Oscillation of Neutral Difference Equations With Positive and Negative Coefficients

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Abstract—In this paper, we establish some oscillation criteria for the neutral difference equation with positive and negative coefficients

\[ \Delta(x_n - c_n x_{n-\gamma}) + p_n x_{n-r} - q_n x_{n-\sigma} = 0 \]  

without the limitation of condition \( c_n + \sum_{i=n-r+\sigma}^{n-1} q_i \leq 1 \), where \( c_n, p_n, \) and \( q_n \) are nonnegative integers with \( r - 1 \geq \max\{\sigma, \gamma\} \), \( \gamma \geq 1 \), \( \sigma \geq 0 \). © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, we are concerned with a class of neutral difference equation with positive and negative coefficients

\[ \Delta(x_n - c_n x_{n-\gamma}) + p_n x_{n-r} - q_n x_{n-\sigma} = 0, \]  

where the forward difference \( \Delta x_n = x_{n+1} - x_n \), and \( c_n, p_n, \) and \( q_n \) are nonnegative real numbers and \( c_n \geq 1, \ r - 1 \geq \max\{\sigma, \gamma\}, \ \gamma \geq 1, \ \sigma \geq 0 \).

The oscillatory behavior of neutral difference equation has been investigated by a number of papers [1–8]. In recent years, many people are devoted to studying oscillation and nonoscillation of equation (1). However, most of these papers consider equation (1) under the hypothesis

\[ c_n + \sum_{i=n-r+\sigma}^{n-1} q_i \leq 1. \]  

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In fact, when condition (3) does not hold, few results for oscillation of equation (1) are obtained until now.

Our aim in this paper is to study the oscillatory behavior without assumption (3) and obtain some more general result than that in the literature.

Let \( \rho_* = \max\{\tau, \sigma, \gamma\} \). By a solution of equation (1) we mean a sequence \( \{x_n\} \) of real numbers which is defined for \( n \geq -\rho_* \) and satisfies equation (1), for \( n = 0, 1, 2, \ldots \). It is easy to see that under the initial conditions

\[
x_n = b_n, \quad n = -\rho_*, \quad \rho_* + 1, \ldots, -1
\]

equation (1) has a unique solution satisfying (4).

As is customary, a nontrivial solution \( \{x_n\} \) of equation (1) is said to be nonoscillatory if the terms \( x_n \) are eventually positive or eventually negative. Otherwise, it is said to be oscillatory.

Throughout this paper, we assume that (2) holds, and let

\[
\rho_* = \max\{\tau, \sigma, \gamma\}, \quad \rho = \min\{\tau, \sigma, \gamma\}, \quad \eta = \min\{\sigma + 1, \gamma\},
\]

\[
h_\lambda^n = p_n \lambda^{-\tau - 1} - q_{n+\sigma-\tau} \lambda^{-\sigma - 1} - c_{n+\gamma-\tau} \lambda^{-\gamma} \left(1 - \frac{1}{\lambda}\right),
\]

and define \( \{y_n\} \) and \( \{z_n\} \) as follows.

\[
y_n = \lambda^{-n} x_n
\]

and

\[
z_n = y_n - c_n \lambda^{-\gamma} y_{n-\gamma} - \sum_{i=n-\tau}^{n-\sigma-1} q_i \lambda^{-\sigma - 1} y_i - \sum_{i=n-\tau}^{n-\gamma-1} c_i \lambda^{-\gamma} \left(1 - \frac{1}{\lambda}\right) y_i,
\]

where \( \lambda \geq 1 \) is a positive constant.

Substituting (6) into (1), we have

\[
\Delta (y_n - c_n \lambda^{-\gamma} y_{n-\gamma}) + \left(1 - \frac{1}{\lambda}\right) (y_n - c_n \lambda^{-\gamma} y_{n-\gamma}) + p_n \lambda^{-\tau - 1} y_{n-\tau} - q_n \lambda^{-\sigma - 1} y_{n-\sigma} = 0.
\]

In this paper, unless specified, we assume that an equality holds for sufficiently large \( n \).

**2. SOME LEMMAS**

**LEMMA 1.** Suppose that

\[
h_\lambda^n > 0.
\]

If the recurrence relation

\[
c_n \lambda^{-\gamma} y_{n-\gamma} + \sum_{i=n-\tau}^{n-\sigma-1} q_i \lambda^{-\sigma - 1} y_i + \sum_{i=n-\gamma}^{n-\gamma-1} c_i \lambda^{-\gamma} \left(1 - \frac{1}{\lambda}\right) y_i
\]

\[
+ \sum_{i=n}^{\infty} \left(1 - \frac{1}{\lambda}\right) y_i + \sum_{i=n-\tau}^{\infty} h_\lambda^n y_i \leq y_n, \quad n \geq n_1,
\]

has a positive solution \( \{\bar{y}_n\} \). Then, recurrence relation

\[
c_n \lambda^{-\gamma} y_{n-\gamma} + \sum_{i=n-\tau}^{n-\sigma-1} q_i \lambda^{-\sigma - 1} y_i + \sum_{i=n-\gamma}^{n-\gamma-1} c_i \lambda^{-\gamma} \left(1 - \frac{1}{\lambda}\right) y_i
\]

\[
+ \sum_{i=n}^{\infty} \left(1 - \frac{1}{\lambda}\right) y_i + \sum_{i=n-\tau}^{\infty} h_\lambda^n y_i = y_n, \quad n \geq n_1,
\]

where \( \lambda > 1 \), has a positive solution \( Y_n \), for \( n \geq n_1 - \rho_* \), and satisfies \( 0 < Y_n \leq \bar{y}_n \).
PROOF. Consider the space $\Omega$ of all sequences $\{\omega_n\}$ for $n \geq n_1 - \rho_*$ that are such that $0 \leq \omega_n \leq \tilde{Y}_n$ for $n \geq n_1 - \rho_*$. Define the operator $S$ on $\Omega$ as follows

$$S\omega_n = \begin{cases} 
  c_n \lambda^{-\gamma} \omega_{n-\gamma} + \lambda^{-\sigma-1} \sum_{i=n-\tau}^{n-\tau-1} q_{i+\sigma} \omega_i + \left(1 - \frac{1}{\lambda}\right) \lambda^{-\gamma} \sum_{i=n-\tau}^{n-\tau-1} c_{i+\gamma} \omega_i \\
  \tilde{Y}_n,
\end{cases}$$

for $n \geq n_1$, $n_1 - \rho_* \leq n < n_1$.

Thus, it is easy to see that $\omega_n^{(1)} \leq \omega_n^{(2)}$ implies that $S\omega_n^{(1)} \leq S\omega_n^{(2)}$. Combining with (10), we may obtain that $S\Omega \subset \Omega$. Define sequences $\{\omega_n^{(k)}\}$ as

$$\omega_n^{(0)} = \tilde{Y}_n, \quad \omega_n^{(k+1)} = S\omega_n^{(k)}, \quad \text{for } k = 0, 1, 2, \ldots.$$

By induction, we easily get

$$0 \leq \omega_n^{(k+1)} \leq \omega_n^{(k)} \leq \cdots \leq \omega_n^{(1)} \leq \tilde{Y}_n.$$

Thus, for any $n \in \mathbb{N}$, $\lim_{k \to \infty} \omega_n^{(k)}$ exists. Set $Y_n = \lim_{k \to \infty} \omega_n^{(k)}$. Obviously, $\{Y_n\}$ satisfies (11). In view of $Y_n = \tilde{Y}_n$ for $n_1 - \rho_* \leq n < n_1$, it follows that $0 < Y_n \leq \tilde{Y}_n$ for $n \geq n_1$. The proof is complete.

**Lemma 2.** Suppose that there exists $\lambda > 1$ such that (9) holds and that

$$c_n \lambda^{-\gamma} + \lambda^{-\sigma-1} \sum_{i=n-\tau}^{n-\tau-1} q_{i+\sigma} \omega_i + \left(1 - \frac{1}{\lambda}\right) \lambda^{-\gamma} \sum_{i=n-\tau}^{n-\tau-1} c_{i+\gamma} \omega_i \leq 1.$$

Let $\{x_n\}$ be an eventually positive solution of the difference inequality

$$\Delta(x_n - c_n x_{n-\tau}) + p_n x_{n-\tau} - q_n x_{n-\sigma} \leq 0.$$

Then, we have eventually have

$$\Delta z_n < 0, \quad z_n > 0, \quad \text{and} \quad \lim_{n \to \infty} z_n = 0.$$

**Proof.** Suppose that $x_{n-\rho_*} > 0$, $n \geq n_1$ for some integer $n_1$. Substituting (6) into (13) and combining with (7), we have

$$\Delta z_n \leq -\left(1 - \frac{1}{\lambda}\right) y_n - h_n^\lambda y_{n-\tau} < 0,$$

which implies that $z_n$ is decreasing for $n \geq n_1$. Hence, $z_n$ is eventually positive or eventually negative. Suppose that on the contrary, $z_n < 0$ eventually, and so there exists an integer $n_2 \geq n_1$ and $\alpha > 0$ such that $z_n \leq -\alpha$ for $n \geq n_2$, that is

$$y_n \leq -\alpha + c_n \lambda^{-\gamma} y_{n-\gamma} + \sum_{i=n-\tau}^{n-\tau-1} q_{i+\sigma} \lambda^{-\sigma-1} y_i + \sum_{i=n-\tau}^{n-\tau-1} c_{i+\gamma} \lambda^{-\gamma} \left(1 - \frac{1}{\lambda}\right) y_i.$$

We consider the following two possible cases.
CASE 1. \{y_n\} is unbounded. Then, there exists a sequence \{n_k\} of integers such that \(y_{n_k} \to \infty\) as \(n_k \to \infty\) and \(y_{n_k} = \max\{y_n \mid n_2 \leq n \leq n_k\}\). It follows that

\[
y_{n_k} \leq -\alpha + c_{nk} \lambda^{-\gamma} y_{n_k-\gamma} + \sum_{n_k-\tau}^{n_k-\gamma-1} q_i \lambda^{-\sigma-1} y_i + \sum_{i=n_k-\tau}^{n_k-\gamma-1} c_i \lambda^{-\gamma} \left(1 - \frac{1}{\lambda}\right) y_i
\]

\[
\leq -\alpha + c_{nk} \lambda^{-\gamma} + \sum_{n_k-\tau}^{n_k-\gamma-1} q_i \lambda^{-\sigma-1} + \sum_{i=n_k-\tau}^{n_k-\gamma-1} c_i \lambda^{-\gamma} \left(1 - \frac{1}{\lambda}\right) y_{nk}
\]

\[
\leq -\alpha + y_{nk},
\]

which is a contradiction.

CASE 2. \{y_n\} is bounded. There exists some \(a > 0\) such that \(|y_n| \leq a < \infty\). We may choose a sequence \{n_k\} such that \(\lim_{k \to \infty} n_k = +\infty\). Let \(\xi_k\) be such that \(y_{\xi_k} = \max\{y_n \mid n - k < n < n_k\}\). Let \(\tilde{\xi}_k = \min\{y_n \mid n - k < n < n_k\}\). So \(\limsup_{k \to \infty} y_{\tilde{\xi}_k} \leq a\). From (7) and (12), we have

\[
y_{n_k} \leq -\alpha + c_{nk} \lambda^{-\gamma} y_{n_k-\gamma} + \sum_{n_k-\tau}^{n_k-\gamma-1} q_i \lambda^{-\sigma-1} y_i + \sum_{i=n_k-\tau}^{n_k-\gamma-1} c_i \lambda^{-\gamma} \left(1 - \frac{1}{\lambda}\right) y_{n_k}
\]

By taking superior limits as \(n_k \to \infty\). We obtain \(a < a - a\), which is also a contradiction.

Next, we prove that \(\lim_{n \to \infty} z_n = 0\). If it is not true, as \(\Delta z_n \leq 0\) and \(z_n > 0\), it follows that \(\lim_{n \to \infty} z_n = l > 0\). Thus, there exists sufficiently large \(n'\) such that \(z_n \geq \frac{l}{2}\) for \(n \geq n'\). Moreover, in view of (7), \(y_n \geq z_n \geq \frac{l}{2}\), \(n \geq n'\). Combining with (14), we obtain that \(\lim_{n \to \infty} z_n = -\infty\), which contradicts to \(z_n \to 0\), the proof is complete.

The proof of the following lemma may be found in [2].

**Lemma 3.** (See [2].) Assume that \(\{p_n\}\) is a nonnegative sequences of real numbers and let \(k\) be a positive integer. Suppose that

\[
\liminf_{n \to \infty} \left[ \frac{1}{k} \sum_{i=n-k}^{n-1} p_i \right] > \frac{k^k}{(k+1)^{k+1}}
\]

hold. Then, difference inequalities

\[
\Delta x_n + p_n x_{n-k} \leq 0, \quad n = 0, 1, 2, \ldots,
\]

\[
\Delta y_n - p_n y_{n+k} \geq 0, \quad n = 0, 1, 2, \ldots,
\]

cannot have eventually positive solution.

### 3. MAIN RESULTS

**Theorem 1.** Suppose that there exists \(\lambda > 1\) such that (9) and (12) hold. Then, equation (1) has no eventually positive solution if and only if inequality (13) has no eventually positive solution.

**Proof.** It is easy to see that the sufficiency is obvious. Next, we prove the necessity. Assume that \(\{x_n\}\) is an eventually positive solution of (13). In view of (6), \(\{y_n\}\) is an eventually positive solution of

\[
\Delta (y_n - c_n \lambda^{-\gamma} y_{n-\gamma}) + \left(1 - \frac{1}{\lambda}\right) (y_n - c_n \lambda^{-\gamma} y_{n-\gamma}) + p_n \lambda^{-\tau-1} y_{n-\tau} - q_n \lambda^{-\sigma-1} y_{n-\sigma} \leq 0. \quad (15)
\]

By Lemma 2 and (6), we see that there exists integer \(n_1\) such that

\[
y_n > 0, \quad z_{n-\rho_n} > 0, \quad n \geq n_1 \quad \text{and} \quad \lim_{n \to \infty} z_n = 0.
\]
By (7) and (15), we get

$$\Delta z_n \leq -\left(1 - \frac{1}{\lambda}\right) y_n - h_n^\lambda y_{n-\tau} < 0, \quad n \geq n_1. \quad (16)$$

Summing above inequality from $n$ to $\infty$, we obtain

$$z_n \geq \sum_{i=n}^{\infty} \left(1 - \frac{1}{\lambda}\right) y_i + \sum_{i=n-\tau}^{\infty} h_i^\lambda y_i,$$

i.e.,

$$c_n \lambda^{-\gamma} y_{n-\gamma} + \sum_{i=n-\tau}^{n-\sigma-1} \lambda^{-\sigma-1} q_i + c_{i+\gamma} y_i + \sum_{i=n-\sigma}^{n-\gamma-1} \left(1 - \frac{1}{\lambda}\right) \lambda^{-\gamma} c_{i+\gamma} y_i + \sum_{i=n}^{\infty} \left(1 - \frac{1}{\lambda}\right) y_i + \sum_{i=n-\tau}^{\infty} h_i^\lambda y_i \leq y_n.$$

By using Lemma 1, we see that (11) has a eventually positive solution which satisfies (15). In view of (6), (1) also has an eventually positive solution. The proof is complete.

**Corollary 1.** Suppose that there exists $r > 0$, such that

$$\sum_{i=n-\tau}^{n-\sigma-1} q_i + c_{i+\gamma} \leq r \quad (17)$$

and

$$h_n^{r^{-1}/\gamma} > 0 \quad (18)$$

hold. Then, the conclusion of Theorem 1 holds.

**Proof.** Let

$$g(\lambda) = c_n \lambda^{-\gamma} + \sum_{i=n-\tau}^{n-\sigma-1} q_i + \lambda^{-\sigma-1} c_{i+\gamma} + \sum_{i=n-\sigma}^{n-\gamma-1} \left(1 - \frac{1}{\lambda}\right) \lambda^{-\gamma} c_{i+\gamma} - 1.$$

Taking $\lambda = r^{1/\eta}$, we have

$$g\left(r^{1/\eta}\right) = c_n r^{-\gamma/\eta} + \sum_{i=n-\tau}^{n-\sigma-1} q_i + r^{-(\sigma+1)/\eta} + \sum_{i=n-\sigma}^{n-\gamma-1} c_{i+\gamma} r^{-\gamma/\eta} \left(1 - r^{-1/\eta}\right) - 1$$

$$\leq \left(\sum_{i=n-\tau}^{n-\sigma-1} q_i + \sum_{i=n-\sigma}^{n-\gamma-1} c_{i+\gamma}\right) \frac{1}{r} - 1$$

$$\leq 0.$$

Thus, all the conditions of Theorem 1 are satisfied. The proof is complete.

Now, consider the difference equations

$$\Delta (x_n - cx_{n-\tau}) + p_n^{(i)} x_{n-\tau} - q_n^{(i)} x_{n-\sigma} = 0, \quad i = 1, 2, \quad (19)$$

where $h_n^{(i)\lambda} = p_n^{(i)} \lambda^{-\gamma-1} - q_n^{(i)} \lambda^{-\sigma-1} - c \lambda^{-\gamma}(1 - 1/\lambda)$, $c > 0$, $p_n^{(i)}$ and $q_n^{(i)}$ are nonnegative sequences, and $\lambda$ is a constant.
THEOREM 2. Suppose that there exists $\lambda > 1$, such that $h_n^{(i)} > 0$ and

$$c\lambda^{-\gamma} + \sum_{n=1}^{n-\sigma-1} q_i^{(i)}(1-1/\lambda)\lambda^{-\gamma} \leq 1, \quad i = 1, 2,$$  \hspace{1cm} (20)

$$p_n^{(2)} \geq p_n^{(1)}, \quad q_n^{(2)} \leq q_n^{(1)}.$$  \hspace{1cm} (21)

If equation (19)$_1$ has no eventually positive solution, then (19)$_2$ also has no eventually positive solution.

PROOF. Suppose on the contrary, that $\{x_n\}$ is an eventually positive solution of equation (19)$_2$. Let $n_0$ be an integer such that $x_n > 0$ for $n \geq n_0$. Then, $y_n > 0$, $n \geq n_0$. Define

$$z_n^{(i)} = y_n - c\lambda^{-\gamma}y_{n-\gamma} - \sum_{j=n-\tau}^{n-\sigma-1} q_j^{(i)}\lambda^{-\sigma-1}y_j - \sum_{j=n-\tau}^{n-\gamma-1} c\lambda^{-\gamma}(1-1/\lambda)y_j, \quad i = 1, 2.$$  \hspace{1cm} (22)

In view of Lemma 2, $z_n^{(i)} > 0$ and $\lim_{n \to \infty} z_n^{(i)} = 0$. Substituting (6) into (19)$_2$, we have

$$\Delta(y_n - c\lambda^{-\gamma}y_{n-\gamma}) + (1-1/\lambda)(y_n - c\lambda^{-\gamma}y_{n-\gamma}) + p_n^{(2)}\lambda^{-\tau-1}y_{n-\tau} - q_n^{(2)}\lambda^{-\sigma-1}y_{n-\sigma} = 0.$$  \hspace{1cm} (23)

By (22) and (23), we have

$$\Delta x_n = -\left(1 - \frac{1}{\lambda}\right)y_n - h_n^{(2)}y_{n-\tau} < 0, \quad n \geq n_1.$$  \hspace{1cm} (24)

Summing above inequality from $n$ to $\infty$, we have

$$y_n \geq c\lambda^{-\gamma}y_{n-\gamma} + \sum_{i=n}^{\infty} \left(1 - \frac{1}{\lambda}\right) y_i - \sum_{i=n}^{\infty} c\left(1 - \frac{1}{\lambda}\right)\lambda^{-\gamma}y_i$$

$$+ \sum_{i=n-\tau}^{\infty} p_i^{(2)}\lambda^{-\tau-1}y_i - \sum_{i=n-\sigma}^{\infty} q_i^{(2)}\lambda^{-\sigma-1}y_i$$

$$\geq c\lambda^{-\gamma}y_{n-\gamma} + \sum_{i=n}^{\infty} \left(1 - \frac{1}{\lambda}\right) y_i - \sum_{i=n}^{\infty} c\left(1 - \frac{1}{\lambda}\right)\lambda^{-\gamma}y_i$$

$$+ \sum_{i=n-\tau}^{\infty} p_i^{(1)}\lambda^{-\tau-1}y_i - \sum_{i=n-\sigma}^{\infty} q_i^{(1)}\lambda^{-\sigma-1}y_i,$$

i.e.,

$$y_n \geq c\lambda^{-\gamma}y_{n-\gamma} + \sum_{i=n-\tau}^{n-\gamma-1} \left(1 - \frac{1}{\lambda}\right)\lambda^{-\gamma}y_i + \sum_{i=n-\tau}^{n-\sigma-1} q_i^{(1)}\lambda^{-\sigma-1}y_i + \sum_{i=n}^{\infty} \left(1 - \frac{1}{\lambda}\right) y_i + \sum_{i=n-\tau}^{\infty} h_i^{(1)\lambda}y_i.$$  \hspace{1cm} (25)

By Lemma 1, the equality

$$y_n = c\lambda^{-\gamma}y_{n-\gamma} + \sum_{i=n-\tau}^{n-\gamma-1} \left(1 - \frac{1}{\lambda}\right)\lambda^{-\gamma}y_i + \sum_{i=n-\tau}^{n-\sigma-1} q_i^{(1)}\lambda^{-\sigma-1}y_i$$

$$+ \sum_{i=n}^{\infty} \left(1 - \frac{1}{\lambda}\right) y_i + \sum_{i=n-\tau}^{\infty} h_i^{(1)\lambda}y_i$$

has an eventually positive solution. Thus, (19)$_1$ has an eventually positive solution which is a contradiction. The proof is complete.
Next, we establish several explicit criterion for oscillation of equation (1).

Define sequences \( f_m^\lambda(n) \) as

\[
f_1(n) = 1 + c_n\lambda^{-\gamma} + \sum_{i=n-\tau}^{n-1} q_{i+\sigma}\lambda^{-\sigma-1} y_i + \sum_{i=n-\tau}^{n-1} c_{i+\gamma}\lambda^{-\gamma} \left( 1 - \frac{1}{\lambda} \right) y_i;
\]

\[
f_m^\lambda(n) = 1 + c_n\lambda^{-\gamma} f_m^\lambda(n-\gamma) + \sum_{i=n-\tau}^{n-1} q_{i+\sigma}\lambda^{-\sigma-1} f_m^\lambda(i)
+ \sum_{i=n-\tau}^{n-1} c_{i+\gamma}\lambda^{-\gamma} \left( 1 - \frac{1}{\lambda} \right)^m f_m^\lambda(i), \quad m = 1, 2, \ldots.
\]

**Theorem 3.** Suppose that there exists a \( \lambda > 1 \) such that (9) and (12) holds. One of following case holds.

(i) \( (1 - 1/\lambda) f_m^\lambda(n) \geq 1 \) for some integer \( m \).

(ii) \( (1 - 1/\lambda) f_m^\lambda(n) < 1 \) for all integers \( m \) and the difference inequality

\[
\Delta u_n + \frac{\prod_{i=n-\tau}^n \left[ 1 - (1 - 1/\lambda) f_m^\lambda(i) \right]}{\prod_{i=n-\tau}^n \left[ 1 - (1 - 1/\lambda) f_m^\lambda(i) \right]} u_{n-\tau} \leq 0 \tag{24}
\]

has no eventually positive solution.

Then, all the solutions of equation (1) oscillates.

**Proof.** Suppose to the contrary that \( \{x_n\} \) is an eventually positive solution of equation (1). By (8)

\[
y_n = z_n + c_n\lambda^{-\gamma} y_{n-\gamma} + \sum_{i=n-\tau}^{n-1} q_{i+\sigma}\lambda^{-\sigma-1} y_i + \sum_{i=n-\tau}^{n-1} c_{i+\gamma}\lambda^{-\gamma} \left( 1 - \frac{1}{\lambda} \right) y_i
\geq z_n + c_n\lambda^{-\gamma} z_{n-\gamma} + \sum_{i=n-\tau}^{n-1} q_{i+\sigma}\lambda^{-\sigma-1} z_i + \sum_{i=n-\tau}^{n-1} c_{i+\gamma}\lambda^{-\gamma} \left( 1 - \frac{1}{\lambda} \right) z_i
\geq \left[ 1 + c_n\lambda^{-\gamma} + \sum_{i=n-\tau}^{n-1} q_{i+\sigma}\lambda^{-\sigma-1} + \sum_{i=n-\tau}^{n-1} c_{i+\gamma}\lambda^{-\gamma} \left( 1 - \frac{1}{\lambda} \right) \right] z_n
= f_1(n)z_n.
\]

Repeating above procedure, we have

\[
y_n \geq z_n + c_n\lambda^{-\gamma} f_1(n-\gamma) z_{n-\gamma} + \sum_{i=n-\tau}^{n-1} q_{i+\sigma}\lambda^{-\sigma-1} f_1(i) z_i + \sum_{i=n-\tau}^{n-1} c_{i+\gamma}\lambda^{-\gamma} \left( 1 - \frac{1}{\lambda} \right) f_1(i) z_i
\geq f_m^\lambda(n) z_n.
\]

By induction, we obtain

\[
y_n \geq f_m^\lambda(n) z_n, \quad m = 1, 2, \ldots.
\]

Substituting it into (14), we have

\[
\Delta z_n \leq - \left( 1 - \frac{1}{\lambda} \right)^m f_m^\lambda(n) z_n - h_n^\lambda f_m^\lambda(n-\tau) z_{n-\tau}. \tag{25}
\]

Let \( u_n = 1/(\prod_{i=1}^{n-1} [1 - (1 - 1/\lambda) f_m^\lambda(i)])z_n \), \( n \geq 2 \), and \( u_1 = z_1 \). Then, \( u_n > 0 \) and substitute it into (25) to get

\[
\Delta u_n + \frac{h_n^\lambda f_m^\lambda(n-\tau)}{\prod_{i=n-\tau}^n [1 - (1 - 1/\lambda) f_m^\lambda(i)]} u_{n-\tau} \leq 0,
\]

which is a contradiction.

By using Lemma 3, we can obtain following corollary.
COROLLARY 2. Suppose that (17) and (18), and there exists $k > 0$ such that one of the following cases hold:

(i) $k \geq r^{-1/\eta}$;
(ii) $k \leq r^{-1/\eta}$ and

$$
\sum_{i=n-\tau}^{n-\gamma} c_{i+\gamma} r^{-\gamma/\eta} + \sum_{i=n-\tau}^{n-\sigma-1} q_{i+\sigma} r^{-(\sigma+1)/\eta} - \sum_{i=n-\tau}^{n-\gamma-1} c_{i+\gamma} r^{-(\gamma+1)/\eta} \geq k,
$$

where $r$ is defined in Corollary 1. Then,

$$
\liminf_{n \to \infty} \frac{h_n^{1/\eta}}{r^{-1/\eta} - k} > \frac{(r+1)^{\tau+1}}{(r+1)^\tau}
$$

is a sufficient condition for oscillation of every solution of equation (1).

PROOF. We define sequence $\{b_m\}$ as

$$
b_1 = k, \hspace{1cm} b_{m+1} = 1 + k b_m, \hspace{1cm} m = 1, 2, \ldots
$$

Taking $\lambda = r^{-1/\eta}$, in view of (26), we have

$$
f_1^{1/\eta}(n) = c_n r^{-\gamma/\eta} + \sum_{i=n-\tau}^{n-\sigma-1} q_{i+\sigma} r^{-(\sigma+1)/\eta} + \sum_{i=n-\tau}^{n-\gamma-1} c_{i+\gamma} r^{-(\gamma+1)/\eta} \geq b_1.
$$

Thus,

$$
f_m^{1/\eta}(n) \geq 1 + b_{m-1} k = b_m, \hspace{1cm} m = 2, 3, \ldots
$$

As $\{b_m\}$ is nondecreasing, it follows that there exists $b \in (0, \infty)$ such that $\lim_{m \to \infty} b_m = b$. From (17) and (26), we have

$$
1 \geq \sum_{i=n-\tau}^{n-\gamma} c_{i+\gamma} r^{-1} + \sum_{i=n-\tau}^{n-\sigma-1} q_{i+\sigma} r^{-1}
> \sum_{i=n-\tau}^{n-\gamma} c_{i+\gamma} r^{-\gamma/\eta} + \sum_{i=n-\tau}^{n-\sigma-1} q_{i+\sigma} r^{-(\sigma+1)/\eta} - \sum_{i=n-\tau}^{n-\gamma-1} c_{i+\gamma} r^{-(\gamma+1)/\eta}
\geq k.
$$

Thus, it is easy to see $b = 1/(1 - k)$. Now, we observe that

$$
\liminf_{n \to \infty} \frac{\sum_{i=n-\tau}^{n-\gamma} c_{i+\gamma} f_m^\lambda(i) - \sum_{i=n-\tau}^{n-\sigma-1} q_{i+\sigma} f_m^\lambda(i)}{1 - (1 - 1/\lambda)f_m^\lambda(i)} = \frac{1}{\tau} \liminf_{n \to \infty} \frac{h_n^{1/\eta}}{r^{-1/\eta} - k} > \frac{(r+1)^{\tau+1}}{(r+1)^\tau}
$$

On the other hand, (17) and (18) imply that (9) and (12) hold. By applying Lemma 3 and Theorem 3, we obtain that every solution of equation (1) is oscillatory. The proof is complete.

For the convenience, in following theorem, we let

$$
p_n = \min_{n-\rho^* \leq i \leq n} p_i, \hspace{1cm} q_n^* = \max_{n-\rho^* \leq i \leq n} q_i.
$$
THEOREM 4. Suppose that $c_n = c$ and there exists $q > 0$, such that

$$q_n \leq q.$$  \hspace{1cm} (27)

Suppose further that there exists $\lambda > 1$, such that $g_\lambda ^\gamma \geq 0$ and

$$q \sigma ^{\gamma - 1} + c(\gamma - 1) \lambda ^{- \gamma} < 1 - \frac{1}{\lambda}.$$  \hspace{1cm} (28)

If the difference inequality

$$\Delta u_n + g_\lambda ^\gamma u_{n-\tau} \leq 0$$  \hspace{1cm} (29)

or

$$\Delta w_n - g_\lambda ^\gamma w_{n+\gamma - \tau} \geq 0,$$  \hspace{1cm} (30)

where

$$g_\lambda ^\gamma = \begin{cases} 
\frac{p_{n} - c q_{\gamma} \lambda ^{\gamma - \sigma}}{c}, & \quad \gamma > \tau, \\
p_{n} - c q_{\gamma} \lambda ^{\gamma - \sigma} - c \left( \frac{1}{\lambda} - 1 \right), & \quad \tau \geq \gamma,
\end{cases}$$

has no eventually positive solution, then all the solutions of equation (1) oscillate.

PROOF. Suppose that on the contrary, that $\{x_n\}$ is an eventually positive solution. Now, we turn equation (8) into the following form

$$\Delta y_n + \left( 1 - \frac{1}{\lambda} \right) y_n + c \lambda ^{- \gamma - 1} y_{n-\gamma} + p_n \lambda ^{- \gamma - 1} y_{n-\tau} - q_n \lambda ^{- \sigma - 1} y_{n-\sigma} - c \lambda ^{- \gamma} y_{n+1-\gamma} = 0.$$  \hspace{1cm} (31)

It follows that

$$\Delta y_n + \left( 1 - \frac{1}{\lambda} \right) y_n + c \lambda ^{- \gamma - 1} y_{n-\gamma} + p_n \lambda ^{- \gamma - 1} y_{n-\tau} - q_n \lambda ^{- \sigma - 1} y_{n-\sigma} - c \lambda ^{- \gamma} y_{n+1-\gamma} \leq 0.$$  \hspace{1cm} (32)

We multiply $q \lambda ^{- \sigma - 1}$ and sum from $n - \sigma$ to $n - 1$ for (31) to get

$$\Delta \left[ \sum_{i=n-\sigma}^{n-1} q \lambda ^{- \sigma - 1} y_i \right] + \left( 1 - \frac{1}{\lambda} \right) \left[ \sum_{i=n-\sigma}^{n-1} q \lambda ^{- \sigma - 1} y_i \right]$$

$$+ c \lambda ^{- \gamma - 1} \left[ \sum_{i=n-\sigma}^{n-1} q \lambda ^{- \sigma - 1} y_{i-\tau} \right] + p_n \lambda ^{- \tau - 1} \left[ \sum_{i=n-\sigma}^{n-1} q \lambda ^{- \sigma - 1} y_{i-\tau} \right]$$

$$- q_n \lambda ^{- \sigma - 1} \left[ \sum_{i=n-\sigma}^{n-1} q \lambda ^{- \sigma - 1} y_{i-\sigma} \right] - c \lambda ^{- \gamma} \left[ \sum_{i=n-\sigma}^{n-1} q \lambda ^{- \sigma - 1} y_{i+1-\gamma} \right] \leq 0.$$  \hspace{1cm} (33)

Similarly, multiply $c \lambda ^{- \gamma}$ and sum from $n + 1 - \gamma$ to $n - 1$ for (31) to get

$$\Delta \left[ \sum_{i=n+1-\gamma}^{n-1} c \lambda ^{- \gamma} y_i \right] + \left( 1 - \frac{1}{\lambda} \right) \left[ \sum_{i=n+1-\gamma}^{n-1} c \lambda ^{- \gamma} y_i \right]$$

$$+ c \lambda ^{- \gamma - 1} \left[ \sum_{i=n+1-\gamma}^{n-1} c \lambda ^{- \gamma} y_{i-\gamma} \right] + p_n \lambda ^{- \tau - 1} \left[ \sum_{i=n+1-\gamma}^{n-1} c \lambda ^{- \gamma} y_{i-\tau} \right]$$

$$- q_n \lambda ^{- \sigma - 1} \left[ \sum_{i=n+1-\gamma}^{n-1} c \lambda ^{- \gamma} y_{i-\sigma} \right] - c \lambda ^{- \gamma} \left[ \sum_{i=n+1-\gamma}^{n-1} c \lambda ^{- \gamma} y_{i+1-\gamma} \right] \leq 0.$$  \hspace{1cm} (34)
Let
\[ v_n = y_n + \sum_{i=n-\sigma}^{n-1} q\lambda^{-\sigma-1} y_i + \sum_{i=n+1-\gamma}^{n-1} c\lambda^{-\gamma} y_i. \] (35)
Combining (31), (33), and (34), we easily see that \( v_n \) satisfies (31) and \( v_n > 0 \). Thus,
\[ \Delta v_n + \left( 1 - \frac{1}{\lambda} \right) v_n + c\lambda^{-\gamma} v_{n-\gamma} + p_{*n}\lambda^{-\tau-1} v_{n-\tau} - q_n^*\lambda^{-\sigma-1} v_{n-\sigma} - c\lambda^{-\gamma} v_{n+1-\gamma} \leq 0. \] (36)
From (32) and (35), we obtain
\[ \Delta v_n \leq -\left( 1 - \frac{1}{\lambda} - q\lambda^{-\sigma-1} - c\lambda^{-\gamma} \right) y_n - (q - q_n^*)\lambda^{-\sigma-1} y_{n-\sigma} - p_{*n}\lambda^{-\tau-1} y_{n-\tau} < 0. \] (37)
**CASE 1.** \( \gamma > \tau \). From (36) and (37), it follows that
\[ \Delta v_n + \left( 1 - \frac{1}{\lambda} \right) v_n - c\lambda^{-\gamma} v_{n-\gamma+1} + c\lambda^{-\gamma-1} v_{n-\gamma} + (p_{*n}\lambda^{-\tau-1} - q_n^*\lambda^{-\sigma-1}) v_{n-\tau} \leq 0. \] (38)
Let
\[ \bar{v}_n = v_n \lambda^n \] (39)
and so \( \bar{v}_n > 0 \). We substitute (39) into (38) to get
\[ \Delta(\bar{v}_n - c\bar{v}_{n+1-\gamma}) + (p_{*n} - q_n^*\lambda^{\gamma-\sigma})\bar{v}_{n-\tau} \leq 0. \] (40)
Let \( \bar{u}_n = \bar{v}_n - c\bar{v}_{n-\gamma} \). We assert that \( \bar{u}_n < 0 \). Otherwise, we assume that \( \bar{u}_n > 0 \) and so \( \bar{v}_n > c\bar{v}_{n-\gamma} \) which implies that that \( \bar{u}_n \) is bounded from below by a positive constant \( m \). From (40), we obtain
\[ \Delta\bar{u}_n \leq -m \left( p_{*n} - q_n^*\lambda^{\gamma-\sigma} \right), \]
which implies that \( \lim_{n \to \infty} \bar{u}_n = -\infty \). This is a contradiction. On the other hand, \( \bar{u}_n > -cu_{n-\gamma} \). We substitute it into (40) to get
\[ \Delta\bar{u}_n - \frac{p_{*n} - q_n^*\lambda^{\gamma-\sigma}}{c}\bar{u}_{n+\gamma-\tau} < 0. \]
Let \( u_n = -\bar{u}_n \) and so \( u_n > 0 \). Hence,
\[ \Delta u_n - \frac{p_{*n} - q_n^*\lambda^{\gamma-\sigma}}{c}u_{n+\gamma-\tau} > 0 \]
or
\[ \Delta u_n - g^\lambda u_{n+\gamma-\tau} \geq 0, \]
which is a contradiction.
**CASE 2.** \( \gamma \geq \tau \). By (36) and (37),
\[ \Delta v_n + \left( 1 - \frac{1}{\lambda} \right) v_n + (c\lambda^{-\gamma-1} - c\lambda^{-\gamma}) v_{n+1-\gamma} + (p_{*n}\lambda^{-\tau-1} - q_n^*\lambda^{-\sigma-1}) v_{n-\tau} \leq 0. \] (41)
Let \( v_n = w_n \lambda^{-n} \) and so \( w_n > 0 \). We substitute it into (41) to get
\[ \Delta w_n + c\left( \frac{1}{\lambda} - 1 \right) w_{n+1-\gamma} + (p_{*n} - q_n^*\lambda^{\gamma-\sigma}) w_{n-\tau} \leq 0. \] (42)
It follows that
\[ \Delta w_n + \left[ p_{*n} - q_n^*\lambda^{\gamma-\sigma} - c\left( \frac{1}{\lambda} - 1 \right) \right] w_{n+1-\gamma} \leq 0. \] (43)
This is a contradiction.
COROLLARY 3. Suppose that \( c_n \equiv c \) and there exists \( \lambda > 1 \), such that (27) holds and \( g_{n}^\lambda \geq 0 \). Then,

\[
\liminf_{n \to \infty} \left[ \frac{1}{\delta} \sum_{n-\delta}^{n-1} g_{n}^\lambda \right] > \frac{\delta}{(\delta + 1)^{(\delta + 1)}}
\]

where

\[
\lambda_0 = q\sigma + c(\gamma - 1) + 1, \quad \delta = \begin{cases} \gamma - \tau, & \gamma > \tau, \\ \gamma - 1, & \tau \geq \gamma, \end{cases}
\]

is a sufficient condition for oscillation of every solution of equation (1).

PROOF. Let

\[
l(\lambda) = q\sigma \lambda^{-\sigma - 1} + c(\gamma - 1)\lambda^{-\gamma} - 1 + \frac{1}{\lambda}.
\]

We take \( \lambda_0 = q\sigma + c(\gamma - 1) + 1 \), and so

\[
l(\lambda_0) = q\sigma \lambda_0^{-\sigma - 1} + c(\gamma - 1)\lambda_0^{-\gamma} - 1 + \frac{1}{\lambda_0} - 1
\leq \left[q\sigma + c(\gamma - 1) + 1\right]\lambda_0^{-1} - 1 = 0.
\]

Hence, (28) holds. By combining with Lemma 3 and Theorem 1, we obtain that all solutions of equation (1) are oscillatory. The proof is complete.

REFERENCES