sum_n \alpha_n$ diverges. Then $\{x_n\}$ converges to a unique fixed point of T in K.

Proof. Let N be the set of all nonnegative integers. For $n, m \in N$ with $0 \leqslant n < m$, we write

$$D_{n,m} = \bigcup_{j=n}^{m} \{x_j, y_j, Tx_j, Ty_j\},$$

and $\delta(D_{n,m}) = \sup\{d(x, y): x, y \in D_{n,m}\}$ denotes the diameter of $D_{n,m}$. By (2.2), we easily show that for all $k, l \in N$ with $n \leqslant k, l \leqslant m$,

$$\max\{d(Tx_k, Tx_l), d(Tx_k, Ty_l), d(Ty_k, Ty_l)\} \leqslant q\delta(D_{n,m}). \tag{2.4}$$

Now we show that

$$\delta(D_{n,m}) = \max\{d(x_n, Tx_j), d(x_n, Ty_j): n \leqslant j \leqslant m\}. \tag{2.5}$$

We consider the following several cases:

(i) $\delta(D_{n,m}) = \max\{d(x_i, Tx_j): n \leqslant i, j \leqslant m\}$. In the case if $\delta(D_{n,m}) = d(x_{k+1}, Tx_l)$ for $n \leqslant k < m$ and $n \leqslant l \leqslant m$, then, by (2.1), (2.3), and (2.4), we have $\delta(D_{n,m}) = d(W(Ty_k, x_k, \alpha_k), Tx_l) \leqslant \alpha_k d(Ty_k, Tx_l) + (1-\alpha_k) d(x_k \cdot Tx_l) \leqslant \alpha_k q\delta(D_{n,m}) + (1-\alpha_k)\delta(D_{n,m})$. It follows that $\alpha_k = 0$ and $\delta(D_{n,m}) = d(x_k, Tx_l)$. By induction we obtain $\delta(D_{n,m}) = d(x_n, Tx_l)$ and so (2.5) holds.

(ii) $\delta(D_{n,m}) = \max\{d(x_i, Ty_j): n \leqslant i, j \leqslant m\}$. Using same argument as in (i), we obtain $\delta(D_{n,m}) = d(x_n, Ty_l)$ with $n \leqslant l \leqslant m$ and so (2.5) holds.

(iii) $\delta(D_{n,m}) = d(y_k, Tx_l)$, $n \leqslant k, l \leqslant m$. In this case, by (2.1), (2.3), and (2.4), we have $\delta(D_{n,m}) = d(W(Tx_k, x_k, \beta_k), Tx_l) \leqslant \beta_k d(Tx_k, Tx_l) +$

$(1-\beta_k)$, $d(x_k, Tx_l) \leqslant \beta_k q\delta(D_{n,m}) + (1-\beta_k)\, d(x_k\ Tx_l) \leqslant \delta(D_{n,m})$, and so $\delta(D_{n,m}) = d(x_k, Tx_l)$. It follows from (i) that (2.5) holds.

(iv) $\delta(D_{n,m}) = d(y_k, Ty_l)$, $n \leqslant k, l \leqslant m$. Using same argument as in (iii), (2.5) holds.

(v) $\delta(D_{n,m}) = \max\{d(x_i, x_j): n \leqslant i, j \leqslant m\}$. In this case there exist $k, l \in N$ with $n \leqslant k \leqslant l < m$ such that $\delta(D_{n,m}) = d(x_k, x_{l+1}) > d(x_k, x_l)$. By (2.1) and (2.3), $\delta(D_{n,m}) = d(x_k, W(Ty_l, x_l, \alpha_l)) \leqslant \alpha_l d(Ty_l, x_k) + (1-\alpha_l)\, d(x_k, x_l) \leqslant \delta(D_{n,m})$. It follows that $\alpha_l = 1$ and $\delta(D_{n,m}) = d(x_k, Ty_l)$. Hence from (ii) it follows that (2.5) holds.

(vi) $\delta(D_{n,m}) = d(y_k, x_l)$, $n \leqslant k, l \leqslant m$. In the case $\delta(D_{n,m}) = d(W(Tx_k, x_k, \beta_k), x_l) \leqslant \beta_k d(Tx_k, x_l) + (1-\beta_k)\, d(x_k, x_l) \leqslant \delta(D_{n,m})$. It follows that $\delta(D_{n,m}) = d(Tx_k, x_l)$ or $d(x_k, x_l)$. By (i) or (v), (2.5) holds.

(vii) $\delta(D_{n,m}) = d(y_k, y_l)$, $n \leqslant k, l \leqslant m$. In this case, using the same argument as in (vi), (2.5) holds. Hence (2.5) is true.

By (2.4) and (2.5), for any $m \in N$ we have

$$\begin{aligned}\delta(D_{0,m}) &= \max\{d(x_0, Tx_j), d(x_0, Ty_j): 0 \leqslant j \leqslant m\}\\ &\leqslant d(x_0, Tx_0) + \max\{d(Tx_0, Tx_j), d(Tx_0, Ty_j): 0 \leqslant j \leqslant m\}\\ &\leqslant d(x_0, Tx_0) + q\delta(D_{0,m})\end{aligned}$$

and so

$$\delta(D_{0,m}) \leqslant \frac{1}{1-q}\, d(x_0, Tx_0).$$

From (2.1), (2.3), and (2.5) we obtain that for any $n, m \in N$ with $0 < n < m$,

$$\begin{aligned}\delta(D_{n,m}) &= \max\{d(x_n, Tx_j), d(x_n, Ty_j): n \leqslant j \leqslant m\}\\ &\leqslant \alpha_{n-1} q\delta(D_{n-1,m}) + (1-\alpha_{n-1})\, \delta(D_{n-1,m})\\ &= (1-(1-q)\,\alpha_{n-1})\, \delta(D_{n-1,m}).\end{aligned}$$

By induction we have

$$\begin{aligned}\delta(D_{n,m}) &\leqslant \prod_{j=0}^{n-1} (1-(1-q)\,\alpha_j)\, \delta(D_{0,m})\\ &\leqslant \frac{d(x_0, Tx_0)}{1-q} \prod_{j=0}^{n-1} (1-(1-q)\,\alpha_j).\end{aligned}$$

Since $1-q > 0$ and $\sum_{j=0}^{\infty} \alpha_j$ diverges, $\prod_{j=0}^{\infty} (1-(1-q)\,\alpha_j) = 0$. Thus

$\lim_{n,m \to \infty} \delta(D_{n,m}) = 0$ and so $\{x_n\}$ is a Cauchy sequence and $\lim_{n \to \infty} d(x_n, Tx_n) = 0$. Letting $x_n \to z \in K$, we have $Tx_n \to z$. By (2.2),

$$d(Tx_n, Tz) \leqslant q \max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), d(x_n, Tz), d(z, Tx_n)\}.$$

Putting $n \to \infty$ in the above inequality, we obtain $d(z, Tz) \leqslant qd(z, Tz)$. Since $0 \leqslant q < 1$, therefore $d(z, Tz) = 0$ and so $z = Tz$, i.e., z is a fixed point of T in K.

Remark 2.1. Theorem 2.1 unifies and generalizes Theorem 1 of [10], Theorem 7 of [11], and Corollaries 1 and 2 of [12] and answers positively the open problem mentioned by Rhoades [13] in the more general setting.

THEOREM 2.2. *Let K be a nonempty closed convex subset of a complete convex metric space X and let $T: K \to K$ is a generalized quasi-contractive mapping, i.e., there exist a constant $q \in [0, 1)$ and a function $n: K \to N^+$ (the set of all positive integers) such that for all $x, y \in K$, $n(x) \mid n(Tx)$ and*

$$d(T^{n(x)}x, T^{n(y)}y) \leqslant q \max\{d(x, y), d(x, T^{n(x)}x), d(y, T^{n(y)}y), d(x, T^{n(y)}y), d(y, T^{n(x)}x)\}. \tag{2.6}$$

Suppose $\{x_n\}$ is an iterative scheme defined by

$$x_0 \in K, \qquad x_{n+1} = W(T^{n(y_n)}y_n, x_n, \alpha_n),$$
$$y_n = W(T^{n(x_n)}x_n, x_n, \beta_n), \qquad n \in N, \tag{2.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions in Theorem 2.1. Then $\{x_n\}$ converges to a unique fixed point of T in K.

Proof. Let $\tilde{T}x = T^{n(x)}x$ for all $x \in K$. Then $\tilde{T}$ and $\{x_n\}$ satisfy the suppositions of Theorem 2.1. By Theorem 2.1, $\{x_n\}$ converges to a unique fixed point z of $\tilde{T}$ in K. Since $n(z) \mid n(Tz)$, we have

$$T^{n(Tz)}z = T^{n(z)}z = \tilde{T}z = z \qquad \text{and} \qquad \tilde{T}(Tz) = Tz$$

and so Tz is also a fixed point of $\tilde{T}$. By the uniqueness of fixed points of $\tilde{T}$, $z = Tz$ and so z is a unique fixed point of T in K.

Remark 2.2. Theorem 2.2 generalizes and unifies Theorems 2 and 3 of [12] and Theorem 2 of [10]. Theorem 2.1 is also a special case of Theorem 2.2.

THEOREM 2.3. *Let K be a nonempty closed convex subset of a complete convex metric space X and let $T: K \to K$. Suppose that there exists a* *nondecreasing upper semicontinuous function* Φ: $R_+^5 - R_+$ (*the set of all nonnegative real numbers*) *such that for all* $t>0$, $\Phi(t, t, t, 0, t)<t$ *and such that for all* $x, y \in K$,

$$d(Tx, Ty) \leqslant \Phi(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)). \quad (2.8)$$

If $\{x_n\}$ *is the iterative sequence defined by* (2.3) *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy* $0 \leqslant \alpha_n, \beta_n \leqslant 1$ *and* $\{\alpha_n\}$ *is bounded away from zero and if* $x_n \to p$, *then* p *is a fixed point of* T.

Proof. From (2.3) and [1, p. 145] it follows that

$$d(x_{n+1}, x_n) = d(W(Ty_n, x_n, \alpha_n), x_n) = \alpha_n d(Ty_n, x_n).$$

Since $x_n \to p$, $d(x_{n+1}, x_n) \to 0$. Since $\{\alpha_n\}$ is bounded away from zero, it follows that $d(Ty_n, x_n) \to 0$. By (2.8) we have

$$\begin{aligned} d(Tx_n, Ty_n) &\leqslant \Phi(d(x_n, y_n), d(x_n, Tx_n), d(y_n, Ty_n), \\ &\qquad d(x_n, Ty_n), d(y_n, Tx_n)), \\ d(x_n, y_n) &= d(x_n, W(Tx_n, x_n, \beta_n)) = \beta_n d(Tx_n, x_n), \\ &\leqslant d(Tx_n, Ty_n) + d(Ty_n, x_n), \\ d(y_n, Ty_n) &= d(W(Tx_n, x_n, \beta_n), Ty_n) \\ &\leqslant \beta_n d(Tx_n, Ty_n) + (1-\beta_n)\, d(x_n, Ty_n) \\ &\leqslant d(Tx_n, Ty_n) + d(x_n, Ty_n), \\ d(y_n, Tx_n) &= d(W(Tx_n, x_n, \beta_n), Tx_n) = (1-\beta_n)\, d(x_n, Tx_n) \\ &\leqslant d(x_n, Ty_n) + d(Ty_n, Tx_n). \end{aligned}$$

If $\limsup d(Tx_n, Ty_n) = d^* > 0$, then from the above inequalities and the assumptions of Φ it follows that

$$d^* = \limsup d(Tx_n, Ty_n) \leqslant \Phi(d^*, d^*, d^*, 0, d^*) < d^*.$$

This is a contradiction and so $\lim d(Tx_n, Ty_n) = 0$ as $n \to \infty$. It follows that $d(x_n, Tx_n) \leqslant d(x_n, Ty_n) + d(Ty_n, Tx_n) \to 0$ and $d(Tx_n, p) \to 0$ as $n \to \infty$. By (2.8), we have

$$\begin{aligned} d(Tp, Tx_n) &\leqslant \Phi(d(p, x_n), d(p, Tp), d(x_n, Tx_n), d(p, Tx_n), d(x_n, Tp)), \\ d(p, Tp) &\leqslant d(p, x_n) + d(x_n, Tx_n) + d(Tx_n, Tp), \\ d(x_n, Tp) &\leqslant d(x_n, Tx_n) + d(Tx_n, Tp). \end{aligned}$$

If $\limsup d(Tp, Tx_n) = t > 0$ as $n \to \infty$, then from above inequalities and the assumptions of Φ it follows that

$$t = \limsup d(Tp, Tx_n) \leqslant \Phi(0, t, 0, 0, t) < t.$$

This a contradiction and so $\lim d(Tp, Tx_n) = 0$ as $n \to \infty$. It follows that $d(p, Tp) \leqslant d(p, x_n) + d(x_n, Tx_n) + d(Tx_n, Tp) \to 0$ as $n \to \infty$ and so $p = Tp$.

COROLLARY 2.1. *Let K be a nonempty closed convex subset of a complete convex metric space X and let $T: K \to K$. Suppose there exists a constant $k \in [0, 1)$ such that for all $x, y \in K$,*

$$d(Tx, Ty) \leqslant k \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty) + d(y, Tx)\}. \tag{2.9}$$

If $\{x_n\}$ is an iterative sequence satisfying the conditions in Theorem 2.3 and $x_n \to p$ as $n \to \infty$, then p is a fixed point of T in K.

Proof. Let $\Phi(t_1, t_2, t_3, t_4, t_5) = k \cdot \max\{t_1, t_2, t_3, t_4 + t_5\}$ for all $(t_1, t_2, t_3, t_4, t_5) \in R^5_+$. Then Φ satisfies the supositions in Theorem 2.3. The conclusion of this corollary follows from Theorem 2.3.

Remark 2.3. Theorem 2.3 improves Theorem 3 of [14]. Corollary 2.1 improves and generalizes Theorem 1.2 of [15] and Theorem 9 of [13].

3. CONVERGENCE OF ITERATES OF QUASI-NONEXPANSIVE MAPPINGS

Let D be a closed subset of a complete convex metric space X. A mapping $T: D \to X$ is said to be conditionally quasi-nonexpansive if the set $F(T)$ of fixed points of T is nonempty and for $x \in D$ and $p \in F(T)$, $d(Tx, p) \leqslant d(x, p)$.

THEOREM 3.1. *Let D be a closed subset of a complete convex metric space X and let $T: D \to X$ be a conditionally quasi-nonexpansive mapping. Suppose that for some $x_0 \in D$, the iterative sequence $\{x_n\}$ defined by*

$$x_{n+1} = W(Ty_n, x_n, \alpha_n), \qquad y_n = W(Tx_n, x_n, \beta_n), \qquad n = 0, 1, 2, ..., \tag{3.1}$$

where $0 < \alpha_n \leqslant 1$ and $0 \leqslant \beta_n \leqslant 1$ satisfies $\{x_n\} \subset D$. Then $\{x_n\}$ converges to a fixed point of T in D if and only if there exists a closed subset G of X such that

(i) *$d(Tx, p) \leqslant d(x, p)$ for $x \in D$ and $p \in G$,*

(ii) *$\liminf d(x_n, G) = 0$ as $n \to \infty$.*

Proof. We first show the necessity. Let $\{x_n\}$ converge to a fixed point x^* of T and let $G = \{x^*\}$. Obviously, G is a closed subset of X. Since T is conditionally quasi-nonexpansive and $x_n \to x^*$, conditions (i) and (ii) are true.

Now we prove the sufficiency. By (3.1) and condition (i) we have that for all $p \in G$,

$$\begin{aligned} d(x_{n+1}, p) = d(W(Ty_n, x_n, \alpha_n), p) &\leqslant \alpha_n d(Ty_n, p) + (1-\alpha_n)\, d(x_n, p) \\ &\leqslant \alpha_n d(y_n, p) + (1-\alpha_n)\, d(x_n, p) \\ &= \alpha_n d(W(Tx_n, x_n, \beta_n), p) + (1-\alpha_n)\, d(x_n, p) \\ &\leqslant \alpha_n \beta_n d(Tx_n, p) + \alpha_n(1-\beta_n)\, d(x_n, p) + (1-\alpha_n)\, d(x_n, p) \\ &\leqslant [\alpha_n\beta_n + \alpha_n(1-\beta_n) + (1-\alpha_n)]\, d(x_n, p) = d(x_n, p). \end{aligned}$$

Hence we have

$$d(x_{n+1}, G) \leqslant d(x_n, G). \tag{3.2}$$

From (3.2) and condition (ii) it follows that

$$\lim d(x_n, G) = \liminf d(x_n, G) = 0. \tag{3.3}$$

Let $\varepsilon > 0$. Then there exists an n_0 such that $d(x_n, G) < \varepsilon/2$ for $n \geqslant n_0$. Hence if $n, m \geqslant n_0$ we have

$$d(x_n, x_m) \leqslant d(x_n, p) + d(x_m, p) \leqslant 2d(x_{n0}, p)$$

for all $p \in G$. It follows that

$$d(x_n, x_m) \leqslant 2d(x_{n0}, G) \leqslant \varepsilon$$

ans so $\{x_n\}$ is a Cauchy sequence. Let $x_n \to x^* \in D$. Since the metric d is continuous and G is closed, it follows from (3.3) that $x^* \in G$. By condition (i) we have $d(Tx^*, x^*) \leqslant d(x^*, x^*) = 0$ and so $x^* = Tx^*$.

Remark 3.1. Theorem 3.1 unifies and generalizes Theorem 5 of [12], the theorem of [16], and Theorems 1.1 and 1.1′ of [17].

THEOREM 3.2. *Let D be a closed convex subset of a complete convex metric space X with W continuous and let $T: D \to D$ be a continuous mapping such that*

(i) $F(T) \neq \varnothing$,

(ii) *T is quasi-nonexpansive, i.e., $d(Tx, p) \leqslant d(x, p)$ for $x \in D$ and $p \in F(T)$ and,*

(iii) *for each* $x \in D \setminus F(T)$, *there exists a* $p_x \in F(T)$ *such that* $d(Tx, p_x) < d(x, p_x)$,

(iv) *there exists an* $x_0 \in D$ *such that the iterative sequence* $\{x_n\}$ *defined by* (3.1) *contains a convergent subsequence* $\{x_{n_j}\}$ *converging to some* $x^* \in D$.

Then $x^* \in F(T)$ *and* $x_n \to x^*$.

Proof. Conditions (i) and (ii) imply that $F(T)$ is a nonempty closed subset of X and $\lim d(x_n, F(T)) = d$ exists. Hence it suffices to show that $d = 0$, for then Theorem 3.1 may be applied. If $x^* \in F(T)$, then $d = 0$. If $x^* \notin F(T)$, then by the condition (iii) there exists a $p = p_{x^*}$ such that $d(Tx^*, p) < d(x^*, p)$. By the continuity of W and T, for each $n \in N$, $T_n x = W(W(Tx, x, \beta_n), x, \alpha_n)$ is also a continuous mapping on D. For all $n \in N$, we have

$$\begin{aligned} d(T_n x^*, p) &= d(W(W(Tx^*, x^*, \beta_n), x^*, \alpha_n), p) \\ &\leqslant \alpha_n d(W(Tx^*, x^*, \beta_n), p) + (1 - \alpha_n)\, d(x^*, p) \\ &\leqslant \alpha_n \beta_n d(Tx^*, p) + \alpha_n(1 - \beta_n)\, d(x^*, p) + (1 - \alpha_n)\, d(x^*, p) \\ &< \alpha_n \beta_n d(x^*, p) + \alpha_n(1 - \beta_n)\, d(x^*, p) + (1 - \alpha_n)\, d(x^*, p) \\ &= d(x^*, p). \end{aligned} \tag{3.4}$$

On the other hand, by the continuity of T_{n_j} and condition (iv), we have

$$\begin{aligned} d(T_{n_j} x^* p) &= d(T_{n_j}(\lim x_{n_j}), p) = \lim d(x_{n_j+1}, p) = \lim d(x_n, p) \\ &= \lim d(x_{n_j}, p) = d(\lim x_{n_j}, p) = d(x^*, p), \qquad \text{as} \quad n \to \infty, \end{aligned}$$

where the middle equalities hold since condition (ii) implies that $\lim d(x_n, p)$ exists as $n \to \infty$. This is a contradiction, hence $x^* \in F(T)$ and the theorem is proven.

Remark 3.2. Theorem 3.2 improves Theorem 6 of [12] and Theorem 1.3 of [17].

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