Cauchy Problem and Initial Trace for a Doubly Degenerate Parabolic Equation with Strongly Nonlinear Sources

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1. INTRODUCTION

In this paper, we consider the Cauchy problem

\[ u_t = \text{div}(|Du|^p - 2Du^m) + u^q, \quad (x, t) \in S_T = \mathbb{R}^N \times (0, T), \quad (1.1) \]

\[ u(x, 0) = u_0(x) \geq 0, \quad x \in \mathbb{R}^N, \quad (1.2) \]

where \( p > 1, m > 0, m(p - 1) > 1, q > 1, N \geq 1, \) and \( u_0 \in L^1_{\text{loc}}(\mathbb{R}^N) \).

Equation (1.1) has been suggested as a mathematical model for a variety of physical problems (see [1, 2]). In particular, the evolution \( p \)-Laplacian equation \( (m = 1) \) and porous media equation \( (p = 2) \) are some special cases of (1.1). In this paper, we are interested in solving the Cauchy problem (1.1)–(1.2) for the largest possible class of initial functions.

**Definition 1.1.** A nonnegative measurable function \( u(x, t) \) defined in \( S_T \) is called a weak solution of (1.1)–(1.2) if for every bounded open set \( \Omega \) with smooth boundary \( \partial \Omega \),

\[ u \in C_{\text{loc}}(S_T), \quad u^m \in L^p_{\text{loc}}(0, T; W^{1,p}(\Omega)), \quad (1.3) \]

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and
\[
\int_{\Omega} u(x, t) \phi(x, t) \, dx + \int_{t_0}^{t} \int_{\Omega} (-u \phi_t + |Du|^p D\phi : D\phi) \, dx \, d\tau \\
= \int_{t_0}^{t} \int_{\Omega} u^q \phi \, dx \, d\tau + \int_{\Omega} u(x, t_0) \phi(x, t_0) \, dx
\]
(1.4)
for all \(0 \leq t_0 \leq t \leq T\) and all test functions \(\phi \in C^1(\overline{\Omega} \times [0, T])\) that equal zero near \(\partial \Omega \times (0, T)\). Moreover,
\[
\lim_{t \to 0^+} \int_{B_R} |u(x, t) - u_0(x)| \, dx = 0 \quad \forall \, R > 0.
\]
(1.5)

Weak subsolutions (resp. supersolutions) are defined in the same way except that the \(=\) in (1.4) is replaced by \(\leq\) (resp. \(\geq\)) and \(\phi\) is taken to be nonnegative.

To describe the class of initial data, we introduce the norm \(\|f\|_{h}\). Let
\[
B_\rho(x) = \{y \in R^N : |x - y| < \rho\}.
\]
For \(f \in L^h_{\text{loc}}(R^N)\), \(h \geq 1\), we define
\[
\|f\|_h = \sup_{x \in R^N} \left( \int_{B_\rho(x)} |f(y)|^h \, dy \right)^{1/h}.
\]

We shall prove the following results.

**Theorem 1.1.** Assume that \(u_0 \geq 0\) and
\[
\|u_0\|_h < \infty,
\]
(1.6)
where \(h = 1\) if \(q < m(p - 1) + p/N\) and \(h > (N/p)(q - m(p - 1))\) if \(q \geq m(p - 1) + p/N\). Then there exist a constant \(C = C(N, p, q, h, m)\) and a positive \(T_0\) defined by
\[
T_0 \|u_0\|_h^{m(p-1)-1} + T_0^{1+N(m(p-1)-q)/(ph)} \|u_0\|_{h^{-1}}^{q-1} = C^{-1}
\]
(1.7)
such that a weak solution \(u\) to (1.1)–(1.2) exists in the strip \(S_{T_0}\) and satisfies, \(\forall \, 0 < t < T_0\),
\[
\|u(\cdot, t)\|_h \leq C \|u_0\|_h, \quad 0 < t < T_0, \quad u(x, t) \leq C t^{-N/\kappa_h}, \quad \kappa_h = N(m(p - 1) - 1) + ph \quad \forall \, x \in R^N, \quad (1.8)
\]
\[
\int_{0}^{t} \int_{B_\rho(x_0)} |Du|^\sigma \, dx \, d\tau \leq C t^{1-\sigma/p-N((m+1)\sigma-p)/(ph)}
\]
\[
\times \left( \sup_{0 < r < t} \int_{B_{2r}(x_0)} u(x, \tau) \, dx \right)^{1+((m+1)\sigma-p)/\kappa}, \quad (1.10)
\]
where \(p/(m+1) < \sigma < p - mN/(mN + 1)\), \(\kappa = N(m(p - 1) - 1) + p\), and \(C\) also depends on \(\sigma\).
Theorem 1.2. Let \( u \) be a supersolution of (1.1) in \( S_T \) and let \( q > m \times (p - 1) \). Then there exists a constant \( C = C(p, q, m, N) \) such that for all \( 0 < t < T \leq 1 \),

\[
\|u(\cdot, t)\|_1 + \int_0^t \|u(\cdot, \tau)\|_q^q \, d\tau \leq C(1 + (T - t)^{-1/(q-1)}),
\]

where the constant \( C \) depends also on \( T \) if \( T > 1 \). If \( 1 < q \leq m(p - 1) \), then there exists a constant \( C = C(p, q, m, N) \) such that for all \( 0 < t < T \),

\[
\|u(\cdot, t)\|_1 \leq C((T - t)^{-1/(m(p-1)-1)} + (T - t)^{(-p + N(m(p-1)))/(p(q-1))}).
\]

Theorem 1.3. Let \( u \) be a weak solution of (1.1) in \( S_T \) and let \( q > m \times (p - 1) \). Then there exists a unique measure \( \mu \) such that

\[
u(\cdot, t) \to \mu \quad \text{in the sense of measure as } t \to 0.
\]

Moreover,

\[
\sup_{x \in \mathbb{R}^N} \int_{B_1(x)} d|\mu| \leq C(1 + T^{-1/(q-1)}).
\]

We now state a uniqueness result of the solution of the Cauchy problem (1.1)–(1.2) in a class \( S \) suggested by the properties of the solution found in Theorem 1.1.

We say that a function \( u: S_T \to \mathbb{R}^+ \) is a solution of class \( S \) to (1.1)–(1.2) if \( u \) satisfies (1.4) and

\[
u \in L^1(S_T), \quad u^m \in L^p_{\text{loc}}(0, T; W^{1, p}(\mathbb{R}^N)),
\]

\[
u_t \in L^1_{\text{loc}}(0, T; L^1(\mathbb{R}^N)), \quad \lim_{t \to 0} \int_{\mathbb{R}^N} |u(x, t) - u_0(x)| \, dx = 0,
\]

\[
\sup_{x \in \mathbb{R}^N} u(x, t) \leq Ct^{-\delta} \quad \forall t \in (0, T),
\]

where \( \delta, C \) are given positive constants (depending on \( u \)) with \( \delta < \frac{1}{q-1} \).

Estimate (1.9) shows that the preceding solution actually satisfies (1.17).

Theorem 1.4. Assume that \( u, v \) are two solutions of class \( S \) to (1.1)–(1.2) corresponding to the same initial datum \( u_0 \in L^1(\mathbb{R}^N) \). Then \( u \equiv v \) in \( S_T \).

Theorems 1.2 and 1.3 show that if \( q > 1 \), then the growth condition (1.6) is optimal. For the porous medium equation \( (p = 2) \) with strongly nonlinear sources, the problem of growth condition on the initial datum \( u_0 \) was
studied in [3]. For the evolution $p$-Laplacian equation ($m = 1$), the anal-
gon problem was studied in [4]. In this paper, some ideas in [3, 4] are used. 
Whereas Eq. (1.1) is degenerate at the points where $|Du| = 0$ and $u = 0$, 
there exist some new difficulties to be overcome. 
We denote by $C = C(a_1, a_2, \ldots, a_n)$ ($n \in N$), a positive constant that can 
be determined a priori in terms of the specified quantities $a_1, a_2, \ldots, a_n$.

2. PROOF OF THEOREM 1.1

2.1. Main Estimates

PROPOSITION 2.1. Let $u$ be any weak subsolution of (1.1) in $S_T$ for some 
$0 < T < \infty$. Then for fixed $h \geq 1$ there exists a constant $C$ depending only on 
$N, p, q, h, m$ such that for every ball $B_{2\rho}(x_0)$ and for all $t \in (0, T)$ satisfying 

$$
\rho^{-p} \|u(\cdot, \tau)\|_{\infty, B_{2\rho}(x_0)}^{(p-1)\frac{1}{p}} + \|u(\cdot, \tau)\|_{\infty, B_{2\rho}(x_0)}^{q-1} \leq \tau^{-1}, \quad \tau \in (0, t),
$$

(2.1)

the estimate

$$
\|u(\cdot, t)\|_{\infty, B_{2\rho}(x_0)} \leq C \left( \frac{1}{t} \int_0^t \int_{B_{2\rho}(x_0)} u^h \, dx \, d\tau \right)^{\frac{p}{p+1}}
$$

(2.2)

holds, where $\kappa_h$ is given by (1.9).

Proof. Let $\rho > 0$, $\sigma \in (0, \frac{1}{2}]$ be fixed, let $k > 0$ remain to be chosen, 
and for $n = 0, 1, 2, \ldots$, set

$$
\rho_n = \rho + \frac{\sigma}{2^n} \rho, \quad t_n = \frac{t}{2} - \left( \frac{\sigma}{2^{n+1}} \right)^p t, \quad k_n = k - \frac{k}{2^{n+1}},
$$

$$
B_n = B_{\rho_n}(x_0), \quad Q_n = B_n \times (t_n, t), \quad 0 < t_n < t \leq T.
$$

Let $\xi_n(x, t)$ be a smooth cutoff function in $Q_n$ such that

$$
\xi_n = 1 \quad \text{in} \ Q_{n+1}, \quad |D\xi_n| \leq C \frac{2^{n+1}}{\sigma \rho}, \quad 0 \leq \frac{\partial \xi_n}{\partial t} \leq C \frac{2^{n+1}}{\sigma^p t}.
$$

Take the testing function $\phi$ in Definition 1.1 as

$$
\phi = (u - k_{n+1})^h \xi_n^p = (\max\{0, u - k_{n+1}\})^h \xi_n^p,
$$
and by a standard Steklov approximating procedure, we get
\[
\frac{1}{h+1} \int_{B_n(t)} (u - k_{n+1})_+^{h+1} \xi_n^p \, dx \\
+ h \left( \frac{P}{h + m(p - 1)} \right)^p \int_{t_n}^{t} \int_{B_n} |D(u - k_{n+1})_+^{(h+m(p-1)/p)} \xi_n^p| \, dx \, d\tau \\
+ m^{p-1} \int_{t_n}^{t} \int_{B_n} (u - k_{n+1})_+^{h(m-1)(p-1)} \xi_n^{p-1} |Du|^{p-2} Du \cdot D\xi_n \, dx \, d\tau \\
\leq \int_{t_n}^{t} \int_{B_n} p(u - k_{n+1})_+^{h+1} \xi_n^{p-1} \xi_n \, dx \, d\tau \\
+ \int_{t_n}^{t} \int_{B_n} (u - k_{n+1})_+^h u^q \xi_n^p \, dx \, d\tau,
\tag{2.3}
\]
where \( t_n < t' < t \).

By the Schwarz inequality,
\[
|p(u - k_{n+1})_+^{h+1} \xi_n^{p-1} |Du|^{p-2} Du \cdot D\xi_n \, dx \, d\tau| \\
\leq \frac{1}{2} h \left( \frac{P}{h + m(p - 1)} \right)^p \int_{t_n}^{t} \int_{B_n} |D(u - k_{n+1})_+^{(h+m(p-1)/p)} \xi_n^p| \, dx \, d\tau \\
+ C \int_{t_n}^{t} \int_{B_n} (u - k_{n+1})_+^{h(m-1)(p-1)} |D\xi_n|^p \, dx \, d\tau.
\]
Notice that
\[
(u - k_{n+1})_+^{h+1} \geq (u - k_{n+1})_+^h \frac{u}{2} \geq C(u - k_{n+1})_+^h u
\]
if \( u/2 > k_n \) and
\[
(u - k_{n+1})_+^{h+1} \geq (u - k_{n+1})_+^h (k_{n+1} - k_{n}) \geq 2^{n-3} u(u - k_{n+1})_+^h
\]
if \( k_{n+1} \leq u \leq 2k_{n+1} \). Thus, we have
\[
\int_{t_n}^{t} \int_{B_n} (u - k_{n+1})_+^h u^q \xi_n^p \, dx \, d\tau \leq C_2^n \int_{t_n}^{t} \int_{B_n} u^q (u - k_{n+1})_+^{h+1} \, dx \, d\tau.
\]
Substituting (a) and (b) into (2.3) leads to
\[
\text{ess sup}_{t_n < \tau < t} \int_{B_n(\tau)} (u - k_{n+1})_+^{h+1} \xi_n^p \, dx \\
+ \int_{Q_\tau} |D((u - k_{n+1})_+^{(h+m(p-1)/p)} \xi_n)|^p \, dx \, d\tau \\
\leq \frac{C_2^n p}{\sigma^p t} (1 + M) \int_{Q_\tau} (u - k_{n})_+^{h+1} \, dx \, d\tau,
\tag{2.4}
\]
From (2.4)–(2.6), we get

\[ M = \sup_{0 < \tau < t} \left\{ \rho^{-p} \| u(\cdot, \tau) \|^{m(p-1)-1}_{\infty, B_\rho(x_0)} + \| u(\cdot, \tau) \|_{p-1}_{\infty, B_\rho(x_0)} \right\}. \]

By the Gagliardo–Nirenberg inequality (see [5, p. 62]),

\[
\int Q_n \xi_n^d (u - k_{n+1})_+^b \, dx \, d\tau \\
\leq C \int Q_n \left| D((u - k_{n+1})_+^{(h+m(p-1)/p)} \xi_n) \right|^p \, dx \, d\tau \\
\times \left( \operatorname{ess sup} \int_{t_n < \tau < t} (u - k_{n+1})_+^{h+1} \xi_n^p \, dx \right)^{p/N},
\]

(2.5)

where \( b = h + m(p - 1) + p(h + 1)/N \) and \( d \) is sufficiently large.

Let \( A_n = \{ (x, t) \in Q_{n-1} : u(x, t) > k_n \}, n = 1, 2, \ldots \), and observe that

\[
\int Q_n (u - k_{n+1})_+^{h+1} \, dx \, d\tau \geq C 2^{-((h+1)n)} |A_{n+1}| k_{n+1}^{b+1}.
\]

(2.6)

From (2.4)–(2.6), we get

\[
\int Q_{n+1} (u - k_{n+1})_+^{h+1} \, dx \, d\tau \\
\leq \int Q_n (u - k_{n+1})_+^{h+1} \xi_n (h+1)/b \, dx \, d\tau \\
\leq |A_{n+1}|^{1-(h+1)/b} \left( \int Q_n (u - k_{n+1})_+^{h+1} \xi_n^d \, dx \, d\tau \right)^{(h+1)/b} \\
\leq C |A_{n+1}|^{1-(h+1)/b} \left( \frac{2n^p}{\sigma \rho t} (1 + M) \int Q_n (u - k_{n+1})_+^{h+1} \, dx \, d\tau \right)^{(1+p/N)(h+1)/b} \\
\leq C k^{-(b-h-1)(h+1)/b} C_0^{(1+p/N)(h+1)/b} (1 + M)^{(1+p/N)(h+1)/b} \\
\times \left( \int Q_n (u - k_{n+1})_+^{h+1} \, dx \, d\tau \right)^{1+(p(h+1)/bN)},
\]

where \( C_0 = 2^{(b+1)(b-h-1+p^2/N)/b} \geq 1. \) Thus, if \( k \) is chosen to satisfy

\[
\int Q_b (u - k/2)_+^{h+1} \, dx \, d\tau \leq C k^{N(h-b-1)/p} \left( \frac{1 + M}{\sigma \rho t} \right)^{(1+N/p)}
\]

then by Lemma 5.6 of [5, p. 95], we get

\[
\lim_{n \to \infty} \int Q_n (u - k_n)_+^{h+1} \, dx \, d\tau = 0,
\]
i.e., \( \|u\|_{\infty, Q_\infty} \leq k \). By the Schwarz inequality, we obtain

\[
\|u\|_{\infty, Q_\infty} \leq C \left( 1 + \frac{M}{\sigma \rho_t} \right)^{(N+p)/(N(b-h-1))} \\
\times \left( \int_{Q_\infty} (u - k/2)^{b+1} \, dx \, d\tau \right)^{p/(N(b-h-1))} \\
\leq C \left( 1 + \frac{M}{\sigma \rho_t} \right)^{(N+p)/(N(b-h-1))} \\
\times \left( \|u\|_{\infty, Q_\infty} \int_{Q_\infty} u^b \, dx \, d\tau \right)^{p/(N(b-h-1))} \\
\leq \frac{1}{2} \|u\|_{\infty, Q_\infty} + C \left( 1 + \frac{M}{\sigma \rho_t} \right)^{(N+p)/\kappa_h} \left( \int_{Q_\infty} u^b \, dx \, d\tau \right)^{p/\kappa_h}.
\]

Hence by [6, p. 161, Lemma 3.1], we get

\[
\|u\|_{\infty, B_\rho(x_0) \times (t/2, t)} \leq C \left( 1 + \frac{M}{\sigma \rho_t} \right)^{(N+p)/\kappa_h} \left( \int_{Q_\infty} u^b \, dx \, d\tau \right)^{p/\kappa_h}.
\]

This implies (2.2).

**Proposition 2.2.** Let the assumptions in Proposition 2.1 hold and let

\[
G(t) = \sup_{0 < \tau < t} \int_{B_{\rho}(x_0)} u(x, \tau) \, dx < \infty.
\]  

(2.7)

Then there exists a constant \( C(N, q, m, \sigma) \) such that for every ball \( B_{2\rho}(x_0) \), \( 0 < t < T \) satisfying (2.1), and \( p/(m+1) < \sigma < p - mN/(mN + 1) \),

\[
\int_0^t \int_{B_{\rho}(x_0)} |D u^m|^{\sigma} \, dx \, d\tau \leq C t^{1-\sigma/p-N((m+1)\sigma-p)/(p\kappa)} (G(t))^{1+((m+1)\sigma-p)/\kappa}.
\]

In particular,

\[
\int_0^t \int_{B_{\rho}(x_0)} |D u^m|^{p-1} \, dx \, d\tau \leq C t^{1/\kappa} (G(t))^{1+(m(p-1)-1)/\kappa},
\]

(2.8)

where \( \kappa = N(m(p - 1) - 1) + p \).

**Proof.** Let \( \xi(x) \) be a piecewise smooth cutoff function in \( B_{3\rho/2}(x_0) \) such that \( \xi = 1 \) in \( B_{\rho}(x_0) \) and \( |D \xi| \leq 2\rho^{-1} \). The calculations to follow are formal.
in which \( u \) is required to be strictly positive. They can be made rigorous by replacing \( u \) with \( u + \varepsilon \) and letting \( \varepsilon \to 0 \). By the Hölder inequality,

\[
\int_0^t \int_{B_{\rho}(x_0)} u^{m-1}\sigma |Du|^\sigma \, dx \, d\tau
\]

\[
= \int_0^t \int_{B_{\rho}(x_0)} \tau^\beta u^{(m-1)\sigma(p-1)/p-\delta} |Du|^{\sigma}\tau^{-\beta} u^{(m-1)\sigma/p+\delta} \, dx \, d\tau
\]

\[
\leq \left( \int_0^t \int_{B_{\rho}(x_0)} \tau^{\beta p/\sigma} u^{(m-1)(p-1)} |Du|^p \, dx \, d\tau \right)^{\sigma/p}
\]

\[
\times \left( \int_0^t \int_{B_{\rho}(x_0)} \tau^{-\beta p/(p-\sigma)} u^{((m-1)\sigma+p\delta)/(p-\sigma)} \, dx \, d\tau \right)^{1-\sigma/p}.
\]

By (2.2), we can take the testing function

\[
\phi = t^{\beta p/\sigma} u^{1-p\delta/\sigma} \xi^p(x), \quad p\delta = p - m\sigma
\]
in (1.4) to obtain

\[
\int_0^t \int_{B_{\rho}(x_0)} \tau^{\beta p/\sigma} u^{(m-1)(p-1)} |Du|^p \, dx \, d\tau
\]

\[
\leq C \rho^{-\rho} \int_0^t \int_{B_{\rho/2}(x_0)} \tau^{\rho\delta/\sigma} u^{\rho + (m-1)(p-1) - p\delta/\sigma} \, dx \, d\tau
\]

\[
+ C \int_0^t \int_{B_{\rho/2}(x_0)} \tau^{\rho\delta/\sigma - 1} u^{2-p\delta/\sigma} \, dx \, d\tau
\]

\[
+ \int_0^t \int_{B_{\rho/2}(x_0)} \tau^{\rho\delta/\sigma} u^{2-p\delta/\sigma} \, dx \, d\tau
\]

\[
\leq C(1 + M) \int_0^t \int_{B_{\rho/2}(x_0)} \tau^{\rho\delta/\sigma - 1} u^{2-p\delta/\sigma} \, dx \, d\tau
\]

\[
\leq C(1 + M) G(t) \int_0^t \int_{B_{\rho/2}(x_0)} u^{(m(p-1)-(p-1))} \|u(\cdot, \tau)\|_{\infty, B_{\rho/2}(x_0)}^{1-p\delta/\sigma} d\tau,
\] (2.9)

where

\[
M = \sup_{0 < \tau < t} \tau \left\{ \rho^{-\rho} \|u(\cdot, \tau)\|_{\infty, B_{\rho/2}(x_0)}^{m(p-1)-1} + \|u(\cdot, \tau)\|_{\infty, B_{\rho/2}(x_0)}^{q-1} \right\}.
\]

Notice that by (2.2),

\[
\|u(\cdot, t)\|_{\infty, B_{\rho/2}(x_0)} \leq C t^{-(N+p)/\kappa} \left( \int_0^t \int_{B_{\rho/2}(x_0)} u \, dx \, d\tau \right)^{p/\kappa} \leq C t^{-N/\kappa}(G(t))^{p/\kappa}
\]
and
\[
\int_0^t \int_{B_r(x_0)} \tau^{-\beta p/(p-\sigma)} u^{(m-1)\sigma + p\delta}/(p-\sigma) \, dx \, d\tau \\
\leq \int_0^t \int_{B_r(x_0)} \tau^{-\beta p/(p-\sigma)} u \, dx \, d\tau \\
\leq C t^{1-\beta p/(p-\sigma)} G(t). \tag{2.10}
\]

From (2.9) and (2.10), we have
\[
\int_0^t \int_{B_r(x_0)} |Du|^m^\sigma \, dx \, d\tau \\
\leq C(G(t))^{1+(m+1)\sigma-p}/\kappa/\beta/(1-p\sigma)/(1-\sigma/p)+\delta/(m+1)\sigma-p/N/(p\kappa).
\]

Thus, Proposition 2.2 follows by choosing \( \beta \) such that
\[
\frac{(m+1)N\sigma}{p\kappa} - \frac{N}{\kappa} < \beta < 1 - \frac{\sigma}{p}.
\]

2.2. Proof of Theorem 1.1

Consider the approximating Cauchy problem
\[
\begin{align*}
&u_t = \text{div}(|Du|^m) + \min\{u^\delta, n\}, \\
u(x,0) = u_{0\text{n}}(x),
\end{align*}
\tag{2.11}
\]

where \( u_{0\text{n}} \in C_0^\infty(R^N) \) satisfies
\[
\lim_{n \to \infty} \int_{B_r} |u_{0\text{n}} - u_0|^{\rho} = 0 \quad \forall \rho > 0
\]

and
\[
\|u_{0\text{n}}\|_h \leq C\|u_0\|_h.
\]

It is well known that there exists a solution \( u_n \) to (2.11) with \( u_n \in C(\overline{S_T}) \cap L^\infty(S_T) \) and \( Du_n^{m} \in L^\rho(S_T) \). Therefore, \( \forall t > 0, \)
\[
\sup_{0 < \tau < t} \sup_{y \in R^N} \left[ u_n^{q-1}(y, \tau) \right] \leq C(n) \tag{2.12}
\]

for a qualitative constant \( C(n) \), depending on \( n \). Theorem 1.1 will follow by a standard limiting process via the compactness results (see [7]) whence we show estimates (1.7)–(1.10) with \( u \) and \( u_0 \) replaced by \( u_n \) and \( u_{0\text{n}} \), respectively, and with the constant \( C \) independent of \( n \). It follows from (1.9) and [7] that \( u \in C_{\text{loc}}(S_T) \). To prove these estimates, we will work with (2.11) and drop the subscript \( n \).
Then (2.14) implies that
\[ \forall \xi \text{ constant} \]
\[ \kappa_h \]
\[ \therefore \]
Thus, we can choose a \( \psi \) and observe that
\[ \text{By (2.12), } \tau > 0. \]
\[ \therefore \]
Thus for \( \delta > 0 \) to be chosen, let
\[ \tau^* = \sup \{ t > 0 : t^h \psi^{m(p-1)-1}(t) + t^{(1/p)(\kappa_h-N(q-1))} \psi^{q-1}(t) \leq \delta \}. \]
Notice that \( \kappa_h - N(q - 1) > 0 \). Therefore, for all \( 0 < t < \min \{ \tau, \tau^* \} \),
\[ t \sup_{x \in \mathbb{R}^N} u^{m(p-1)-1}(x, t) + t^{q-1}(x, t) \leq C\delta^{p/h}. \]
Thus, we can choose a \( \delta = \delta(p, q, h, N, m) \) small enough to insure that \( \tau^* < \tau \). Let \( \xi \) be a nonnegative smooth cutoff function in \( B_{2\rho}(x) \) such that \( \xi = 1 \) in \( B_{\rho}(x) \), \( |D\xi| \leq C\rho^{-1} \), and we use \( u^{p-1} \xi \) as the testing function in
If \( h > 1 \), we get
\[
\int_{B_{\rho}(x)} u^h(y, t) \xi^p \, dy + h(h - 1) \int_0^t \int_{B_{\rho}(x)} \xi^p u^{(m-1)(p-1)+h-2} |Du|^p \, dy \, d\tau
\]
\[
\leq \int_{B_{\rho}(x)} u^h_0(y) \xi^p \, dy + phm^{p-1}
\]
\[
\times \int_0^t \int_{B_{\rho}(x)} u^{(m-1)(p-1)+h-1} |Du|^{p-1} |D\xi|^{p-1} \, dy \, d\tau
\]
\[
+ h \int_0^t \int_{B_{\rho}(x)} u^{q+h-1} \, dy \, d\tau.
\]
Notice that \( \rho \geq 1 \), and by the Hölder and Young inequalities,
\[
\int_{B_{\rho}(x)} u^h(y, t) \, dy \leq \int_{B_{\rho}(x)} u^h_0(y) \, dy + C \sup_{0 < \tau < t} \sup_{x \in R^N} \int_{B_{\rho}(x)} u^h(y, \tau) \, dy
\]
\[
\times \left\{ \int_0^t \tau^{-\left(N(m(p-1)-1)/\kappa_h \right)} \psi_p(m(p-1)-1)/\kappa_h(\tau) \, d\tau
\]
\[
+ \int_0^t \tau^{-\left(N(q-1)/\kappa_h \right)} \psi_q(q-1)/\kappa_h(\tau) \, d\tau \right\}
\]
\[
\leq C \| u_0 \|_{H}^{h} + C\psi(t)
\]
\[
\times \left\{ t^h \psi^{m(p-1)-1}(t) + t^{(1/p)(\kappa_h-N(q-1))} \psi^{-1}(t) \right\}^{p/\kappa_h}
\]
\[
\leq C \| u_0 \|_{H}^{h} + C\delta^{p/\kappa_h} \psi(t).
\]
By using (2.15) and (2.17), we obtain \( \forall \, t \in (0, t^*) \),
\[
\int_{B_{\rho}(x)} u^h(y, t) \, dy \leq \int_{B_{\rho}(x)} u^h_0(y) \, dy + C \sup_{0 < \tau < t} \sup_{x \in R^N} \int_{B_{\rho}(x)} u^h(y, \tau) \, dy
\]
\[
\times \left\{ \int_0^t \tau^{-\left(N(m(p-1)-1)/\kappa_h \right)} \psi_p(m(p-1)-1)/\kappa_h(\tau) \, d\tau
\]
\[
+ \int_0^t \tau^{-\left(N(q-1)/\kappa_h \right)} \psi_q(q-1)/\kappa_h(\tau) \, d\tau \right\}
\]
\[
\leq C \| u_0 \|_{H}^{h} + C\psi(t)
\]
\[
\times \left\{ t^h \psi^{m(p-1)-1}(t) + t^{(1/p)(\kappa_h-N(q-1))} \psi^{-1}(t) \right\}^{p/\kappa_h}
\]
\[
\leq C \| u_0 \|_{H}^{h} + C\delta^{p/\kappa_h} \psi(t).
\]
If \( h = 1 \), we get by Proposition 2.2,
\[
\int_{B_{\rho}(x)} u(y, t) \, dy \leq \int_{B_{\rho}(x)} u_0(y) \, dy + C \int_0^t \int_{B_{\rho}(x)} \xi^p |Du|^p \, dy \, d\tau
\]
\[
+ \int_0^t \int_{B_{\rho}(x)} u^{q-1} u \, dy \, d\tau
\]
\[
\leq \int_{B_{\rho}(x)} u_0(y) \, dy + C t^{1/\kappa} \left( \sup_{0 < \tau < t} \int_{B_{\rho}(x)} u(y, \tau) \, dy \right)^{1+(m(p-1)-1)/\kappa}
\]
\[
+ \int_0^t \int_{B_{\rho}(x)} u^{q-1} u \, dy \, d\tau.
\]
We use (2.15) and (2.17) to get \( \forall t \in (0, t^*) \),
\[
\int_{B_2(x)} u(y, t) \, dy \leq \int_{B_2(x)} u_0(y) \, dy + C \sup_{0 < \tau < t} \sup_{x \in \mathbb{R}^N} \int_{B_2(x)} u(y, \tau) \, dy
\]
\[
\times \left\{ t^{1/\kappa} \psi^{(m(p-1)-1)/\kappa}(t) + \int_0^t \tau^{-N(q-1)/\kappa} \psi^{p(q-1)/\kappa} \, d\tau \right\}
\]
\[
\leq C \|u_0\|_h^h + C\delta^{p/\kappa} \psi(t).
\] (2.21)

By (2.19) and (2.21), we can determine \( \delta = \delta(p, q, h, m, N) \) such that
\[
\psi(t) \leq C \|u_0\|_h^h \quad \forall 0 < t < t^*.
\] (2.22)

The number \( t^* \) is still only qualitatively known. A quantitative lower bound can be found by substituting (2.22) into the definition (2.16) of \( t^* \). Thus (2.22) holds for all \( 0 < t < T_0 \), where \( T_0 \) is the smallest root of
\[
T_0 \|u_0\|_h^{m(p-1)-1} + T_0^{1+(m(p-1)-q)/(ph)} \|u_0\|_h^{q-1} = C^{-1}
\]
for some constant \( C = C(p, q, h, N, m) \). Substituting (2.22) into (2.15), we get (1.9). Inequality (1.10) follows from Proposition 2.2.

3. PROOF OF THEOREM 1.2

First, we prove (1.11). Let \( \xi(x, t) \in C^1(\overline{B}_2(x) \times [t_0, T]) \) with \( 0 \leq \xi \leq 1 \), \( \xi = 1 \) on \( B_1(x) \times (t_0, (T + t_0)/2) \), \( \xi(x, t) = 0 \) if \( x \in \partial B_2(x) \) or \( t = T \), \( |D\xi| \leq C \), and \( |\xi_t| \leq C(T - t_0)^{-1} \). By Definition 1.1 of supersolutions of (1.1), we have
\[
\int_{B_2(x)} u(y, t_0)\xi^\alpha \, dy + \int_{B_2(x)} u^q \xi^\alpha \, dy \, d\tau
\]
\[
\leq -\alpha \int_{t_0}^T \int_{B_2(x)} u(y, \tau)\xi_t \xi^{\alpha-1} \, dy \, d\tau
\]
\[
+ \int_{t_0}^T \int_{B_2(x)} |Du|^p |D\xi|^{\alpha-1} \, dy \, d\tau,
\] (3.1)
where \( \alpha \geq pq/(q - (m + 1 - s)(p - 1)) \) and \( s > ((m + 1)(p - 1) - q)/(p - 1) \) for some \( s \in (0, 1) \).

By the Schwarz inequality,
\[
\alpha \int_{t_0}^T \int_{B_2(x)} u(y, \tau)|\xi_t| \xi^{\alpha-1} \, dy \, d\tau
\]
\[
\leq \frac{1}{4} \int_{t_0}^T \int_{B_2(x)} u^q \xi^\alpha \, dy \, d\tau + C(T - t_0)^{-1/2q}.
\] (3.2)
By the Hölder inequality, \( \forall s \in (0, 1) \),
\[
\int_0^T \int_{B_1(x)} \alpha |Du^{m-1}|D\xi |\xi^{-s-1} dy d\tau \\
\leq am^{p-1} \left( \int_0^T \int_{B_1(x)} \xi^a |Du|^p u^{(m-1)(p-1)+s-2} dy d\tau \right)^{(p-1)/p} \\
\times \left( \int_0^T \int_{B_1(x)} u^{(p-1)(m+1-s)} \xi^a p |D\xi| p dy d\tau \right)^{1/p} \\
\equiv am^{p-1} (I_1(t_0))^{(p-1)/p} (I_2(t_0))^{1/p}.
\]
To estimate \( I_1(t_0) \), we take the testing function \( \phi = \xi^a u^{p-1} \) in (1.4) to obtain
\[
\int_0^T \int_{B_2(x)} u^{m-1(p-1)+s-2} |Du|^p \xi^a dy d\tau \\
\leq C \int_0^T \int_{B_2(x)} \xi^{p-1} u^{m(p-1)+s+1} dy d\tau + \alpha \int_0^T \int_{B_2(x)} u^s \xi^{a-1} |\xi| dy d\tau \\
\equiv H_1 + H_2.
\]
Estimating \( H_1 \) and \( H_2 \) separately, we get
\[
H_1 \leq C \left( \int_0^T \int_{B_2(x)} \xi^a u^{s} dy d\tau \right)^{(m(p-1)+s-1)/q} \\
\times (T - t_0)^{(q-m(p-1)-s+1)/q}, \quad (3.5)
\]
\[
H_2 \leq C \left( \int_0^T \int_{B_2(x)} u^s \xi^{a} dy d\tau \right)^{s/q} (T - t_0)^{-s/q}. \quad (3.6)
\]
By the Hölder inequality,
\[
I_2(t_0) \leq C \left( \int_0^T \int_{B_2(x)} u^s \xi^{a} dy d\tau \right)^{(p-1)(m+1-s)/q} \\
\times (T - t_0)^{(q-(p-1)(m+1-s))/q}. \quad (3.7)
\]
From (3.1)–(3.7), we get
\[
\int_{B_1(x)} u(y, t_0) dy + \int_0^T \int_{B_1(x)} u^d dy d\tau \leq C((T - t_0)^{-1/(q-1)} + 1).
\]
Whereas \( t_0 \in (0, T) \) is arbitrary, we deduce (1.11).
Second, we prove the Hanack inequality for the case \( 1 < q \leq m(p-1) \).
It can be verified that the function
\[
z(x, t) = a(T_0 - t)^{-1/(q-1)} f(x, t)^{(p-1)/(m(p-1)-1)}, \quad (3.8)
\]
where \( f(x, t) = \{1 - b|x - x_0|^{p(T_0 - t)^{(m(p-1)-q)/(q-1)}}\}_+ \) satisfies
\[
z_t - \text{div}(\|Dz^m\|^{p-2}Dz^m) \leq \sigma z^a \quad \text{in } S_{T_0}, \quad \forall \sigma > 0, \tag{3.9}
\]
if the constants \( a \) and \( b \) are suitably chosen depending on \( N, m, p, q, \sigma \).

Indeed, noticing that
\[
z_t = \frac{a}{q-1}(T_0 - t)^{-q/(q-1)} f(x, t)^{(p-1)/(m(p-1)-1)}
\times \left[ 1 - b \left( 1 - \frac{(p-1)(m(p-1)-q)}{m(p-1)-1} \right) \right]
\times |x - x_0|^p(T_0 - t)^{(m(p-1)-q)/(q-1)}
\times f(x, t)^{1/(m(p-1)-1)},
\]
we have
\[
z_t - \text{div}(\|Dz^m\|^{p-2}Dz^m)
= \frac{a}{q-1}(T_0 - t)^{-q/(q-1)} f(x, t)^{(p-1)/(m(p-1)-1)}
\times \left[ 1 - b \left( 1 - \frac{(p-1)(m(p-1)-q)}{m(p-1)-1} \right) \right] |x - x_0|^p(T_0 - t)^{(m(p-1)-q)/(q-1)}
- a^{m(p-1)m(p-1)} \left( \frac{bp(p-1)}{m(p-1)-1} \right)^p |x - x_0|^{p(p-1)}
\times (T_0 - t)^{(m(p-2)-q)/(q-1)+m(p-1)-q)/(q-1)}
\times f(x, t)^{(p-1)/(m(p-1)-1)-1}
+ (N + p(p-2)) \left( \frac{a^{m}bm(p-1)}{m(p-1)-1} \right)^{p-1}
\times (T_0 - t)^{(p-1)(m(p-2)-q)/(q-1)}|x - x_0|^{p(p-2)} f(x, t)^{(p-1)/(m(p-1)-1)}.
\]
Hence (3.9) follows by considering the cases
\[
|x - x_0|^p(T_0 - t)^{(m(p-1)-q)/(q-1)} \leq \delta b^{-1}
\]
and
\[
\delta b^{-1} \leq |x - x_0|^p(T_0 - t)^{(m(p-1)-q)/(q-1)} \leq b^{-1}, \quad \delta \in (0, 1),
\]
separately. We take \( \sigma = 1 \) in the following discussion.
To derive (1.12), we may assume, modulo a translation in time and space, that $u$ is defined and continuous up to $t = 0$. Accordingly it suffices to estimate the quantity

$$E_0 = \int_{B_1(0)} u(x, 0) \, dx.$$ 

Let $v$ be the unique solution of

$$
\begin{align*}
&v_t - \text{div}(|Dv|^p Dv) = 0 \quad \text{in } S_{\infty}, \\
&v(x, 0) = u(x, 0)\chi_{B_1(0)}, \quad x \in \mathbb{R}^N.
\end{align*}
$$

(3.10)

By the comparison principle,

$$u \geq v \quad \text{in } S_T. \quad (3.11)$$

Next let

$$k = \left( C_0^{-1} E_0^{m(p-1)-1} T \right)^{1/\kappa}, \quad \kappa = N(m(p-1) - 1) + p,$$

where $C_0 = C_0(N, p, m)$. It is shown in [8] that for a suitable $C_0$, either $k \leq 2$ and

$$E_0 \leq C T^{-1/(m(p-1)-1)}, \quad (3.12)$$

or $k > 2$ and

$$v(x, \frac{1}{2} T) \geq \frac{1}{4} k^{-N} E_0 \quad \forall x \in \{|x - x_0| < e_1 k\} \quad (3.13)$$

for some $x_0 \in \mathbb{R}^N$ and $e_1 = e_1(N, p, m)$. From (3.11) and (3.13) it follows that either (3.12) holds or

$$u(x, \frac{1}{2} T) \geq \frac{1}{4} k^{-N} E_0, \quad |x - x_0| < e_1 k. \quad (3.14)$$

Without loss of generality, we may assume $T_0 > T/2$ and observe that the subsolution in (3.9) satisfies

$$
\begin{align*}
z(x, \frac{T}{2}) &\leq a \left( T_0 - \frac{T}{2} \right)^{-1/(q-1)} \\
&\quad \text{if } |x - x_0|^p < b^{-1} \left( T_0 - \frac{T}{2} \right)^{(q-m(p-1))/(q-1)} \quad (3.15) \\
z(x, \frac{T}{2}) &\equiv 0 \quad \text{if } |x - x_0|^p \geq b^{-1} \left( T_0 - \frac{T}{2} \right)^{(q-m(p-1))/(q-1)}.
\end{align*}
$$
Choose $T_0 > T/2$ such that
\[ a \left( T_0 - \frac{T}{2} \right)^{-1/(q-1)} = \frac{1}{4} k^{-N} E_0, \] (3.16)
and consider separately the cases
\[ b^{-1/p} \left( T_0 - \frac{T}{2} \right)^{(q-m(p-1))/(p(q-1))} \leq \varepsilon_1 k, \] (3.17)
\[ b^{-1/p} \left( T_0 - \frac{T}{2} \right)^{(q-m(p-1))/(p(q-1))} > \varepsilon_1 k. \] (3.18)
If the latter holds, we get from the definition of $k$, (3.16), and (3.18) that
\[
E_0^{(m(p-1)-1)/\kappa} = (C_0^{-1} T)^{-1/\kappa} k \leq C_1 T^{-1/\kappa} \left( T_0 - \frac{T}{2} \right)^{(q-m(p-1))/(p(q-1))} \\
= C_2 T^{-1/\kappa} \left( E_0^{N(q-m(p-1)-1)/\kappa} T^{-N/\kappa} E_0 \right)^{(m(p-1)-q)/p} \\
= C_2 E_0^{(m(p-1)-q)/\kappa} T^{-1/(\kappa+N(q-m(p-1))/(\kappa p))}.
\]
This implies
\[ E_0 \leq CT^{-(p-N(q-m(p-1)))/(p(q-1))}. \] (3.19)
Assume now (3.17) holds. Then
\[ z(x, t) \leq u(x, t), \quad x \in \mathbb{R}^N, \quad T/2 < t < T. \]
Indeed, $z(x, T/2) \leq u(x, T/2)$ follows from (3.14)–(3.17).
Because $\lVert z(\cdot, t) \rVert_{\infty, \mathbb{R}^N} \to \infty$ as $t \to T_0$, we must have $T_0 \geq T$. Hence from (3.16),
\[ a \left( T_0 - \frac{T}{2} \right)^{-1/(q-1)} \geq \frac{1}{4} k^{-N} E_0. \]
Substituting this inequality in the definition of $k$, we find again (3.19). Estimates (3.12) and (3.19) yield (1.12).

4. PROOF OF THEOREM 1.3

It follows from Theorem 1.2 that we can find a sequence $\{t_j\} \to 0$ and a Radon measure such that $\{u(\cdot, t_j)\} \to \mu$ in the sense of measure. In view of (1.11), the measure $\mu$ will satisfy (1.14). Assume another Radon measure $\nu$ has the property that $u(\cdot, s_k) \to \nu$ as $s_k \to 0$ for a suitable sequence $\{s_k\} \to 0$. 
We may assume that $s_k < t_j$. Take any $\eta \in C^\infty_0(B_R)$ as the testing function in (1.4) to obtain
\[
\int_{R^n} u(x, t_j) \eta(x) \, dx - \int_{R^n} u(x, s_k) \eta(x) \, dx
\leq C \int_{t_k}^{t_j} \int_{B_R} |Du|^m \, dx \, d\tau + C \int_{t_k}^{t_j} \int_{B_R} u^\beta \, dx \, d\tau,
\]
where $B_R$ contains the support of $\eta$ and $C$ depends on $N, p, q, m$ as well as $R$ and $\eta$. Letting $s_k \to 0$ and then $t_j \to 0$, and interchanging the role of $s_k$ and $t_j$, we obtain
\[
\left| \int_{R^n} \eta \, d(\mu - \nu) \right| \leq C \lim_{t \to 0} \int_{t_0}^{t} \int_{B_R} |Du|^m \, dx \, d\tau. \quad (4.1)
\]
By the following lemma (Lemma 4.1),
\[
\lim_{t \to 0} \int_{t_0}^{t} \int_{B_R} |Du|^m \, dx \, d\tau = 0.
\]
Therefore, $\mu = \nu$ and Theorem 1.3 is proved.

**Lemma 4.1.** Let $u$ be a solution of (1.1) in $S_T$ and let $q > m(p - 1)$. Then $|Du|^m \in L^1(0, T; L^1_{\text{loc}}(R^N))$.

**Proof.** Without loss of generality, we assume $u > 0$. Then by the Schwarz inequality, for every $B_R = B_R(0), a > 0$, and $0 < t < T/2$, we have
\[
\int_{0}^{t} \int_{B_R} |Du|^m \, dx \, d\tau \leq \frac{1}{p} \int_{0}^{t} \int_{B_R} u^a(\beta) \, dx \, d\tau
\]
\[
+ \frac{p - 1}{p} \int_{0}^{t} \int_{B_R} |Du|^m u^{-a} \, dx \, d\tau. \quad (4.2)
\]
Let $\xi$ be a cutoff function in $B_{2R}$ with $\xi = 1$ on $B_R$ and take $\phi = u^{-\beta} \xi^p$ as the testing function in (1.4); here $\beta \in (0, 1)$ is to be chosen. After standard calculations, we find
\[
\int_{0}^{t} \int_{B_R} |Du|^m u^{-\beta - m} \, dx \, d\tau \leq C \int_{B_{2R}} u^{1-\beta}(x, t) \, dx
\]
\[
+ C \int_{0}^{t} \int_{B_R} u^{m(1-\beta)} \, dx \, d\tau. \quad (4.3)
\]
From this inequality and Theorem 1.2, there exists $C = C(N, p, q, m, \beta, T, R)$ such that
\[
\int_{0}^{t} \int_{B_R} |Du|^m u^{-\beta - m} \, dx \, d\tau \leq C. \quad (4.4)
\]
Indeed, we observe from Theorem 1.2 that
\[
\int_{0}^{t} \int_{B_{2R}} u^\beta(x, \tau) \, dx \, d\tau < \infty. \quad (4.5)
\]
Choose $\beta$ such that $(m + \beta)(p - 1) \leq q$. Then by choosing $a = m + \beta$ and invoking again (4.5), we have the sought after estimate.
5. PROOF OF THEOREM 1.4

Let \( g_\varepsilon : \mathbb{R} \to [-1, 1] \) be a monotone increasing smooth approximation of the function \( s \mapsto \text{sign}(s) \equiv s|s|^{-1}, \) \( s \neq 0, \) \( g_\varepsilon(0) = 0. \) Let also \( \xi \) denote a nonnegative smooth cutoff function in \( B_{2\rho} \equiv \{|x| < 2\rho\}, \rho \geq 1, \) such that
\[
\xi \equiv 1 \text{ in } B_\rho, \quad |D\xi| \leq C\rho^{-1}.
\]
Let \( \eta_j(t) \in C^1_0(0, T), 0 \leq \eta_j \leq 1, \) satisfying \( \eta_j \to \eta \) almost everywhere in \((t_1, t_2)\) as \( j \to \infty, \) where \( \eta \) is the characteristic function of \((t_1, t_2)(0 < t_1 < t_2 < T).\)

We denote \( w = u - v \) and take the testing function
\[
\phi = g_\varepsilon(u^m - v^m)\eta_j(t)\xi(x)
\]
in (1.4) to obtain
\[
\int_0^T \int_{B_{2\rho}} g_\varepsilon(u^m - v^m)\eta_j(t)\xi(x)w_t \, dx \, dt
\]
\[
+ \int_0^T \int_{B_{2\rho}} (|Du^m|^{p-2}Du^m - |Dv^m|^{p-2}Dv^m) \cdot (Du^m - Dv^m)g_\varepsilon' \eta_j \xi \, dx \, dt
\]
\[
+ \int_0^T \int_{B_{2\rho}} (|Du^m|^{p-2}Du^m - |Dv^m|^{p-2}Dv^m) \cdot D\xi g_\varepsilon\eta_j \, dx \, dt
\]
\[
= \int_0^T \int_{B_{2\rho}} (u^m - v^m)g_\varepsilon \eta_j \xi \, dx \, dt. \quad (5.1)
\]
Whereas \( u^m, v^m \in L^p_{\text{loc}}(0, T; W^{1, p}(\mathbb{R}^N)), \) that is, \( \forall 0 < \tau_1 < \tau_2 < T, \)
\[
u^m, Du^m, Dv^m \in L^p(\mathbb{R}^N \times (\tau_1, \tau_2),
\]
it follows from the Hölder and Young inequalities that
\[
\left| \int_0^T \int_{B_{2\rho}} (|Du^m|^{p-2}Du^m - |Dv^m|^{p-2}Dv^m) \cdot D\xi g_\varepsilon\eta_j \, dx \, dt \right|
\]
\[
\leq C\rho^{-1} \int_0^T \int_{B_{2\rho}} (|Du^m|^{p-1} + |Dv^m|^{p-1})|g_\varepsilon'|_{\infty}|u^m - v^m| \eta_j \, dx \, dt
\]
\[
\leq C\rho^{-1} \left( \int_0^T \int_{\mathbb{R}^N} (|Du^m|^p + |Dv^m|^p) \eta_j \, dx \, dt
\]
\[
+ \int_0^T \int_{\mathbb{R}^N} (u^m + v^m)^p \eta_j \, dx \, dt \right) \quad (5.2)
\]
and
\[
\int_0^T \int_{B_{2\rho}} (|Du^m|^{p-2}Du^m - |Dv^m|^{p-2}Dv^m) \cdot (Du^m - Dv^m)g_\varepsilon' \eta_j \xi \, dx \, dt \geq 0.
\]
Thus by letting $\rho \to \infty$ in (5.1) and (5.2), we obtain
\[
\int_0^T \int_{\mathbb{R}^N} g_\varepsilon(u^m - v^m) \eta_j(t) w \, dx \, dt \leq \int_0^T \int_{\mathbb{R}^N} (u^q - v^q) g_\varepsilon(u^m - v^m) \eta_j \, dx \, dt.
\]

Letting $\varepsilon \to 0$ and $j \to \infty$, standard calculations give
\[
\int_{\mathbb{R}^N} |w(x, t_2)| \, dx \leq \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |u^q - v^q| \, dx \, dt + \int_{\mathbb{R}^N} |w(x, t_1)| \, dx.
\]  

(5.3)

We get from (1.17) that
\[
\int_{t_1}^{t_2} \int_{\mathbb{R}^N} |u^q - v^q| \, dx \, dt \leq C \int_{t_1}^{t_2} t^{-\delta(q-1)} \int_{\mathbb{R}^N} |w(x, t)| \, dx \, dt.
\]

This inequality and (5.3) lead to
\[
\int_{\mathbb{R}^N} |w(x, t_2)| \, dx \leq C \int_{t_1}^{t_2} t^{-\delta(q-1)} \int_{\mathbb{R}^N} |w(x, t)| \, dx \, dt + \int_{\mathbb{R}^N} |w(x, t_1)| \, dx.
\]

Notice that $\delta(q - 1) < 1$, and by letting $t_1 \to 0$ and using the Gronwall inequality,
\[
\int_{\mathbb{R}^N} |w(x, t)| \, dx \equiv 0 \quad \forall \, t \in (0, T).
\]

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