New crack elements for boundary element analysis of elastostatics considering arbitrary stress singularities

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New crack elements are proposed for fracture mechanics analysis of two- and three-dimensional elastostatics by the boundary element method (BEM). In the proposed crack elements, arbitrary singularities of the stress or the traction near the crack front can be taken into account, and the behavior of displacements near the crack front can be simulated in a reasonable manner. Three different kinds of interpolation functions are used for modeling the boundary element geometry and the traction and displacement variations near the crack front, so the nodal points of the crack element are located at their regular positions. An attempt is made to apply the proposed crack elements to the stress-intensity-factor computation of some typical surface-crack problems. Some interesting features inherent to the surface-crack problem are revealed through comparison of the results obtained with other solutions available in the literature.

Keywords: boundary element analysis, stress singularity, surface crack, crack element, three-dimensional elastostatics

Introduction

The stress intensity factor of a cracked elastic body is one of the most important parameters for assessing its fracture mechanics strength. There have been many investigations of two-dimensional crack problems subject to a wide variety of boundary conditions. In practical applications, however, the crack has a three-dimensional form, and its two-dimensional idealization fails in most cases to be a good approximation. In particular, the surface-crack problem is such a case, because the stress singularities and the behavior of displacements near the surface points at which the crack front intersects the free surfaces depend, in general, on the form of the cracked surface and on the elastic moduli. There are still obstacles to be removed in order to obtain an accurate numerical computation of the stress intensity factors in surface-crack problems.

Finite element methods have been widely applied to the stress-intensity-factor computation of cracked bodies, and a large amount of useful knowledge has been accumulated for a rational design based on fracture mechanics. In the last decades, however, boundary element methods have been successfully applied to such computations. The authors' previous paper has been concerned with a new crack element considering the so called square-root stress singularity for fracture mechanics analysis of three-dimensional elastostatics by the boundary element method. In principle, this crack element can be valid only for the buried-crack problem.

In this paper a new family of crack elements is
where \( a \) and \( b \) are constants, and \( \lambda \) and \( \gamma \) are coefficients that depend on the angle \( \alpha \) and Poisson's ratio \( \nu \). Takakuda has obtained the values of \( \lambda \) and \( \gamma \) under various conditions. In this study we use his results for deriving a new family of crack elements.

### Two-dimensional crack element

First, we discuss our crack element for two-dimensional problems. We assume that the element geometry \( x_i \), the displacement \( u_i \), and the traction \( t_i \) can be interpolated by using the nodal values in such a way that

\[
x_i = \sum_{j=1}^{n} \phi_j(x) x_{ij} \quad u_i = \sum_{j=1}^{n} \phi_j(x) u_{ij} \quad t_i = \sum_{j=1}^{n} \phi_j(x) t_{ij}
\]

where \( \phi \) denotes the so-called shape function, and \( \phi' \) and \( \phi'' \) are the interpolation functions for the displacement and the traction, respectively.

\[u_2 = ar^\gamma \quad t_2 = br^{-\gamma}\]
dimensional coordinate \( \xi \) by
\[
\xi = \frac{x - (\bar{x}_1 + \bar{x}_d)/2}{(\bar{x}_2 - \bar{x}_1)/2}
\] (4)

where \( \bar{x}_1 \) is the coordinate of the crack tip. Then the distance from the crack tip, denoted by \( r \), is related to \( \xi \) by
\[
r = (l/2)(1 + \xi)
\] (5)

Note that \( r \) is proportional to \( 1 + \xi \). This fact guarantees that equally spaced nodal points in the global coordinate \( x \) are mapped into equally spaced nodal points in the nondimensional coordinate \( \xi \).

The shape functions in equation (3) can be obtained from a Lagrangian family of the interpolation functions and expressed as\(^{10}\)
\[
\phi_j(\xi) = \sum_{i=1}^{n} \frac{\xi_i - \xi_j}{\xi_i - \xi_j} (i \neq j)
\] (6)
The interpolation functions for the displacement, which show the expected behavior as in equation (2), are
\[
\phi_j(\xi) = (1 + \xi)^{-1} \phi_j(\xi)
\]
\[
\phi_j(\xi) = (1 + \xi)^{-1} \phi_j(\xi) \quad \text{for} \ j \neq 1
\] (8)

By using equations (7) and (8), we can interpolate the displacement and traction components in the crack element as follows:
\[
u_i = c_{i1} + c_{i2} r^k + \cdots + c_{in} r^{(n-1)k}
\]
\[
t_i = d_{i1} r^k + d_{i2} r^{k+1} + \cdots + d_{nk} r^{(n-1)k}
\] (9)
where \( c_{ik} \) and \( d_{ik} \) are constants.

### Three-dimensional crack element

The idea for the two-dimensional crack element in the previous section will be extended to derive the three-dimensional crack element. We shall consider a general element with \( n \) nodal points as shown in Figure 4, where \( n \) is a multiple of the number 4. The edge 1–2 coincides with the crack front, from which the distance \( r \) is measured. We assume that the global coordinates \( x_i \), the displacement \( u_i \), and the traction \( t_i \) can be interpolated in the following manner:
\[
x_i = \sum_{j=1}^{n} \phi_j(\xi_1, \xi_2)x_{ij}
\]
\[
u_i = \sum_{j=1}^{n} \phi_j(\xi_1, \xi_2)u_{ij}
\]
\[
t_i = \sum_{j=1}^{n} \phi_j(\xi_1, \xi_2)t_{ij}
\] (10)
The shape functions \( \phi_j \) and the interpolation functions \( \phi_j \) for the displacement and \( \phi_j \) for the traction can be obtained again from a Lagrangian family and expressed as follows:
\[
\phi_j(\xi_1, \xi_2) = \frac{1}{2} (1 + \xi_1/\xi_j)(1 + \xi_2/\xi_j)
\]
\[
- \sum_{i=5}^{n} \frac{1}{4} (1 + \xi_{1i}/\xi_j)(1 + \xi_{2j}/\xi_i)\phi_i
\]
\[\text{for} \ j = 1, 2, 3, 4\]
\[
\phi_j(\xi_1, \xi_2) = \frac{1 + \xi_{2j}/\xi_j}{2} \sum_{i=1}^{n} \frac{\xi_i - \xi_{1j}}{\xi_j - \xi_{1j}}
\]
\[\text{on edge 1–2} \ j = 5, 9, \ldots, n - 3\]
\[\text{on edge 3–4} \ j = 7, 11, \ldots, n - 1\]
\[
\phi_j(\xi_1, \xi_2) = \frac{1 + \xi_{2j}/\xi_j}{2} \sum_{i=1}^{n} \frac{\xi_i - \xi_{1j}}{\xi_j - \xi_{1j}}
\]
\[\text{on edge 2–3} \ j = 6, 10, \ldots, n - 2\]
\[\text{on edge 4–1} \ j = 8, 12, \ldots, n\]
\[
\phi_j(\xi_1, \xi_2) = \frac{1 + \xi_{2j}/\xi_j}{2} \sum_{i=1}^{n} \frac{\xi_i - \xi_{1j}}{\xi_j - \xi_{1j}}
\]
\[\text{on edge 2–3} \ j = 6, 10, \ldots, n - 2\]
\[\text{on edge 4–1} \ j = 8, 12, \ldots, n\]
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Figure 5 Crack element applied to surface points

\[
\phi_j(\xi_1, \xi_2) = (1 + \xi_2)^{-j}(1 + \xi_2)^{-j} \text{ on side 1-2, } j = 1, 5, 9, \ldots, n - 3, 2 \\
\phi_j(\xi_1, \xi_2) = (1 + \xi_2)^{j-1}(1 + \xi_2)^{-j} \text{ on edge 2-3, } j = 6, 10, \ldots, n - 2 \\
\phi_j(\xi_1, \xi_2) = (1 + \xi_2)^{j-1}(1 + \xi_2)^{-j} \text{ on edge 3-4, } j = 3, 7, 11, \ldots, n - 1, 4 \\
\phi_j(\xi_1, \xi_2) = (1 + \xi_2)^{j-1}(1 + \xi_2)^{-j} \text{ on edge 4-1, } j = 8, 12, \ldots, n
\]

In the above equations \(k\) and \(h\) denote the first and the last nodal points included in the edge under consideration, except for nodes 1, 2, 3, and 4.

One can easily show that by using the interpolation functions in equations (12) and (13) the displacement and traction components in the crack element are expressible in the following form:

\[
u_i = c_{1i} + c_{2i} r^k + c_{3i} r^h + \cdots + c_{mi} r^{(m - 1)k}
\]

\[
t_i = d_{1i} r^{-\gamma} + d_{2i} r^{-\gamma + 1} + \cdots + d_{mi} r^{-\gamma + (m - 1)}
\]

where \(c_{ki}\) and \(d_{ki}\) are again constants, and \(m\) denotes the number of nodal points along one edge of the element.

For the region including the surface point where the crack front intersects the free surface of the cracked body under consideration, we may employ the triangular crack element shown in Figure 5. We can easily obtain it from the crack element of Figure 4, if we assume the following relations:

\[
x_{1i} = x_{5i} = \cdots = x_{(n - 3)i} = x_{2i}
\]

\[
u_{1i} = u_{5i} = \cdots = u_{(n - 3)i} = u_{2i}
\]

\[
t_{1i} = t_{5i} = \cdots = t_{(n - 3)i} = t_{2i}
\]

In this case the \(r\)-direction coincides with the \(\xi_2\)-direction as shown in Figure 5, and the element can show the same singularity and the same behavior for the traction and displacement components, respectively, as those of the quadrilateral crack element of Figure 4.

**Numerical results and discussion**

A few surface-crack problems are computed by incorporating the present crack element into the boundary element computer program developed previously by the authors.\(^5\) Small regions including the crack front and the surface points at which the crack front intersects the free surface are discretized into the crack elements, and the other boundary portion is subdivided into a series of regular 8-node isoparametric quadrilateral elements. In all the computations Poisson’s ratio is assumed to be \(v = 0.3\).

In Figure 6 the arrow indicates the \(r\)-direction in the crack elements near the surface point. The exponents \(\lambda\) and \(\gamma\) used are summarized in Table 1. These values correspond to those obtained by Benthem\(^8\) and Takanakuda\(^9\).

The stress intensity factors are calculated from the boundary element analysis by the following relations:

\[
u_2 = \frac{k + 1}{2G} (A_1 r_1^{1/2} - A_2 r_2^{1/2} + A_3 r_4^{1/2} - A_4 r_5^{1/2})
\]

\[
t_2 = A_1 r_1^{-1/2} + 3A_2 r_2^{-1/2} + 5A_3 r_4^{-1/2} + 7A_4 r_5^{-1/2}
\]

where \(G\) is the shear modulus and \(k = 3 - 4v\). The nodal values obtained for the elements \(c\) in Figure 6 are substituted into equation (16), and the coefficient \(A_1 = K_1/\sqrt{2\pi}\) is finally calculated for the crack-opening area.
mode. Similar procedures are applicable to the other cracking modes.

First, we analyze the semieliptical surface crack embodied in a rectangular parallelepiped. Figure 7 illustrates the notation used for the cross section, including the cracked plane. The aspect ratio \( a/c \) denotes the ellipticity of the crack shape, and the case \( a/c = 1 \) corresponds to a semicircular surface crack.

For comparison with other available solutions we first analyze the semicircular surface crack when the parallelepiped is subjected to uniform tension at its ends. We also assume that the cracked plane is perpendicular to the axis of the parallelepiped. Because of symmetry a quarter portion is discretized into crack elements and boundary elements, as shown in Figure 8. Two mesh patterns of 80 elements and 55 elements in totality for the quarter portion are used. In Figure 9 the results obtained are compared with the finite element solutions by Raju and Newman, using coarse and fine finite element meshes. Near the surface point the present solution shows a similar variation to the fine-mesh solution of Raju and Newman.

The values of \( K_1 \) near the free-surface point are calculated for various points besides the nodes of the crack elements. The interpolation functions of the crack element are used to obtain the displacement and traction components in equation (16). The same procedure will be applied to the following numerical examples. Triangles, squares, and circles, shown in the figures as the BEM solution, denote the computed values at the nodal points of the crack and boundary elements. Moreover, since some discontinuous jumps arise between the crack elements \( b \) and \( c \) in Figure 6, the \( K_1 \) values are calculated only for the element \( c \).

In Figure 10 a comparison is made between the computational results obtained for the two cases of the square-root singularity and the general \( \gamma \)-\( \gamma \) singularity proposed in this study. For both cases the 80-element mesh division is used in the boundary element computation.

Next, we analyze two cases of the semieliptical surface crack. In one example the aspect ratios are assumed such that \( a/c = 0.6, a/t = 0.2 \), and \( b/t = 1.5 \), in the other example \( a/c = 1.67, a/t = 0.2 \), and \( b/t = 1.5 \). The element discretization of the cracked surface in the former case is illustrated in Figure 11, where a total of 84 elements is used for a quarter portion of the parallelepiped. In Figure 12 the results obtained are compared with the Raju Newman finite element solution using a coarse mesh and the Kisu–Yuuki–Kitagawa boundary element solution. A larger discrepancy between three solutions can be recognized, particularly near the
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Figure 7  Element subdivision of cracked surface

Figure 12  Comparison between results on shallow elliptical crack surface point. Note that $Q$ in the figure is defined as follows:

$$Q = [E(\rho)]^2$$

$$E(\rho) = \sqrt{1 - \rho^2 \sin^2 \theta}$$

$$\rho^2 = 1 - (a/c)^2 \quad (a \leq c)$$

$$\rho^2 = 1 - (c/a)^2 \quad (a > c)$$

(17)

Figure 13 shows the results obtained for a deep semi-elliptical surface crack. The stress intensity factor of this case becomes larger and fluctuates much near the surface point. A refinement of the boundary element subdivision confirms the same tendency.

Concluding remarks

We have proposed a new family of crack elements for boundary element analysis of fully three-dimensional elastostatic problems. The proposed crack element can take into account arbitrary properties of the stress and the displacement near the crack front, which is most important for fracture mechanics analysis of the surface-crack problem. The potential usefulness of the proposed method is demonstrated through the numerical computation of a few surface-crack problems. It is also revealed that the value of the stress intensity factor varies much near the surface point at which the crack front intersects the free surface.

This paper was concerned only with a few surface-crack problems, in which the crack-opening fracture mode is dominant. Since the proposed method is not restricted to such cases, further application to more complicated crack geometries is recommended for future research work.

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