A Result on the Equivalence Problem for Deterministic Pushdown Automata*

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This paper shows that it is decidable whether or not two deterministic pushdown automata, one of which is nonsingular (introduced by L. G. Valiant), are equivalent (in the sense that they accept the same languages by empty stack). This is an extension of Valiant's result that the equivalence of two nonsingular deterministic pushdown automata is decidable.

1. INTRODUCTION

Theoretically and practically, the equivalence problem for deterministic pushdown automata (dpda) is very important. However, it has not been solved for general dpda's. For some subfamilies of dpda's or deterministic context-free languages, the following results on the decidability (1°) to (4°) are already known.

(1°) Equivalence of two $LL(k)$ languages is decidable [2].

Valiant [3] defined the class of nonsingular dpda's, which is a subclass of dpda's with empty stack acceptance. The class of languages accepted by nonsingular dpda's contains properly the class of $LL(k)$ languages.

(2°) Equivalence of two nonsingular dpda's is decidable [3].

(3°) Equivalence of two deterministic one-counter automata is decidable [3, 4].

(4°) Equivalence of two finite-turn dpda's is decidable [3, 5].

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The purpose of this paper is to prove the following result.

(5') Equivalence of two dpda's (with empty stack acceptance), one of which is nonsingular, is decidable.

This is an extension of the previous result (2').

To prove (5'), we construct a single stack machine $M'$ which simulates given dpda $M$ and nonsingular dpda $\overline{M}$. This simulating machine is analogous to Valiant's alternating stack machine in [3] which was used to simulate two nonsingular dpda's. However, in our simulating machine $M'$, there is no symmetry between $M$ and $\overline{M}$. $M'$ has the stack of the form $w_0[\overline{w}_0] w_1[\overline{w}_1] \cdots w_n[\overline{w}_n]$, where $w_0, w_1, \ldots, w_n$ and $\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_n$ are the stacks of $M$ and $\overline{M}$, respectively, at that time. If $M$ and $\overline{M}$ are equivalent, then for each input string which is the proper prefix of some accepted string, $M'$ produces the stack $w_0[\overline{w}_0] w_1[\overline{w}_1] \cdots w_n[\overline{w}_n]$ with the property that (i) each $w_i$ ($0 \leq i \leq n$) is not null, (ii) the topmost stack segment of $\overline{M}$, $\overline{w}_n$, is not null, and (iii) each $\overline{w}_i$ ($0 \leq i \leq n$) is of bounded size. Then, we can construct a (nondeterministic) pda $M''$ with the property that $M''$ accepts an input iff $M$ and $\overline{M}$ are inequivalent. Since the emptiness of $M''$ is decidable, it follows that the equivalence of $M$ and $\overline{M}$ is decidable. We present two key lemmas which will be used to prove that such a simulating machine $M'$ can be constructed. In Lemma 2, we introduce some new ideas, whereas Lemma 1 is not really different from Valiant's result.

2. Definitions

Since our proof techniques are similar to those of [3] for nonsingular dpda's, we use the same notation as [3] unless stated otherwise.

Let $M = (\Sigma, \Gamma, Q, F, A, \epsilon)$ be a deterministic pushdown automaton (dpda), where $\Sigma, \Gamma, Q$ are finite sets of input symbols, stack symbols, and states, respectively, and $A$ is a finite set of transition rules. Let $\epsilon$ and $A$ denote the null elements of $\Sigma^*$ and $\Gamma^*$, respectively. A configuration $c = (s, w)$ is an element of $Q \times (\{\Omega\} \cup \Gamma^*)$ and describes the state and stack content of the machine at some instant. Here, $\Omega$ is a special empty stack symbol. The mode of a configuration $c$ is an element from $Q \times (\{\Omega\} \cup \Gamma^*)$ and describes the state and top stack symbol of $c$. $F \subseteq Q \times (\{\Omega\} \cup \Gamma^*)$ is a set of distinguished accepting modes. The machine $M$ makes the move $(s, wA) \rightarrow^\alpha (s', ww')$ from the one configuration to the other if and only if one of the transition rules is $(s, A) \rightarrow^\alpha (s', w')$, where $\pi \in \Sigma \cup \{\epsilon\}$. A derivation $c \rightarrow^\alpha c'$ is a sequence of such moves through successive configurations where $\alpha$ is the concatenation of the symbols read by the constituent moves. This also allows for the null sequence of moves. The height $|c|$ of a configuration $c$ is the length of its stack.

We denote the set of strings which can take the machine from a configuration $c$ to accepting modes, by $L(c)$. Let $c_0 \in Q \times (\{\Omega\} \cup \Gamma^*)$ be the starting configuration of $M$. 
For a machine $M$, we denote $L(c_0)$ by $L(M)$. For two dpda's $M_1$ and $M_2$, if $L(M_1) = L(M_2)$, then they are said to be equivalent.

Let $D$ be a family of dpda's. We define the subclass $D_0$ by imposing the restrictions

(a) $F \subseteq Q \times \{\Omega\}$, and
(b) no rule is defined for modes in $Q \times \{\Omega\}$.

Valiant [3] defined the class $N_0$ of nonsingular machines as follows.

$M \in D_0$ is nonsingular iff there exists $m \geq 0$ such that for any $w, w' \in \Gamma^*$ and $s, s' \in Q$ where $|w|$ (the length of $w$) $> m$, if $L(s, w'w) = L(s', w')$, then it holds that $L(s', w') = \emptyset$.

Note 1. By the conditions described above, if $M$ is nonsingular, then the appropriate constant $m$ sets up an upper bound on the amount that the height of a configuration can change in the course of any sequence of $\epsilon$-moves. Hence, we can find another equivalent dpda with no $\epsilon$-moves. Thus we consider that $M$ in $N_0$ has no $\epsilon$-moves.

An input string $\alpha$ is said to be live for a dpda $M$ iff it is the proper prefix of some accepted string, and a configuration $c$ is said to be live iff $L(c) \neq \emptyset$. By the definitions, if $M$ is in $D_0$ and if for some live input $\alpha, c, c, \rightarrow^\alpha c$ in $M$, then $L(c) \neq \emptyset$.

The derivation $c \rightarrow^\alpha c'$ is written as $c \uparrow (\alpha) c'$ if $|c| \leq |c'|$ and every intermediate configuration in the derivation has height $\geq |c|$. Note that this is different from [3], since $c \uparrow (\alpha) c'$ now allows for $|c| = |c'|$, and also for the null sequence of moves.

The following definitions are new. The following $A_M$ and $Z_M$ correspond to $z$ in the proof of [3, Theorem 3.2]. But we treat two dpda's, one in $D_0$ and the other in $N_0$, so in order to make the discussions clear, we introduce the notation $A_M$ and $Z_M$, where $M$ is a dpda in $D_0$ or $N_0$. For a dpda $M$ in $D_0$ (or $N_0$), and for $s, s' \in Q, Z, Z' \in \Gamma$, let $P_M(s, Z, s') = \{\alpha | \text{there is a derivation } (s, Z) \rightarrow^\alpha (s', \Omega) \text{ in } M\}$ and let $S_M(s, Z, s', Z') = \{\alpha | \text{there is a derivation } (s, Z) \rightarrow^\alpha (s', Z') \text{ in } M\}$. Let us define

$$A_M = \max_{s, s' \in Q, Z \in \Gamma, P_M(s, Z, s') \neq \emptyset} \{\min \{||\alpha|| | \alpha \in P_M(s, Z, s')\}\}$$

and

$$\tilde{A}_M = \max_{s, s' \in Q, Z \in \Gamma, S_M(s, Z, s', Z') \neq \emptyset} \{\min \{||\alpha|| | \alpha \in S_M(s, Z, s', Z')\}\}.$$ Note that $A_M$ and $\tilde{A}_M$ are some constants which are determined by $M$. For $M \in D_0$ (or $N_0$), let

$$h_M = \max \{||\alpha'|| | \text{ M has a rule } (s, Z) \rightarrow^\pi (s', \alpha') \text{ for some } s, s' \in Q, Z \in \Gamma, \pi \in \Sigma \cup \{\epsilon\}\} - 1.$$

For a dpda in $D_0$, the number of consecutive $\epsilon$-moves which cause decrease in stack height is not bounded. We will consider the stack height decrease by $\epsilon$-moves by...
introducing the predicate $\text{Pop}(c, n)$ defined below. For $M \in D_0$, we define the predicate

$$\text{Pop}(c, n),$$

where $c$ is a live configuration and $n$ is a nonnegative integer. Define $\text{Pop}(c, n)$ to be true iff there exist $\alpha \in \Sigma^*$ and live configurations $c'$ and $c''$ such that $c \xrightarrow{\alpha} c'$, $|c| = |c'|$, $c' \xrightarrow{e} c''$ and $|c''| = |c| - n$. Note that $\text{Pop}(c, 0)$ is always true for each live configuration $c$.

### 3. Lemmas

In this section, we present two key lemmas. We distinguish the notation for $M \in D_0$ and $M \in N_0$ by overlining everything concerned with the latter. Without loss of generality, we can assume that for $M \in D_0$, $h_M \leq 1$, that is, if $(s, Z) \xrightarrow{e} (s', w')$ is a rule of $M$, then $|w'| \leq 2$.

**Lemma 1.** For $M \in D_0$ and $\overline{M} \in N_0$, assume that $L(M) = L(\overline{M})$. Then there exists $l_1 \geq 0$ such that if, for live input string $a_1 a_2$, $c_0 \xrightarrow{a_1} c_1$ and $c_1 \xrightarrow{a_2} c_2$ in $M$ and $\overline{c}_0 \xrightarrow{a_1} \overline{c}_1$ and $\overline{c}_1 \xrightarrow{a_2} \overline{c}_2$ in $\overline{M}$, then it holds that $|\overline{c}_1| - |\overline{c}_2| \leq l_1$.

The proof of this lemma is not really different from Valiant's result [3, Theorem 3.2]. For convenience, the proof is presented in the Appendix. The following lemma introduces some new ideas.

**Lemma 2.** For $M \in D_0$ and $\overline{M} \in N_0$, assume that $L(M) = L(\overline{M})$. Then there exists $l_2 \geq 0$ with the following property.

If, for live input string $a_1 a_2$, $c_0 \xrightarrow{a_1} c_1$ and $c_1 \xrightarrow{a_2} c_2$ in $M$, $\overline{c}_0 \xrightarrow{a_1} \overline{c}_1$ and $\overline{c}_1 \xrightarrow{a_2} \overline{c}_2$ in $\overline{M}$, and $\text{Pop}(c_2, |c_2| - |c_1|)$ is true, then for any $\alpha_3$ such that $c_2 \xrightarrow{\alpha_3} c_3$, $|c_2| = |c_3|$ and $\alpha_3 \alpha_2 \alpha_3$ is live, it holds that $|\overline{c}_3| - |\overline{c}_2| \leq l_2$, where $\overline{c}_2 \xrightarrow{a_3} \overline{c}_3$ in $\overline{M}$.

**Proof.** Let $l' = m - l_1 - 1 + \Delta_M(h_M \overline{A}_M + l_1 + 1) + \overline{A}_M$, $l'' = m - l_1 - 1 + \Delta_M(h_M \overline{A}_M + l_1 + 1)$ and $l_2 = l' + l''$. Here $m$ is the nonsingularity constant of $M$ and $l_1$ is the constant shown in Lemma 1. We shall show that

(i) $|\overline{c}_2| - |\overline{c}_1| \leq l'$, and

(ii) $|\overline{c}_3| - |\overline{c}_2| \leq l''$.

Then, evidently we have $|\overline{c}_3| - |\overline{c}_1| \leq l_2 = l' + l''$.

To prove (i), let us assume the contrary. That is, let us assume that

$$|\overline{c}_2| - |\overline{c}_1| > l' = m - l_1 - 1 + \Delta_M(h_M \overline{A}_M + l_1 + 1) + \overline{A}_M.$$  \hspace{1cm} (1)

By the definition that $\text{Pop}(c_2, |c_2| - |c_1|)$ is true, there exist $\alpha_3 \in \Sigma^*$ and live configurations $c_3$ and $c_4$ such that $c_2 \xrightarrow{\alpha_3} c_3$, $|c_2| = |c_3|$, $\alpha_3 \xrightarrow{e} c_4$, $|c_4| = |c_1|$. 

Let \( \gamma_1 \) and \( \gamma_2 \) be shortest strings such that \( c_2 \uparrow (\gamma_1) c_3 \) and \( c_1 \uparrow (\gamma_2) c_4 \), respectively. (See Fig. 1.) Then, since \( |c_2| = |c_3| \) and \( |c_1| = |c_4| \), by definitions \( |\gamma_1| \leq \bar{A}_M \) and \( |\gamma_2| \leq \bar{A}_M \). Let \( \bar{c}_2 \rightarrow^{\gamma_1} \bar{c}_3 \). Then, since \( \bar{M} \) has no \( \epsilon \)-moves (Note 1),

\[
|\bar{c}_2| - |\bar{c}_3| \leq |\gamma_1| \leq \bar{A}_M. \tag{2}
\]

Let \( \bar{c}_1 \rightarrow^{\gamma_2} \bar{c}_4 \). Then, by the definition of \( h_{\bar{M}} \), we have

\[
|\bar{c}_4'| - |\bar{c}_1| \leq h_{\bar{M}} |\gamma_2| \leq h_{\bar{M}} \bar{A}_M, \tag{3}
\]

since \( \bar{M} \) has no \( \epsilon \)-moves.

Let \( \beta \) be a shortest string in \( L(\bar{c}_4') \) (note that \( L(\bar{c}_4') \neq \emptyset \)), and let \( \beta_1 \) be the shortest prefix of \( \beta \) such that \( \bar{c}_4' \rightarrow^{\beta_1} \bar{c}_5' \) and

\[
|\bar{c}_5'| = |\bar{c}_1| - l_1 - 1. \tag{4}
\]

Note that \( |\bar{c}_4'| > |\bar{c}_1| - l_1 - 1 \) by Lemma 1. Clearly, \( |\beta_1| \leq \Delta_M(|\bar{c}_4'| - |\bar{c}_5'|) \).

By using (3) and (4),

\[
|\beta_1| \leq \Delta_M(h_{\bar{M}} \bar{A}_M + l_1 + 1). \tag{5}
\]

If \( M \) and \( \bar{M} \) are equivalent, then by the definitions, \( L(\bar{c}_4') = L(c_4) \) and \( L(\bar{c}_3) = L(c_3) \).

Since \( L(c_4) = L(c_3) \) by condition \( c_3 \rightarrow^* c_4 \), \( L(\bar{c}_4') = L(\bar{c}_3) \). Hence, there exists a derivation starting from \( \bar{c}_3 \) and reading \( \beta_1 \). Let \( \bar{c}_3 \rightarrow^{\beta_1} \bar{c}_5 \). Then, \( L(\bar{c}_5') = L(\bar{c}_3) \), and this is not empty. Since \( \bar{M} \) has no \( \epsilon \)-moves, we have

\[
|\bar{c}_5| \geq |\bar{c}_3| - |\beta_1| \\
\geq |\bar{c}_2| - \bar{A}_M - |\beta_1| \quad \text{(by (2))} \\
|\bar{c}_1| + m - l_1 - 1 \quad \text{(by (1) and (5))} \\
= |\bar{c}_5'| + m \quad \text{(by (4))}.
\]
By Lemma 1, every intermediate configuration of \( \tilde{e}_1 \rightarrow \gamma \tilde{e}_4 \) has height \( > |\tilde{e}_4'| \) and every intermediate configuration of \( \tilde{e}_1 \rightarrow \omega \tilde{e}_3 \) has height \( > |\tilde{e}_3'| \). By the definition of \( \beta_1 \), every intermediate configuration of \( \tilde{e}_4' \rightarrow \bar{\beta}_1 \tilde{e}_0' \) except for the last configuration \( \tilde{e}_0' \) has height \( > |\tilde{e}_0'| \). Also, every intermediate configuration \( \epsilon_i \rightarrow \tilde{e}_3 \rightarrow \tilde{e}_5 \) has height \( > |\tilde{e}_5'| \), since \( |\epsilon_i| \geq |\tilde{e}_3| - |\beta_1| > |\tilde{e}_5'| + m \) by (6). Therefore, the contents of the pushdown stack of \( \tilde{e}_5' \) are contained at the bottom of the pushdown stack of \( \tilde{e}_0' \). These results show that we have a contradiction to the nonsingularity condition of \( \tilde{M} \). This completes the proof of (i).

The proof of (ii) is similar to that of (i) described above. ((ii) corresponds to (i) in which \( |\epsilon_1| = |\epsilon_2| \) and \( \epsilon_1 = \epsilon_3 = \epsilon_4 \). So the proof is omitted. Q.E.D.

4. SIMULATION OF \( M \) AND \( \tilde{M} \)

We construct a single stack machine \( M' \) for simulating two dpda's \( M \in D_0 \) and \( \tilde{M} \in N_0 \) together. \( M' \) has stack alphabet \( \Gamma \cup [\Gamma, ] \) (\( \Gamma \) and \( [ \) are new symbols) and state set \( Q \times \bar{Q} \). A typical configuration of \( M' \) is described by

\[ c' = ([s, s], w_0[w_0]\cdots w_n[w_n]), \]

where

1. \( w_i \in \Gamma^* \) and \( w_i \neq \Lambda \) for each \( i \) (\( 0 \leq i \leq n \)), and

2. \( w_i \in \Gamma^* \) for each \( i \) (\( 0 \leq i \leq n \)) and \( w_n \neq \Lambda \). (If \( n = 0 \), then \( w_n \) may be null.)

The configurations of \( M \) and \( \tilde{M} \) at that time are \( c = (s, w_0w_1\cdots w_n) \) and \( \tilde{c} = (s, \bar{w}_0\bar{w}_1\cdots \bar{w}_n) \), respectively.

This single stack machine \( M' \) is analogous to Valiant's alternating stack machine in [3]. However, our machine \( M' \) has the following differences from Valiant's alternate stacking.

1. In Valiant's machine, which simulates two dpda's \( M \) and \( \tilde{M} \) in \( N_0 \), there is symmetry, that is, \( w_0, \bar{w}_0, \ldots, w_{n-1}, \bar{w}_{n-1} \) and \( w_n \) are not null and only \( \bar{w}_n \) may be null. In our machine, \( \bar{w}_0, \ldots, \bar{w}_{n-1} \) may be null but \( \bar{w}_n \) is not null (for \( n = 0 \), \( \bar{w}_n = \bar{w}_0 \) may be null), whereas \( w_0, \ldots, w_n \) are not null.

2. Each of stack segments \( w_0, \bar{w}_0, \ldots, w_n, \bar{w}_n \) produced by Valiant's simulating machine for live inputs is of bounded size. In our machine, each \( \bar{w}_i \) (\( 0 \leq i \leq n \)) is of bounded size, but the size of each \( w_i \) (\( 0 \leq i \leq n \)) is not bounded.

3. In our machine, the segmentation of the stack is determined by the conditions on \( M \) as explained below.

Let

\[ \epsilon_1' = ([s_1, \bar{s}_1], w_0[w_0]\cdots w_{n-1}[\bar{w}_{n-1}] w_n[\bar{w}_n]), \]

\[ w_n = \bar{\xi}A, A \in \Gamma, \]

\[ \bar{w}_n = \bar{\xi}A, A \in \Gamma, \]
be a configuration of $M'$. We have to define the next transition $c_1' \to^\sigma c_2'$, where $\sigma \in \Sigma \cup \{\varepsilon\}$. For convenience, we consider the intermediate configuration $c'$ of the form

$$c' = (s', w_0[\bar{w}_0] \cdots w_{n-1}[\bar{w}_{n-1}] w_n'[\bar{w}_n']) .$$

Here, $w_n'$ and/or $\bar{w}_n'$ may be the null string $\Lambda$. The transition $c_1' \to^\sigma c_2'$ is performed by two transitions $c_1' \to^\sigma c'$ and $c' \to^\sigma c_2'$.

The transition $c_1' \to^\sigma c'$ is defined as follows.

(I) If $(s_1, \Lambda) \to^\sigma (s_2, \eta)$ is a rule of $M$, then let $\pi = \varepsilon$, $s' = [s_2, \bar{s}_1]$, $w_n' = \xi\eta$, and $\bar{w}_n' = \bar{w}_n$.

(II) If for some $a \in \Sigma$, $(s_1, \Lambda) \to^a (s_2, \eta)$ and $(s_1, \bar{A}) \to^a (s_2, \bar{\eta})$ are rules of $M$ and $\bar{M}$, respectively, then let $\pi = a$, $s' = [s_2, \bar{s}_2]$, $w_n' = \xi\eta$ and $\bar{w}_n' = \bar{\xi}\bar{\eta}$.

The configuration $c_2'$ is defined as follows.

(a) The case where $w_n' \neq \Lambda$. Let $c_2 = (s_2, w_0 \cdots w_{n-1}w_n')$ be a configuration of $M$ in $c'$.

(a.1) If Pop$(c_2, |w_n'| - 1)$ is true, then let $c_2' = c'$. (The transition $c' \to^\sigma c_2'$ is not needed.)

(a.2) If not, then let $c_2'$ be $c'$ replacing $w_n'[\bar{w}_n']$ by $\xi'[\bar{\xi}_2] B[\bar{\xi}_2]$, where $\xi' B = w_n'$, $B \in \Gamma'$, $\bar{\xi}_1 \bar{\xi}_2 = \bar{w}_n'$ and

$$|\bar{\xi}_2| = l_1 + 1, \quad \text{if } |w_n'| > l_1 + 1, \quad \bar{\xi}_1 = \Lambda \text{ and } \bar{\xi}_2 = \bar{w}_n', \quad \text{otherwise.}$$

Here, $l_1$ is the constant shown in Lemma 1. Note that $\xi' \neq \Lambda$.

(b) The case where $w_n' = \Lambda$. Let $c_2 = (s_2, w_0 \cdots w_{n-1})$ be a configuration of $M$ in $c'$.

(b.1) If Pop$(c_2, |w_n|-1)$ is true, then let $c_2'$ be $c'$ replacing $w_{n-1}[\bar{w}_{n-1}]w_n'[\bar{w}_n']$ by $w_{n-1}[\bar{w}_{n-1}\bar{w}_n']$.

(b.2) If not, then let $c_2'$ be $c'$ replacing $w_{n-1}[\bar{w}_{n-1}]w_n'[\bar{w}_n']$ by $\xi'[\bar{\xi}_2] B[\bar{\xi}_2]$, where $\xi' B = w_{n-1}$, $B \in \Gamma'$, $\bar{\xi}_1 \bar{\xi}_2 = \bar{w}_{n-1}\bar{w}_n'$ and

$$|\bar{\xi}_2| = l_1 + 1, \quad \text{if } |\bar{w}_{n-1}\bar{w}_n'| > l_1 + 1, \quad \bar{\xi}_1 = \Lambda \text{ and } \bar{\xi}_2 = \bar{w}_{n-1}\bar{w}_n', \quad \text{otherwise.}$$

Here, $l_1$ is the constant shown in Lemma 1. Note that $\xi' \neq \Lambda$.

Note 2. $M'$ can be modified by a standard technique to recognize whether Pop$(c_2, |w_n| - 1)$ (or Pop$(c_2, |w_{n-1}| - 1)$ in case (b)) is true or not. The idea is that $M'$ can maintain two subsets of states of $M$ added to each pushdown symbol of $M$ with the following properties.
Let \((A_{i0}, Q_{i0}, \Pi_{i0})(A_{i1}, Q_{i1}, \Pi_{i1}) \cdots (A_{il}, Q_{il}, \Pi_{il})\) be the \(i\)th segment of the pushdown stack of \(M\), where \(A_{i0}A_{i1} \cdots A_{il} = \omega_i\). Then, for each \(j\) (\(0 \leq j \leq l\)), \(Q_{ij} = \{s | (s, \omega_0, \omega_{i-1}A_{i0} \cdots A_{ij})\) is a live configuration of \(M\), and for each \(j\) (\(1 \leq j \leq l\)), \(Q'_{ij} = \{s' | \text{for some } s \in Q_{ij}, (s', A_{i1} \cdots A_{ij}) \rightarrow^* (s, \Omega) \text{ in } M\} \).

Note 3. Let
\[
([s, x], \omega_0[\omega_0] \cdots \omega_{n-1}[\omega_{n-1}]\omega_n[\omega_n])
\]
be a configuration of \(M'\) (corresponding to \(c_1'\) or \(c_2'\), but not to \(c'\) in the construction described above), and let
\[
c = (s, \omega_0 \cdots \omega_{n-1}\omega_n)
\]
be a configuration of \(M\) at that time. Then, \(\text{Pop}(c, [\omega_n] = 1)\) is true. (Note that \(\omega_n \neq \Lambda\).) This follows from the assumption \(h_M \leq 1\) and the construction of \(M'\) described above.

By the following theorem we can ensure that the simulating machine is itself a pushdown automaton. This theorem corresponds to [3, Theorem 3.2].

**Theorem 1.** Assume that \(M \in D_0\) and \(\overline{M} \in N_0\) are equivalent. For any live input \(\alpha\), \(M'\) constructed as described above produces the stack \(\omega_0[\omega_0] \cdots \omega_n[\omega_n]\) which satisfies the following conditions.

(i) \(\omega_i \neq \Lambda\) for each \(i\) (\(0 \leq i \leq n\)).

(ii) \(\omega_n \neq \Lambda\).

(iii) For some \(l \geq 0\), \(|\omega_i| \leq l\) for each \(i\) (\(0 \leq i \leq n\)).

**Proof.** (i) Evidently by constructions of \(M'\).

(ii) Let us assume the contrary. That is, let us assume that, for some live input \(\alpha\), there is a derivation \(c_\alpha'\) (the starting configuration of \(M'\)) \(\rightarrow^* c_2'\) where
\[
c_2' = ([s_2, \xi_2], \omega_0[\omega_0] \cdots \omega_n[\omega_n])
\]
and \(\overline{\omega_n} = \Lambda\). Assume that the topmost segment of pushdown stack of \(\overline{M}\) in any intermediate configuration of \(c_\alpha' \rightarrow^* c_2'\) is not null. Let \(c_2\) and \(\overline{c}_2\) be the configurations of \(M\) and \(\overline{M}\) in \(c_\alpha'\), respectively.

Let \(\overline{\omega}_i\) be the segment of pushdown stack of \(\overline{M}\) in \(c_\alpha'\) such that \(\overline{\omega}_i \neq \Lambda\) and \(\overline{\omega}_j = \Lambda\) for each \(j\) (\(i < j \leq n\)). Since \(\alpha \notin L(M)\), there does exist such \(\overline{\omega}_i\) (\(i \geq 0\)). Consider the intermediate configuration \(c_1'\) of \(\alpha \rightarrow^* c_\alpha'\) at the moment when \(\overline{\omega}_i\) actually appeared. \(c_1'\) has the form
\[
c_1' = ([s_1, \xi_2], \omega_0[\omega_0] \cdots \omega_j[\omega_j] \omega_{j+1}[\omega_{j+1}]).
\]
Since \(\overline{\omega}_i \neq \Lambda\) (\(i \geq 0\)), \(|\omega_{i+1}| = l_1 + 1\) by the construction of \(M'\). (In cases (a.2) and (b.2) of the construction of \(M'\), \(\xi_1 \neq \Lambda\) implies \(|\xi_2| = l_1 + 1\).) Let \(c_1\) and \(\overline{c}_1\) be the
configurations of \( M \) and \( \bar{M} \) in \( c_1' \), respectively. Since \(| \bar{c}_1 | = | \bar{w}_0 \cdots \bar{w}_i \bar{w}_{i+1}^0 | \) and \(| \bar{c}_2 | = | \bar{w}_0 \cdots \bar{w}_n | = | \bar{w}_0 \cdots \bar{w}_i | \), we have

\[
| \bar{c}_1 | - | \bar{c}_2 | = | \bar{w}_{i+1}^0 | = l_1 + 1.
\]

Let \( \beta \) be the input which has taken \( M' \) from \( c_1' \) to the present configuration \( c_2' \). Then, \( c_1 \uparrow (\beta) c_2 \) in \( M \) (note that \(| \bar{w}_{i+1}^0 | = 1 \) by the construction of \( M' \)) and \( \bar{c}_1 \rightarrow^\beta \bar{c}_2 \) in \( \bar{M} \). By the assumption, \( \alpha \) is live. Therefore, by Lemma 1, \(| \bar{c}_1 | - | \bar{c}_2 | \leq l_1 \). This contradicts (7).

(iii) Let \( l' \) be the larger of \( l_2 + l_1 + 1 \) and \( l_2 + h \), where \( l_1 \) and \( l_2 \) are the constants shown in Lemmas 1 and 2, respectively. We shall show that \(| \bar{w}_i | \leq l' \) for each \( i \) (0 \( \leq i \leq n \)).

Let us assume the contrary. That is, let us assume that, for some live input \( \alpha \), there is a derivation \( c_2' \rightarrow^\alpha c_2 \) such that some segment \( \bar{w}_i \) in \( c_2' \) has length > \( l' \). Assume that any segment \( \bar{w}_j \) of the pushdown stack of \( \bar{M} \) in any intermediate configuration of \( c_2' \rightarrow^\alpha c_2 \) has length \( \leq l' \). Let \( c_2 \) and \( \bar{c}_3 \) be the configurations of \( M \) and \( \bar{M} \) in \( c_2' \), respectively.

(A) The case where \( c_2' \) is defined by (a.1). Let

\[ w_0[\bar{w}_0] \cdots w_{n-1}[\bar{w}_{n-1}] w_n[\bar{w}_n] \]

be the tape of \( c_2' \). Then, by the assumption that some segment \( \bar{w}_i \) of length > \( l' \) has appeared in \( c_2' \) for the first time and by the construction in (a.1), such segment \( \bar{w}_i \) of length > \( l' \) must be \( \bar{w}_n = \bar{w}_n' \). Consider the configuration \( c_1' \) (let \( c_1 \) and \( c_1 \) be the configurations of \( M \) and \( \bar{M} \), respectively, at that time) such that \( c_2' \rightarrow^\alpha c_1', c_1' \rightarrow^\alpha c_2' \), where \( \alpha = \alpha_1 \alpha_2 \), and \( \bar{c}_1' \) has the form

\[
([s_0, 3_1], w_0[\bar{w}_0] \cdots w_{n-1}[\bar{w}_{n-1}] w_n[\bar{w}_n]),
\]

where \(| w_0 \alpha_1 | = 1 \) and \(| w_{n-1} \alpha_2 | \leq l_1 + 1 \), and the pushdown head of \( M \) does not see the symbols in \( w_{n-1} \) in the course of \( c_1 \rightarrow^\alpha c_2 \). (Note that such a configuration \( c_1 \) does exist. \( \bar{w}_0 \alpha_1 \) and \( \bar{w}_{n-1} \alpha_2 \) correspond to \( B \) and \( \bar{c}_2 \), respectively, in (a.2) or (b.2) in the construction of \( M' \) or the initial pushdown stacks of \( M \) and \( \bar{M} \), respectively.) By \(| w_0 \alpha_1 | = 1 \) and the conditions on \( c_1 \rightarrow^\alpha c_2 \), \( c_1 \uparrow (\alpha_2) c_2 \) in \( M \). Since \( c_2' \) is defined by (a.1), \( \text{Pop}(c_2, | w_n | - 1) \) is true. Here, \(| c_0 | - | c_1 | = | w_n | - | w_0 \alpha_1 | = | w_n | - 1 \), and \( \alpha_2 \alpha_2 \) is a live input. Therefore, from Lemma 2 (let \( \alpha_4 \alpha_1 = \epsilon \) and \( \bar{c}_2 = \bar{c}_3 \) in Lemma 2), it follows that

\[
| \bar{c}_2 | - | \bar{c}_1 | \leq l_2.
\]

On the other hand,

\[
| \bar{c}_2 | - | \bar{c}_1 | = | w_n | - | \bar{w}_n \alpha_1 | > l' - (l_1 + 1) \geq l_2.
\]

This contradicts (8).
(B) The case where $c_3'$ is defined by (a.2). Let $c_3' \rightarrow^\sigma c'$, $c' \rightarrow^\tau c_2'$ be the last two transitions in the derivation $c_3' \rightarrow^\sigma c_2'$. The transition $c_3' \rightarrow^\sigma c'$ is defined by (I) or (II) and the transition $c' \rightarrow^\tau c_2'$ is defined by (a.2). Let

$$c_3' = ([s_0, s_0], w_0[\bar{w}_0] \cdots w_{n-1}[\bar{w}_{n-1}] w_n[\bar{w}_n])$$

(let $c_3$ and $\bar{c}_3$ be the configurations of $M$ and $\bar{M}$ in $c_3'$, respectively), and let

$$w_0[\bar{w}_0] \cdots w_{n-1}[\bar{w}_{n-1}] w_n[\bar{w}_n]$$

be the tape of $c'$. Then, it must hold that

$$|\bar{w}_n'| > l' + l_1 + 1.$$ 

(If $|\bar{w}_n'| \leqslant l' + l_1 + 1$, then by the construction in (a.2), $|\bar{e}_1| \leqslant l'$ and $|\bar{e}_2| = l_1 + 1 \leqslant l'$, and therefore any segment $\bar{w}_2$ such that $|\bar{w}_2| > l'$ does not appear in $c_3'$.)

As in (A), consider the configuration $c_0'$ (let $c_0$ and $\bar{c}_0$ be the configurations of $M$ and $\bar{M}$, respectively, at that time) such that

$$c_0 \rightarrow^\tau c_0', c_0' \rightarrow^\tau c_3',$$

where $|\bar{w}_n| = 1$ and $|\bar{w}_n'| \leqslant l_1 + 1$, and the pushdown head of $M$ does not see the symbols in $w_{n-1}$ in the course of $c_0 \rightarrow^\tau c_3$. By $|\bar{w}_n| = 1$ and the conditions on $c_0 \rightarrow^\tau c_3$, $c_0 \uparrow (\alpha_2) c_3$ in $M$. By Note 3, $\text{Pop}(s_3, |w_n| - 1)$ is true ($\bar{w}_n$ is the topmost stack segment of $c_3$). Here, $|c_3| - |c_0| = |w_n| - |w_n' - 1| = |w_n| - 1$, and $\alpha_3\alpha_2$ is a live input. Therefore, from Lemma 2 (let $\alpha_3' = \varepsilon$ in Lemma 2), it follows that

$$|\bar{e}_3| = |\bar{e}_0| \leqslant h_{\bar{M}}.$$ 

Clearly, $|\bar{e}_2| \leqslant |\bar{e}_3| + h_{\bar{M}}$. Thus,

$$|\bar{e}_2| = |\bar{e}_0| \leqslant l_2 + h_{\bar{M}}. \quad (9)$$

On the other hand,

$$|\bar{e}_2| = |\bar{e}_0| = |\bar{w}_n'| - |\bar{w}_n|$$

$$> (l' + l_1 + 1) - (l_1 + 1) = l'.$$

Since $l' > l_2 + h_{\bar{M}}$, this contradicts (9).

(C) The case where $c_3'$ is defined by (b.1). The proof is similar to that of case (A) described above. In this case, we have to consider $\alpha_3'$ in Lemma 2. The details are omitted.

(D) The case where $c_3'$ is defined by (b.2). The proof is similar to that of case (B) described above. In this case, too, we have to consider $\alpha_3'$ in Lemma 2. The details are omitted. Q.E.D.
5. Conclusion

Since the previous section has established that for $M \in D_0$ and $\bar{M} \in N_0$, we can construct a simulating machine of $M$ and $\bar{M}$ which satisfies the conditions in Theorem 1, we can deduce our main result.

**Theorem 2.** It is decidable for $M \in D_0$ and $\bar{M} \in N_0$ whether they are equivalent or not.

**Proof.** The proof is essentially the same as that of [3, Theorem 3.3]. First suppose that we know the bound $l$ shown in Theorem 1. Then we can effectively construct a (nondeterministic) pda $M^*$ with the property that $L(M^*) = \emptyset$ iff $M$ and $\bar{M}$ are equivalent. $M^*$ mimics the simulating machine $M'$ for $M$ and $\bar{M}$ by encoding the top stack segment $[w_n]$ in its finite state control. As long as $\bar{w}_n'$ in (a.1), $\bar{\xi}_1$ in (a.2), $\bar{w}_{n-1}\bar{w}_n'$ in (b.1), and $\bar{\xi}_1$ in (b.2) in the construction of $M'$ never get larger than the given bound $l$ (if so, any stack segment $\bar{w}_i$ has length $\leq l$), $M^*$ simulates $M$ and $\bar{M}$, and $M^*$ accepts the input iff exactly one of $M$ and $\bar{M}$ accepts the input. In the case where a transition is defined for one of $M$ and $\bar{M}$, but no transition is defined for the other, $M^*$ continues the simulation of the first pda and accepts it does. When the length of $\bar{w}_n'$, $\bar{w}_{n-1}\bar{w}_n'$, or $\bar{\xi}_1$ mentioned above exceeds the bound $l$, $M^*$ proceeds nondeterministically to mimic one of $M$ or $\bar{M}$, and accepts if the appropriate machine does.

Assume that $M$ and $\bar{M}$ are equivalent. Then, (i) $\bar{w}_n'$, $\bar{w}_{n-1}\bar{w}_n'$, and $\bar{\xi}_1$ mentioned above will have length $\leq l$ for live inputs; (ii) the bound $l$ will only be exceeded once nothing more can be accepted by $M$ and $\bar{M}$; and (iii) if no transition is defined for one of $M$ and $\bar{M}$, then nothing more can be accepted by the other pda. Thus, $L(M^*) = \emptyset$ by construction. Conversely, if $L(M^*) = \emptyset$, then clearly no input can produce different behavior in $M$ and $\bar{M}$, which are therefore equivalent.

It is decidable whether $L(M^*) = \emptyset$ or not [1]. Thus, if we have an a priori bound, we can test equivalence by constructing this pda and testing it for emptiness. However, even if we do not know this bound, by enumerating and testing for emptiness the possible candidate machines, we can obtain a partial decision procedure. That is, we construct pda of the form $M^*$ for assumed bounds of 1, 2, ... successively. If $M$ and $\bar{M}$ are not equivalent, then none of these constructed machines can be empty, while if they are, then one of them must be. We therefore have partial decidability of equivalence of $M$ and $\bar{M}$. On the other hand, the inequivalence of $M$ and $\bar{M}$ is partially decidable. Hence, we have the decidability of equivalence. Q.E.D.

**Corollary.** It is decidable for $M \in D$ and $\bar{M} \in N_0$ whether they are equivalent or not.

**Proof.** For $M \in D$ and $\bar{M} \in N_0$, we can easily construct dpda's $M' \in D_0$ and $\bar{M}' \in N_0$ such that $L(M') = L(M)\#$ and $L(\bar{M}') = L(\bar{M})\#$, respectively, where $\#$ is an end-marker ($\#$ is a symbol not in the input alphabet of $M$ and $\bar{M}$). Then, $M$ and $\bar{M}$ are
equivalent iff $M'$ and $\bar{M}'$ are equivalent. By Theorem 2, the equivalence of $M'$ and $\bar{M}'$ is decidable. Hence, the equivalence of $M$ and $\bar{M}$ is decidable. Q.E.D.

APPENDIX: Proof of Lemma 1 [3]

Let $l_1 = m + \bar{A}_M + \bar{A}_{\bar{M}}$. Here $m$ is the nonsingularity constant of $\bar{M}$. We shall show that $|\hat{e}_1| - |\hat{e}_2| \leq l_1 = m + \bar{A}_M + \bar{A}_{\bar{M}}$.

Let us assume that

$$|\hat{e}_1| - |\hat{e}_2| > m + \bar{A}_M + \bar{A}_{\bar{M}}.$$ (A1)

Without loss of generality, we can assume that every intermediate configuration of $\hat{e}_1 \rightarrow^{*z} \hat{e}_2$ except for the last configuration $\hat{e}_2$ has height $> |\hat{e}_2|$.

Let $\beta$ be a shortest string in $L(e_2)$ ($L(e_2) \neq \emptyset$) and let $\beta_1$ be the shortest prefix of $\beta$ such that $e_2 \rightarrow^{\beta_1} e_3$ and $|e_3| = |e_1|$. (See Fig. 2.) Also, $\beta$ is a string in $L(\bar{e}_2)$. Then $\beta$ is the concatenation of segments induced by the popping derivation in $\bar{M}$, each one taking some $(\hat{s}_1, Z)$ to some $(\hat{s}_2, A)$. Since $\beta$ is shortest in $L(\bar{e}_2)$, each such segment is of length no more than $\bar{A}_{\bar{M}}$. Thus $\beta_1$ consists of a sequence of such segments terminated possibly by a proper prefix of another such segment. Let $\beta_2$ be the completion of this last segment. Let $\bar{e}_2 \rightarrow^{\beta_1} \bar{e}_3$, $\bar{e}_3 \rightarrow^{\beta_2} \bar{e}_4$ and $\bar{e}_3 \rightarrow^{\beta_2} \bar{e}_4$. Clearly

$$|\bar{e}_2| - |\bar{e}_4| \geq |\beta_1|/\bar{A}_{\bar{M}}$$ (A2)

and every intermediate configuration of $\bar{e}_2 \rightarrow^{\beta_1, \beta_2} \bar{e}_4$ except for the last configuration $\bar{e}_4$ has height $> |\bar{e}_4|$.

![Fig. 2. Derivations in the proof of Lemma 1.](image)
Now consider a shortest string $\gamma$ taking $M$ from $c_1$ to $c_4$. The definitions of $c_3$ and $c_4$ ensure that $|\gamma| \leqslant \Delta_M + |\beta_2| \leqslant \Delta_M + \Delta_M$. Let $\bar{c}_1 \rightarrow^* \bar{c}_4'$. Then, since $M$ has no $\epsilon$-moves (Note 1),

$$|\bar{c}_1| - |\bar{c}_4'| \leqslant |\gamma| \leqslant \Delta_M + \Delta_M \quad \text{(A3)}$$

and also for every intermediate configuration $\bar{c}_i$ of $\bar{c}_1 \rightarrow^* \bar{c}_4'$, $|\bar{c}_1| - |\bar{c}_i| \leqslant |\gamma| \leqslant \Delta_M + \Delta_M$. From (A2) and (A3), it follows that

$$|\bar{c}_4'| - |\bar{c}_4| \geqslant |\bar{c}_1| - |\bar{c}_2| + |\beta_1|/\Delta_M - (\Delta_M + \Delta_M).$$

By assumption (A1) and the fact $|\beta_1|/\Delta_M \geqslant 0$, we have

$$|\bar{c}_4'| - |\bar{c}_4| \geqslant m.$$ 

By the definitions of $\bar{c}_4$ and $\bar{c}_4'$, $L(\bar{c}_4) = L(\bar{c}_4') \neq \emptyset$ and the contents of the pushdown stack of $\bar{c}_4$ are contained at the bottom of the pushdown stack of $\bar{c}_4'$. These results show that we have a contradiction to the nonsingularity condition of $M$.

**References**