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Stochastic integration for Lévy processes with values in Banach spaces

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Abstract

A stochastic integral of Banach space valued deterministic functions with respect to Banach space valued Lévy processes is defined. There are no conditions on the Banach spaces or on the Lévy processes. The integral is defined analogously to the Pettis integral. The integrability of a function is characterized by means of a radonifying property of an integral operator associated with the integrand. The integral is used to prove a Lévy–Itô decomposition for Banach space valued Lévy processes and to study existence and uniqueness of solutions of stochastic Cauchy problems driven by Lévy processes.

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1. Introduction

Lévy processes are an extensively studied class of stochastic processes. They play an important role in models of evolutionary phenomena perturbed by noise as for example in financial mathematics. In many models the complexity of the dynamics under consideration is often captured more effectively using stochastic processes with values in infinite dimensional spaces; see for example Carmona and Teranchi [8] for models in financial mathematics. However,

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a central issue in the use of infinite dimensional spaces in stochastic analysis is a convenient stochastic integral.

A real valued stochastic integral with respect to a real valued Wiener process can be defined in the classical sense of K. Itô. By augmenting only a small amount of operator theory this approach can be easily generalized to integrands with values in Hilbert spaces and Hilbert space valued Wiener processes, which is accomplished in Da Prato and Zabczyk [9]. Their approach has been extended to Lévy processes by Peszat and Zabczyk in [14]. For Banach spaces, even in the case of Wiener processes, there seemed to be no general method for introducing a rigorously defined stochastic integral without making special assumptions on the geometry of the Banach space. But more recently, van Neerven and Weis introduced in [18] for deterministic Banach space valued integrands a stochastic integral with respect to Wiener processes on Banach spaces without any conditions on the underlying Banach space; see also [7,17]. The main point in their construction is the case of a Banach space valued integrand and a scalar Wiener process, which is then extended to Banach space valued Wiener processes. Together with Veraar they continued this work in [19] for random integrands on UMD Banach spaces. But already the integral for deterministic integrands turned out to be very helpful for dealing with evolution equations on infinite dimensional spaces.

In this work we develop in the same spirit a stochastic integral for deterministic Banach space valued integrands with respect to Lévy processes on Banach spaces. This integral can be interpreted as a stochastic version of a Pettis integral or of a weak integral. In the case of a Wiener process a key feature in constructing the integral is the equivalent condition for the existence of the integral in terms of γ-radonifying operators associated with the integrands. We introduce an analogous definition of "martingale-radonifying" operators which turns out to be of the same significance for the integration theory as the γ-radonifying operators in the Gaussian case.

The usefulness of the integral is demonstrated by two important applications: a pathwise decomposition of Lévy processes on Banach spaces, the so-called *Lévy–Itô decomposition*, and *evolution equations driven by Lévy processes*.

For finite dimensional Lévy processes the pathwise decomposition into its continuous and jump parts is well known and often used. In this decomposition, the small jumps are represented by an integral with respect to the compensated random Poisson measure. The definition of this integral is based on the fact that for a finite dimensional Lévy process the Lévy measure ν satisfies

$$\int_{|\beta|\leqslant 1} \min\{1, |\beta|^p\}\nu(\mathrm{d}\beta) < \infty \tag{1.1}$$

for $p = 2$. But for infinite dimensional Lévy processes this condition does not hold any longer in general. Consequently, the decomposition cannot be derived by a direct generalization of the integral to an integral in the Bochner sense.

There are few works in the literature concerning the pathwise decomposition in infinite dimensions. Linde achieved in [11] the decomposition as a limit but this limit was not associated with a possible integral definition. This shortcoming limits the utility of the pathwise decomposition in comparison to the decomposition in finite dimensional spaces where properties of the integral are often used. In [1] Albeverio and Rüdiger filled this gap by introducing a new integral, but not every Lévy process or Banach space satisfies the necessary conditions for the existence of the integral. The power of our construction of the integral lies in the fact that it gives exactly the desired decomposition by means of a proper defined integral without any further condition on the underlying Banach space or Lévy process.

A very nice review of the results mentioned above on the pathwise decomposition and some additional results on the integration with respect to a Lévy process in a Banach space can be found in [4].

Among the simplest stochastic evolutionary equations – but nonetheless objects of extensive study in the last few years – are equations of the form

$$\mathrm{d}Y(t) = AY(t)\mathrm{d}t + F\mathrm{d}X(t) \quad \text{for } t \geqslant 0,$$

where A is a generator of a C_0 semigroup on a linear space V and $F : U \to V$ is a linear bounded operator on a linear space U. If the stochastic process $X = (X(t) : t \geqslant 0)$ is a Wiener process and U and V are Hilbert spaces then these equations are covered by a comprehensive theory introduced in the monograph [9] by Da Prato and Zabczyk. For a Banach space V and a Wiener process X with values in a Banach space U the work [18] gives a complete answer for the existence of a solution to the equation. Our construction for an integral with respect to Lévy processes allows us to deal with such equations on Banach spaces U and V and a Lévy process X with values in U. We derive necessary and sufficient conditions for the existence of a solution in terms of the semigroup generated by A. Furthermore, the pathwise decomposition by means of our weak integral allows even that the operator F acts differently on the continuous and jump parts of the Lévy process.

2. Martingale valued measures

Let $(\Omega, \mathcal{A}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t\geqslant 0}$. In this work we will consider mainly random variables on this probability space with values in a separable real Banach space U. The dual space of U is denoted by U^* and the dual pairing by $\langle u, u^*\rangle$ for $u \in U$ and $u^* \in U^*$. The Borel σ-algebra is denoted by $\mathcal{B}(U)$.

If $(S, \mathcal{S})$ is a measurable space and μ a measure on $\mathcal{S}$, then $L^p(S, \mu)$ defines the Banach space of all measurable, real valued functions f with

$$\int_S |f(s)|^p \, \mu(\mathrm{d}s) < \infty.$$

If V is another Banach space then we call a function $f : S \to V$ *V-weakly $L^p(S, \mu)$* if the function $s \mapsto \langle f(s), v^*\rangle$ is in $L^p(S, \mu)$ for all $v^* \in V^*$.

If S is a set and $\mathcal{R}$ a ring of subsets of S, then a *martingale valued measure* M on $(S, \mathcal{R})$ is a collection of real valued random variables $(M(t, A) : t \geqslant 0,\ A \in \mathcal{R})$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P)$ such that:

(a) $M(0, A) = 0$ a.s. and $E[M(t, A)^2] < \infty$ for all $t \geqslant 0$ and all $A \in \mathcal{R}$;
(b) $M(t, \emptyset) = 0$ a.s. and for any mutually disjoint sets $A_1, A_2, \ldots \in \mathcal{R}$ with $\bigcup_{j=1}^{\infty} A_j \in \mathcal{R}$ one has

$$M\left(t, \bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} M(t, A_j) \quad \text{a.s. for all } t \geqslant 0;$$

(c) for every $A \in \mathcal{R}$, $(M(t, A) : t \geqslant 0)$ is a martingale with respect to $(\mathcal{F}_t)_{t\geqslant 0}$.

For a Banach space U and a closed set C of U let $\mathcal{R}$ be the ring of all $A \in \mathcal{B}(U)$ with $\bar{A} \cap C = \emptyset$. A martingale valued measure on U with *forbidden set* C is a martingale valued measure $(M(t, A) : t \geqslant 0,\ A \in \mathcal{R})$ on $(U \setminus C, \mathcal{R})$. We assume in the sequel that martingale valued measures M also satisfy:

(d) for any mutually disjoint sets $A_1, \dots, A_n \in \mathcal{R}$ the random variables $M(t_1, A_1), \dots, M(t_n, A_n)$ are independent for all $t_1, \dots, t_n \in \mathbb{R}_+$;
(e) $M(t, A) - M(s, A)$ is independent of $\mathcal{F}_s$ for all $0 \leqslant s \leqslant t$ and all $A \in \mathcal{R}$;
(f) there exists a σ-finite measure ρ on $\mathcal{B}(\mathbb{R}_+ \times (U \setminus C))$ for which

$$E[M(t, A)^2] = \rho([0, t] \times A) \quad \text{for all } t \geqslant 0 \text{ and all } A \in \mathcal{R}.$$

We call ρ the *square mean measure*.

Due to (d) and (f), the a.s. convergence of the series in (b) is equivalent to convergence in $L^2(\Omega, P)$. Indeed, by dominated convergence the a.s. convergence yields L^2-convergence and the converse direction follows from [20, Thm. V.2.3]. Thus, in our setting the definition of martingale valued measures corresponds to that of Walsh [21] (see also [3]).

Example 2.1. A *real valued Wiener process* $(B(t) : t \geqslant 0)$ is an adapted stochastic process with continuous paths starting in 0 and with independent, stationary increments $B(t) - B(s)$ which are normally distributed with expectation $E[B(t) - B(s)] = 0$ and variance $\text{Var}(B(t) - B(s)) = c\,|t - s|$ for a constant $c > 0$. If $c = 1$ we call B a *standard* real valued Wiener process.

An adapted stochastic process $B := (B(t) : t \geqslant 0)$ with values in a separable Banach space U is called a *Wiener process* if

(a) $B(0) = 0$ a.s.;
(b) B has independent increments;
(c) for any $u^* \in U^*$ the stochastic process $(\langle B(t), u^* \rangle : t \geqslant 0)$ is a real valued Wiener process;
(d) B has a.s. continuous paths.

Condition (c) yields by Pettis's measurability theorem ([15, Thm. 1.1] or [20, Prop. I.1.10]) that B has stationary increments. Moreover, there exists a covariance operator $R : U^* \to U$ which gives the covariance of two components $\langle B(t), u^* \rangle$ and $\langle B(t), v^* \rangle$ for $u^*, v^* \in U^*$; see [20, Thm. III.2.1] or [6]. Although Kolmogorov's continuity theorem implies only by conditions (b) and (c) that there exists a version of B with continuous paths we include condition (d) in our definition to avoid the necessity of considering versions of B.

If B is a Wiener process with values in $\mathbb{R}$, then $M(t, A) := B(t)\delta_0(A)$ for $A \in \mathcal{B}(\mathbb{R})$ defines a martingale valued measure M with empty forbidden set. The Dirac measure δ_0 provides the condition (b) on the σ-additivity in the definition of a random measure.

Example 2.2. An adapted, U valued process $(L(t) : t \geqslant 0)$ is called a *Lévy process* if

(a) $L(0) = 0$ a.s.;
(b) L has stationary, independent increments;
(c) L has càdlàg paths.

We call a set $\Lambda \in \mathcal{B}(U \setminus \{0\})$ *bounded from below* if $0 \notin \bar{\Lambda}$. For every $\Lambda \in \mathcal{B}(U \setminus \{0\})$ bounded from below and $t \geqslant 0$ we define

$$N(t, \Lambda) := \sum_{s \in [0,t]} \mathbb{1}_\Lambda(\Delta L(s)),$$

where $\Delta L(s) := L(s) - L(s-)$. Note, that because L has càdlàg paths there are only finitely many jumps of size larger than a positive constant and thus only finitely many in the set Λ. Then $(N(t, \Lambda) : t \geqslant 0)$ defines a Poisson process on $\mathbb{N}_0$. Moreover, let $\nu(\Lambda) := E[N(1, \Lambda)]$ be the so-called *Lévy measure*. Then ν extends to a σ-finite measure on $\mathcal{B}(U \setminus \{0\})$, which is finite for every set bounded from below. The so-called *compensated Poisson random measure* is defined by

$$\tilde{N}(t, \Lambda) := N(t, \Lambda) - t\nu(\Lambda)$$

for every $\Lambda \in \mathcal{B}(U \setminus \{0\})$ bounded from below. Thus $\tilde{N}$ defines a martingale valued measure on U with the forbidden set $C = \{0\}$. The square mean measure is given by

$$E[\tilde{N}(t, \Lambda)^2] = t\nu(\Lambda).$$

These properties of a Lévy process may be established as in the finite dimensional case.

3. Integration of Banach space valued functions

In this section let U and V be arbitrary Banach spaces and assume V separable. Moreover, let M denote a martingale valued measure on U with forbidden set $\emptyset$ or $\{0\} \subseteq U$ and let $\mathcal{R}$ be the ring of all sets $A \in \mathcal{B}(U)$ with closure contained in the complement of the forbidden set. We fix a set $B \in \mathcal{B}(U)$ which should be bounded from below if the forbidden set is $\{0\}$.

We introduce in this section a stochastic integral for functions $F : [0, T] \times B \to V$ with respect to the martingale valued measure M. If V is finite dimensional, that is $V = \mathbb{R}^d$, then the integral

$$\int_{[0,T]\times B} F(s, u)\, M(\mathrm{d}s, \mathrm{d}u) \tag{3.1}$$

can be defined in the standard way by step functions F and then by extension to functions F which are square integrable relative to the square mean measure ρ of M by use of the Itô isometry

$$\mathbb{E}\left|\int_{[0,T]\times B} F(s, u) M(\mathrm{d}s, \mathrm{d}u)\right|^2 = \int_{[0,T]\times B} |F(s, u)|^2 \rho(\mathrm{d}s, \mathrm{d}u). \tag{3.2}$$

This is carried out for finite dimensional U in [2] and can easily be generalized to the case of infinite dimensional spaces U, as long as the range of the function F is finite dimensional. This definition of the integral is used in the sequel without any further notice.

In general, if V is infinite dimensional, one may try to follow a similar approach to define the integral (3.1) for a Bochner square integrable function $F : [0, T] \times B \to V$,

$$\int_{[0,T]} \|F(t, u)\|^2\ \rho(\mathrm{d}t, \mathrm{d}u) < \infty.$$

For Hilbert spaces this approach is accomplished in [3]. However, it is well known that the strong integral can be only defined in Banach spaces under certain geometric conditions.

Instead of the strong integral we consider a definition more similar to the Pettis integral in Banach spaces, originally in a deterministic setting introduced in [15]. The stochastic analog for martingale valued measures reads as follows.

Definition 3.1. A function $F : [0, T] \times B \to V$ is called *stochastically integrable on* $[0, T] \times B$ *with respect to* M if it is V-weakly $L^2([0, T] \times B, \rho)$ and there exists a V valued random variable Y such that for all $v^* \in V^*$ we have

$$\langle Y, v^* \rangle = \int_{[0,T]\times B} \langle F(t, u), v^* \rangle\, M(\mathrm{d}t, \mathrm{d}u) \quad \text{a.s.}$$

In this situation we write

$$Y = \int_{[0,T]\times B} F(t, u)\, M(\mathrm{d}t, \mathrm{d}u).$$

A similar idea is used by Rosiński in [16] to define a 'weak random integral' in a more abstract setting. Some further properties of the integral and the class of integrable functions can be found there. We consider our more specific situation as it seems more appropriate for our purposes. In fact we will use the stochastic integral to study stochastic Cauchy problems and the Banach space valued Lévy–Itô decomposition. Our terminology is closely related to that of [18].

One of the major achievements in the work of van Neerven and Weis [18] for Wiener processes is the equivalent condition for the existence of the integral in terms of γ-radonifying operators associated with the integrands. In the following we generalize this property of an operator and we will also set this operator in relation to the existence of the integral. But further consideration of this operator and its properties will be the subject of forthcoming work.

Definition 3.2. We call a linear continuous map $R : L^2([0,T] \times B, \rho) \to V$ *M-radonifying* if there exists an orthonormal basis $(f_n) \subseteq L^2([0,T] \times B, \rho)$ and a V valued random variable Y such that

$$\langle Y, v^* \rangle = \sum_{n=1}^{\infty} \gamma_n \langle R f_n, v^* \rangle \quad \text{in } L^2(\Omega, P) \text{ for all } v^* \in V^*, \tag{3.3}$$

where $\gamma_n := \int_{[0,T]\times B} f_n(t,u)\, M(\mathrm{d}t, \mathrm{d}u)$.

Remark 3.3. (a) The definition of M-radonifying depends on the martingale valued measure M but not on the choice of the orthonormal basis. Indeed, let $(g_n) \subseteq L^2([0,T] \times B, \rho)$ be another orthonormal basis. Then we have a.s.

$$\begin{aligned} \sum_{n=1}^{\infty} \gamma_n \langle R f_n, v^* \rangle &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \langle R g_k, v^* \rangle \gamma_n \langle f_n, g_k \rangle \\ &= \sum_{k=1}^{\infty} \langle R g_k, v^* \rangle \int_{[0,T]\times B} \sum_{n=1}^{\infty} f_n(t,u) \langle f_n, g_k \rangle\, M(\mathrm{d}t, \mathrm{d}u) \\ &= \sum_{k=1}^{\infty} \langle R g_k, v^* \rangle \delta_k, \end{aligned}$$

where $\delta_k = \int_{[0,T]\times B} g_k(t,u)\, M(\mathrm{d}t, \mathrm{d}u)$.

(b) If M is a Wiener process, that is, $M(t, A) = W(t)\delta_0(A)$ for a real valued Wiener process $(W(t) : t \geqslant 0)$ and the forbidden set is $\emptyset$, then M-radonifying is equivalent to γ-radonifying. For, the sequence $(\gamma_n)_{n\in\mathbb{N}}$ defines a sequence of independent, identically standard-normal distributed random variables. In this case the convergence of the series in (3.3) is equivalent to the fact that the operator R is γ-radonifying; see [18]. The stochastic integral of Definition 3.1 with $B = U$ then coincides with the stochastic integral of [18] of $F(\cdot, 0)$ with respect to W. The latter integral is the key construction of [18], which is there subsequently extended to an integral with respect to a Banach space valued Wiener process. The same approach could be taken in our setting to extend to a Banach space valued Wiener process with the aid of its Cameron–Martin space and Hilbert space valued martingale valued measures (as defined in [3]). Instead, we will simply use the stochastic integral of [18] whenever we want to integrate with respect to a Banach space valued Wiener process.

(c) The family of random variables γ_n, $n \in \mathbb{N}$, is orthonormal in $L^2(\Omega, P)$, as follows from the Itô isometry (3.2) and the polarization formula.

Lemma 3.4. *For a V-weakly $L^2([0,T]\times B,\rho)$ function $F:[0,T]\times B\to V$ we define*

$$\langle I_F f, v^*\rangle = \int_{[0,T]\times B} \langle F(t,u), v^*\rangle f(t,u)\rho(\mathrm{d}t,\mathrm{d}u), \quad f\in L^2([0,T]\times B,\rho)$$

for all $v^*\in V^*$. In this way we obtain a bounded linear operator $I_F : L^2([0,T]\times B,\rho)\to V$.

Proof. We begin to show that $I_F f\in V^{**}$ for which we use that

$$G: V^*\to L^2([0,T]\times B,\rho), \qquad G(v^*) = \langle F(\cdot,\cdot), v^*\rangle$$

is closed. Indeed, if $v_n^*\to v^*$ in V^* and $\langle F(\cdot,\cdot), v_n^*\rangle\to g$ in $L^2([0,T]\times B,\rho)$ then there exists a subsequence such that

$$g(t,u) = \lim_{k\to\infty}\langle F(t,u), v_{n_k}^*\rangle = \langle F(t,u), v^*\rangle \quad \text{for } \rho\text{-a.e. } (t,u)\in[0,T]\times B.$$

Hence, the operator G is closed and the closed graph theorem implies

$$\left|\langle I_F f, v^*\rangle\right| = \langle G(v^*), f\rangle_{L^2(\rho)} \leqslant \left\|G(v^*)\right\|_{L^2(\rho)} \|f\|_{L^2(\rho)} \leqslant c\left\|v^*\right\| \|f\|_{L^2(\rho)} \tag{3.4}$$

for a constant $c>0$.

We proceed to establish that $I_F f$ is actually in V. Because F is strongly measurable by Pettis's measurability theorem [15, Theorem 1.1], the map $(t,u)\mapsto \|F(t,u)\|$ is measurable. We define

$$A_n := \{(t,u)\in[0,T]\times B : \|F(t,u)\|\leqslant n\}.$$

Let $f\in(L^1\cap L^2)([0,T]\times B,\rho)$ be such that its essential support is contained in A_n for some $n\in\mathbb{N}$. Then for $v_k^*\in V^*$ we have

$$\left|\langle F(t,u), v_k^*\rangle f(t,u)\right| \leqslant \|F(t,u)\| \left\|v_k^*\right\| |f(t,u)| \leqslant n\,|f(t,u)| \sup_{k\in\mathbb{N}}\left\|v_k^*\right\|$$

and thus the dominated convergence theorem can be applied to conclude $\langle I_F f, v_k^*\rangle\to 0$ for $v_k^*\to 0$ weakly*. A corollary [12, Cor. 2.7.10] to the Krein–Smulyan theorem yields that $I_F f\in V$ as V is separable.

Next for arbitrary $f\in L^2([0,T]\times B,\rho)$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $(L^1\cap L^2)([0,T]\times B,\rho)$ such that $h_n := f_n \mathbb{1}_{A_n}\to f$ in $L^2([0,T]\times B,\rho)$ and $I_F h_n\in V$ for all $n\in\mathbb{N}$. Because (3.4) implies

$$\|I_F(h_n - f)\|_{V^{**}} = \sup_{\|v^*\|\leqslant 1}\left|\langle I_F(h_n-f), v^*\rangle\right| \leqslant c\,\|h_n - f\|_{L^2(\rho)},$$

it follows that $I_F h_n\to I_F f$ in V^{**} and hence $I_F f\in V$ since V is closed in V^{**}. Boundedness of I_F follows from (3.4). □

Remark 3.5. Note that inequality (3.4) gives the estimate

$$\begin{aligned}
&E\left|\int_{[0,T]\times B}\langle F(t,u), v^*\rangle\, M(\mathrm{d}t,\mathrm{d}u)\right|^2 = \int_{[0,T]\times B}\langle F(t,u), v^*\rangle^2\rho(\mathrm{d}t,\mathrm{d}u)\\
&= \left\|\langle F(\cdot,\cdot), v^*\rangle\right\|^2_{L^2(\rho)}\\
&= \sup_{\|f\|_{L^2(\rho)}=1}\left(\int_{[0,T]\times B}\langle F(t,u), v^*\rangle f(t,u)\rho(\mathrm{d}t,\mathrm{d}u)\right)^2\\
&\leqslant c^2\left\|v^*\right\|^2.
\end{aligned}$$

By means of the operator I_F introduced in Lemma 3.4 we find an equivalent condition guaranteeing the stochastic integrability of a function.

Theorem 3.6. *Let $F : [0, T] \times B \to V$ be a V-weakly $L^2[(0, T] \times B, \rho)$ function. Then the following are equivalent:*

(a) *F is stochastically integrable on $[0, T] \times B$ with respect to M.*
(b) *I_F is M-radonifying.*

In this situation we have

$$\left\langle \int_{[0,T]\times B} F(t,u)\, M(\mathrm{d}t, \mathrm{d}u), v^* \right\rangle = \sum_{n=1}^{\infty} \gamma_n \langle I_F f_n, v^* \rangle \quad \text{for every } v^* \in V^*,$$

and an arbitrary orthonormal basis $(f_n) \subseteq L^2([0, T] \times B, \rho)$, where the γ_n are as in Definition 3.2.

Proof. By Itô's isometry we obtain

$$\begin{aligned}\sum_{n=1}^{N} \gamma_n \langle I_F f_n, v^* \rangle &= \sum_{n=1}^{N} \int_{[0,T]\times B} f_n(t,u)\, M(\mathrm{d}t, \mathrm{d}u) \\ &\quad \times \int_{[0,T]\times B} \langle F(s,v), v^* \rangle f_n(s,v) \rho(\mathrm{d}s, \mathrm{d}v) \\ &= \int_{[0,T]\times B} \sum_{n=1}^{N} \langle \langle F, v^* \rangle, f_n \rangle_{L^2(\rho)} f_n(t,u)\, M(\mathrm{d}t, \mathrm{d}u) \\ &\to \int_{[0,T]\times B} \langle F(t,u), v^* \rangle\, M(\mathrm{d}t, \mathrm{d}u) \quad \text{in } L^2(\Omega, P) \text{ for } N \to \infty.\end{aligned}$$

(b) $\Rightarrow$ (a): In this case the assumption yields

$$\sum_{n=1}^{N} \gamma_n \langle I_F f_n, v^* \rangle \to \langle Y, v^* \rangle \quad \text{in } L^2(\Omega, P)$$

for a V valued random variable Y. Consequently, the function F is stochastically integrable.

(a) $\Rightarrow$ (b): There exists a V valued random variable Y such that

$$\langle Y, v^* \rangle = \int_{[0,T]\times B} \langle F(t,u), v^* \rangle\, M(\mathrm{d}t, \mathrm{d}u)$$

for every $v^* \in V^*$ and therefore

$$\langle Y, v^* \rangle = \sum_{n=1}^{\infty} \gamma_n \langle v^*, I_F f_n \rangle \quad \text{in } L^2(\Omega, P),$$

which completes the proof. □

In Section 4 the next lemma will be useful in addition to Theorem 3.6.

Lemma 3.7. *Let $F : [0, T] \times B \to V$ be a V-weakly $L^2([0, T] \times B, \rho)$ function and (f_n) be an orthonormal basis of $L^2([0, T] \times B, \rho)$. If a V valued random variable Y exists such that*

$$\langle Y, v^* \rangle = \sum_{n=1}^{\infty} \gamma_n \langle I_F f_n, v^* \rangle \quad \text{in } L^2(\Omega, P)$$

for all v^* in a sequentially weak dense subspace of V^*, then I_F is M-radonifying and*

$$Y = \int_{[0,T]\times B} F(t,u)\, M(\mathrm{d}t, \mathrm{d}u) \quad a.s.$$

Proof. The proof follows [18]. Let $v^* \in V^*$ and let (v_m^*) be a sequence in V^* such that v_m^* converges weakly* to v^* and $\langle Y, v_m^*\rangle = \sum_{n=1}^\infty \gamma_n \langle I_F f_n, v_m^*\rangle$ a.s. for all m. The adjoint operator $I_F^* : V^* \to L^2([0,T]\times B, \rho)$ is continuous with respect to the weak* topologies, so $I_F^* v_m^* \to I_F^* v^*$ weakly* and hence weakly in $L^2([0,T]\times B,\rho)$. Then there exist convex combinations w_m^* of v_k^* with $k \geqslant m$ such that $\|I_F^* w_m^* - I_F^* v^*\|_{L^2(\rho)} \to 0$ and $w_m^* \to v^*$ weakly* in V^* as $m\to\infty$. Consequently, $\langle Y, w_m^*\rangle \to \langle Y, v^*\rangle$ a.s. and

$$\begin{aligned}\mathbb{E}\left|\langle Y, w_m^*\rangle - \sum_{n=1}^\infty \gamma_n\langle I_F f_n, v^*\rangle\right|^2 &= \mathbb{E}\left|\sum_{n=1}^\infty \gamma_n \langle I_F f_n, w_m^* - v^*\rangle\right|^2 = \sum_{n=1}^\infty \langle I_F f_n, w_m^* - v^*\rangle^2 \\ &= \sum_{n=1}^\infty \langle f_n, I_F^*(w_m^* - v^*)\rangle^2 = \|I_F^*(w_m^*-v^*)\|_{L^2(\rho)}^2 \to 0.\end{aligned}$$

For a subsequence we obtain $\langle Y, w_{m_\ell}^*\rangle \to \sum_{n=1}^\infty \gamma_n\langle I_F f_n, v^*\rangle$ a.s., so that $\langle Y, v^*\rangle = \sum_{n=1}^\infty \gamma_n \langle I_F f_n, v^*\rangle$ a.s. Thus I_F is M-radonifying. □

4. Cauchy problem

In this section we apply our previous results to a stochastic Cauchy problem with respect to a martingale valued measure M. In order to avoid technicalities we assume that the square mean measure ρ is time-homogeneous, i.e. $\rho(\mathrm{d}s,\mathrm{d}u) = \mathrm{d}s\,\nu(\mathrm{d}u)$ for a measure ν. We consider

$$\begin{aligned}&\mathrm{d}Y(t) = AY(t)\mathrm{d}t + \int_B G(u)\, M(\mathrm{d}t,\mathrm{d}u), \qquad t\in[0,T],\\ &Y(0) = y_0,\end{aligned} \tag{4.1}$$

where A is the generator of a C_0-semigroup $(T(t))_{t\geqslant 0}$ on V and $G : B \to V$ a function which is V-weakly in $L^2([0,T]\times B,\rho)$. Here we interpret G also as a function on $[0,T]\times B$ which is constant in the first variable. The initial condition y_0 is assumed to be an element of V. The function G models the influence of the noise depending on its size. We use the set B in order to include martingale valued measures which have a nonempty forbidden set; see the next section.

The paths of a solution turn out to have some regularity property.

Definition 4.1. A V valued process $(X(t) : t\geqslant 0)$ is called *weakly Bochner regular* if for every sequence $(H_n)_{n\in\mathbb{N}}$ of functions $H_n \in C([0,T],V^*)$ we have

$$\sup_{s\in[0,T]} \|H_n(s)\|_{V^*} \to 0 \Rightarrow \int_0^T \left|\langle X(s), H_{n_k}(s)\rangle\right| \mathrm{d}s \to 0 \quad \text{a.s. for } k\to\infty,$$

for a subsequence $\{H_{n_k}\}_{k\in\mathbb{N}} \subseteq \{H_n\}_{n\in\mathbb{N}}$.

Note that if X has a.s. Bochner integrable paths on $[0,T]$ the process X is also weakly Bochner regular.

Definition 4.2. A V valued process $(Y(t, y_0))_{t\in[0,T]}$ is called a *weak solution of* (4.1) *on* $[0, T]$ if it is weakly progressively measurable and weakly Bochner regular and for every $v^* \in \mathrm{D}(A^*)$ and $t \in [0, T]$ we have, almost surely,

$$\langle Y(t, y_0), v^*\rangle = \langle y_0, v^*\rangle + \int_0^t \langle Y(s, y_0), A^* v^*\rangle\, \mathrm{d}s + \int_{[0,t]\times B} \langle G(u), v^*\rangle\, M(\mathrm{d}s, \mathrm{d}u). \quad (4.2)$$

Note that the condition on G yields the existence of the latter integral with respect to M.

Theorem 4.3. *The following are equivalent:*

(a) *there exists a weak solution* $(Y(t, y_0))_{t\in[0,T]}$ *of* (4.1) *on* $[0, T]$*;*
(b) *the function* $(s, u) \mapsto T(s)G(u)$ *is stochastically integrable on* $[0, T] \times B$ *with respect to* M*.*

In this situation, the solution is represented by

$$Y(t, y_0) = T(t)y_0 + \int_{[0,t]\times B} T(t-s)G(u)\, M(\mathrm{d}s, \mathrm{d}u) \quad \textit{for } t \in [0, T]$$

almost surely.

Proof. Due to the linearity of Eq. (4.1) we may assume $y_0 = 0$ and we write $Y(t) := Y(t, 0)$.
(a) $\Rightarrow$ (b): We begin to establish for $f \in C^1([0, T])$ the integration by parts formula

$$\begin{aligned}&\int_0^t f'(s) \int_{[0,s]\times B} \langle G(u), v^*\rangle\, M(\mathrm{d}r, \mathrm{d}u)\, \mathrm{d}s \\ &\quad = f(t) \int_{[0,t]\times B} \langle G(u), v^*\rangle M(\mathrm{d}r, \mathrm{d}u) - \int_{[0,t]\times B} \langle G(u), f(s)v^*\rangle\, M(\mathrm{d}s, \mathrm{d}u). \qquad (4.3)\end{aligned}$$

For a simple function of the form $G = g \otimes \mathbb{1}_C$ with $C \in \mathcal{B}(U) \cap B$ and $g \in V$ we obtain

$$\begin{aligned}&\int_0^t \int_{[0,s]\times B} \langle G(u), f'(s)v^*\rangle\, M(\mathrm{d}r, \mathrm{d}u)\, \mathrm{d}s \\ &\quad = \int_0^t \langle g, f'(s)v^*\rangle \int_{[0,s]\times C} M(\mathrm{d}r, \mathrm{d}u)\, \mathrm{d}s \\ &\quad = M(t, C)\langle g, f(t)v^*\rangle - \int_{[0,t]\times C} \langle g, f(s)v^*\rangle\, M(\mathrm{d}s, \mathrm{d}u) \\ &\quad = f(t) \int_{[0,t]\times B} \langle G(u), v^*\rangle M(\mathrm{d}r, \mathrm{d}u) - \int_{[0,t]\times B} \langle G(u), f(s)v^*\rangle\, M(\mathrm{d}s, \mathrm{d}u),\end{aligned}$$

where we applied in the second to last line the integration by parts formula for Lebesgue–Stieltjes integrals. This result can be generalized to an arbitrary V-weakly $L^2([0, T] \times B, \rho)$ function G by approximation with simple functions. Next we can follow the lines in [18] and show that for all $v^* \in \overline{\mathrm{D}(A^*)}$ we have, almost surely,

$$\langle Y(t), v^*\rangle = \int_{[0,t]\times B} \langle T(t-s)G(u), v^*\rangle\, M(\mathrm{d}s, \mathrm{d}u). \quad (4.4)$$

For that, let $f \in C^1([0, T])$ and $v^* \in \mathrm{D}(A^*)$ and observe that integration by parts and Eq. (4.3) yield

$$\int_0^t f'(s)\langle Y(s), v^*\rangle\, \mathrm{d}s$$

$$= \int_0^t f'(s) \left(\int_0^s \langle Y(r), A^* v^* \rangle \mathrm{d}r \right) \mathrm{d}s + \int_0^t f'(s) \int_{[0,s]\times B} \langle G(u), v^* \rangle \, M(\mathrm{d}r, \mathrm{d}u) \, \mathrm{d}s$$
$$= f(t) \int_0^t \langle Y(s), A^* v^* \rangle \, \mathrm{d}s - \int_0^t f(s) \langle Y(s), A^* v^* \rangle \, \mathrm{d}s$$
$$+ f(t) \int_{[0,t]\times B} \langle G(u), v^* \rangle \, M(\mathrm{d}r, \mathrm{d}u) - \int_{[0,t]\times B} \langle G(u), f(s) v^* \rangle \, M(\mathrm{d}s, \mathrm{d}u).$$

By multiplying (4.2) with $f(t)$ and putting $F = f \otimes v^*$ we therefore obtain

$$\langle Y(t), F(t) \rangle = \int_0^t \langle Y(s), F'(s) + A^* F(s) \rangle \mathrm{d}s + \int_{[0,t]\times B} \langle G(u), F(s) \rangle \, M(\mathrm{d}s, \mathrm{d}u). \tag{4.5}$$

We can find a sequence $F_n \in \operatorname{span}\{f \otimes w^* : f \in C^1([0,t]), w^* \in \mathrm{D}(A^*)\}$ such that F_n converges to $F := T^*(t - \cdot) v^*$ in $C^1([0,t], V^*) \cap C([0,t], \mathrm{D}(A^*))$. The weak Bochner regularity implies that

$$\int_0^t \left| \langle Y(s), F'_{n_k}(s) + A^* F_{n_k}(s) \rangle \right| \mathrm{d}s \to 0 \quad \text{a.s. for } k \to \infty, \tag{4.6}$$

for some subsequence. For the second integral in (4.5) we have, for $H_n := F_n - F$,

$$E \left| \int_{[0,t]\times B} \langle G(u), H_n(s) \rangle \, M(\mathrm{d}s, \mathrm{d}u) \right|^2$$
$$= \int_{[0,t]\times B} \langle G(u), H_n(0) + \int_0^s H_n'(r) \mathrm{d}r \rangle^2 \rho(\mathrm{d}s, \mathrm{d}u)$$
$$\leqslant 2 \int_{[0,t]\times B} \langle G(u), H_n(0) \rangle^2 \rho(\mathrm{d}s, \mathrm{d}u) + 2t \int_0^t \int_{[r,t]\times B} \langle G(u), H_n'(r) \rangle^2 \rho(\mathrm{d}s, \mathrm{d}u) \mathrm{d}r$$
$$\leqslant 2c^2 \|H_n(0)\|^2 + 2c^2 t \int_0^t \left\| H_n'(r) \right\|^2 \mathrm{d}r$$
$$\to 0 \quad \text{for } n \to \infty, \tag{4.7}$$

where we used in the last inequality the estimate in Remark 3.5. Together (4.6) and (4.7) imply that (4.5) holds true for F, which results in equality (4.4). As in the proof of Theorem 3.6 it follows from (4.4) that $\langle Y(t), v^* \rangle = \sum_{n=1}^{\infty} \gamma_n \langle v^*, I_F f_n \rangle$ for all $v^* \in \overline{D(A^*)}$, where $F(s,u) = T(t-s)G(u)$. Since $\overline{D(A^*)}$ is weak* sequentially dense in V^*, Lemma 3.7 establishes assertion (b).

(b) $\Rightarrow$ (a) As the function $(s,u) \mapsto T(t-s)G(u)$ is stochastically integrable we can define

$$Y(t) := \int_{[0,t]\times B} T(t-s) G(u) M(\mathrm{d}s, \mathrm{d}u)$$

for all $t \in [0,T]$. We start to verify that Y is weakly Bochner regular. For $H_n \in C([0,T], V^*)$ we have by Remark 3.5

$$E \left(\int_0^T |\langle Y(t), H_n(t) \rangle| \, \mathrm{d}t \right)^2 = E \left(\int_0^T \left| \int_{[0,t]\times B} \langle T(t-r) G(u), H_n(t) \rangle \, M(\mathrm{d}r, \mathrm{d}u) \right| \mathrm{d}t \right)^2$$
$$\leqslant T \int_0^T \int_{[0,t]\times B} \langle T(t-r) G(u), H_n(t) \rangle^2 \rho(\mathrm{d}r, \mathrm{d}u) \mathrm{d}t$$

$$= T \int_0^T \int_{[0,t]\times B} \langle T(r)G(u), H_n(t)\rangle^2 \mathrm{d}r\,\nu(\mathrm{d}u)\mathrm{d}t$$
$$\leqslant T \int_0^T c^2 \, \|H_n(t)\|^2 \, \mathrm{d}t,$$

which proves that Y is weakly Bochner regular.

By Fubini's theorem we obtain

$$\begin{aligned}
\int_0^t \langle Y(s), A^*v^*\rangle \mathrm{d}s &= \int_0^t \int_{[0,s]\times B} \langle T(s-v)G(u), A^*v^*\rangle \, M(\mathrm{d}v, \mathrm{d}u)\mathrm{d}s \\
&= \int_{[0,t]\times B} \int_v^t \langle T(s-v)G(u), A^*v^*\rangle \, \mathrm{d}s \, M(\mathrm{d}v, \mathrm{d}u) \\
&= \int_{[0,t]\times B} \langle G(u), T^*(t-v)v^* - v^*\rangle M(\mathrm{d}v, \mathrm{d}u) \\
&= \langle Y(t), v^*\rangle - \int_{[0,t]\times B} \langle G(u), v^*\rangle M(\mathrm{d}v, \mathrm{d}u).
\end{aligned}$$

The application of Fubini's theorem is justified because

$$\int_0^t \int_{[0,s]\times B} \big(\langle T(s-r)G(u), A^*v^*\rangle\big)^2 \, \mathrm{d}r\,\nu(\mathrm{d}u)\,\mathrm{d}s < \infty. \tag{4.8}$$

A proof of Fubini's theorem for the special case of Poisson random measures can be found in Theorem 5 in [3] which can be generalized easily to the case of martingale valued measures. Finally, there is for each $C \in \mathcal{B}(U)$ a modification of the martingale $(M(t, C) : t \geqslant 0)$ with càdlàg paths and thus, each of these martingales is measurable. Consequently, for each $v^* \in V^*$ the stochastic process $(\langle Y(t), v^*\rangle : t \geqslant 0)$ is measurable and, thus, also progressively measurable because it is also adapted; see [13]. □

5. Integration relative to Lévy processes

By means of the stochastic integration with respect to martingale valued measures we can define a stochastic integral for Banach space valued functions with respect to Lévy processes in Banach spaces. Such an integral for Wiener processes is contained in [18]. Therefore we will focus on integration with respect to the 'jumps' of the Lévy process and then combine the two integrals in Section 7 after establishing a Lévy–Itô decomposition in Section 6.

Let $(L(t) : t \geqslant 0)$ be a Lévy process with values in a separable Banach space U and define

$$N(t, \Lambda) := \sum_{s\in[0,t]} \mathbb{1}_\Lambda(\Delta L(s))$$

for every $\Lambda \in \mathcal{B}(U \setminus \{0\})$ bounded from below. With the Lévy measure $\nu(\Lambda) := E[N(1, \Lambda)]$ the compensated Poisson random measure is defined by

$$\tilde{N}(t, \Lambda) := N(t, \Lambda) - t\nu(\Lambda).$$

This defines a martingale valued measure on U with the forbidden set $C = \{0\}$ and $E[\tilde{N}(t, \Lambda)^2] = t\nu(\Lambda)$; see Example 2.2. To keep our notation from the previous sections we define $\rho(\mathrm{d}t, \mathrm{d}u) := \mathrm{d}t\,\nu(\mathrm{d}u)$.

Remark 5.1. Formally, a Lévy measure on a Banach space is defined in the following way [11, Section 5.4]. A symmetric, σ-finite measure μ on $\mathcal{B}(U \setminus \{0\})$ is called a *Lévy measure* if the function

$$\varphi : U^* \to \mathbb{R}, \quad \varphi(u^*) = \exp\left(\int_{U\setminus\{0\}} (\cos(\langle u, u^*\rangle) - 1)\mu(\mathrm{d}u) \right)$$

is the characteristic function of a measure on $\mathcal{B}(U)$. An arbitrary σ-finite measure μ is called a *Lévy measure* if $\mu + \bar{\mu}$ with $\bar{\mu}(\Lambda) := \mu(-\Lambda)$ is a Lévy measure.

In contrast to the finite dimensional case, the condition

$$\int_{U\setminus\{0\}} \min\{1, \|u\|^2\}\mu(\mathrm{d}u) < \infty$$

is in general neither necessary nor sufficient for a σ-finite measure μ on $\mathcal{B}(U \setminus \{0\})$ to be a Lévy measure; see [5,10]. To be precise, if and only if U is of cotype 2, this integral is finite for every Lévy measure.

For the measure ν defined above for a Lévy process it can be proved that ν is a Lévy measure according to the above definition.

The integration with respect to N is defined as the *Poisson integral*

$$\int_{[0,T]\times B} F(t, u)\, N(\mathrm{d}t, \mathrm{d}u) := \sum_{0\leqslant s\leqslant T} F(s, \Delta L(s))\mathbb{1}_B(\Delta L(s))$$

for every measurable function $F : [0, T] \times U \to V$ and set $B \in \mathcal{B}(U)$ bounded from below. This integral is finite because the Lévy process has càdlàg paths.

We proceed with the integration with respect to $\tilde{N}$. For a function $F : [0, T] \times B \to V$ with $B \in \mathcal{B}(U)$ which is stochastically integrable with respect to $\tilde{N}$ we have introduced in Definition 3.1 the integral

$$\int_{[0,T]\times B} F(t, u)\, \tilde{N}(\mathrm{d}t, \mathrm{d}u).$$

Note that only on sets B bounded from below is the Poisson random measure $\tilde{N}$ finite and, thus, only in this case is the integral well defined according to our Definition 3.1. In a moment we will extend the integration domain. For a Pettis integrable integrand, the stochastic integral with respect to $\tilde{N}$ can be expressed using the Poisson integral.

Proposition 5.2. *If B is a set in $\mathcal{B}(U)$ bounded from below then every function $F : [0, T]\times B \to V$ which is Pettis integrable with respect to ρ is stochastically integrable with respect to $\tilde{N}$ and we have*

$$\int_{[0,T]\times B} F(t, u)\, \tilde{N}(\mathrm{d}t, \mathrm{d}u) = \int_{[0,T]\times B} F(t, u)\, N(\mathrm{d}t, \mathrm{d}u) - \int_{[0,T]\times B} F(t, u)\mathrm{d}t\,\nu(\mathrm{d}u),$$

where the integrals on the right hand side are a Poisson and a Pettis integral, respectively.

Proof. Define Y to be the right hand side, which exists a.s. As both integrals commute with functionals in V^*, we obtain by [2, p.206]

$$\langle Y, v^*\rangle = \left\langle \int_{[0,T]\times B} F(t, u)N(\mathrm{d}t, \mathrm{d}u), v^* \right\rangle - \left\langle \int_{[0,T]\times B} F(t, u)\mathrm{d}t\,\nu(\mathrm{d}u), v^* \right\rangle$$

$$= \int_{[0,T]\times B} \langle F(t,u), v^*\rangle\, N(\mathrm{d}t,\mathrm{d}u) - \int_{[0,T]\times B} \langle F(t,u), v^*\rangle \mathrm{d}t\,\nu(\mathrm{d}u)$$
$$= \int_{[0,T]\times B} \langle F(t,u), v^*\rangle\, \tilde N(\mathrm{d}t,\mathrm{d}u).$$

Therefore, F is stochastically integrable on $[0,T]\times B$ and the required equality holds. □

For the pathwise decomposition and also for dealing with differential equations, we need the stochastic integration on the set

$$D := \{u \in U : 0 < \|u\| < 1\},$$

which is not bounded from below. We extend the integral as follows. Let (f_n) be an orthonormal basis of $L^2([0,T]\times D,\rho)$. As in the proof of [2, Thm. 2.4.11] one can verify that for

$$\gamma_{n,k} := \int_{[0,T]\times D_k} f_n(t,u)\, \tilde N(\mathrm{d}t,\mathrm{d}u)$$

and $D_k := \{u \in U : \frac{1}{k} \leqslant \|u\| < 1\}$, the sequence $(\gamma_{n,k})_{k\in\mathbb{N}}$ converges in $L^2(\Omega,P)$ for $k\to\infty$. We denote the limit by

$$\gamma_n := \int_{[0,T]\times D} f_n(t,u)\, \tilde N(\mathrm{d}t,\mathrm{d}u) := \lim_{k\to\infty} \int_{[0,T]\times D_k} f_n(t,u)\, \tilde N(\mathrm{d}t,\mathrm{d}u) \quad \text{in } L^2(\Omega,P).$$

More generally, for every function $f \in L^2([0,T]\times D,\rho)$ we define

$$\int_{[0,T]\times D} f(t,u)\, \tilde N(\mathrm{d}t,\mathrm{d}u) := \lim_{k\to\infty} \int_{[0,T]\times D_k} f(t,u)\, \tilde N(\mathrm{d}t,\mathrm{d}u) \quad \text{in } L^2(\Omega,P). \tag{5.1}$$

Now, because the integral with respect to $\tilde N$ on D is well defined for every real function $f \in L^2([0,T]\times D,\rho)$ Definition 3.1 reads as before when replacing the set B by D. The same applies to Definition 3.2 and also to the definition of the operator I_F in Lemma 3.4.

Summarizing, for a V-weakly $L^2([0,T]\times D,\rho)$ function $F : [0,T]\times D \to V$ the operator I_F is $\tilde N$-radonifying for the set D if there exists an orthonormal basis $(f_n) \subseteq L^2([0,T]\times D,\rho)$ and a V valued random variable Y such that

$$\langle Y, v^*\rangle = \sum_{n=1}^{\infty} \gamma_n \langle I_F f_n, v^*\rangle \quad \text{in } L^2(\Omega,P) \text{ for all } v^* \in V^*, \tag{5.2}$$

where $\gamma_n := \int_{[0,T]\times D} f_n(t,u)\, \tilde N(\mathrm{d}t,\mathrm{d}u)$.

Furthermore, Theorem 3.6 may be formulated analogously for the set D. Indeed, as for every function $f \in L^2([0,T]\times D,\rho)$ we have

$$E\left|\int_{[0,T]\times D} f(t,u)\, \tilde N(\mathrm{d}t,\mathrm{d}u)\right|^2 = \lim_{k\to\infty} E\left|\int_{[0,T]\times D_k} f(t,u)\tilde N(\mathrm{d}t,\mathrm{d}u)\right|^2$$
$$= \int_{[0,T]\times D} |f(t,u)|^2\, \rho(\mathrm{d}t,\mathrm{d}u),$$

and it follows also here that

$$\sum_{n=1}^{N} \gamma_n \langle v^*, I_F f_n\rangle = \int_{[0,T]\times D} \sum_{n=1}^{N} \langle\langle F, v^*\rangle, f_n\rangle_{L^2(\rho)} f_n(t,u)\tilde N(\mathrm{d}t,\mathrm{d}u)$$

$$\to \int_{[0,T]\times D} \langle F(t,u), v^*\rangle \tilde{N}(\mathrm{d}t,\mathrm{d}u) \quad \text{for } N\to\infty \text{ in } L^2(\Omega,P).$$

Hence, the proof can be completed as in Theorem 3.6. In addition, a stochastically integrable function $F : [0,T]\times D \to V$ satisfies

$$\left\langle \int_{[0,T]\times D} F(t,u)\,\tilde{N}(\mathrm{d}t,\mathrm{d}u), v^* \right\rangle = \int_{[0,T]\times D} \langle F(t,u), v^*, \rangle \tilde{N}(t,\mathrm{d}u)$$
$$= \lim_{k\to\infty} \int_{[0,T]\times D_k} \langle F(t,u), v^*\rangle\,\tilde{N}(t,\mathrm{d}u) \quad \text{in } L^2(\Omega,P).$$

6. Lévy–Itô decomposition

In this section we apply our previous results to obtain a pathwise decomposition of the Lévy process. For this purpose, let $V = U$. We are led to consider the stochastic integrability of the function $\mathrm{Id}_D : D \to U$, $\mathrm{Id}_D(u) = u$. If the function Id_D is stochastically integrable we define for simplicity

$$\int_D u\,\tilde{N}(t,\mathrm{d}u) := \int_{[0,T]\times D} u\tilde{N}(\mathrm{d}t,\mathrm{d}u)$$

and use analogous notation for the Poisson integral with respect to N. The condition that Id_D is U-weakly in $L^2([0,T]\times D,\rho)$ is satisfied according to Proposition 5.4.5 in Linde [11], which asserts

$$\sup_{\|u^*\|\leqslant 1} \int_D \langle u,u^*\rangle^2 \nu(\mathrm{d}u) < \infty. \tag{6.1}$$

Moreover, we even have stochastic integrability of Id_D.

Proposition 6.1. *The function* Id_D *is stochastically integrable with respect to* $\tilde{N}$.

Proof. We follow here arguments in [10]. Let $t \geqslant 0$ be fixed. The Poisson integral

$$J_n(t) := \int_{D_n} uN(t,\mathrm{d}u) = \sum_{s\in[0,t]} \Delta L(s) \mathbb{1}_{D_n}(\Delta L(s))$$

for $D_n = \{u \in U : \frac{1}{n} \leqslant \|u\| < 1\}$ is a random variable with the compound Poisson distribution

$$P_{J_n(t)}(\Lambda) = \mathrm{e}^{-t\nu(D_n)} \sum_{k=0}^{\infty} \frac{t^k}{k!} \nu^{*k}(\Lambda\cap D_n) \quad \text{for all } \Lambda \in \mathcal{B}(U).$$

By Proposition 5.2 the random variables

$$I_n(t) := \int_{D_n} u\tilde{N}(t,\mathrm{d}u) = \int_{D_n} uN(t,\mathrm{d}u) - t\int_{D_n} u\nu(\mathrm{d}u)$$

are well defined for all $n\in\mathbb{N}$ and have the distributions

$$P_{I_n(t)} = P_{J_n(t)} * \delta_{x_n} \quad \text{with } x_n := -t\int_{D_n} u\nu(\mathrm{d}u)$$

and the characteristic functions

$$\varphi_{I_n(t)}(u^*) = E\left[e^{i\langle I_n(t),u^*\rangle}\right] = \exp\left(t\int_{D_n} K(u,u^*)\nu(du)\right) \quad \text{for } u^* \in U^*,$$

where $K(u,u^*) := \exp(i\langle u,u^*\rangle) - 1 - i\langle u,u^*\rangle$ [11, Rem. below Thm. 5.3.11]. By [11, Cor. 5.4.6] the sequence $\{P_{I_n(t)} : n \in \mathbb{N}\}$ is tight and because the characteristic functions $\varphi_{I_n(t)}(u^*)$ are convergent for every $u^* \in U^*$ by [11, Proof of Thm. 5.4.8] it follows by [11, Prop. 1.8.2] that the laws of $I_n(t)$ converge weakly to a probability measure on U. Since $I_n(t)$ equals the sum of the mutually independent random variables $\int_{D_n\setminus D_{n-1}} u\,N(t,du) - t\int_{D_n\setminus D_{n-1}} u\nu(du)$ the Itô–Nisio Theorem [20, Thm. V.2.3] implies that $I_n(t)$ converges a.s. to a random variable $I(t)$. Consequently, we have for all $u^* \in U^*$

$$\langle I_n(t), u^*\rangle \to \langle I(t), u^*\rangle \quad \text{a.s.}$$

But on the other hand, we have

$$\langle I_n(t), u^*\rangle = \int_{D_n} \langle u,u^*\rangle\, \tilde{N}(t,du) \to \int_D \langle u,u^*\rangle\, \tilde{N}(t,du) \quad \text{in } L^2(\Omega,P),$$

due to (6.1). Therefore, we obtain

$$\langle I(t), u^*\rangle = \int_D \langle u,u^*\rangle\, \tilde{N}(t,du) \quad \text{a.s. for all } u^* \in U^*,$$

which shows the stochastic integrability of Id_D. □

A Lévy process $L = (L(t) : t \geqslant 0)$ is said to have *jumps bounded by a constant* $c > 0$ if

$$\sup_{t\geqslant 0} \|\Delta L(t)\| \leqslant c.$$

A Lévy process L with bounded jumps is called *centered* if $E[L(t)] = 0$ for all $t \geqslant 0$.

Proposition 6.2. *If $L := (L(t) : t \geqslant 0)$ is a centered Lévy process with jumps bounded by 1 then there is a version $(I(t) : t \geqslant 0)$ of*

$$\int_{[0,T]\times D} u\,\tilde{N}(ds,du)$$

which has the following properties:

(a) *$I := (I(t) : t \geqslant 0)$ is a Lévy process.*
(b) *$B(t) := L(t) - I(t)$ defines a Wiener process $B := (B(t) : t \geqslant 0)$ on U.*
(c) *I and B are independent.*

Proof. Let us first fix $u^* \in U^*$ with $\|u^*\| = 1$. As $L_{u^*}(t) := \langle L(t), u^*\rangle$ defines a Lévy process the Lévy–Itô decomposition for finite dimensional processes [2, Thm. 2.4.16] yields

$$L_{u^*}(t) = a_{u^*}t + \sigma_{u^*}^2 B_{u^*}(t) + \int_{[0,t]\times(-1,0)\cup(0,1)} \beta\,\tilde{N}_{u^*}(ds,d\beta) + \int_{[0,t]\times(-\infty,-1]\cup[1,\infty)} \beta N_{u^*}(ds,d\beta),$$

where $a_{u^*} \in \mathbb{R}$, $\sigma_{u^*}^2 \in \mathbb{R}_+$, B_{u^*} is a standard real valued Wiener process and N_{u^*} is a Poisson random measure and $\tilde{N}_{u^*}$ its compensated Poisson random measure.

For $C \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ one obtains

$$\begin{aligned} N_{u^*}(t, C) &= \sum_{0 \leqslant s \leqslant t} \mathbb{1}_C(\Delta L_{u^*}(s)) \\ &= \sum_{0 \leqslant s \leqslant t} \mathbb{1}_C(\langle \Delta L(s), u^* \rangle) \\ &= \sum_{0 \leqslant s \leqslant t} \mathbb{1}_{(u^*)^{-1}(C)}(\Delta L(s)) \\ &= N(t, (u^*)^{-1}(C)), \end{aligned}$$

which allows us to conclude that

$$\begin{aligned} L_{u^*}(t) &= \alpha_{u^*} t + \sigma_{u^*}^2 B_{u^*}(t) + \int_{[0,t]\times(-1,0)\cup(0,1)} \beta \, \tilde{N}_{u^*}(\mathrm{d}s, \mathrm{d}\beta) \\ &\quad + \int_{[0,t]\times(-\infty,-1]\cup[1,\infty)} \beta \, N_{u^*}(\mathrm{d}s, \mathrm{d}\beta) \\ &= \alpha_{u^*} t + \sigma_{u^*}^2 B_{u^*}(t) + \int_{[0,t]\times D_{u^*}} \langle u, u^* \rangle \, \tilde{N}(\mathrm{d}s, \mathrm{d}u) \\ &\quad + \int_{[0,t]\times D_{u^*}^c \setminus\{0\}} \langle u, u^* \rangle N(\mathrm{d}s, \mathrm{d}u) \end{aligned}$$

with $D_{u^*} := \{u \in U : 0 < |\langle u, u^* \rangle| < 1\}$. Because $D_{u^*}^c \setminus \{0\} \subseteq D^c$ and the support of $N(s, \cdot)$ is in D for all $s \geqslant 0$ we have also $\operatorname{supp} \tilde{N}(s, \cdot) \subseteq D$ and

$$\begin{aligned} L_{u^*}(t) &= \alpha_{u^*} t + \sigma_{u^*}^2 B_{u^*}(t) + \int_{[0,t]\times D_{u^*}} \langle u, u^* \rangle \tilde{N}(\mathrm{d}s, \mathrm{d}u) \\ &= \alpha_{u^*} t + \sigma_{u^*}^2 B_{u^*}(t) + \int_{[0,t]\times D} \langle u, u^* \rangle \tilde{N}(\mathrm{d}s, \mathrm{d}u) + \int_{[0,t]\times D_{u^*}\setminus D} \langle u, u^* \rangle \, \tilde{N}(\mathrm{d}s, \mathrm{d}u) \\ &= \alpha_{u^*} t + \sigma_{u^*}^2 B_{u^*}(t) + \int_{[0,t]\times D} \langle u, u^* \rangle \, \tilde{N}(\mathrm{d}s, \mathrm{d}u). \end{aligned}$$

The constant α_{u^*} can be calculated from the scalar decomposition in the following way:

$$\alpha_{u^*} = -E\left[\int_{|\beta|\geqslant 1} \beta \, N_{u^*}(1, \mathrm{d}\beta)\right] = -E\left[\int_{D_{u^*}^c\setminus\{0\}} \langle u, u^* \rangle \, N(1, \mathrm{d}u)\right] = 0,$$

which yields

$$L_{u^*}(t) = \sigma_{u^*}^2 B_{u^*}(t) + \int_{[0,t]\times D} \langle u, u^* \rangle \tilde{N}(\mathrm{d}s, \mathrm{d}u).$$

The same representation follows for arbitrary $u^* \in U^*$ by means of considering $u^*/\|u^*\|$. Consequently, we obtain for all $u^* \in U^*$,

$$\langle B(t), u^* \rangle = \langle L(t) - I(t), u^* \rangle = \sigma_{u^*}^2 B_{u^*}(t),$$

where $(B_{u^*}(t) : t \geqslant 0)$ is a real valued Wiener process. Hence B is a Gaussian process.

By applying the two-dimensional Lévy–Itô decomposition to $((\langle L(t), u^* \rangle, \langle L(t), v^* \rangle) : t \geqslant 0)$ we obtain that B_{u^*} is the first component of a two-dimensional Gaussian process $((B_{u^*}(t), B_{v^*}(t)) : t \geqslant 0)$ with independent increments. In particular, for all $u^*, v^* \in U^*$ and all $0 \leqslant s \leqslant t$ the random variables $\langle B(s), u^* \rangle$ and $\langle B(t) - B(s), v^* \rangle$ are independent. Therefore, the σ-algebras generated by $\{\langle B(s), u^*\rangle : u^* \in U^*\}$ and $\{\langle B(t) - B(s), u^*\rangle : u^* \in U^*\}$ are independent for all $0 \leqslant s \leqslant t$, which yields the independence of the increments of B as these σ-algebras coincide with the σ-algebras generated by $B(s)$ and $B(t) - B(s)$, respectively. Now we can choose a continuous version of B, which has thus been identified as a Wiener process with values in U.

In the same way one verifies that I is a stochastic process with independent, stationary increments. Because the process L has càdlàg paths and B has continuous paths, I has also a version with càdlàg paths which is therefore a Lévy process.

Applying the decomposition for the two-dimensional Lévy process $((\langle L(t), u^*\rangle, \langle L(t), v^*\rangle) : t \geqslant 0)$ for some $u^*, v^* \in U^*$ yields that $\langle I, u^*\rangle$ and $\langle L - I, v^*\rangle$ and therefore I and $L - I$ are independent. □

In the sequel we will choose the version of $\int_{[0,T]\times D} u\tilde{N}(t, \mathrm{d}u)$ which has càdlàg paths.

Theorem 6.3. *For every Lévy process $(L(t) : t \geqslant 0)$ there exist a constant $b \in U$ and a Wiener process $B := (B(t) : t \geqslant 0)$ with values in U such that*

$$L(t) = bt + B(t) + \int_D u\,\tilde{N}(t, \mathrm{d}u) + \int_{\|u\|\geqslant 1} uN(t, \mathrm{d}u) \quad \textit{for all } \ t \geqslant 0 \ \textit{ a.s.},$$

where the first integral is the stochastic integral with respect to $\tilde{N}$.

Moreover, the Wiener process B and $(N(t) : t \geqslant 0)$ are independent.

Proof. We define the random variable

$$Y(t) := \int_{[0,t]\times\{u:\,\|u\|\geqslant 1\}} u\,N(\mathrm{d}s, \mathrm{d}u) \quad \text{for } t \geqslant 0.$$

Then

$$Z(t) := L(t) - Y(t) - E[L(t) - Y(t)] = L(t) - Y(t) - tE[L(1) - Y(1)] \quad \text{for } t \geqslant 0$$

defines a centered Lévy process $(Z(t) : t \geqslant 0)$ with jumps bounded by 1.

As the compensated Poisson random measure of Z coincides with $\tilde{N}$ on the set D, Proposition 6.2 implies that

$$B(t) := Z(t) - \int_{[0,t]\times D} u\tilde{N}(\mathrm{d}s, \mathrm{d}u)$$

defines a Wiener process $(B(t) : t \geqslant 0)$ with values in U.

The independence of B and N can be proved as in the proof of Proposition 6.2 by the analogous result for finite dimensional Lévy processes and by using the fact that in separable Banach spaces the Borel σ-algebra coincides with the cylindrical σ-algebra. □

7. Cauchy problem driven by a Lévy process

For analyzing stochastic differential equations driven by Lévy processes with values in a separable Banach space U, the pathwise decomposition of Theorem 6.3 allows us to consider random perturbations which differ in the continuous and jump parts of the Lévy process. Thus we assume for a given Lévy process $L = (L(t) : t \geqslant 0)$ the decomposition

$$L(t) = bt + B(t) + \int_D u\,\tilde{N}(t, \mathrm{d}u) + \int_{\|u\|\geqslant 1} u\,N(t, \mathrm{d}u),$$

where b is a constant in U, $(B(t) : t \geqslant 0)$ is a Wiener process with values in U and $(N(t) : t \geqslant 0)$ is the associated Poisson process with the compensated Poisson random measure $\tilde{N}$. The measure ρ denotes as before the mean square measure of $\tilde{N}$.

We consider stochastic differential equations on a separable Banach space V driven by the Lévy process $L = (L(t) : t \geqslant 0)$ of the following form for $t \in [0, T]$:

$$\begin{aligned} \mathrm{d}Y(t) &= AY(t)\mathrm{d}t + F\mathrm{d}B(t) + \int_D G(u)\,\tilde{N}(\mathrm{d}t, \mathrm{d}u) + \int_{\|u\|\geqslant 1} H(u)N(\mathrm{d}t, \mathrm{d}u) \\ Y(0) &= y_0. \end{aligned} \tag{7.1}$$

Here A is the generator of a C_0 semigroup $(T(t))_{t\geqslant 0}$ on V and $F : U \to V$ is a linear bounded operator and the initial condition y_0 is in V. The function $G : U \to V$ is assumed to be V-weakly in $L^2([0, T] \times D, \rho)$ and $H : U \to V$ is assumed to be Borel measurable. Both functions are interpreted as before as functions on $[0, T] \times B$ which are constant in the first variable.

Definition 7.1. A V valued process $(Y(t, y_0))_{t\in[0,T]}$ is called a *weak solution of* (7.1) *on* $[0, T]$ if it is weakly progressively measurable and weakly Bochner regular and for every $v^* \in \mathrm{D}(A^*)$ and $t \in [0, T]$ we have, almost surely,

$$\begin{aligned} \langle Y(t, y_0), v^*\rangle &= \langle y_0, v^*\rangle + \int_0^t \langle Y(s, y_0), A^*v^*\rangle\,\mathrm{d}s + \langle FB(t), v^*\rangle \\ &\quad + \int_{[0,t]\times D} \langle G(u), v^*\rangle\,\tilde{N}(\mathrm{d}s, \mathrm{d}u) + \int_{[0,t]\times\{u:\,\|u\|\geqslant 1\}} \langle H(u), v^*\rangle\,N(\mathrm{d}s, \mathrm{d}u). \end{aligned} \tag{7.2}$$

In the following theorem we derive a representation of the solution of (7.1). The Gaussian part of the Lévy process L gives rise to a stochastic integral with respect to the Banach space valued Wiener process $(B(t) : t \geqslant 0)$. This integral is to be understood in the sense of van Neerven and Weis in [18] to which we refer here.

Theorem 7.2. *The following are equivalent:*

(a) *there exists a weak solution* $(Y(t, y_0))_{t\in[0,T]}$ *of* (7.1) *on* $[0, T]$*;*
(b) *the function* $t \mapsto T(t)F$ *is stochastically integrable with respect to* B *and the function* $(t, u) \mapsto T(t)G(u)$ *is stochastically integrable on* $[0, T] \times D$ *with respect to* $\tilde{N}$.

In this situation, the solution is represented by

$$\begin{aligned} Y(t, y_0) &= T(t)y_0 + \int_0^t T(t-s)F\,B(\mathrm{d}s) \\ &\quad + \int_{[0,t]\times D} T(t-s)G(u)\,\tilde{N}(\mathrm{d}s, \mathrm{d}u) + \int_{[0,t]\times\{u:\,\|u\|\geqslant 1\}} T(t-s)H(u)\,N(\mathrm{d}s, \mathrm{d}u) \end{aligned}$$

almost surely for all $t \in [0, T]$.

Proof. By linearity we may assume $y_0 = 0$.

(a) $\Rightarrow$ (b): The integration by parts formula (4.3) can be extended to the set D and therefore may be applied to $\tilde{N}$. Hence

$$\begin{aligned} &\int_0^t f'(s) \int_{[0,s]\times D} \langle G(u), v^*\rangle\,\tilde{N}(\mathrm{d}r, \mathrm{d}u)\mathrm{d}s \\ &\quad = f(t) \int_{[0,t]\times D} \langle G(u), v^*\rangle \tilde{N}(\mathrm{d}r, \mathrm{d}u) - \int_{[0,t]\times D} \langle G(u), f(s)v^*\rangle\,\tilde{N}(\mathrm{d}s, \mathrm{d}u), \end{aligned}$$

and we can read off a similar formula from [18] for B

$$\int_0^t f'(s)\langle FB(s), v^*\rangle \,\mathrm{d}s = f(t)\langle FB(t), v^*\rangle - \int_0^t \langle FB(\mathrm{d}s), f(s)v^*\rangle$$

for $f \in C^1([0,T],\mathbb{R})$. Similarly, for the Poisson integral we obtain for $H : U \to V$ Borel measurable that

$$\begin{aligned}
&\int_0^t f'(s)\int_{[0,s]\times\{u:\|u\|\geqslant 1\}} \langle H(u), v^*\rangle N(\mathrm{d}r,\mathrm{d}u)\,\mathrm{d}s\\
&\quad = \int_0^t f'(s)\sum_{0\leqslant r\leqslant t}\langle H(\Delta L(r)), v^*\rangle \mathbb{1}_{\{u:\,\|u\|\geqslant 1\}}(\Delta L(r))\mathbb{1}_{[0,s]}(r)\mathrm{d}s\\
&\quad = \sum_{0\leqslant r\leqslant t}\mathbb{1}_{\{u:\|u\|\geqslant 1\}}(\Delta L(r))\int_r^t \langle H(\Delta L(r)), f'(s)v^*\rangle\,\mathrm{d}s\\
&\quad = \sum_{0\leqslant r\leqslant t}\mathbb{1}_{\{u:\|u\|\geqslant 1\}}(\Delta L(r))(\langle H(\Delta L(r)), f(t)v^*\rangle - \langle H(\Delta L(r)), f(r)v^*\rangle)\\
&\quad = f(t)\int_{[0,t]\times\{u:\|u\|\geqslant 1\}}\langle H(u), v^*\rangle\, N(\mathrm{d}r,\mathrm{d}u)\\
&\qquad - \int_{[0,t]\times\{u:\|u\|\geqslant 1\}}\langle H(u), f(s)v^*\rangle\, N(\mathrm{d}s,\mathrm{d}u).
\end{aligned}$$

Thus we can proceed as in the proof of Theorem 4.3 to obtain

$$\begin{aligned}
\langle Y(t), v^*\rangle = {}& \int_0^t \langle T(t-s)FB(\mathrm{d}s), v^*\rangle + \int_{[0,t]\times D}\langle T(t-s)G(u), v^*\rangle\,\tilde{N}(\mathrm{d}s,\mathrm{d}u)\\
&+ \int_{[0,t]\times\{u:\,\|u\|\geqslant 1\}}\langle T(t-s)H(u), v^*\rangle\,N(\mathrm{d}s,\mathrm{d}u)
\end{aligned} \tag{7.3}$$

for all $v^* \in \overline{D(A^*)}$. The first integral defines by

$$Z : V^* \to L^2(\Omega, P), \qquad Z(v^*) := \int_0^t \langle T(t-s)FB(\mathrm{d}s), v^*\rangle$$

a cylindrical random variable which induces a symmetric cylindrical measure. The first integral is independent of the other integral terms in (7.3) by Theorem 6.3 and therefore it follows from Proposition 3.4 in [20, Ch. VI] that this cylindrical measure can be extended to a Radon measure on $\mathcal{B}(U)$. Consequently, the cylindrical random variable Z is induced by a random variable $Y : \Omega \to V$ (see [20, Thm. VI.3.1]) which yields that $s \mapsto T(t-s)F$ is stochastically integrable with respect to the Wiener process B. Eq. (7.3) implies that $s \mapsto T(t-s)G(u)$ is stochastically integrable with respect to $\tilde{N}$.

(b) $\Rightarrow$ (a) The argument in the proof of Theorem 4.3 can be generalized to this situation. □

Example 7.3. Heat equation with Lévy noise. Let $d \in \mathbb{N}$ and let $\mathcal{O}$ be a nonempty bounded open subset of $\mathbb{R}^d$. Consider the heat equation on $\mathcal{O}$ with Dirichlet boundary conditions,

$$\begin{aligned}
&\mathrm{d}y(t,x) = \Delta u(t,x)\mathrm{d}t + \sigma(x)\mathrm{d}L(t), \quad (t,x)\in[0,\infty)\times\mathcal{O},\\
&y(t,x) = 0, \quad (t,x)\in[0,\infty)\times\partial\mathcal{O},\\
&y(0,x) = y_0(x), \quad x\in\mathcal{O},
\end{aligned}$$

where $\sigma, y_0 \colon \mathcal{O} \to \mathbb{R}$ are given, and where $(L(t) : t \geqslant 0)$ is a Lévy process in $\mathbb{R}$. Let $p \geqslant 2$, take $V = L^p(\mathcal{O})$, and assume $\sigma, y_0 \in L^p(\mathcal{O})$. The Laplace operator Δ with Dirichlet boundary conditions generates a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on V with generator A, which has domain $W_0^{1,p}(\mathcal{O})$ [22, Section 5.10]. More generally, we can consider a Lévy process $L = (L_i)_{i\in\mathbb{N}}$ in $U = \ell^2$, where ℓ^2 denotes the space of square summable real sequences. Let $\sigma_i \in L^p(\mathcal{O})$, $i \in \mathbb{N}$, and consider

$$\mathrm{d}y(t,x) = \Delta u(t,x)\mathrm{d}t + \sum_{i=1}^{\infty} \alpha_i \sigma_i(x)\mathrm{d}L_i(t), \quad (t,x) \in [0,\infty) \times \mathcal{O}, \tag{7.4}$$

with Dirichlet boundary conditions and initial condition $y_0 \in L^p(\mathcal{O})$. Here $\alpha_i \in \mathbb{R}$ are such that $(\alpha_i \|\sigma_i\|_{L^p})_{i\in\mathbb{N}} \in \ell^2$.

Consider the Lévy–Itô decomposition of L as in Theorem 6.3. The maps $F, G, H : U \to V$ defined by $F(u) = G(u) = H(u) = \sum_i \alpha_i u_i \sigma_i$ for $u = (u_i)_{i\in\mathbb{N}} \in \ell^2$ are Borel and (7.4) leads to

$$\mathrm{d}Y(t) = (AY(t) + b)\mathrm{d}t + F\mathrm{d}B(t) + \int_D G(u)\tilde{N}(\mathrm{d}t,\mathrm{d}u) + \int_{\|u\|\geqslant 1} H(u)\, N(\mathrm{d}t,\mathrm{d}u). \tag{7.5}$$

Due to [18, Theorem 4.7] the function $T(\cdot)F(\cdot)$ is stochastically integrable with respect to B. If we assume that G is Bochner integrable with respect to ν on D, then $(t,u) \mapsto T(t)G(u)$ is stochastically integrable with respect to $\tilde{N}$. Indeed, for $m \geqslant n$ and $D_{m,n} := [0,T] \times (D_n \setminus D_m)$ Proposition 5.2 yields that

$$\begin{aligned}
&E\left\|\int_{D_{m,n}} T(s)G(u)\,\tilde{N}(\mathrm{d}s,\mathrm{d}u)\right\|^2 \\
&\leqslant 2E\left\|\int_{D_{m,n}} T(s)G(u)N(\mathrm{d}s,\mathrm{d}u)\right\|^2 + 2\left\|\int_{D_{m,n}} T(s)G(u)\rho(\mathrm{d}s,\mathrm{d}u)\right\|^2 \\
&\leqslant 2E\left(\int_{D_{m,n}} \|T(s)G(u)\|N(\mathrm{d}s,\mathrm{d}u)\right)^2 + 2\left(\int_{D_{m,n}} \|T(s)G(u)\|\rho(\mathrm{d}s,\mathrm{d}u)\right)^2 \\
&\leqslant 2E\left(\|T(s)G(u)\|\,\tilde{N}(\mathrm{d}s,\mathrm{d}u) - \int_{D_{m,n}} \|T(s)G(u)\|\rho(\mathrm{d}s,\mathrm{d}u)\right)^2 \\
&\quad + 2\left(\int_{D_{m,n}} \|T(s)G(u)\|\rho(\mathrm{d}s,\mathrm{d}u)\right)^2 \\
&\leqslant 4M\int_{D_{m,n}} \|G(u)\|^2\rho(\mathrm{d}s,\mathrm{d}u) + 6M\left(\int_{D_{m,n}} \|G(u)\|\rho(\mathrm{d}s,\mathrm{d}u)\right)^2,
\end{aligned}$$

where $M = \sup_{s\in[0,T]} \|T(s)\|^2$. Hence

$$Y(t) := \lim_{n\to\infty} \int_{[0,t]\times D_n} T(s)G(u)\,\tilde{N}(\mathrm{d}s,\mathrm{d}u)$$

exists in the Bochner space $L^2(\Omega, V)$, since $\int_D \|u\|^2 \nu(\mathrm{d}u) < \infty$ by [5]. Then by (5.1),

$$\int_{[0,t]\times D} \langle T(s)G(u), v^* \rangle \, \tilde{N}(\mathrm{d}s, \mathrm{d}u) = \langle Y(t), v^* \rangle \quad \text{a.s.}$$

It follows from Theorem 7.2 that there exists a weak solution of (7.5) and therefore of (7.4). A typical choice of σ_i would be an orthonormal basis in $L^2(\mathcal{O})$ consisting of eigenfunctions of Δ. If $d \leqslant 2$ or $p \leqslant 2d/(d-2)$, then the Sobolev embedding theorem yields that $W_0^{1,2}(\mathcal{O})$ is contained in $L^p(\mathcal{O})$ and

$$\|w\|_{L^p} \leqslant C\|w\|_{W^{1,2}} \quad \text{for all } w \in W_0^{1,2}(\mathcal{O}),$$

for some constant C. If $-\lambda_i$ are the corresponding eigenvalues, then

$$\|\sigma_i\|_{L^p} \leqslant C\|\sigma_i\|_{W^{1,2}} = C\left(\int_{\mathcal{O}} \sigma_i^2 + \nabla\sigma_i \cdot \nabla\sigma_i \mathrm{d}x\right)^{1/2} C\sqrt{1+\lambda_i}\ .$$

According to Weyl's theorem,

$$\lim_{n\to\infty} \frac{1}{n}\lambda_n^{d/2} = \frac{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)}{|\mathcal{O}|},$$

from which an appropriate choice of α_i can be computed.

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