Global bifurcation of limit cycles in a family of polynomial systems

Guanghui Xiang ∗ and Maoan Han

Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, PR China

Received 19 May 2003
Available online 28 May 2004
Submitted by Z.S. Athanassov

Abstract

In this paper, we study the number of limit cycles in a family of polynomial systems. Using bifurcation methods and Melnikov functions, we obtain the maximal number of limit cycles in global bifurcation.

© 2004 Elsevier Inc. All rights reserved.

1. Preliminary lemmas

Consider a planar system of the form

\[
\begin{align*}
\dot{x}(t) &= H_y + \epsilon f(x, y, \epsilon, a), \\
\dot{y}(t) &= -H_x + \epsilon g(x, y, \epsilon, a),
\end{align*}
\]

(1.1)

where \( H, f, g \) are \( C^\infty \) functions in a region \( G \subset \mathbb{R}^2 \), \( \epsilon \in \mathbb{R} \) is a small parameter and \( a \in D \subset \mathbb{R}^n \) with \( D \) compact. For \( \epsilon = 0 \), Eq. (1.1) becomes Hamiltonian with the Hamiltonian function \( H(x, y) \). Suppose there exists a constant \( H_0 > 0 \) such that for \( 0 < h < H_0 \), the equation \( H(x, y) = h \) defines a smooth closed curve \( L_h \subset G \) surrounding the origin and shrinking to the origin as \( h \to 0 \). Hence \( H(0, 0) = 0 \) and for \( \epsilon = 0 \) (1.1) has a center at the origin.

∗ Research supported by the National Natural Science Foundation of China (10371072).
∗ Corresponding author.
E-mail address: zhanghm@sjtu.edu.cn (G. Xiang).
Let
\[ \Phi(h,a) = \oint_{L_h} (g \, dx - f \, dy) \varepsilon = \oint_{L_h} (H_y g + H_x f) \varepsilon = 0, \]
which is called the first-order Melnikov function or Abelian integral of (1.1). This function plays an important role in the study of limit cycle bifurcations. In the case that (1.1) is a polynomial system, a well-known problem is to determine the least upper bound of the number of zeros of \( \Phi \). This is called the weakened Hilbert 16th problem, see [1–11].

In this paper, we first state some preliminary lemmas which will be used to find the maximal number of limit cycles by using zeros of \( \Phi \). These lemmas are already known or are easy corollaries of known results. Then we study the global bifurcations of limit cycles for some polynomial systems, and obtain the least upper bound of the number of limit cycles.

Now we give some lemmas. First, for Hopf bifurcation we have the following:

**Lemma 1.1** [4]. Let \( H(x, y) = K(x^2 + y^2) + O(|x, y|^3) \) with \( K > 0 \) for \((x, y)\) near the origin. Then the function \( \Phi \) is of class \( C^\infty \) in \( h \) at \( h = 0 \). If \( \Phi(h, a_0) = K_1(a_0) h^{k+1} + O(h^{k+2}) \), \( K_1(a_0) \neq 0 \) for some \( a_0 \in D \), then (1.1) has at most \( k \) limit cycles near the origin for \(|\varepsilon| + |a - a_0| \) sufficiently small.

The following lemma is well known (see [10] for example).

**Lemma 1.2.** If \( \Phi(h, a_0) = K_2(a_0)(h-h_0)^k + O(|h-h_0|^{k+1}) \), \( K_2(a_0) \neq 0 \) for some \( a_0 \in D \) and \( h_0 \in (0, H_0) \), then (1.1) has at most \( k \) limit cycles near \( L_{h_0} \) for \(|\varepsilon| + |a - a_0| \) sufficiently small.

Let \( L_0 \) denote the origin and set
\[ S = \bigcup_{0 < h < H_0} L_h. \tag{1.3} \]
It is obvious that \( S \) is a simply connected open subset of the plane. We suppose that the function \( \Phi \) has the following form:
\[ \Phi(h, a) = I(h) N(h, a), \tag{1.4} \]
where \( I \in C^\infty \) for \( h \in [0, H_0) \) and satisfies
\[ I(0) = 0, \quad I'(0) \neq 0 \quad \text{and} \quad I(h) \neq 0 \quad \text{for} \ h \in (0, H_0). \tag{1.5} \]
Using above two lemmas, we can prove

**Lemma 1.3.** Let (1.4) and (1.5) hold. If there exists a positive integer \( k \) such that for every \( a \in D \) the function \( N(h, a) \) has at most \( k \) zeros in \( h \in [0, H_0) \) (multiplicities taken into account), then for any given compact set \( V \subset S \), there exists \( \varepsilon_0 = \varepsilon_0(V) > 0 \) such that for all \( 0 < |\varepsilon| < \varepsilon_0 \), \( a \in D \) the system (1.1) has at most \( k \) limit cycles in \( V \).
2. The number of limit cycles in a family of polynomial systems

Proof. Suppose the conclusion of the lemma is not true. Then for a given compact set \( V \subseteq S \), there exists a series \( \{(\varepsilon_l, a^{(l)})\} \) satisfying \( \varepsilon_l \to 0 \) (as \( l \to \infty \)), \( a^{(l)} \in D \), such that there are \( k+1 \) limit cycles in \( V \) when \( (\varepsilon, a) = (\varepsilon_l, a^{(l)}) \). Notice that \( V \) is compact. There must exist integers \( r \geq 0 \), \( k_j \geq 0 \), \( j = 0, 1, \ldots, r \) and numbers \( 0 \leq h_0 < h_1 < \cdots < h_r < H_0 \) satisfying

\[
0 + k_1 + k_2 + k_3 + \cdots + k_r = k + 1,
\]

such that (1.1) has \( k_j \) limit cycles which tend to \( L_{h_j} \) when \( (\varepsilon, a) = (\varepsilon_l, a^{(l)}) \) and \( l \to \infty \).

Since \( D \) is compact, we can suppose \( a^{(l)} \to a_0 \in D \). Consider function \( \Phi(h, a_0) = I(h)N(h, a_0) \). From Lemmas 1.1 and 1.2, we can see that each \( h_j \) is a root of \( N(h, a_0) \) of multiplicity at least \( k_j \). Hence, from (1.6) it follows that the total multiplicity of roots of function \( N(h, a_0) \) is at least \( k + 1 \). This is a contradiction. The proof is completed. \( \square \)

Remark 1.1. As we know, if there exists \( a_0 \in D \) such that the function \( N(h, a_0) \) has exactly \( k \) simple zeros \( 0 < h_1 < \cdots < h_k < H_0 \) with \( N(0, a_0) \neq 0 \), then for any compact set \( V \) satisfying \( L_{h_k} \subset V \) and \( V \subset S \), there exists \( \varepsilon_0 > 0 \) such that for all \( 0 < |\varepsilon| < \varepsilon_0 \), \( |a - a_0| < \varepsilon_0 \), (1.1) has precisely \( k \) limit cycles in \( V \).

Remark 1.2. The conclusions of Lemmas 1.1 and 1.2 are local with respect to both parameter \( a \) and the set \( S \) while the conclusion of Lemma 1.3 is global because it holds in any compact set of \( S \) and uniformly in \( a \in D \).

2. The number of limit cycles in a family of polynomial systems

In this section, we consider a polynomial system of degree \( n \) of the form

\[
\dot{x} = y(1 + Bx + Ax^2) + \varepsilon \sum_{0 \leq i + j \leq n} a_{ij} x^i y^j,
\]

\[
\dot{y} = -x(1 + Bx + Ax^2) + \varepsilon \sum_{0 \leq i + j \leq n} b_{ij} x^i y^j,
\]

(2.1)

where \( A, B, a_{ij}, b_{ij} \) are real and \( A \neq 0 \), \( |a_{ij}| \leq K, |b_{ij}| \leq K \) with \( K \) a positive constant and \( n \) a positive integer. Let \( a = (a_{ij}, b_{ij})_{0 \leq i + j \leq n} \) and \( B_K = \{ a \mid |a_{ij}| \leq K, |b_{ij}| \leq K \} \).

On the region \( \Omega = \{(x, y) \mid 1 + Bx + Ax^2 > 0\} \), (2.1) is equivalent to

\[
\dot{x} = y + \frac{\varepsilon}{1 + Bx + Ax^2} \sum_{0 \leq i + j \leq n} a_{ij} x^i y^j,
\]

\[
\dot{y} = -x + \frac{\varepsilon}{1 + Bx + Ax^2} \sum_{0 \leq i + j \leq n} b_{ij} x^i y^j.
\]

(2.2)

Let \( \Phi(h) \) denote the first-order Melnikov function of (2.2) for \( 0 \leq h < H_0 \). Then we have the following main results.

Theorem 2.1. Suppose \( B^2 - 4A \neq 0 \). Then for any \( K > 0 \) and compact set \( V \) in \( \Omega \), if \( \Phi(h) \) is not identically zero for \( a = (a_{ij}, b_{ij}) \) varying in a compact set \( D \) in \( B_K \), there exists an
\( \varepsilon_0 > 0 \) such that for \( 0 < |s| < \varepsilon_0, \ a \in D, \) (2.1) or (2.2) has at most \( 2n + 2 - (-1)^n \) limit cycles in \( V. \)

**Theorem 2.2.** Suppose \( B^2 - 4A = 0, \) Then for any \( K > 0 \) and compact set \( V \) in \( \Omega, \) if \( \Phi(h) \) is not identically zero for \( a = (a_{ij}, b_{ij}) \) varying in a compact set \( D \) in \( B_k, \) then there exists an \( \varepsilon_0 > 0 \) such that for \( 0 < |s| < \varepsilon_0, \ a \in D, \) (2.1) or (2.2) has at most \( n \) limit cycles in \( V. \)

**Theorem 2.3.** Suppose \( B^2 - 4A = 0, \) Then for any \( K > 0 \) and compact set \( V \) in \( \Omega, \) there exists an \( \varepsilon_0 > 0 \) and \( \alpha^0 = (\alpha_{ij}^0, b_{ij}^0) \in B_k, \) such that for all \( 0 < |s| < \varepsilon_0, \ |a - \alpha^0| < \varepsilon_0 \) (2.1) or (2.2) has precisely \( n \) limit cycles in \( V. \)

Before proving above theorems, we first give some lemmas.

Let
\[
I_{i,j} = \oint_{L_h} \frac{x^i y^j dt}{(1 + Bx + Ax^2)} = \oint_{L_h} \frac{x^i y^j dt}{(1 - \alpha_1 x)(1 - \alpha_2 x)}, \quad (B^2 - 4A \neq 0),
\]
(2.3)
\[
I_{i,j}^{(2)} = \oint_{L_h} \frac{x^i y^j dt}{(1 + Bx + Ax^2)} = \oint_{L_h} \frac{x^i y^j dt}{(1 - \alpha_1 x)2}, \quad (B^2 - 4A = 0),
\]
(2.4)
\[
\Phi_{ij} = a_{ij} I_{i+1,j} + b_{ij} I_{i,j+1}, \quad (2.5)
\]
\[
\Phi_{ij}^{(2)} = a_{ij} I_{i+1,j}^{(2)} + b_{ij} I_{i,j+1}^{(2)}, \quad (2.6)
\]
where
\[
L_h: \ x = \sqrt{h} \sin t, \quad y = \sqrt{h} \cos t.
\]

Let
\[
K_0 = 2\pi, \quad K_{2j} = \frac{(2j - 1)!!}{(2j)!!} 2\pi, \quad C_j^k = \frac{k!}{j!(k-j)!}.
\]

**Lemma 2.1.** We have
\[
J_1 = \oint_{L_h} \frac{dt}{(1 - \alpha_1 x)} = \frac{2\pi}{\sqrt{1 - \alpha_1^2 h}}, \quad (2.7)
\]
\[
J_2 = \oint_{L_h} \frac{dt}{(1 - \alpha_2 x)} = \frac{2\pi}{\sqrt{1 - \alpha_2^2 h}}, \quad (2.8)
\]
\[
I_{0,0}^{(2)} = \oint_{L_h} \frac{dt}{(1 - \alpha_1 x)^2} = \frac{2\pi}{(1 - \alpha_1^2 h)^3}, \quad (2.9)
\]

and
\[
I_{0,0} = \oint_{L_h} \frac{dt}{(1 - \alpha_1 x)(1 - \alpha_2 x)} = \frac{\alpha_1 J_1}{\alpha_1 - \alpha_2} + \frac{\alpha_2 J_2}{\alpha_2 - \alpha_1}. \quad (2.10)
\]
Proof. We use the residue theorem to compute these integrals.

By the definition of $L_h$, we have

$$J_1 = \oint_{L_h} \frac{dt}{1 - \alpha_1 x} = \frac{2\pi}{1 - \alpha_1 \sqrt{h} \sin t}.$$  

Let $e^{it} = z$. Then $dt = (1/iz)\,dz$, $\sin t = (z^2 - 1)/(2iz)$ where $i = \sqrt{-1}$. The above formula becomes

$$J_1 = \frac{2}{\alpha_1 \sqrt{h}} \oint_{|z|=1} \frac{dz}{z^2 - \frac{2}{\alpha_1 \sqrt{h}} z - 1} = \frac{2}{\alpha_1 \sqrt{h}} 2\pi i \text{Res} \left[ \frac{1}{z^2 - \frac{2}{\alpha_1 \sqrt{h}} z - 1}, z_1 \right]$$

where $z_{1,2} = (1 \pm \sqrt{1 - \alpha_1^2 h^2})/(\alpha_1 \sqrt{h})i$. Hence (2.7) follows and (2.8)–(2.10) can be obtained in the same way. \hfill \Box

Further, we have

**Lemma 2.2.** If $B^2 - 4A \neq 0$, then

$$I_{1,0} = \frac{J_1}{\alpha_1 - \alpha_2} + \frac{J_2}{\alpha_2 - \alpha_1}, \quad (2.11)$$

and

$$I_{i,0} = A_1^{(i)} J_1 + A_2^{(i)} J_2 + a_4^{(i)} h + \cdots + a_{i/2}^{(i)} h^{[i/2]}, \quad i > 1, \quad (2.12)$$

where $[r]$ denotes the integer part of $r$, $a_j^{(i)}$ are real for $j \geq 0$ and $A_1^{(i)}$ and $A_2^{(i)}$ are real numbers for $B^2 - 4A > 0$ or a pair of conjugate complex numbers for $B^2 - 4A < 0$. Also $I_{i,0} = 0$ at $h = 0$ for $i \geq 1$.

**Proof.** By the definition of $I_{i,0}$ we have

$$I_{i,0} = \oint_{L_h} \frac{x^i \, dt}{(1 - \alpha_1 x)(1 - \alpha_2 x)} = \frac{1}{\alpha_2^i} \oint_{L_h} \frac{(\alpha_2 x)^i \, dt}{(1 - \alpha_1 x)(1 - \alpha_2 x)}$$

$$= \frac{1}{\alpha_2^i} \left( \oint_{L_h} \frac{dt}{(1 - \alpha_1 x)(1 - \alpha_2 x)} - \oint_{L_h} \frac{1 + \alpha_2 x + \cdots + (\alpha_2 x)^{i-1}}{(1 - \alpha_1 x)(1 - \alpha_2 x)} \, dt \right)$$

$$= \frac{1}{\alpha_2^i} \left( I_{0,0} - J_1 (1 + \beta + \cdots + \beta^{i-1}) + K_0(\beta + \beta^2 + \cdots + \beta^{i-1}) 
+ \alpha_2 h (\beta^3 + \beta^4 + \cdots + \beta^{i-1}) + \cdots \right)$$
where
\[ \beta = \frac{\alpha_2}{\alpha_1}, \quad A_1^{(i)} = \frac{1}{(\alpha_1 - \alpha_2)\alpha_1^{i-1}}, \quad A_2^{(i)} = \frac{1}{(\alpha_2 - \alpha_1)\alpha_2^{i-1}}. \]

It is obvious that \( A_1^{(i)} \) and \( A_2^{(i)} \) are real for \( B^2 - 4A > 0 \) or a pair of conjugate complex numbers for \( B^2 - 4A < 0 \). Hence (2.12) follows. Obviously for \( i \geq 1 \) and \( h = 0 \) we have

\[ I_{i,0} = \frac{2\pi}{(\alpha_1 - \alpha_2)\alpha_1^{i-1}} + \frac{2\pi}{(\alpha_2 - \alpha_1)\alpha_2^{i-1}} + \frac{\alpha_1^{i-1} - \alpha_2^{i-1}}{(\alpha_1\alpha_2)^{i-1}(\alpha_1 - \alpha_2)} K_0 = 0. \]

The proof is completed. \( \square \)

Further, we have similarly

**Lemma 2.3.** If \( B^2 - 4A = 0 \), then

\[ I_{1,0}^{(2)} = \alpha_1 h I_{0,0}^{(2)}, \quad (2.13) \]

and

\[ I_{1,0}^{(2)} = (B_0^{(i)} + B_1^{(i)} h) I_{0,0}^{(2)} + b_1^{(i)} h + \cdots + b_i^{(i)} \frac{h^{[\frac{i}{2}]}}{[\frac{i}{2}]} \], \quad i > 1, \quad (2.14) \]

where \( B_0^{(i)}, B_1^{(i)}, b_i^{(i)} \) are all real.

**Lemma 2.4.** For \( i \geq 0, \ k > 0 \), we have

\[ I_{i,2k-1} = 0, \quad (2.15) \]

\[ I_{i,2k-1}^{(2)} = 0, \quad (2.16) \]

and
$I_{i,2k} = \sum_{j=0}^{k} (-1)^j C_k^j I_{i+j,0} h^{k-j},$  \hspace{1cm} (2.17)

$I_{i,2k}^{(2)} = \sum_{j=0}^{k} (-1)^j C_k^j I_{i+j,0}^{(2)} h^{k-j}.$  \hspace{1cm} (2.18)

**Proof.** By using (2.3), we can prove easily

$$I_{i,2k-1} = \oint_{L_h} \frac{x^i y^{2k-1}}{(1-\alpha_1 x)(1-\alpha_2 x)} \, dt = 0$$

and

$$I_{i,2k} = \oint_{L_h} \frac{x^i y^{2k}}{(1-\alpha_1 x)(1-\alpha_2 x)} \, dt = \oint_{L_h} \frac{x^i (h-x^2)^k}{(1-\alpha_1 x)(1-\alpha_2 x)} \, dt.$$  

Hence, (2.15) and (2.17) follow. Formulas (2.16) and (2.18) can be proved in the same way. The proof is completed. \(\square\)

**Lemma 2.5.** Suppose $B^2 - 4A \neq 0$. Then for $k > 0$ we have

$$\sum_{i+j=2k-1} \Phi_{ij} = (B_{2k-1,k-1} h^{k-1} + \cdots + B_{2k-1,1} h + B_{2k-1,0})$$

$$+ J_1 (\alpha_{2k-1,k} h^k + \cdots + \alpha_{2k-1,1} h + \alpha_{2k-1,0})$$

$$+ J_2 (\beta_{2k-1,k} h^k + \cdots + \beta_{2k-1,1} h + \beta_{2k-1,0}),$$  \hspace{1cm} (2.19)

and

$$\sum_{i+j=2k} \Phi_{ij} = (B_{2k,k-1} h^{k-1} + \cdots + B_{2k,1} h + B_{2k,0})$$

$$+ J_1 (\alpha_{2k,k} h^k + \cdots + \alpha_{2k,1} h + \alpha_{2k,0})$$

$$+ J_2 (\beta_{2k,k} h^k + \cdots + \beta_{2k,1} h + \beta_{2k,0}),$$  \hspace{1cm} (2.20)

where $\alpha_{2k-1,j}$, $\beta_{2k-1,j}$, and $B_{2k-1,j}$ are linear combinations of coefficients $(a_{ij}, b_{ij})$ with $i + j = 2k - 1$ while $\alpha_{2k,j}$, $\beta_{2k,j}$, and $B_{2k,j}$ are linear combinations of $(a_{ij}, b_{ij})$ with $i + j = 2k$. Also $\alpha_{2k-1,j}$, $\beta_{2k-1,j}$, $\alpha_{2k,j}$, and $\beta_{2k,j}$ are real for $B^2 - 4A > 0$, $\alpha_{2k-1,j}$ and $\beta_{2k-1,j}$, $\alpha_{2k,j}$ and $\beta_{2k,j}$ are pairs of conjugate complex numbers for $B^2 - 4A < 0$ and $B_{2k-1,j}$, $B_{2k,j}$ are real in both cases. Moreover, for $k > 0$, and $h = 0$,

$$\sum_{i+j=2k-1} \Phi_{ij} = 0, \quad \sum_{i+j=2k} \Phi_{ij} = 0.$$

**Proof.** By the definition (2.5) of $\Phi_{ij}$ and Lemma 2.4, we have

$$\sum_{i+j=2k} \Phi_{ij} = \sum_{i=1}^{k} (\Phi_{2k-2i,2i} + \Phi_{2k-2i+1,2i-1}) + \Phi_{2k,0} = \sum_{i=0}^{k} a_{2k,i} I_{2k-2i+1,2i},$$  \hspace{1cm} (2.21)
where
\[ \tilde{a}_{2k,j} = \begin{cases} 
\alpha_{2k-2j,2j} + b_{2k-2j+1,2j-1}, & j \neq 0, \\
\alpha_{2k,0}, & j = 0.
\end{cases} \]

By the formula (2.17), (2.21) becomes
\[
\sum_{i+j=2k} \Phi_{ij} = \tilde{a}_{2k,i} I_{1,0}h^k + \left( \tilde{a}_{2k,k} - C_j^i \tilde{a}_{2k,k} I_{3,0}h^{k-1} + \cdots \right. \\
+ \left( \tilde{a}_{2k,0} - \tilde{a}_{2k,1} + \cdots + (-1)^k \tilde{a}_{2k,k} I_{2k+1,0} \right) \\
= \tilde{b}_{2k,i} I_{1,0}h^k + \cdots + \tilde{b}_{2k,1} I_{2k-1,0}h + \tilde{b}_{2k,0} I_{2k+1,0}. \tag{2.22}
\]
where
\[
\tilde{b}_{2k,i} = \sum_{j=i}^{k} (-1)^{j-i} C_j^i \tilde{a}_{2k,j}, \quad 0 \leq i \leq k.
\]

Hence the formula (2.20) follows from (2.22) and (2.12). Formula (2.19) can be proved in the same way. The proof is completed. \( \square \)

**Lemma 2.6.** Suppose \( B^2 - 4A = 0 \). Then for \( k > 0 \) we have
\[
\sum_{i+j=2k-1} \Phi_{ij}^{(2)} = \left( A_{2k-1,k-1} h^{k-1} + \cdots + A_{2k-1,1} h + A_{2k-1,0} \right) \\
+ \cdot I_{0,0}^{(2)} \left( \alpha_{2k-1,1} h^{k} + \cdots + \alpha_{2k-1,1} h + \alpha_{2k-1,0} \right), \tag{2.23}
\]

and
\[
\sum_{i+j=2k} \Phi_{ij}^{(2)} = \left( A_{2k,k-1} h^{k-1} + \cdots + A_{2k,1} h + A_{2k,0} \right) \\
+ \cdot I_{0,0}^{(2)} \left( \alpha_{2k,k+1} h^{k+1} + \cdots + \alpha_{2k,1} h + \alpha_{2k,0} \right). \tag{2.24}
\]

where
\begin{align*}
\alpha_{2k-1,k} &= \tilde{b}_{2k-1,k} + \tilde{b}_{2k-1,1} B_1^{(2k)}, \\
\alpha_{2k-1,0} &= \tilde{b}_{2k-1,0} B_0^{(2k)} , \\
\alpha_{2k-1,j} &= \tilde{b}_{2k-1,j} B_0^{(2k-2j)} + \tilde{b}_{2k-1,j-1} B_1^{(2k-2j+2)}, \quad 0 < j \leq k - 1, \\
A_{2k-1,j} &= \sum_{i=0}^{j} \tilde{b}_{2k-1,j-i} B_0^{(2k-2j+2i)}, \quad 0 \leq j \leq k - 1, \\
\alpha_{2k,k+1} &= \tilde{b}_{2k,k} a_1 , \\
\alpha_{2k,k} &= \tilde{b}_{2k,k-1} B_1^{(3)} , \\
\alpha_{2k,0} &= \tilde{b}_{2k,0} B_0^{(2k+1)} , \\
\alpha_{2k,j} &= \tilde{b}_{2k,j} B_0^{(2k+1-2j)} + \tilde{b}_{2k,j-1} B_1^{(2k+3-2j)}, \quad 0 < j \leq k - 1 ,
\end{align*}
\[ A_{2k, j} = \sum_{i=0}^{j} \tilde{b}_{2k, j-i} b_i^{(2k-2j+2i+1)}, \quad 0 \leq j \leq k - 1, \]

\[ \tilde{b}_{2k-1, j} = \sum_{i=j}^{k} (-1)^{j-i} C_i^j \tilde{a}_{2k-1, i}, \quad 0 \leq j \leq k - 1, \]

\[ \tilde{b}_{2k, j} = \sum_{i=j}^{k} (-1)^{j-i} C_i^j \tilde{a}_{2k, i}, \quad 0 \leq j \leq k, \]

\[ \tilde{a}_{2k-1, j} = \begin{cases} a_{2k-1-2j, 2j} + b_{2k-2j, 2j-1}, & j \neq 0, \\ d_{2k-1, 0}. & j = 0, \end{cases} \]

\[ \tilde{a}_{2k, j} = \begin{cases} a_{2k-2j, 2j} + b_{2k-2j+1, 2j-1}, & j \neq 0, \\ d_{2k, 0}. & j = 0, \end{cases} \tag{2.25} \]

and \( B_0^{(1)}, B_1^{(1)}, b_j \) are given in (2.14).

**Proof.** By the definition (2.6) of \( \Phi_{ij}^{(2)} \) and Lemma 2.4, we have

\[ \sum_{i+j=2k} \Phi_{ij}^{(2)} = \sum_{i=1}^{k} \Phi_{2k-2i, 2i}^{(2)} + \Phi_{2k-2i+1, 2i-1}^{(2)} + \Phi_{2k, 0}^{(2)} = \sum_{i=0}^{k} \tilde{a}_{2k, i} I_{1, 0}^{(2)} h^k + \cdots + \tilde{a}_{2k, 0} I_{2, 0}^{(2)} h^{2k-1} + \cdots + \tilde{a}_{2k-1, 0} I_{2k-1, 0}^{(2)} h^{2k-1} + \cdots + \tilde{a}_{2k, 0} I_{2k, 0}^{(2)} h^{2k} \]

From (2.12) and the above formula, we can prove (2.24) easily. Similarly, the formula (2.23) can be proved, too. The proof is completed. \( \square \)

Now we are in position to prove the main results.

**Proof of Theorem 2.1.** In the following, we suppose \( n = 2s \) first. In this case, by (1.2), the Melnikov function \( \Phi(h) \) of the system (2.2) has the following form:

\[ \Phi(h) = \frac{1}{L_h} \sum_{0 \leq i+j \leq 2s} \left( a_{ij} x^{i+1} y^j + b_{ij} x^i y^{j+1} \right) dt \]

\[ = \sum_{0 \leq i+j \leq 2s} \Phi_{ij} = \sum_{k=1}^{s} \left( \sum_{i+j=2k-1} \Phi_{ij} + \sum_{i+j=2k} \Phi_{ij} \right) + \Phi_{00}. \]

By Lemma 2.5 the above formula becomes

\[ \Phi(h) = J_1 (a_1 h^2 + \cdots + a_k h + a_0) + J_2 (b_2 h^2 + \cdots + b_1 h + b_0) + \cdots + \Phi_{00}. \tag{2.26} \]
where \( a_j, b_j, B_j \) are linear combinations of \( a_{ij}, b_{ij} \) with \( 0 \leq i + j \leq 2s \) and \( B_j \) \( (j \geq 0) \) are always real. For \( B^2 - 4A > 0, a_j \) and \( b_j \) are real; for \( B^2 - 4A < 0, a_j \) and \( b_j \), \( J_1 \) and \( J_2 \) are pairs of conjugate complex numbers. From (2.26), \( \Phi(h) = 0 \) at \( h = 0 \).

Obviously, all the zeros of (2.26) satisfy

\[
(J_1(a_sh^s + \cdots + a_1h + a_0) + J_2(b_sh^s + \cdots + b_1h + b_0))^2 \quad = (B_{s-1}h^{s-1} + \cdots + B_1h + B_0)^2.
\]

Further, the above is equivalent to

\[
\sqrt{(1-a_1^2h)(1-a_2^2h)} P_{2s}(h) = Q_{2s+1}(h),
\]

where \( P_{2s}(h) \) and \( Q_{2s+1}(h) \) are two real coefficient polynomials of \( h \) of degree \( 2s \) and \( 2s + 1 \) respectively. Hence the number of zeros of \( \Phi(h) \) are not larger than \( 4s + 2 \). For the case of \( n = 2s - 1 \), similarly we can prove that the number of zeros of \( \Phi(h) \) are not larger than \( 4s + 2 \).

Notice that \( \Phi(h) = 0 \) at \( h = 0 \) in (2.26). From Lemma 1.3, we know that there exists an \( \varepsilon_0 > 0 \) such that for \( 0 < \varepsilon < \varepsilon_0, a = (a_{ij}, b_{ij}) \) with \( |a_{ij}| \leq K, |b_{ij}| \leq K \), the system (2.2) has at most \( 2n + 2 - (-1)^n \) limit cycles. The proof is completed.

**Proof of Theorem 2.2.** Let \( n = 2s \) first. From Lemma 2.6, we know that the function \( \Phi(h) \) has the form

\[
\Phi(h) = (B_{s-1}h^{s-1} + \cdots + B_1h + B_0) + I_0^{(2)}(A_{s+1}h^{s+1} + \cdots + A_1h + A_0), \quad (2.27)
\]

where

\[
\begin{align*}
B_{s-1} &= \sum_{j=2s-1}^{2s} A_{j,s-1}, & A_{s+1} &= \alpha_{2s,s+1}, \\
B_{s-2} &= \sum_{j=2s-3}^{2s} A_{j,s-2}, & A_s &= \sum_{j=2s-2}^{2s} \alpha_{j,s}, \\
&\vdots & & \vdots \\
B_1 &= \sum_{j=3}^{2s} A_{j,1}, & A_2 &= \sum_{j=2}^{2s} \alpha_{j,2}, \\
B_0 &= \sum_{j=1}^{2s} A_{j,0}. & A_1 &= \sum_{j=1}^{2s} \alpha_{j,1} + a_0\alpha_1, \\
& & A_0 &= \sum_{j=1}^{2s} \alpha_{j,0}. \quad (2.28)
\end{align*}
\]

Here \( \alpha_{j,i} \) and \( A_{j,i} \) are given in (2.25). Let \( \sqrt{1 - a_1^2h} = r \). Then (2.27) becomes
\( \Phi(h) = \frac{2\pi}{r} \left( c_{2m+2}r^{2m+2} + c_{2m+1}r^{2m+1} + \cdots + c_2 r^2 + c_0 \right) \)
\[ = \frac{2\pi}{r} P_{2s+2}(r), \tag{2.29} \]

where \( P_{2s+2}(r) \) is a real coefficient polynomial of \( r \) of degree \( 2s + 2 \). Notice that the polynomial \( P_{2s+2}(r) \) has only \( 2s + 2 \) items. From the Rolle’s theorem we have that \( P_{2s+2}(r) \) has at most \( 2s + 1 \) positive zeros. Obviously, \( \Phi(h) = 0 \) at \( h = 0 \) in (2.27). That is, \( P_{2s+2}(r) = 0 \) at \( r = 1 \) in (2.29). So the polynomial \( P_{2s+2}(r)/(1-r) \) has at most \( 2s \) positive zeros. From Lemma 1,3, we know that there exists an \( \varepsilon_0 > 0 \) such that for \( 0 < |\varepsilon| < \varepsilon_0 \), \( a = (a_{ij}, b_{ij}) \) with \( |a_{ij}| \leq K, |b_{ij}| \leq K \), the system (2.2) has at most \( 2s \) limit cycles.

For the case of \( n = 2s - 1 \), we can prove the theorem similarly. The proof is completed.

**Proof of Theorem 2.3.** Let \( n = 2s \) first. From Theorem 2.2, we know that the function \( \Phi(h) \) has the form of (2.27) and (2.28). From (2.25), (2.27), and (2.28) we can deduce that
\[ \text{rank} \left( \frac{\partial(B_{s-1}, \ldots, B_1, B_0, A_{s+1}, \ldots, A_1)}{\partial(a_{ij}, b_{ij})_{0 \leq i+j \leq 2s}} \right) = 2s + 1. \tag{2.30} \]

Let \( \sqrt{1 - a_i^2} h = r \) in (2.27). Then (2.27) becomes (2.29) with \( P_{2s+2}(r) = 0 \) at \( r = 1 \), where the coefficients \( c_{2s+2}, c_{2s+1}, \ldots, c_3, c_2, c_0 \) satisfy
\[ \text{rank} \left( \frac{\partial(c_{2s+2}, c_{2s+1}, \ldots, c_3, c_2, c_0)}{\partial(B_{s-1}, \ldots, B_1, B_0, A_{s+1}, \ldots, A_1)} \right) = 2s + 1. \tag{2.31} \]

From the proof of Theorem 2.2 we have that the polynomial \( P_{2s+2}(r)/(1-r) \) has at most \( 2s \) positive simple zeros. From (2.30) and (2.31), we have that there exists \( a^0 = (a_{ij}^0, b_{ij}^0) \) such that \( P_{2s+2}(r)/(1-r) \) has exactly \( 2s \) positive simple zeros. Then from Lemma 1.3 and its Remark 1.1, we have proved that there exists an \( \varepsilon_0 > 0 \) such that for \( 0 < |\varepsilon| < \varepsilon_0 \) and \( 0 < \varepsilon < \varepsilon_0 \), the system (2.2) has precisely \( 2s \) limits cycles.

For the case of \( n = 2s - 1 \), the theorem can be proved in the same way. This finishes the proof.

**References**