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NOTE

# A NOTE ON PATH-PERFECT GRAPHS

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In this paper we explore the concept of factoring a graph into non-isomorphic paths. Let $P_i$ denote the path of length $i$. We say that a graph $G$ having $\frac{1}{2}n(n+1)$ edges is path-perfect if $E(G)$ can be partitioned as $E_1 \cup E_2 \cup \cdots \cup E_n$ such that the subgraph of $G$ induced by $E_i$ is isomorphic to $P_i$, for $1 \leq i \leq n$. It is noted that the graphs $K_n$, $K(r, 2r-1)$ and $K(r, 2r+1)$ are path-perfect. Also some results are given concerning the existence of regular path-perfect graphs.

In this paper we follow the terminology of Behzad and Chartrand [1], except that $T_i$ shall denote a tree of size $i$.

If a graph has $\frac{1}{2}n(n+1)$ edges for some positive integer $n$, it may be possible to decompose the graph into mutually edge-disjoint trees; one for each size $i$, $1 \leq i \leq n$. In this paper we consider this notion, where the trees in the decomposition are all paths. Specifically, we define a graph $G$ having $\frac{1}{2}n(n+1)$ edges to be *path-perfect* if $E(G)$ can be partitioned into $E_1 \cup E_2 \cup \cdots \cup E_n$ so that the subgraph of $G$ induced by $E_i$ is isomorphic to $P_i$, for $1 \leq i \leq n$.

As an example, consider the famous Petersen graph, shown in Fig. 1. This graph is path-perfect; one possible decomposition of it would be

$$P_1 = v_7v_8, \qquad P_2 = v_4v_9v_{10}, \qquad P_3 = v_5v_1v_8v_9,$$

$$P_4 = v_3v_7v_6v_{10}v_2, \quad \text{and} \quad P_5 = v_1v_2v_3v_4v_5v_6.$$

The problem of decomposing a complete graph into trees of different size has been studied by Gyárfás and Lehel [3], Straight [5, 6] and Zaks and Liu [8]. Gyárfás and Lehel showed that $K_n$ can be decomposed into $T_1, T_2, \ldots, T_{n-1}$ provided that each $T_i$ is a path or a star, or if each $T_i$, with at most two exceptions, is a star. The first of these results was also discovered by Zaks and Liu, who gave a nice proof. Straight extended these results by showing that a decomposition exists if for each $i$

(1) $T_i$ is a star, a path, or a comb, or
(2) $\Delta(T_i) \geq i-1$, with at most two exceptions, or
(3) $\Delta(T_i) \geq i-2$, with at most one exception.

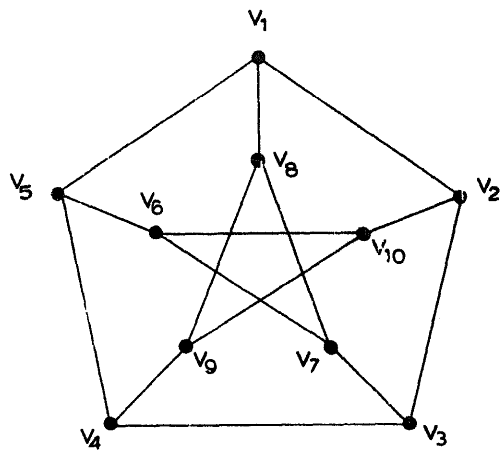


Fig. 1

It follows that $K_n$ can be decomposed into $T_1, T_2, \ldots, T_{n-1}$, for $n \leq 7$.

From the above results it follows that any complete graph is path-perfect. However, because our proof of this fact is much easier than the proofs of the more general results, and also non-constructive, we give it here.

**Theorem 1.** *For each positive integer $n \geq 2$, the complete graph $K_n$ is path-perfect.*

**Proof.** If $n$ is odd, it is well-known that $K_n$ can be factored into $\frac{1}{2}(n-1)$ hamiltonian cycles (see for example [4, p. 89]). Partitioning the $i$th cycle into $P_i$ and $P_{n-i}$, $1 \leq i \leq \frac{1}{2}(n-1)$, yields paths of the desired lengths.

For even $n$, Beineke [2] has shown that $K_n$ can be factored into $\frac{1}{2}n$ hamiltonian paths. We may partition the $i$th path into $P_i$ and $P_{n-i-1}$ for $1 \leq i \leq \frac{1}{2}n-1$, and use the remaining path as $P_{n-1}$. □

The complete bipartite graphs $K(s, t)$ form another important and often studied family of graphs. This graph has $st$ edges and hence is potentially path-perfect if $2st = n(n+1)$ for some $n$. We state the next result without proof

**Theorem 2.** *Let $r$ be a positive integer. Then the graphs $K(r, 2r-1)$ and $K(r, 2r+1)$ are path-perfect.*

This result has been extended by Zaks and Liu [8], to allow stars in the decomposition.

Suppose $2st = n(n+1)$. If $s$ is too small, namely, if $s < \frac{1}{2}(n-1)$, then it is impossible for $K(s, t)$ to be path-perfect. For then the longest path in $K(s, t)$ has length $2s$, and $2s < n$. Theorem 2 tells us, however, that in the "worst" possible case, when $s = \{\frac{1}{2}(n+1)\}$, that $K(s, t)$ is path-perfect. We would conjecture that $K(s, t)$ is path-perfect whenever $2st = n(n+1)$ and $\frac{1}{2}(n-1) \leq s \leq t$.

Another class of potentially path-perfect graphs is formed by graphs whose complements are trees. Deciding whether or not these graphs are path-perfect would solve another special case of the problem of factoring $K_n$ into trees.

The girth of a graph is the length of the smallest cycle which it contains. An $n$-cage is a graph of minimum order which is 3-regular and has girth $n$. The complete graph $K_4$ is the unique 3-cage and the Petersen graph is known to be the only 5-cage. We have shown both of these graphs are path-perfect. Are any other cages path-perfect? We next turn our attention to the question of the existence of regular path-perfect graphs.

If $G$ is an $r$-regular graph of order $p$ and size $\frac{1}{2}n(n+1)$, then

$$pr = n(n+1). \tag{1}$$

Hence $r$ must divide $n(n+1)$. We first consider the case where $r$ is odd and $r$ divides $n$ or $r$ divides $n+1$.

**Theorem 3.** *Let $r$ be an odd positive integer. There exists an $r$-regular path-perfect graph of order $m(mr+1)$ if, and only if, $m$ equals 1. Also, there exists an $r$-regular path-perfect graph of order $m(mr-1)$ if, and only if, $m$ equals 2.*

**Proof.** Let $r$ be odd and suppose $G$ is an $r$-regular path-perfect graph of order $p$ with $\frac{1}{2}n(n+1)$ edges. Then $pr = n(n+1)$. Since each vertex of $G$ must be an end vertex of at least one path, $p$ is at most $2n$. We consider two cases.

*Case* 1. If $p$ equals $m(mr+1)$, then $n$ equals $mr$. Thus we have $m(mr+1) \leqslant 2mr$, which implies that $mr+1 \leqslant 2r$, whence $m$ equals 1. Conversely, $K_{r+1}$ is an $r$-regular path-perfect graph of order $r+1$.

*Case* 2. If $p$ equals $m(mr-1)$, then $n$ equals $mr-1$. Thus $m(mr-1) \leqslant 2(mr-1)$, whence $m \leqslant 2$. Also as $p$ is at least $r+1$, $m$ is at least 2. Therefore, $m$ equals 2.

Conversely, we must exhibit an $r$-regular path-perfect graph of order $2(2r-1)$ for each odd integer $r$. If $r$ is equal to 1 or 3, we can use as examples $K_2$ and the Petersen graph, respectively. For $r \geqslant 5$, we define a graph $H_r$ as follows. Let $V(H_r) = X \cup Y \cup U \cup V$, where

$$X = \{x_1, x_2, \ldots, x_r\}, \qquad Y = \{y_1, y_2, \ldots, y_r\},$$
$$U = \{u_1, u_2, \ldots, u_{r-1}\} \text{ and } V = \{v_1, v_2, \ldots, v_{r-1}\}.$$

Form the complete bipartite graphs on $X$ and $Y$, and on $U$ and $V$. $H_r$ results by removing the edge $x_1y_r$ and adding the edges $x_1v_1$, $v_{r-1}y_r$, $u_1u_2$, $u_3u_4, \ldots u_{r-2}u_{r-1}$, and $v_2v_3$, $v_4v_5, \ldots v_{r-3}v_{r-2}$. Clearly $H_r$ is $r$-regular and has $2(2r-1)$ vertices. However, the verification that $H_r$ is path-perfect will not be given here. It relies heavily on the fact that $K(r, r)$ can be factored into the odd-length paths $P_1, P_3, \ldots, P_{2r-1}$.

We have from (1) that $r$ must divide the product $n(n+1)$. If $r$ is a power of a prime integer, then $r$ must divide either $n$ or $n+1$. We thus obtain the following corollary.

**Corollary 1.** *If $G$ is an $r$-regular path-perfect graph and $r$ is a power of an odd prime number, then either $G$ equals $K_{r+1}$, or the order of $G$ is $2(2r-1)$.*

It follows from Corollary 1 that $K_4$ and the Petersen graph are the only path-perfect cages.

If $r$ is even, (1) can be written in the form

$$p(\tfrac{1}{2}r) = \tfrac{1}{2}n(n+1) \tag{2}$$

Suppose $n(n+1) = 2ts$, where $t$ and $s$ are integers and $t \geqslant n$. In [7] it was shown that $\{1, 2, \ldots, n\}$ could then be partitioned into $s$ pairwise-disjoint subsets so that the sum of the elements in any subset is $t$. This result was then applied to show the following.

**Theorem 4.** *Let $p$ and $n$ be positive integers and let $r$ be an even positive integer such that $p \geqslant r+1$ and $pr = n(n+1)$. Then there exists an $r$-regular path-perfect graph of order $p$ and size $\frac{1}{2}n(n+1)$.*

We are left with one unsolved case regarding the existence of regular path-perfect graphs – when $r$ is odd and $r$ divides neither $n$ nor $n+1$.

For example, does there exist a 15-regular path-perfect graph of order 28 and size 210?

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