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GROUP-LIKE STRUCTURES AND THE WHITEHEAD PRODUCT IN PRO-HOMOTOPY AND SHAPE

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The notion of 'H-space' is of considerable importance in the homotopy theory of CW-complexes. This paper studies a similar notion in the framework of pro-homotopy and shape theories. This is achieved by following the general plan set forth by Eckmann and Hilton. Examples of shape H-space are also given; it is observed that every compact connected topological monoid is a shape H-space. The Whitehead product is defined and studied in the pro-homotopy and shape categories; and, it is shown that this Whitehead product vanishes on an H-object in pro-homotopy. These results are the natural extensions of some well-known classical results in the homotopy theory of CW-complexes.

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pro-homotopy	simple space
shape	multiplicative structures
Whitehead product	H-space.

1. Introduction

The notion of 'H-space' is of considerable importance in the homotopy theory of CW-complexes. It is well-known that homotopy theoretic notions break down when applied to spaces with irregular local structure. In order to overcome this difficulty, K. Borsuk [2] was led to develop 'shape theory'; see also [6, 8]. We emphasize that the notion of H-space is also not very useful on spaces with local pathologies, and hence, an analogous notion of 'shape H-space' must be carefully formulated. A similar need exists in pro-homotopy; this is discussed below.

Artin and Mazur [1] have extensively discussed the pro-homotopy theory in their study of 'etale homotopy type' of locally Noetherian schemes. They give analogues of many theorems from classical algebraic topology in the setting of pro-homotopy. Pro-homotopy provides a natural setting in which 'completions' are studied; see [1] for pro-finite completions. Bousfield and Kan [3] have obtained their R-completion by first, constructing a pro-homotopy type like Artin and Mazur [1], and then 'collapsing' it to ordinary homotopy type. Dror [4] has pointed out that it is more advantageous to work with the pro-homotopy type, i.e., not to collapse. These considerations provide motivation for pro-homotopy; it is also worth pointing out that pro-homotopy theory includes shape theory.

The business at hand is, as stated above, to provide an all encompassing definition of multiplicative objects in pro-homotopy and shape. This is done by following a general plan set forth by Eckmann–Hilton [7]; a quick review of [7] is presented in Section 2. In Section 3, we concretely discuss 'group-like structures' in pro-categories; this section is of independent interest. These discussions are then applied in Section 4 to obtain 'group-like structures' in pro-homotopy and shape. In Section 5, we have presented a brief study of the Whitehead products in pro-homotopy and shape. This study terminates with our major theorem which roughly states that the extended Whitehead products vanish on H-objects in pro-homotopy; see Theorem 6.1 for a precise statement. This theorem represents convincing evidence that all the notions are correctly and naturally formulated.

We have demonstrated in Section 4 that there are plenty of examples of shape-H-spaces. There are numerous examples of H-objects which naturally arise in pro-homotopy in the context of completions or Moore–Postnikov decompositions associated with H-spaces. A complete study of these examples with additional results is postponed because of length; a good reference for these matters is [3] where other references may also be found. As a concluding remark, we emphasize that H-objects are particularly interesting in pro-homotopy since we have extended Dror's [4] celebrated generalization of the Whitehead theorem to pro-homotopy; see [18, 19]. Some related shape theoretic results are given in [17].

2. Group-like structures in categories

2.1. A review of Eckmann–Hilton [7]. All categories considered in this section are assumed to possess zero-maps. Let $\mathscr{C}$ be a category and let P be the direct product of objects $A_1, A_2, \ldots, A_n$ of $\mathscr{C}$. A system of maps $\{f_j: X \to A_j: 1 \leq j \leq n\}$ from an object X in $\mathscr{C}$ determines a unique map $f: X \to P$ satisfying $p_j f = f_j$ where $p_j: P \to A_j$, $1 \leq j \leq n$, is the projection. The map f is often denoted by $\{f_1, f_2, \ldots, f_n\}$ where f_j's are called the *components* f. We follow [7] for notation and terminology whenever appropriate; this allows us to be brief concerning these matters. A category is called a *D-category* if it has finite direct products. The operation of forming direct products is commutative and associative; more specifically, there are canonical equivalences $\tau: A \times B \to B \times A$ (τ is called *the switching map*) and $a: (A \times B) \times C \to A \times (B \times C)$, see [7] for details.

Let $\mathscr{C}$ be a D-category. An *M-structure* or *multiplication* on an object A of $\mathscr{C}$ is simply a map $m: A \times A \to A$ in $\mathscr{C}$; and, the pair (A, m) is called an *M-object* of $\mathscr{C}$. The following is a list of axioms that can be imposed on an M-object (A, m) of $\mathscr{C}$.

I (Zero as Unit). $m\{1, 0\} = m\{0, 1\} = 1: A \to A \times A \to A$ where 1 and 0 denote the identity map and the zero-map.

II (Associativity). The maps $m(m\times 1)$ and $m(1\times m)a$ are equal whenever appropriately interpreted from the diagram

$$(A\times A)\times A\to A\times(A\times A)\to A\times A\to A$$

where a is a canonical equivalence.

III (Inverse). There exists a map $s:A\to A$ in $\mathscr{C}$ such that $m\{1,s\}=m\{s,1\}=0:A\to A\times A\to A$.

IV (Commutativity). The map $m\tau:A\times A\to A$ equals to the map $m:A\times A\to A$ where $\tau:A\times A\to A\times A$ is the switching map.

2.1.1. Definition. An M-structure $m:A\times A\to A$ in $\mathscr{C}$ is called *H-structure, AH-structure, G-structure, CG-structure*, or *ACH-structure* if it satisfies:

(a) The axiom I;
(b) the axioms I and II;
(c) the axioms I, II, and III;
(d) the axioms I, II, III, and IV; or
(e) the axioms I, II, and IV, respectively.

The following is a theorem of Eckmann and Hilton [7].

2.1.2. Theorem. *Let (A,m) be an M-object of $\mathscr{C}$ and let $H(X,A)$, the set of morphisms in $\mathscr{C}$ from X into A, have the induced M-structure. Then (A,m) satisfies axiom K, K = I, II, III, or IV if and only if the induced M-structures on $H(X,A)$'s, for varying X, satisfy axiom K. Moreover, zero is a right (left) unit for (A,m) if and only if it is a right (left) unit for $H(X,A)$'s; and, a right (left) inverse exists for (A,m) if and only if it exists for $H(X,A)$'s.*

2.1.3. Examples. We shall briefly review some examples from [7]. *Example*: In the category of pointed sets, G-objects and CG-objects are just the groups and the abelian groups, respectively. *Example*: In the category of pointed topological spaces and pointed homotopy classes of maps, the H-objects are the well-known H-spaces and all topological groups are G-objects. A detailed discussion of these two examples is given in [7] where other examples may also be found.

2.2. Pro-categories. The concept of a pro-category is due to Grothendieck [11, 12]; see Artin and Mazur [1; Appendix] for an excellent treatment. A concrete description of a morphism in a pro-category can be found in [6], [8], or [15]; we assume familiarity with Mardešić [15], and Edwards and Hastings [8; p. 4–8].

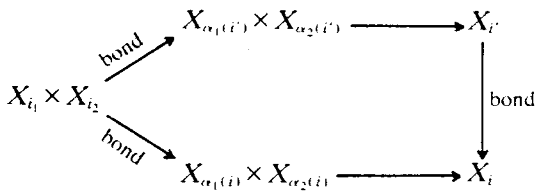
3. Group-like structures in pro-categories

The purpose of this section is to concretize the notions of M-structure, H-structure, etc., given in (2.1), in suitable pro-categories. We begin with the following proposition.

3.1. Proposition. *If $\mathscr{C}$ is a D-category with zero-maps, then pro-$\mathscr{C}$ is also a D-category with zero maps.*

It is easy to see that pro-$\mathscr{C}$ has zero-maps if $\mathscr{C}$ does; and, pro-$\mathscr{C}$ is a D-category when $\mathscr{C}$ is a D-category (this follows from [1], see page 164). Moreover, the product $X \times Y$ is isomorphic to $\{X_i \times Y_j \colon i \in I, j \in J\}$ where $X = \{X_i \colon i \in I\}$, $Y = \{Y_j \colon j \in J\}$, and $I \times J$ is directed by $(i, j) \leq (i', j')$ if and only if $i \leq i'$ and $j \leq j'$.

3.2. *M*-objects in pro-$\mathscr{C}$. Suppose $\mathscr{C}$ is a D-category with zero-maps. Suppose X admits an M-structure $m \colon X \times X \to X$. The map $m \colon X \times X \to X$ can be interpreted as follows. There is a map $\alpha \colon I \to I \times I$ defined by $\alpha(i) = (\alpha_1(i), \alpha_2(i))$ and morphisms $\{X_{\alpha_1(i)} \times X_{\alpha_2(i)} \xrightarrow{m_i} X_i \colon i \in I\}$ such that if $i \leq i'$ in I, then there is a commutative diagram

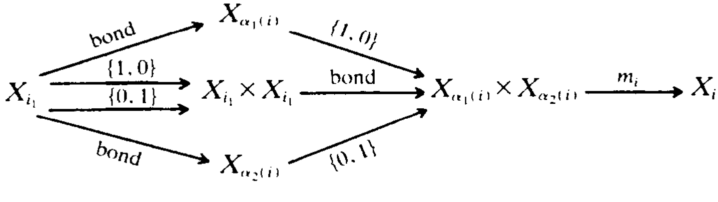
3.2.1 

where $(i_1, i_2) \geq (\alpha_1(i), \alpha_2(i))$ and $(\alpha_1(i'), \alpha_2(i'))$. Clearly, (i_1, i_2) may be chosen so that $i_1 = i_2$.

3.3. *H*-structure in pro-$\mathscr{C}$. With notation as above in 3.1, let (X, m) be an M-object which satisfies axiom I of (2.1), i.e., (X, m) is an H-object. This means that the two composites in the diagram

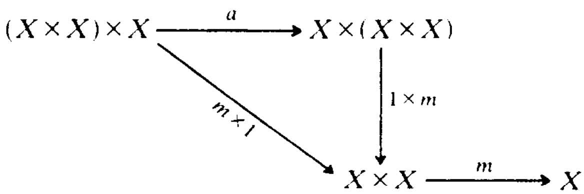
$$X \underset{\{1,0\}}{\overset{\{0,1\}}{\rightrightarrows}} X \times X \xrightarrow{m} X$$

are equal to the identity, i.e., $m\{0, 1\} = m\{1, 0\} = 1$. This is discussed as follows. For each i in I, there exists an $i_1 \geq i$ such that the diagram

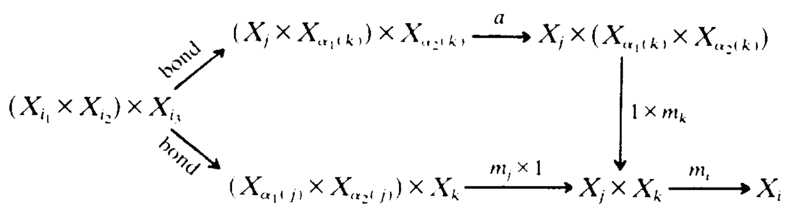
3.3.1 

has the properties: (a) Each composite starting from X_{i_1} and ending at X_i is equal to the bond $X_{i_1} \to X_i$; and (b) the upper and lower triangles commute, i.e., for the upper triangle, bond{1, 0} = {1, 0}bond, and similarly for the lower triangle. This finishes our discussions of an H-structure and we next discuss AH-structure.

3.4. AH-structure in pro-$\mathscr{C}$. With notation as above in 3.1–3.2, suppose an M-object (or H-object) (X, m) satisfies axiom II of 2.1. We study this axiom as follows, i.e., we study the diagram

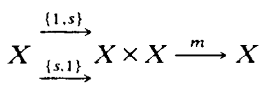


where the two composite maps from $(X\times X)\times X$ to X are equal. Consider the map $X_{\alpha_1(i)}\times X_{\alpha_2(i)} \xrightarrow{m_i} X_i$ and put $j=\alpha_1(i)$ and $k=\alpha_2(i)$. The following diagram explains the situation:

3.4.1 

where both the composites from $(X_{i_1}\times X_{i_2})\times X_{i_3}$ to X_i are equal and $(i_1, i_2, i_3) \geqslant (\alpha_1(j), \alpha_2(j), k)$ and $(j, \alpha_1(k), \alpha_2(k))$; and furthermore, we may choose (i_1, i_2, i_3) such that $i_1 = i_2 = i_3$. In the case when (X, m) is an H-object, our choice of i_1 in 3.4.1 and our choice of i_1 in 3.3.1 can be assumed to be the same (this is the reason for using i_1 in both cases).

3.5. G-structures in pro-$\mathscr{C}$. With notation as above in 3.2–3.4, suppose an M-object (or AH-object) satisfies axiom III. We study this axiom as follows. Suppose there exists a map $s: X\to X$ such that both the composites in the diagram



are equal to the zero-map, i.e., $m\{1, s\} = m\{s, 1\} = 0$. Let $\beta: I\to I$ denote the map given in the definition of the map s. The following diagram contains all the relevant information:

3.5.1

$$\begin{array}{ccccc} X_{i_1}\times X_{i_2} & \xrightarrow{\text{bond}} & X_{\beta(j)}\times X_k & & \\ \Big\downarrow{\scriptstyle \text{bond}} & & \Big\downarrow{\scriptstyle s_j\times 1} & & \\ X_j\times X_{\beta(k)} & \xrightarrow{1\times s_k} & X_j\times X_k & \xrightarrow{m_i} & X_i \end{array}$$

where both the composites from $X_{i_1}\times X_{i_2}$ to X_i are equal to the zero-map, $j=\alpha_1(i)$, $k=\alpha_2(i)$, and $(i, i_2)\geq(j,\beta(k))$ and $(\beta(j), k)$; and furthermore, we may choose $i_1=i_2$. In the case (X, m) is G-object, we may choose i_1 such that i_1 simultaneously works for 3.3.1, 3.4.1, and 3.5.1. This finishes our discussions of G-objects.

3.6. CG-objects in pro-$\mathscr{C}$. With notation as above in 3.2–3.5, suppose an M-object (or G-object) (X, m) satisfies the axiom IV of 2.1. This means that the two maps

$$X\times X \underset{m_\tau}{\overset{m}{\rightrightarrows}} X$$

are equal where τ is 'the switching map' τ: $X\times X\to X\times X$. We study this situation as follows. The following commutative diagram explains the situation:

3.6.1

$$\begin{array}{ccccc} & \nearrow^{\text{bond}} & X_{\alpha_2(i)}\times X_{\alpha_1(i)} & \xrightarrow{\tau} & X_{\alpha_1(i)}\times X_{\alpha_2(i)} \\ X_{i_1}\times X_{i_2} & & & & \Big\downarrow{\scriptstyle m_i} \\ & \searrow_{\text{bond}} & X_{\alpha_1(i)}\times X_{\alpha_2(i)} & \xrightarrow{m_i} & X_i \end{array}$$

where $(i_1, i_2)\geq(\alpha_1(i), \alpha_2(i))$ and $(\alpha_2(i), \alpha_1(i))$; and furthermore, we may choose $i_1=i_2$. In the case (X, m) is a CG-object, we may choose i_1 such that i_1 simultaneously works for all the diagrams 3.3.1, 3.4.1, 3.5.1, and 3.6.1. This finishes our discussions of CG-objects.

3.7. ACH-objects in pro-$\mathscr{C}$. With notation as above, an M-object (X, m) is an ACH-object if the M-structure on X satisfies the axioms I, II, and IV of 2.1. This can be interpreted by combining our discussions given in 3.4 with 3.6.

The following is easy to prove:

3.8. Theorem. *A retract of an H-object of pro-$\mathscr{C}$ is also an H-object of pro-$\mathscr{C}$.*

4. Group-like structures in pro-homotopy and shape theories

4.1. Pro-homotopy. Let $\mathscr{H}$ denote the pointed homotopy category of pointed and connected: (a) CW-complexes, (b) Kan complexes, or (c) topological spaces (satisfying some property). Observe that $\mathscr{H}$ is a D-category with zero-maps. By Proposition 3.1, the pro-homotopy category pro-$\mathscr{H}$ is a D-category with zero-maps. A *pro-homotopy K-object* is by definition an object of pro-$\mathscr{H}$ which is a K-object of pro-$\mathscr{H}$ in the sense of Section 2 and Section 3, where $K = M, H, AH, G, CG$, or ACH. More specifically, put $\mathscr{H}$ equal to $\mathscr{C}$ in Section 3 and define a *pro-homotopy M-object, H-object,...* to be the respective M-object, H-object,... of pro-$\mathscr{C}$ as discussed in 3.2–3.7.

4.2. Shape. Let $\mathscr{W}_0$ denote the pointed homotopy category of pointed CW-complexes. A pointed topological space A is a *shape K-space* if and only if there exists a K-object $\{A_i: i \in I\}$ of pro-$\mathscr{W}_0$ associated with A in the sense of Morita [16], (see [6, 15] for some relevant discussions), where $K = M, H, AH, G, CG$, or ACH.

Group-like structures are commonly studied in the context of the homotopy theory under the broad title 'H-spaces'. Our notions of H-object, shape-H-space, etc., are natural extensions of the notion of H-space.

4.3. Examples of pro-homotopy H-objects and shape H-spaces. We now give several examples. We merely point out the sources of examples and leave the details to the reader.

4.3.1. Example. Every object $\{X_i: i \in I\}$ of pro-$\mathscr{H}$ (see 4.1 for the definition of $\mathscr{H}$) such that each X_i is an H-space and each bond is structure preserving. In particular, any inverse system of Lie groups whose bonds are continuous homomorphisms is a pro-homotopy H-object (or even a G-object).

4.3.2. Example. For each object $\{X_i: i \in I\}$ of pro-$\mathscr{H}$, the object $\{\Omega X_i: i \in I\}$ is an H-object of pro-$\mathscr{H}$ where ΩX_i denotes the suitable loop-space of X_i of pro-$\mathscr{H}$.

4.3.3. Example. Any object of pro-$\mathscr{H}$ whose bonds are the zero-maps in $\mathscr{H}$ is an H-object of pro-$\mathscr{H}$.

4.3.4. Example. A Moore–Postnikov system of an H-space in $\mathscr{H}$ is an H-object of pro-$\mathscr{H}$.

4.3.5. Example. For each object $\{G_i: i \in I\}$ of pro-(topological groups), the object $\{BG_i: i \in I\}$ is an H-object of pro-$\mathscr{H}$ where BG_i is the suitable classifying space of G_i; there are many well-known constructions of the classifying spaces (see [9] for related references, in particular, see G. Segal's construction cited in [9]).

4.3.6. Example. The tower of fibrations $\{R_s X\}$ used to construct the R-completion in [3] is a pro-homotopy H-object when X is an H-space.

4.3.7. Example. Given G a compact metric group, $\{G_n\}$ be a Lie series for G, and $\{BG_n\}$ the inverse system of classifying spaces corresponding to $\{G_n\}$ as in [9]. Edwards and Hastings have shown that the inverse limit of $\{BG_n\}$ is useful for classifying open principal G-fibrations over compact metric spaces; see [9] for a precise statement. It follows that the pro-homotopy information of $\{BG_n\}$ can be useful in the classification of G-fibrations as above; we emphasize that $\{BG_n\}$ must be regarded as an H-object while studying its pro-homotopy.

4.3.8. Example. Let X be a compact metric semigroup with identity such that X is a subset of the Hilbert cube Q or an absolute neighborhood retract (Abbreviate: ANR) A. Observe that for any nbd. U of X in Q or A there exists a nbd. V of X contained in U such that the multiplication $X \times X \xrightarrow{m} X$ suitably extends to a map $V \times V \xrightarrow{\hat{m}} U$ satisfying the maps $\hat{m}(\ ,*)$, $\hat{m}(*,\ ) : V \to U$ (defined by sending x to $\hat{m}(x,*)$ or $\hat{m}(*,x)$, respectively) are homotopic to the inclusion $V \to U$ rel. the base-point $*$. Use this fact to inductively construct a nest $\{X_n\}$ such that: (a) each X_n is an nbd. of X in Q or A, (b) each X_n is an ANR which is compact for Q and which may be non-compact for A, (c) the intersection of $\{X_n\}$ is X (this implies $\{X_n\}$ is associated with X in Morita's sense), and (d) $\{X_n\}$ is a pro-homotopy H-object in our sense. *Thus, every compact metric semigroup with identity is a shape H-space.* In fact, the following more general result can be easily deduced from Keesling [14].

Theorem. *Every compact Hausdorff semigroup with identity or an H-space is a shape H-space.*

We conclude this example with the remark that one must carefully develop shape theory for topological semigroups which classifies topological semigroup structures on a space upto shape.

4.4. Shape H-spaces: geometric examples. Here we are interested in the specific constructions of examples of shape H-spaces. The following propositions follow immediately.

4.4.1. Proposition. *Any space X having the shape of a point is a shape K-space, where $K = H$, AH, G, CG, or ACH.*

4.4.2. Proposition. *If X is a space having the shape of a shape K-space, $K = H$, AH, G, CG, or ACH, then X is a shape H-space.*

4.4.3. Proposition. *The Warsaw circle W, see Fig. 1, is a shape H-space; moreover, it is shape CG-space. This follows, since W has the shape of the circle S^1.*

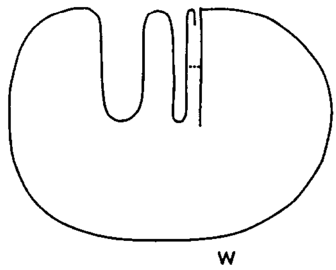


Fig. 1.

4.4.4. Example. The space $\hat{W}$, see Fig. 2, has also the shape of S^1; and hence, $\hat{W}$ is a shape CG-space. Although, $\hat{W}$ is not an H-space in the usual sense (or even an H-space up to homotopy).

We obtain $\hat{W}$ from W by attaching two copies of cylinder $S^1\times[0,\infty)$ along an infinite ray of W as in Fig. 2. Of course, any finite number and even countably many cylinders can be similarly attached. It is easy to see that $\hat{W}$ has the shape of a circle. Now, $\hat{W}$ is not an H-space since $\pi_1 \hat{W}$ is nonabelian (it is a free group on two generators).

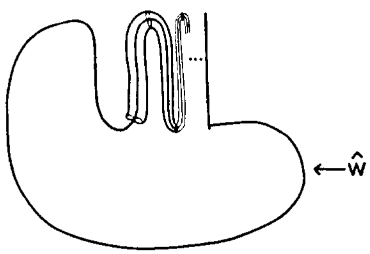


Fig. 2.

4.4.5. Example. Let P be any polyhedron (or more generally any space). Let A equal to the following Warsaw interval, see Fig. 3, with end-points a and b. Construct a space $\hat{P}$ by identifying the subset $P\times\{b\}$ of $P\times A$ to a point. The space $\hat{P}$ will be called a 'wiggly cone' over P. It is easy to see that $\hat{P}$ has the shape of a point; although, the inclusion $P\to\hat{P}$ induces an isomorphism of homotopy groups. This shows that the homotopy groups of a shape H-space X may be quite arbitrary even when X has trivial shape (see [10] for a more general result in this direction).

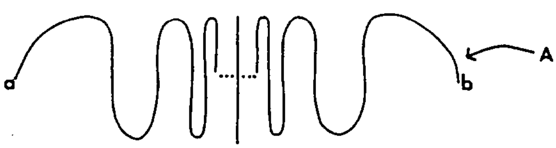


Fig. 3.

4.4.6. Example. Let X be any H-space which is a CW-complex. The following operation will generate many interesting examples of shape H-spaces: *Remove any open n-cell bounded by $(n-1)$-sphere Σ and glue in the 'wiggly cone' over Σ.* This operation can be carefully repeated to construct rather complicated looking shape H-spaces, which are not H-spaces, each of which has the shape of X but not the homotopy type of X.

5. The Whitehead product in pro-homotopy

The main purpose of this section is to carefully develop an analogue of the Whitehead product (Abbreviate: W-product) in pro-homotopy. We assume familiarity with the homotopy notions of W-product and the action of the fundamental group of a space on its homotopy groups. Our development of the W-product in pro-homotopy and its formalization has necessitated the definitions of several auxiliary categories; thus, this development appears to be more complicated than it actually is.

5.1. A category of pro-actions. By a *left action* of a group H on a group G we mean a map $H\times G \xrightarrow{\xi} G$ satisfying the usual properties $e\cdot g = g$, $h\cdot(gg') = (h\cdot g)(h\cdot g')$, and $(hh')\cdot g = h\cdot(h'\cdot g)$ where g, g' is in G, h, h' is in H, e is the identity of H, and $\xi(h, g)$ is denoted by $h\cdot g$. We form a category of 'left actions' $\mathcal{LA}$ whose objects are maps of the form ξ and a morphism $(\phi, \psi): \xi \to \xi'$ is a commutative diagram

$$\begin{array}{ccc} H\times G & \xrightarrow{\xi} & G \\ \Big\downarrow{\scriptstyle\phi} & & \Big\downarrow{\scriptstyle\psi} \\ H'\times G' & \xrightarrow{\xi'} & G' \end{array}$$

where ϕ and ψ are homomorphisms. The category $\mathcal{RA}$ of 'right actions' is similarly defined. The corresponding pro-categories pro-$\mathcal{LA}$ or pro-$\mathcal{RA}$ will be called the category of *pro-actions.*

5.2. The pro-action of the fundamental pro-group. Let $\mathscr{C}$ equal to $\mathscr{HT}_0$ or $\mathscr{HW}_0$ throughout the following. Suppose $\{X_i: i\in I\}$ is an object of pro-$\mathscr{C}$. The fundamental pro-group $\pi_1X = \{\pi_1X_1: i\in I\}$ *pro-acts* (we often drop the prefix 'pro') on $\pi_pX = \{\pi_pX_i: i\in I\}$ as follows: The morphisms $\{\pi_1X_i\times\pi_pX_i \xrightarrow{\xi_p} \pi_pX_i: i\in I\}$ determine an object $\pi_1X\times\pi_pX \xrightarrow{\xi_p} \pi_pX$ of pro-$\mathcal{LA}$ since for each $j\geq i$ the diagram

$$\begin{array}{ccc} \pi_1 X_j \times \pi_p X_j & \xrightarrow{\xi_p} & \pi_p X_j \\ \big\downarrow & & \big\downarrow \\ \pi_1 X_i \times \pi_p X_i & \xrightarrow{\xi_p} & \pi_p X_i \end{array}$$

commutes where the vertical maps are induced by the bond $X_j \to X_i$.

5.2.1. Simplicity. With notation as above, X is *p-simple* in pro-$\mathscr{C}$ if and only if for each i in I there exist $j = j(i) \geq i$ in I such that $\xi_p(x, y) = y$ for all (x, y) in the image of the homomorphism $\pi_1 X_j \times \pi_p X_j \to \pi_1 X_i \times \pi_p X_i$ induced by the bond $X_j \to X_i$. We say X is *simple* if and only if it is p-simple for all $p \geq 1$. We say X is *uniformly simple* if and only if for each i in I there exists $j = j(i) \geq i$ in I such that $\xi_p(x, y) = y$ for all $p \geq 1$ and all (x, y) as above. In case there is ambiguity, we may use the terminology *pro-p-simple, pro-simple,* and *pro-(uniformly simple)* instead of p-simple, simple, and uniformly simple.

5.3. A pro-category of bilinear maps. Suppose G, H, and K are abelian groups. A *bilinear* map $G \times H \to K$ is a map of sets which is linear in each coordinate separately. Let $\mathscr{BL}$ denote the category of 'bilinear maps' whose objects are bilinear maps and whose morphism are pairs (ϕ, ψ) of homomorphisms such that the diagram

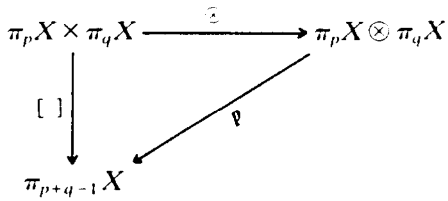
$$\begin{array}{ccc} G' \times H' & \longrightarrow & K' \\ \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \psi} \\ G \times H & \longrightarrow & K \end{array}$$

commutes where the horizontal maps are two objects of $\mathscr{BL}$. The objects of pro-$\mathscr{BL}$ are called 'pro-bilinear'.

5.4. The Whitehead product in pro-homotopy. Suppose p and q are greater than 1 until further notice. With notation as above, the set of bilinear maps $\{\pi_p X_i \times \pi_q X_i \xrightarrow{[\,]} \pi_{p+q-1} X_i \colon i \in I\}$ form an object $\pi_p X \times \pi_q X \xrightarrow{[\,]} \pi_{p+q-1} X$ of pro-$\mathscr{BL}$ since for each $j \geq i$ the diagram

$$\begin{array}{ccc} \pi_p X_j \times \pi_q X_j & \xrightarrow{[\ ]} & \pi_{p+q-1} X_j \\ \big\downarrow & & \big\downarrow \\ \pi_p X_i \times \pi_p X_i & \xrightarrow{[\ ]} & \pi_{p+q-1} X_i \end{array}$$

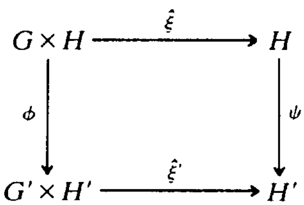
commutes where the vertical maps are induced by the bond $X_j \to X_i$; and furthermore, the set $\{\pi_p X_i \otimes \pi_q X_i \xrightarrow{p} \pi_{p+q-1} X_i : i \in I\}$ of homomorphisms determines a map $\pi_p X \otimes \pi_q X \xrightarrow{p} \pi_{p+q-1} X$ of pro-groups such that the diagram



commutes. The *pro-Whitehead product* or *pro-W-product* (or W-product when there is no confusion) is defined to be the object $\pi_p X \times \pi_q X \xrightarrow{[\,]} \pi_{p+q-1} X$ of pro-$\mathscr{BL}$ or the morphism p of pro-groups whichever is convenient.

In the case $p = q = 1$, the *pro-W-product* is defined to be the object $\pi_1 X \times \pi_1 X \xrightarrow{\xi_1} \pi_1 X$ of pro-$\mathscr{LA}$, see 5.4. We often use '[ ]' for 'ξ_1'. We need the following auxiliary category before proceeding further.

5.4.1. An auxiliary category. Let $G \times H \xrightarrow{\xi} H$ be a group action where G is a (possibly nonabelian) group and H is an abelian group. Define a new map $G \times H \xrightarrow{\hat{\xi}} H$ by setting $\hat{\xi}(g, h)$ equal to $[\xi(g, h) - h]$ for every (g, h) in $G \times H$. Define a category $\mathscr{P}$ whose objects are maps of the form $\hat{\xi}$ and a morphism $(\phi, \psi) : \hat{\xi} \to \hat{\xi}'$ of $\mathscr{P}$ is the commutative diagram



where ϕ and ψ are homomorphisms.

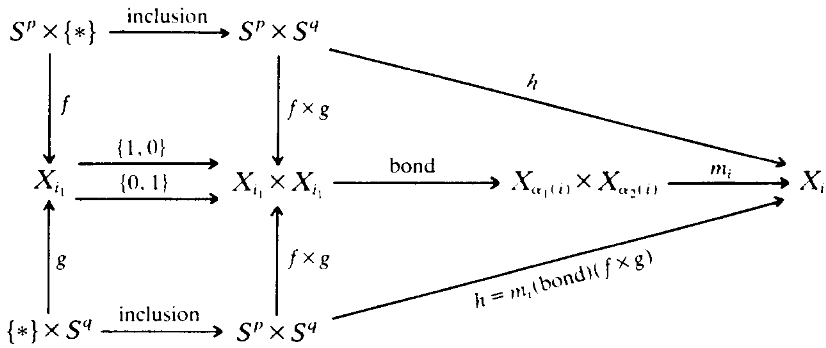
Suppose $p = 1 < q$ until further notice. The *pro-W-product* is defined to be the object $\pi_1 X \times \pi_q X \xrightarrow{\hat{\xi}_q} \pi_q X$ of pro-$\mathscr{P}$ determined by the object $\pi_1 X \times \pi_q X \xrightarrow{\xi_q} \pi_q X$ of pro-$\mathscr{LA}$. We often use '[ ]' for '$\hat{\xi}_p$'. Now $\hat{\xi}_q$ determines the *pro-W-product* $\pi_q X \times \pi_1 X \xrightarrow{[\,]} \pi_q X$ as follows: For each i in I define $\pi_q X_i \times \pi_1 X_i \to \pi_q X_i$ by sending (x, y) to $(-1)^q[\xi_q(y, x) - x]$. This finishes our definition of the Whitehead product in the pro-homotopy setting.

6. Pro-Whitehead product vanishes on H-objects in pro-homotopy

It is well-known that the Whitehead products vanish on H-spaces. The purpose of this section is to prove an analogous result in pro-homotopy and shape theory. Suppose $X = \{X_i : i \in I\}$ is an object of pro-$\mathscr{C}$, $\mathscr{C}$ as in 5.2. Let us observe that the W-product, see 5.4, of the pro-groups $\pi_p X$ and $\pi_q X$ *vanishes* if and only if for each i in I there exists $j = j(i) \geq i$ in I such that the map $\pi_{p+q-1} X_j \to \pi_{p+q-1} X_i$ induced by the bond takes the W-product of two elements of $\pi_p X_j$ and $\pi_q X_i$ onto the zero element.

6.1. Theorem. *If $X = \{X_i : i \in I\}$ is an H-object of pro-$\mathscr{C}$, then the (pro-) Whitehead product of pro-groups $\pi_p X$ and $\pi_q X$ vanishes for all p and $q \geq 1$.*

Proof. With notation as in 3.3, we consider the following diagram



satisfying: (a) the map f and g represent elements α in $\pi_p X_{i_1}$ and β in $\pi_q X_{i_1}$, respectively; (b) $\{1, 0\} f = (f \times g)$ inclusion; (c) $\{0, 1\} g = \text{inclusion}(f \times g)$; and (d) the middle of the diagram comes from the diagram 3.3.1. This suffices to prove that the image of $[\alpha, \beta]$ under the bond $X_{i_1} \to X_i$ is zero. This completes our proof. $\square$

6.2. Corollary. *If $X = \{X_i : i \in I\}$ is an H-object of pro-$\mathscr{C}$, then X is uniformly simple* (*see* 5.2.1).

This follows immediately from Theorem 6.1.

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