Permutable subgroups of a direct product

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Abstract

A subgroup $S$ of a group $G$ is a permutable subgroup of $G$ if for all subgroups $X$ of $G$, $SX = XS$. In this article, we establish necessary and sufficient conditions for a subgroup of the finite group $G \times H$ to be permutable. Then, we attempt to improve this theorem by making conjectures that would simplify these conditions. Counterexamples to these conjectures are presented, demonstrating that in some way, the aforementioned characterization is the best one possible. We conclude by showing how our conjectures do provide further insight into permutability in some special cases.

Keywords: Permutable subgroup; Direct product

1. Introduction

When $M$ and $S$ are two subgroups of a group $G$, $MS$ is also a subgroup of $G$ if and only if $MS = SM$. In such a case, we say that $M$ and $S$ permute. Furthermore, $M$ is a permutable subgroup of $G$, or $M$ is permutable in $G$, if $M$ permutes with every subgroup of $G$. Permutable subgroups were first studied by Ore [9], who called them quasinormal, in 1939. While it is clear that a normal subgroup is permutable, Ore proved that a permutable subgroup of a finite group is subnormal.

This article considers subgroup permutability in a direct product. A well-known characterization of normal subgroups of a direct product states that $N$ is normal in $G \times H$ if and only if $\pi_G(N)/(N \cap G) \leq Z(G/(N \cap G))$ and $\pi_H(N)/(N \cap H) \leq Z(H/(N \cap H))$ where $\pi_G$ and $\pi_H$ are the natural projections of $G \times H$ onto $G$ and $H$, respectively. Additionally, the work of P. Hauck [6] relates subnormal subgroups of direct products.
to the Fitting subgroups of the direct factors. Since with respect to set containment, the set of permutable subgroups is between the set of subnormal and set of normal subgroups in a finite group, it seems interesting to study the permutable subgroups of $G \times H$.

Our previous work has led to characterizations of certain types of permutable subgroups of $G \times H$. In [3], we suppose that $A \leq G$ and $B \leq H$, and provide necessary and sufficient conditions for $A \times B$ to be permutable in $G \times H$. We then write about permutable diagonal-type subgroups in [4]. A subgroup $D$ of $G \times H$ is a diagonal-type subgroup if $D \cap G$ and $D \cap H$ are both trivial. We prove that a diagonal-type subgroup is permutable if and only if it is contained in the norm of $G \times H$. First studied by R. Baer [2], the norm of a group $G$, denoted by $N(G)$, is $\{g \in G \mid \text{for all } X \leq G, g \in N_G(X)\}$. Because permutable subgroups behave well under correspondence, this result characterizes a permutable subgroup of $G \times H$ whose intersections with the direct factors are normal.

Ultimately, our goal has been to provide necessary and sufficient conditions for an arbitrary subgroup of $G \times H$ to be permutable. A famous result of R. Maier and P. Schmid [8] allows us to understand finite groups by limiting ourselves to the case that $G \times H$ is a $p$-group. In Section 4 of this article, we are able to apply results of [3,4] to obtain such a characterization. This is done in Theorem 4.2, which is stated here.

**Theorem.** Let $M$ be a subgroup of the finite $p$-group $G \times H$. Without loss of generality, assume that $\exp(G/\text{Core}_G(M \cap G)) \geq \exp(H/\text{Core}_H(M \cap H))$. Then, $M$ is a permutable subgroup of $G \times H$ if and only if for all $(g,h) \in G \times H$, one of the following two statements is true:

1. $M \leq N_{G \times H}(((M \cap G) \times (M \cap H))(\langle g, h \rangle))$, or
2. $\left|\langle g \rangle/(\langle g \rangle \cap M)\right| > \exp(H/\text{Core}_H(M \cap H))$, and there is a nonnegative integer $i$ so that $h^i \in \pi_H((\langle g \rangle \times \langle h \rangle) \cap M)$ and

$$M \leq N_{G \times H}(((M \cap G) \times (M \cap H))(\langle 1, h^i \rangle)(\langle g, h \rangle)).$$

One might hope that it is possible to simplify this characterization by modifying Conditions (1) and (2). In Section 5, we study this by considering two possible modifications, which are presented in Conjectures 5.1 and 5.2. While both of these conditions may seem more natural, counterexamples to their validity are also presented in Section 5. All of this shows that in some way, Theorem 4.2 is the best possible characterization of permutable subgroups of a finite direct product.

In a couple of special cases, it is still possible to use Conjecture 5.1 to characterize permutable subgroups. We show in Theorems 5.6 and 5.7 that Conjecture 5.1 is true for subgroups of $G \times H$ whose projections onto the direct factors are core-free or cyclic.

Furthermore, as part of this conjecture, we speculate that the permutability of a subgroup in $G \times H$ implies that the direct product of the intersections of that subgroup with the direct factors is also permutable in $G \times H$. While Example 5.4 shows that this is false in general, in Theorem 5.9, we prove that this is indeed true when we stipulate that $G$ and $H$ are groups of odd order with modular subgroup lattices. This result is then especially germane to this article since it demonstrates that we may not use modular groups of odd order for both direct factors when constructing Example 5.4.
2. Notation

The functions $\pi_G$ and $\pi_H$ are the natural projections of $G \times H$ onto $G$ and $H$, respectively. We write functions on the left.

We use $\mathbb{N}$ to represent the natural numbers. If $m$ and $n$ are integers, then $(m, n)$ is the greatest common divisor of $m$ and $n$.

We denote the norm of a group by $N(G)$. The hypercenter of $G$ is represented by $Z_\infty(G)$. We will see $\exp(G)$ to stand for the exponent of $G$. Group theoretic notations that are not explained here are consistent with those used in [10].

3. Basic results

The following results are applied in this article. Sources are provided for many, and a proof is only given for 3.8.

3.1. Let $M$ and $N$ be subgroups of $G$ such that $N \subseteq M$ and $N \trianglelefteq G$. Then, $M$ is a permutable subgroup of $G$ if and only if $M/N$ is a permutable subgroup of $G/N$.

3.2. If $M$ is a permutable subgroup of $G$, and $S$ is a subgroup of $G$, then $M \cap S$ is a permutable subgroup of $S$.

3.3. Suppose that $M$ is a core-free permutable subgroup of the finite $p$-group $G$, and that $G = MX$ where $X$ is cyclic. Then,

(a) $M \cap X = 1$, and
(b) $\exp(G) = \exp(X)$.

Any of results 3.1–3.3 can be found in [12]. Both 3.1 and 3.2 are on page 202, and while 3.3 was proved by Gross [5], we found it in [12] as part of Theorem 5.2.8.

3.4. Let $G$ and $H$ be finite groups. Suppose that $M$ is a subgroup of $G \times H$. Then, $\langle (g, h) \rangle$ permutes with $M$ in $G \times H$ if and only if $g \in N_G(\langle g^{\alpha(h)} \rangle M)$.

3.5. Let $M$ be a subgroup of $G \times H$ such that both $M \cap G \trianglelefteq G$ and $M \cap H \trianglelefteq H$. Then, $M$ is a permutable subgroup of $G \times H$ if and only if $M/((M \cap G) \times (M \cap H))$ is contained in $N((G \times H)/((M \cap G) \times (M \cap H)))$.

3.6. Let $G$ and $H$ be any two groups. If $M$ is a permutable subgroup of $G \times H$, then for all $(g, h) \in G \times H$, both $\pi_G(M) \subseteq N_G(\langle g \rangle (M \cap G))$ and $\pi_H(M) \subseteq N_H(\langle h \rangle (M \cap H))$.

Result 3.4 is Corollary 5.2 in [3], while both 3.5 and 3.6 come from [4]. Corollaries 5.5 and 5.6 can be combined to form 3.5, and 3.6 is Lemma 5.1.

3.7. Let $G$ and $H$ be finite groups. If $M$ is a permutable subgroup of $G \times H$, then $M/((\text{Core}_G(M \cap G) \times \text{Core}_H(M \cap H))$ is contained in $Z_\infty((G \times H)/((\text{Core}_G(M \cap G) \times \text{Core}_H(M \cap H))))$. 

To prove 3.7, first use the Maier–Schmid Theorem [8] to show that \( ((M \cap G) \times (M \cap H))/\text{Core}_G(M \cap G) \times \text{Core}_H(M \cap H) \) is in \( Z_{\infty}((G \times H)/(\text{Core}_G(M \cap G) \times \text{Core}_H(M \cap H))) \). To conclude, apply 3.6 and a result of Schenkman [11] stating that \( N(G) \leq Z_2(G) \).

3.8. If \( p \) is an odd prime, and \( a \in \mathbb{N} \), then \( p^2 \) does not divide \( 1 + a + a^2 + \cdots + a^{p-1} \).

**Proof.** The proof is by contradiction. Notice that \( a \mod p = 1 \), and then apply the binomial theorem to obtain the contradiction. \( \square \)

4. A characterization of permutable subgroups of a direct product

In this section, we prove Theorem 4.2, which provides necessary and sufficient conditions for a subgroup of the finite \( p \)-group \( G \times H \) to be permutable. The reader should carefully consider 3.7, which demonstrates that this result can in fact be applied to characterize a permutable subgroup of any finite group. We begin here with Lemma 4.1, where it is shown that if \( G \times H \) is a finite \( p \)-group and \( M \) is a permutable subgroup of \( G \times H \), then at least one of the intersections of \( M \) with the direct factors is normal.

**Lemma 4.1.** Let \( G \times H \) be a finite \( p \)-group. If \( M \) is a permutable subgroup of \( G \times H \) and \( \exp(G/\text{Core}_G(M \cap G)) \leq \exp(H/\text{Core}_H(M \cap H)) \), then \( M \cap G \triangleleft G \).

**Proof.** Let \( G \times H \) be a counterexample of minimal order, and let \( M \) be a subgroup of \( G \times H \) for which the result fails. As a consequence of 3.1 and the minimality of \( |G \times H| \), we have that \( \text{Core}_G(M \cap G) \) and \( \text{Core}_H(M \cap H) \) are both trivial. By 3.2 and the minimality of \( |G \times H| \), one sees that \( G = (M \cap G)/(g) \) where \( g \) does not normalize \( \text{Core}_G(M \cap G) \). Furthermore, it follows from the fact that \( \text{Core}_H(M \cap H) = 1 \), together with 3.2 and the minimality of \( |G \times H| \), that \( H = \langle h \rangle \) where \( o(h) = \exp(G) \).

As a result of 3.3, \( (g) \cap (M \cap G) = 1 \), and \( o(g) = o(h) \). Since \( M \cap H = 1 \) and \( (M \cap G) \cap (g) = 1 \), it follows from Goursat’s description of the subgroup lattice of a direct product (see [1, p. 25] for a nice outline of this) that \( M = \{(g^{ap}, h^{bp})\} \cap (M \cap G) \) for some \( a, \ i \in \mathbb{N} \) such that \( (i, p) = 1 \). Of course, there exists an automorphism \( \phi \) of \( H \) such that \( \phi(h^i) = h \). Thus, we may assume that \( M = \{(g^{ap}, h^{bp})\} \cap (M \cap G) \).

Let \( x \in M \cap G \). Since \( M \) is permutable in \( G \times H \),

\[
(x, 1)(g, h) = (g^{j}, h^j)(g^{kp^a}, h^{kp^b})(y, 1)
\]

for some \( j, k \in \mathbb{N} \) and \( y \in M \cap G \). Hence, \( M \cap G \) permutes with \( \langle (g, h) \rangle \). Therefore, it follows from 3.4 that \( g \) normalizes \( M \cap G \), which is a contradiction. \( \square \)

**Theorem 4.2.** Let \( M \) be a subgroup of the finite \( p \)-group \( G \times H \). Without loss of generality, assume that \( \exp(G/\text{Core}_G(M \cap G)) \geq \exp(H/\text{Core}_H(M \cap H)) \). Then, \( M \) is a permutable subgroup of \( G \times H \) if and only if for all \( (g, h) \in G \times H \), one of the following two statements is true:
Conjecture 5.1. Let $M \subseteq NG_{xH}(((M \cap G) \times (M \cap H))(g, h))$, or $\langle (g)/(g \cap M) \rangle > \exp(H/\text{Core}_H(M \cap H))$, and there is a nonnegative integer $i$ so that $h^{p^i} \in \pi_H(\langle (g) \times \langle h \rangle \rangle \cap M)$ and

$$M \subseteq NG_{xH}(((M \cap G) \times (M \cap H))(\langle 1, h^{p^i} \rangle)(g, h)).$$

Proof. To prove the forward direction, suppose that Condition (1) does not hold for the element $(g, h)$. As a result of Lemma 4.1, $M \cap H < H$. We claim that $\langle (g)/(g \cap M) \rangle > \exp(H/\text{Core}_H(M \cap H))$. Suppose not. By 3.2, $M \cap ((M \cap G)(g) \times H)$ is permutable in $(M \cap G)(g) \times H$. We then apply 3.1, 3.3, and finally Lemma 4.1 to conclude that $g \in NG_G(M \cap G)$. This contradicts 3.5.

Now we may let $\langle (g)/(g \cap M) \rangle = p^a$ and $\langle (h)/(h \cap M) \rangle = p^b$ where $a, b \in \mathbb{N}$ with $a > b$. By applying Goursat's description of subgroups of a direct product, we obtain that $M \cap ((M \cap G)(g) \times (M \cap H)(h)))$ is equal to $((M \cap G) \times (M \cap H))(\langle g^{p^a}, h^{p^b} \rangle)$, where $a$ is a nonnegative integer and $c_1 \in \mathbb{N}$ with $(c_1, p) = 1$.

Let $(m_1, m_2) \in M$. By the permutability of $M$, we have $(m_1, m_2)^{-1}((g, h)\langle m_1, m_2 \rangle = (\overline{m}_1, \overline{m}_2)(g, h)^a$ for some $(\overline{m}_1, \overline{m}_2) \in M$ and $a \in \mathbb{N}$. It then follows from 3.6 that $(\overline{m}_1, \overline{m}_2)$ is in $M \cap ((M \cap G)(g) \times (M \cap H)(h))$. Thus, $(m_1, m_2)^{-1}((g, h)(m_1, m_2)$ is an element of $((M \cap G) \times (M \cap H))(\langle g^{p^a}, h^{p^b} \rangle)$, and since $a = b + 0$, $h^{p^a} \leq (h^{p^b})$, and $g^{p^a}h^{p^b} = (1, h^{p^b})(g, h)g^{p^a}$ for some $r \in \mathbb{N}$. Hence, $(g^{p^a}, h^{p^b})$ is actually contained in $((1, h^{p^b})(g, h))$. Since $[M, (1, h^{p^b})] \leq (M \cap H)(1, h^{p^b})$ by 3.6, Condition (2) holds for $(g, h)$, completing the proof of the forward direction.

In order to prove the converse, notice that if Condition (1) is true, then $M$ clearly permutes with $(g, h)$. So, assume that Condition (2) holds, but Condition (1) does not.

Let $\langle (g)/(g \cap M) \rangle = p^f$ and $\langle (h)/(h \cap M) \rangle = p^e$ for $e, f \in \mathbb{N}$. Since $f > e$, we can again apply Goursat's result to conclude that there exist a nonnegative integer $i$ and $c_2 \in \mathbb{N}$ with $(c_2, p) = 1$ such that $(g^{c_2p^i}, h^{p^e}) \leq M$ and $M \subseteq NG_{xH}(((M \cap G) \times (M \cap H))(\langle 1, h^{p^e} \rangle)(g, h))$. But $f - e > 0$, and so there is $c_3 \in \mathbb{N}$ such that

$$(g^{c_2p^i}, h^{p^e})\langle 1, h^{p^e} \rangle = (1, h^{p^e} + c_3p^i)^f,$$

which is a generator for $\langle (1, h^{p^e}) \rangle$. Hence, $\langle (1, h^{p^e}) \rangle \leq (g^{c_2p^i}, h^{p^e})(g, h)$. Therefore, $M$ permutes with $(g, h)$.

5. Conjectures, examples, and special cases

Of course, it would be nice to simplify the conditions provided in Theorem 4.2. Conjectures 5.1 and 5.2 reflect attempts at such a simplification. Unfortunately, neither is successful.

**Conjecture 5.1.** Let $M$ be a subgroup of $G \times H$. Then, $M$ is a permutable subgroup of $G \times H$ if and only if $(M \cap G) \times (M \cap H)$ is permutable in $G \times H$ and for all $S \subseteq G \times H$ such that $(M \cap G) \times (M \cap H) \subseteq S$, we have $M \subseteq NG_{xH}(S)$. 
Conjecture 5.2. Let $M$ be a subgroup of the finite $p$-group $G \times H$. Without loss of generality, assume that $\exp(G/\Core_{G}(M \cap G)) \geq \exp(H/\Core_{H}(M \cap H))$. Then, $M$ is permutable in $G \times H$ if and only if for all $(g, h) \in G \times H$ such that $[|g|/(g \cap M)] \leq \exp(H/\Core_{H}(M \cap H))$, we have $M \leq N_{G \times H}(((M \cap G) \times (M \cap H))(g, h))$.

Conjecture 5.1 would be an ideal characterization of permutability in $G \times H$. It is equivalent to stating that $M$ is a permutable subgroup of $G \times H$ if and only if the first condition of Theorem 4.2 holds for all elements in $G \times H$. Considering the aforementioned characterization of normality in a direct product in a primitive way, one sees that a subgroup $G$ of $G \times H$ is normal if and only if for all $(g, h) \in G \times H$, $N$ normalizes the set $((N \cap G) \times (N \cap H))(g, h)$. Of course, in order for $N$ to be normal in $G \times H$, we need $N(g, h) = (g, h)N$. Since $M$ is permutable in $G \times H$ if and only if the permutability of $M \cap G$ and $M \cap H$ is not strong enough to force the proposed condition. So, we instead speculate in Conjecture 5.2, that in order to establish the permutability of $M$, it is necessary to carefully consider the group elements described in the second condition of Theorem 4.2.

Motivation for this conjecture comes from the discussion of the norm of a direct product in [4]. In particular, if $G$ and $H$ are finite $p$-groups with $\exp(G) \geq \exp(H)$, then $(x, y) \in N(G \times H)$ as long as $(x, y) \in N(g \in G \circ (g \leq \exp(H)) \times H)$. So we hypothesize that in a similar way, the prescribed behavior of conjugation by $G \times H$ for which $[(g)/(g \cap M)] \leq \exp(H/\Core_{H}(M \cap H))$, is sufficient to guarantee the permutability of $M$ in $G \times H$. The forward direction of Conjecture 5.2 is indeed true, but the condition used is now too weak to imply the permutability of $M$.

Examples 5.4 and 5.5 serve as counterexamples to Conjectures 5.1 and 5.2, respectively. Both rely on Example 5.3, in which we construct a permutation group that is used as a direct factor in each of the counterexamples. For every odd prime $p$ and natural number $n$ that is at least 3, in Example 5.3 we construct an abelian subgroup $K$ of order $p^{n/2} - p^{n-1}$ such that if $g = (1 \ldots p^n)$, then $K \cap (g \leq \exp(H))$ and $K$ is a core-free permutable subgroup of $K$. While Stonehewer [13] uses permutation groups to present examples of groups containing core-free permutable subgroups that fail to be metabelian, the examples here are different because when studying permutability in direct products, we must examine the behavior of permutable subgroups with respect to normalizers rather than their underlying structures.

Example 5.3. Suppose that $p$ is an odd prime, and $n$ is a natural number that is at least 3. Let $g = (1 \ldots p^n)$. For each $i \in \mathbb{N}$ such that $1 \leq i \leq p$, let $x_i$ be the permutation $(i(p + i) \ldots (p^{n-1} - 1)p + i)$. Let $k_1 = s_{p-1}x_{p-1} \ldots x_3$, and for each $j \in \mathbb{N}$ such that $2 \leq j \leq p - 1$, let $k_j = x_{j+1}$. Construct $K$, a subgroup of $S_{p^n}$ by letting $K = (k_1, \ldots, k_{p-1})$. Then, $(g)K$ is a group, and $K$ is a core-free permutable subgroup of this group.

Observe that for $i \in \mathbb{N}$ such that $1 \leq i \leq p$, $g^{-1}x_i g = x_{i+1}$, and $g^{-1}x_p g = x_1$. This fact is used throughout the example. First, we will show that $(g)K$ is a group. It is sufficient to show that $g$ normalizes $(g^n)K$. We recognize that $g^{-1}k_1 g = x_{3}x_{4} \ldots x_{p-2}x_{1}^{-1}$. So,
Example 5.4. Let $G$ be the group $(g)K$ from Example 5.3, and let $H = \langle h \rangle$ be a cyclic group of order $p^{n-1}$. Construct $S$ so that $S = K((g)p,h))$. Then, $S$ is a permutable subgroup of $G \times H$, but $(S \cap G) \times (S \cap H)$ is not permutable in $G \times H$.

Clearly, $S$ is a subgroup of $G \times H$ since $(g)p,h) \in Z(G \times H)$. Let $a, b \in \mathbb{N}$ and $w, v \in K$, and consider $(g^aw, h^b)$, an element of $G \times H$.

Since $(g^aw, h^b) \in Z(G \times H)$, if $(a, p) = p$, then $S$ and $(g^aw, h^b)$ satisfy Condition (1) of Theorem 4.2. On the other hand, if $(a, p) = 1$, then $[(g^aw)/((g^aw) \cap S)] = p^n$, which is greater than $\exp(H/\text{Core}_T(S \cap H))$. Furthermore, $\pi_H((S \cap (g^aw)) \times (h^b))) = (h^b)$. Clearly, $S$ normalizes $(S \cap G)/(1,h^b)((g^aw, h^b))$. Therefore, $S$ and $(g^aw, h^b)$ satisfy Condition (2) of Theorem 4.2, completing the proof that $S$ is permutable in $G \times H$.

Finally, notice that $g^{-1}k_1g$ is not in $(g^p)K$. Thus, $g$ does not normalize $(g^p)K$. By 3.4, $K$, which is in fact $(S \cap G) \times (S \cap H)$, does not permute with $(g^p,h))$.

Example 5.4 is particularly interesting in light of the fact that the product of a diagonal-type subgroup, $(g^p,h)$, and $K$, a subgroup of a direct factor, while $S$ is permutable in $G \times H$, $(g^p,h))$ is normal, $K$ still fails to be permutable in $G \times H$.

Example 5.5. Let $G$ be the group $(g)K$ from Example 5.3, but assume that $n \geq 4$. Furthermore, let $H = \langle h \rangle$ be a cyclic group of order $p^{n-1}$. Construct $X$ so that $X = \langle (g)p^j \rangle \in \langle (g)p, K \rangle$. Then, $X$ is a subgroup of $G \times H$, and $(X \cap G) \times (X \cap H)$ is not permutable in $G \times H$.
K((g^p, h^p^2)). For all u ∈ G such that |⟨u⟩/(⟨u⟩ ∩ X)| ≤ exp(H/\text{Core}_H(X ∩ H)), we have X ≤ N_{G \times H}/((X ∩ G)/(⟨u, v⟩)), but X is not permutable in G × H.

Since (g^p, h^p^2) ∈ Z(G × H), it follows that X ≤ G × H, and it is clear that for all (u, v) ∈ G × H such that |⟨u⟩/(⟨u⟩ ∩ X)| ≤ exp(H/\text{Core}_H(X ∩ H)), we have X ≤ N_{G \times H}/((X ∩ G)/(⟨u, v⟩)). But (k_1, 1)^{-1}(g, h)(k_1, 1) is not in (X ∩ G)/((1, h^p^2))/⟨(g, h)⟩. Therefore, by Theorem 4.2, X is not permutable in G × H.

Conjecture 5.1 does appear to be the more natural of the two conjectures. Theorems 5.6 and 5.7 demonstrate that it can be used to characterize the permubility of two types of subgroups of G × H.

**Theorem 5.6.** Let M be a subgroup of the finite group G × H where Core_G(π_G(M)) = 1 and Core_H(π_H(M)) = 1. Then, M is a permutable subgroup of G × H if and only if (M ∩ G) × (M ∩ H) is permutable in G × H and for all S ≤ G × H such that ((M ∩ G) × (M ∩ H)) ≤ S, we have M ≤ N_{G \times H}(S).

**Proof.** The converse is clearly true, and so we prove only the forward direction. To do this, it is sufficient to prove that for all (g, h) ∈ G × H of prime power order, M ≤ N_{G \times H}/((M ∩ G) × (M ∩ H))/⟨(g, h)⟩. So, let (m_G, m_H) ∈ M, and let (g, h) ∈ G × H have order p^k where p is a prime and n ∈ \mathbb{N}.

Without loss of generality, assume that o(g) = p^k and o(h) = p^l where k, n ∈ \mathbb{N}, but n ≥ k. Since Core_H(π_H(M)) = 1, there exists \bar{h} ∈ H such that o(\bar{h}) = p^k and (\bar{h}) ∩ π_H(M) = 1. It then follows from Theorem 4.2 that there are (x_1, y_1) ∈ ((M ∩ G) × (M ∩ H)) and j ∈ \mathbb{N} such that m_G^{-1}gm_G = x_1g^j and m_H^{-1}hm_H = y_1h^j.

But π_G(M) is also core-free, and thus there is \bar{g} ∈ G with o(\bar{g}) = p^k, but (\bar{g}) ∩ π_G(M) = 1. So, by a similar argument, we see that for some (x_2, y_2) ∈ ((M ∩ G) × (M ∩ H)) and r ∈ \mathbb{N}, we have m_G^{-1}\bar{g}m_G = x_2\bar{g}^r and m_H^{-1}hm_H = y_1h^r, while also m_H^{-1}hm_H = y_2h^r.

Now notice that j = r + c_1p^k for some integer c_1. Therefore, m_G^{-1}hm_G = y_2h^j, completing the proof. □

**Theorem 5.7.** Let G and H be finite groups, and let M be a subgroup of G × H such that both π_G(M) and π_H(M) are cyclic. Then, M is a permutable subgroup of G × H if and only if (M ∩ G) × (M ∩ H) is a permutable subgroup of G × H, and for all subgroups S of G × H such that S contains ((M ∩ G) × (M ∩ H)), we have M ≤ N_{G \times H}(S).

**Proof.** Once again, it is clear that the converse is true. Suppose that G × H is a counterexample to the forward direction that has minimal order, and let M be a subgroup of G × H for which this result fails. As a result of 3.7, 3.1, and the minimality of |G × H|, G × H is a p-group. By Lemma 4.1, 3.1, and the minimality of |G × H|, we may assume without loss of generality that M ∩ H = 1.

So there is (g, h) ∈ G × H, such that M does not normalize (M ∩ G)/⟨(g, h)⟩. By Condition (2) of Theorem 4.2, (g) ∩ π_G(M) fails to be contained in (M ∩ G). Since π_G(M) is a cyclic p-group, (M ∩ G) ≤ (g) ∩ π_G(M). But then (M ∩ G) ≤ π_G(M)/⟨g⟩. This contradicts 3.5, completing the proof. □
A group $G$ has a modular subgroup lattice if each of its subgroups is a modular subgroup. The subgroup $S$ of $G$ is modular if for all subgroups $X$ and $Z$ of $G$ with $X \leq Z$, we have $(X(S \cap G)) = (X,S) \cap Z$, and for all subgroups $Y$ and $Z$ of $G$ with $S \leq Z$, we have $(S(Y \cap Z)) = (S,Y) \cap Z$. A subgroup of a finite group is permutable if and only if it is both subnormal and modular (see [12, Theorem 5.1.1]), leading to our interest in groups with modular subgroup lattices.

Groups with modular subgroup lattices are of great interest in the study of permutable subgroups. A finite $p$-group has a modular subgroup lattice if and only if all of its subgroups are permutable. Iwasawa classified such groups in [7], and a complete presentation of this material is contained in [12]. These groups provide the most accessible examples of permutable subgroups that are not normal.

As part of Conjecture 5.1, we claim if $M$ is a permutable subgroup of $G \times H$, then $(M \cap G) \times (M \cap H)$ is also a permutable subgroup of $G \times H$. Example 5.4 serves as a counterexample, but in Theorem 5.9, we prove that this result is true in the special case that both $G$ and $H$ are groups of odd order with modular subgroup lattices. Most of the work needed to prove this is done in Lemma 5.8.

**Lemma 5.8.** Let $G$ and $H$ be groups of odd order. If $M$ is a permutable subgroup of $G \times H$ such that both $M \cap G$ and $M \cap H$ are cyclic, then $(M \cap G) \times (M \cap H)$ is also permutable in $G \times H$.

**Proof.** Let $G \times H$ be a counterexample to the statement that has minimal order, and let $M$ be a subgroup of $G \times H$ for which the result fails. By 3.1 and the minimality of $|G \times H|$, both $Core_G(M \cap G)$ and $Core_H(M \cap H)$ are trivial. It then follows from 3.7 that for the same odd prime $p$, $G$ and $H$ are $p$-groups.

As a consequence of Lemma 4.1, we may assume without loss of generality that $M \cap H = 1$ and $\exp(H) < \exp(G)$. By applying 3.2 and the minimality of $|G \times H|$, we conclude that $G = (M \cap G)(g)$ and $H = \langle h \rangle$ where $M \cap G$ does not permute with $\langle (g,h) \rangle$. It follows from 3.4 that $g$ is not in $N_G((g^{\exp(h)})(M \cap G))$.

Now notice that by 3.3(b), $o(g) = \exp(G)$. Thus, we may assume $o(g) = p^n$ and $o(h) = p^k$ for $n,k \in \mathbb{N}$ such that $n > k$. By the minimality of $|G \times H|$ and 3.2, $M \cap G$ is permutable in $(M \cap G)(g) \times (h^p)$. So, as a result of 3.5, $g \in N_G((g^{p^{k-1}})(M \cap G))$. Let $M \cap G = \langle m \rangle$. Then, $g^{-1}mg = g^{p^{k-1}}m^a$ where $c,a \in \mathbb{N}$, but $(c,p) = 1$.

By once again applying the minimality of $|G \times H|$, we see that $M \cap G$ must be permutable in $(M \cap G)(g^p) \times \langle h \rangle$. As a consequence of 3.4, $g^p \in N_G((g^{p^{k+1}})(M \cap G))$. Iterated calculations show that $g^{-p}mg^p = g^{c(1+a+a^2+\cdots+a^{p-1})}g^{wp^{k+1}}m^v$ for some $v,w \in \mathbb{N}$. Since $(g) \cap (m) = 1$ as a result of 3.3(a), we have that $p^2$ divides $c(1+a+a^2+\cdots+a^{p-1})$. But $p$ is an odd prime, contradicting 3.8. □

**Theorem 5.9.** Let $G$ and $H$ be groups of odd order with modular subgroup lattices. If $M$ is a permutable subgroup of $G \times H$, then $(M \cap G) \times (M \cap H)$ is a permutable subgroup of $G \times H$.

**Proof.** Let $G \times H$ be a counterexample to the statement of minimal order, and let $M$ be a subgroup of $G \times H$ for which the result fails. By arguments similar to those used
in Lemma 5.8, $G \times H$ is a $p$-group for an odd prime $p$, and without loss of generality, we may assume that both $\text{Core}_G (M \cap G)$ and $M \cap H$ are trivial. Furthermore, there is $(g, h) \in G \times H$ such that $G = (M \cap G) \langle g \rangle$ and $H = \langle h \rangle$ where $M \cap G$ fails to permute with $(g, h)$. As a consequence of 3.3(a), $(M \cap G) \cap \langle g \rangle$ is trivial. But every subgroup of $G$ is permutable in $G$. Therefore, it follows from the minimality of $|G \times H|$ and 3.2 that $M \cap G$ is cyclic, contradicting Lemma 5.8. □

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References