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A uniformly convergent scheme for a system of reaction–diffusion equations ☆

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Abstract

In this work a system of two parabolic singularly perturbed equations of reaction–diffusion type is considered. The asymptotic behaviour of the solution and its partial derivatives is given. A decomposition of the solution in its regular and singular parts has been used for the asymptotic analysis of the spatial derivatives. To approximate the solution we consider the implicit Euler method for time stepping and the central difference scheme for spatial discretization on a special piecewise uniform Shishkin mesh. We prove that this scheme is uniformly convergent, with respect to the diffusion parameters, having first-order convergence in time and almost second-order convergence in space, in the discrete maximum norm. Numerical experiments illustrate the order of convergence proved theoretically.

MSC: 65N12; 65N30; 65N06

Keywords: Singular perturbation; Reaction–diffusion problems; Uniform convergence; Coupled system; Shishkin mesh

Contents

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1. Introduction

We consider parabolic singularly perturbed boundary value problems given by

$$\begin{cases} L_{\vec{\varepsilon}}\vec{u} \equiv \dfrac{\partial \vec{u}}{\partial t} + L_{x,\vec{\varepsilon}}\vec{u} = \vec{f}, & (x,t) \in Q = \Omega \times (0,T] = (0,1) \times (0,T], \\ \vec{u}(0,t) = \vec{g}_0(t), \quad \vec{u}(1,t) = \vec{g}_1(t) \ \ \forall t \in [0,T], \\ \vec{u}(x,0) = \vec{0} \quad \forall x \in \Omega, \end{cases} \tag{1}$$

where

$$L_{x,\vec{\varepsilon}} \equiv \begin{pmatrix} -\varepsilon_1 \dfrac{\partial^2}{\partial x^2} & \\ & -\varepsilon_2 \dfrac{\partial^2}{\partial x^2} \end{pmatrix} + A, \quad A = \begin{pmatrix} a_{11}(x,t) & a_{12}(x,t) \\ a_{21}(x,t) & a_{22}(x,t) \end{pmatrix}.$$

We shall denote $\Gamma_0 = \{(x,0) \mid x \in \Omega\}$, $\Gamma_1 = \{(x,t) \mid x = 0, 1, \ t \in [0,T]\}$, $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2)^{\mathrm{T}}$, with $0 < \varepsilon_1 \leqslant \varepsilon_2 \leqslant 1$, the vectorial singular perturbation parameter. The coupling matrix A satisfies the positivity condition

$$a_{i1} + a_{i2} \geqslant 0, \quad a_{ii} > 0, \quad i = 1, 2, \tag{2a}$$

$$a_{ij} \leqslant 0 \quad \text{if } i \neq j. \tag{2b}$$

Otherwise, we consider the transformation $\vec{v}(x,t) = \vec{u}(x,t)\mathrm{e}^{-\alpha_0 t}$ with $\alpha_0 > 0$ sufficiently large in order to transform diagonal entries such that (2a) holds.

Also, we assume that enough regularity and compatibility conditions hold for data of problem (1) in order that $\vec{u} \in C^{4,2}(\bar{Q})$, i.e., the spatial partial derivatives of the solution are continuous up to fourth order and the time partial derivatives are continuous up to second order. For instance, we will suppose the conditions

$$\begin{aligned} &\vec{g}_i^{(k)}(0) = \vec{0}, \quad i = 0, 1, \quad k = 0, 1, 2, \\ &\frac{\partial^{k+k_0} \vec{f}}{\partial x^k \partial t^{k_0}}(0,0) = \frac{\partial^{k+k_0} \vec{f}}{\partial x^k \partial t^{k_0}}(1,0) = \vec{0}, \quad 0 \leqslant k + 2k_0 \leqslant 2, \end{aligned} \tag{3}$$

which are an extension of compatibility conditions for the scalar case (see [7]).

The classical linear double-diffusion model for saturated flow in fractured porous media (Barenblatt system) introduced in [1] is an example of systems of type (1). The first equation describes the flow in the porous material and the second one the flow in the fracture system. The coupling terms are given by the matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ which describes the interchange of fluid between pores and fractures. The permeabilities ε_1 and ε_2 in these equations could be very small and with different magnitudes. In geological models typical values are $\varepsilon_1 = 10^{-7}$ and $\varepsilon_2 = 10^{-4}$. The Barenblatt system can be also used to model diffusion process in bones (see, for example, [4]) which can be studied as a double porosity model where the typical values are $\varepsilon_1 = 10^{-13}$ and $\varepsilon_2 = 10^{-9}$.

These small parameters cause a multiscale character of the solution. Narrow regions, the boundary layers, appear close to the boundary where the solution has strong gradients and in the rest of the domain the solution varies smoothly. To approximate efficiently the solution it is necessary to employ special numerical methods for which the number of mesh points is independent of the singular perturbation parameter. In recent years fitted mesh methods have been used extensively. In these methods a graded mesh is defined according to the behaviour of the solution. In particular, we shall use here meshes of Shishkin type (see [5,14,18,19] and the references therein) which are piecewise uniform meshes. To design these meshes it is necessary to know a priori the asymptotic behaviour of the solution and its partial derivatives.

Singularly perturbed problems for elliptic and parabolic problems of reaction–diffusion type have been extensively studied. We refer to [5,14] for 1D stationary problems, to [6,15] for 1D parabolic problems, to [2] for 2D stationary problems and to [3] for 2D evolutionary problems.

The theoretical analysis and the numerical approximation is more complicated when systems are considered. Recently, some papers consider problems of type (1) in the stationary case under the hypothesis $a_{i1} + a_{i2} > 0$, $i = 1, 2$,

$a_{ii} > 0$, $a_{ij} \leqslant 0$, $i \neq j$. These hypothesis guarantee that the maximum principle holds. To approximate the solution, the classical central difference operator is defined on a special Shishkin mesh, proving the uniform convergence in the maximum norm.

Three cases can be distinguished depending on the relation between the singular perturbation parameters ε_1 and ε_2

(i) $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = 1$,
(ii) $\varepsilon_1 = \varepsilon_2 = \varepsilon$,
(iii) $\varepsilon_1, \varepsilon_2$ arbitrary.

In [12,11] first-order uniform convergence was proved in cases (ii) and (iii), respectively. In [13,8,10] second-order uniform convergence was obtained for cases (i)–(iii), respectively. Finally, in [9], the authors consider Bakhavalov and Shishkin meshes, proving in both cases second-order convergence in norm L_2 and first-order convergence in the standard energy norm.

In this paper, Section 2 is devoted to establish the asymptotic behaviour of the solution of problem (1). Some results in this direction has been given in [20] without proofs. Also, we use a decomposition of the solution into regular and singular components which is different to the one given in [20]. In Section 3 we use the implicit Euler scheme to discretize in time and the central difference scheme on a fitted Shishkin mesh to discretize in space. We analyse the uniform convergence of this scheme, proving first-order convergence in time and almost second-order convergence in space. The analysis given in this work for problem (1) relies on the steady problem, standing out the paper [11] because the ideas of the authors have been extended to the evolutionary case. Finally, in Section 4 we show some numerical experiments that confirm in practice the theoretical results.

We write $\vec{v} \leqslant \vec{w}$ if $v_i \leqslant w_i$, $i = 1, 2$, $|\vec{v}| = (|v_1|, |v_2|)^{\mathrm{T}}$, $\vec{c} = (c, c)^{\mathrm{T}}$, where c is a constant, $\|f\|_H$ is the maximum norm of f on H, $\|\vec{f}\|_H = \max\{\|f_1\|_H, \|f_2\|_H\}$ and for simplicity we use f_z or $\partial f/\partial z$ (analogously for higher order partial derivatives). Henceforth, any positive constant is independent of the diffusion parameters ε_1, ε_2 and the discretization parameters N and Δt.

2. Asymptotic behaviour of the solution

In this section some bounds of the exact solution and its partial derivatives are deduced. We shall use systematically the maximum principle (see [7,17]). This principle for system (1) is given by:

Theorem 1 (*Maximum principle*). *Let $L_{\vec{\varepsilon}}$ be the differential operator given in* (1) *and we assume that the coefficients of matrix A satisfy the positivity conditions* (2a) *and* (2b). *If $\vec{\psi} \geqslant \vec{0}$ on Γ and $L_{\vec{\varepsilon}}\vec{\psi} \geqslant \vec{0}$ in Q, then $\vec{\psi} \geqslant \vec{0}$ for all $(x, t) \in \bar{Q}$.*

Proof. We write $\vec{\psi} = \mathrm{e}^{\lambda t}\vec{\varphi}$, with λ a positive constant. Note that sgn $\psi_i =$ sgn φ_i, $i = 1, 2$ and

$$L_{\vec{\varepsilon}}\vec{\psi} = \mathrm{e}^{\lambda t}\left(\frac{\partial \vec{\varphi}}{\partial t} + \begin{pmatrix} -\varepsilon_1 & \\ & -\varepsilon_2 \end{pmatrix} \frac{\partial^2 \vec{\varphi}}{\partial x^2} + (A + \lambda I)\vec{\varphi}\right) \geqslant \vec{0}.$$

We shall prove that $\vec{\varphi} \geqslant \vec{0}$ for all $(x, t) \in \bar{Q}$. We suppose that there exists $(x_0, t_0) \in \bar{Q} \backslash \Gamma$ such that

$$\min\{\varphi_1(x_0, t_0), \varphi_2(x_0, t_0)\} = \min\left\{\min_{(x,t)\in\bar{Q}} \varphi_1(x, t), \min_{(x,t)\in\bar{Q}} \varphi_2(x, t)\right\} < 0.$$

Without loss of generality, we assume that $\varphi_1(x_0, t_0) \leqslant \varphi_2(x_0, t_0)$. At this point

$$\frac{\partial \varphi_1}{\partial t}(x_0, t_0) \leqslant 0, \quad \frac{\partial^2 \varphi_1}{\partial x^2}(x_0, t_0) \geqslant 0,$$

and the first component of $L_{\vec{\varepsilon}}\vec{\psi}$ satisfies

$$[L_{\vec{\varepsilon}}\vec{\psi}(x_0,t_0)]_1 = \mathrm{e}^{\lambda t_0}\left(\frac{\partial\varphi_1}{\partial t}(x_0,t_0) - \varepsilon_1\frac{\partial^2\varphi_1}{\partial x^2}(x_0,t_0) + (a_{11}(x_0,t_0)+\lambda)\varphi_1(x_0,t_0) + a_{12}(x_0,t_0)\varphi_2(x_0,t_0)\right)$$
$$\leqslant \mathrm{e}^{\lambda t_0}\left(a_{11}(x_0,t_0)\varphi_1(x_0,t_0) + a_{12}(x_0,t_0)\varphi_2(x_0,t_0) + \lambda\varphi_1(x_0,t_0)\right)$$
$$\leqslant \mathrm{e}^{\lambda t_0}\lambda\varphi_1(x_0,t_0) < 0,$$

using the hypothesis (2a) and (2b). This negative bound contradicts the hypothesis of this theorem and therefore $\vec{\psi} \geqslant \vec{0}$ for all $(x,t) \in \bar{Q}$. □

The following result is an immediately consequence of the maximum principle.

Corollary 2 (*Comparison principle*). *Let $L_{\vec{\varepsilon}}$ be the differential operator given in* (1) *and we assume that the coefficients of matrix A satisfy the positivity conditions* (2a) *and* (2b). *If $|\vec{\psi}| \leqslant \vec{\varphi}$ on Γ and $|L_{\vec{\varepsilon}}\vec{\psi}| \leqslant L_{\vec{\varepsilon}}\vec{\varphi}$* in Q, *then $|\vec{\psi}| \leqslant \vec{\varphi}$ for all $(x,t) \in \bar{Q}$.*

Lemma 3. *The solution of problem* (1) *satisfies*

$$\left\|\frac{\partial^i\vec{u}}{\partial t^i}\right\|_{\bar{Q}} \leqslant C, \quad i = 0, 1, 2.$$

Proof. The barrier function $\vec{\psi} = (t+1)\vec{C}$, with C a sufficiently large positive constant, proves that $\vec{u}$ is bounded. Now, we denote $\vec{p} = \vec{u}_t$. On Γ_1 this function satisfies

$$|\vec{p}(x,t)| \leqslant \max_{t\in[0,T]}\left\{|\vec{g}\,'_0(t)|, |\vec{g}\,'_1(t)|\right\} \leqslant \vec{C},$$

and on Γ_0, by continuity, we have

$$\|\vec{p}(x,0)\|_{\bar{\Omega}} = \|\vec{f}(x,0)\|_{\bar{\Omega}} \leqslant C.$$

Differentiating (1) w.r.t. t, we have

$$L_{\vec{\varepsilon}}\vec{p} = \vec{f}_t - A_t\vec{u}, \quad (x,t) \in Q,$$

where $A_t = (\partial a_{ij}/\partial t)$. The same barrier function as before $\vec{\psi} = (t+1)\vec{C}$, proves $\|\vec{p}\|_{\bar{Q}} = \|\vec{u}_t\|_{\bar{Q}} \leqslant C$. The analysis of the function $\vec{q} = \vec{u}_{tt}$ is similar. Now, it holds

$$|\vec{q}(x,t)| \leqslant \vec{C}, \quad (x,t) \in \Gamma_1,$$
$$\|\vec{q}(x,0)\|_{\bar{\Omega}} = \|\vec{f}_t(x,0) - A_t\vec{u} - L_{x,\vec{\varepsilon}}\vec{f}(x,0)\|_{\bar{\Omega}} \leqslant C,$$
$$L_{\vec{\varepsilon}}\vec{q} = \vec{f}_{tt} - A_{tt}\vec{u} - 2A_t\vec{u}_t, \quad (x,t) \in Q,$$

where $A_{tt} = (\partial^2 a_{ij}/\partial t^2)$. The results follows by using again the barrier function $\vec{\psi} = (t+1)\vec{C}$. □

In the following, we shall use a decomposition of the exact solution

$$\vec{u} = \vec{v} + \vec{w},$$

where $\vec{v}$, the regular component, is the solution of the problem

$$\begin{cases} L_{\vec{\varepsilon}}\vec{v} = \vec{f} & \text{in } Q, \\ \vec{v}(x,0) = \vec{0} & \text{on } \Gamma_0, \\ \vec{v} = \vec{z} & \text{on } \Gamma_1, \end{cases} \tag{4}$$

where $\vec{z}$ satisfies the following IVP

$$\begin{cases} \vec{z}_t + A\vec{z} = \vec{f}, & (x,t) \in \{0,1\} \times (0,T], \\ \vec{z}(x,0) = \vec{0}, & x \in \{0,1\}. \end{cases} \tag{5}$$

The singular component is the solution of the problem

$$\begin{cases} L_{\vec{\varepsilon}}\vec{w} = \vec{0} & \text{in } Q, \\ \vec{w} = \vec{u} - \vec{v} & \text{on } \Gamma. \end{cases} \tag{6}$$

Note that from the hypothesis on the function $\vec{f}$ given in (3) and that $\vec{z}$ is solution of problem (5), we have that $\vec{z}(x,0) = \vec{z}_t(x,0) = \vec{z}_{tt}(x,0) = \vec{0},\ x \in \{0,1\}$ and therefore $\vec{v} \in C^{4,2}(\bar{Q})$. In addition $\vec{w} \in C^{4,2}(\bar{Q})$.

Lemma 4. *The regular component satisfies*

$$\left\| \frac{\partial^i \vec{v}}{\partial t^i} \right\|_{\bar{Q}} \leqslant C, \quad i = 0, 1, 2. \tag{7}$$

Proof. Similarly to Lemma 3. □

Lemma 5. *The regular component satisfies*

$$\left\| \frac{\partial^i \vec{v}}{\partial x^i} \right\|_{\bar{Q}} \leqslant C, \quad i = 1, 2.$$

Proof. From (4) and (5), we have

$$\vec{v}_{xx}(x,0) = \vec{0}, \ \ x \in \bar{\Omega}, \quad \vec{v}_{xx}(x,t) = \vec{0}, \ \ (x,t) \in \Gamma_1. \tag{8}$$

Differentiating (4) twice w.r.t. x, we have

$$|L_{\vec{\varepsilon}}\vec{v}_{xx}| = |(\vec{f}_{xx} - 2A_x\vec{v}_x - A_{xx}\vec{v})| \leqslant (1 + \|\vec{v}_x\|_{\bar{Q}})\vec{C}, \tag{9}$$

where $A_x = (\partial a_{ij}/\partial x)$ and $A_{xx} = (\partial^2 a_{ij}/\partial x^2)$. Then, the comparison principle proves

$$|\vec{v}_{xx}| \leqslant t(1 + \|\vec{v}_x\|_{\bar{Q}})\vec{C},$$

and therefore

$$\|\vec{v}_{xx}\|_{\bar{Q}} \leqslant C^*(1 + \|\vec{v}_x\|_{\bar{Q}}). \tag{10}$$

Similarly to [11, Lemma 3], using the Mean Value Theorem, we can prove

$$\|\vec{v}_x\|_{\bar{Q}} \leqslant C + \frac{\|\vec{v}_{xx}\|_{\bar{Q}}}{2C^*}. \tag{11}$$

The result follows from (10) and (11). □

Lemma 6. *The regular component satisfies*

$$\left\| \frac{\partial^2 \vec{v}}{\partial t \partial x} \right\|_{\bar{Q}} \leqslant C, \quad \left\| \frac{\partial^3 \vec{v}}{\partial t \partial x^2} \right\|_{\bar{Q}} \leqslant C.$$

Proof. From (8) we have that $\vec{v}_{xxt} = \vec{0},\ (x,t) \in \Gamma_1$. Using that $\vec{v}(x,0) = \vec{0}$ on Γ_0 and differentiating (4) twice w.r.t. x, we deduce $|\vec{v}_{xxt}(x,0)| = |\vec{f}_{xx}(x,0)| \leqslant \vec{C},\ x \in \bar{\Omega}$. Differentiating (4) twice w.r.t. x and once w.r.t. t, we obtain

$$|L_{\vec{\varepsilon}}\vec{v}_{xxt}| = |(\vec{f}_{xxt} - A_{xxt}\vec{v} - 2A_{xt}\vec{v}_x - A_t\vec{v}_{xx} - 2A_x\vec{v}_{xt} - A_{xx}\vec{v}_t)| \leqslant (1 + \|\vec{v}_{xt}\|_{\bar{Q}})\vec{C}. \tag{12}$$

The comparison principle proves $\|\vec{v}_{xxt}\|_{\bar{Q}} \leqslant C(1 + \|\vec{v}_{xt}\|_{\bar{Q}})$ and the proof finishes using the same argument than in previous Lemma 5. □

Lemma 7. *The regular component* $\vec{v} = (v_1, v_2)^{\mathrm{T}}$ *satisfies*

$$\left\| \frac{\partial^i v_1}{\partial x^i} \right\|_{\bar{Q}} \leqslant C\varepsilon_1^{1-i/2}, \quad \left\| \frac{\partial^i v_2}{\partial x^i} \right\|_{\bar{Q}} \leqslant C\varepsilon_2^{1-i/2}, \quad i = 3, 4.$$

Proof. We only consider the component v_1 since the proof is similar for both components. Differentiating (4) twice w.r.t. x, for any $t \in (0, T]$, it holds

$$\begin{cases} -\varepsilon_1 \dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial^2 v_1}{\partial x^2}\right) + a_{11}\dfrac{\partial^2 v_1}{\partial x^2} = -a_{12}\dfrac{\partial^2 v_2}{\partial x^2} + \dfrac{\partial^2 f_1}{\partial x^2} - \dfrac{\partial^3 v_1}{\partial t \partial x^2} - \left[A_{xx}\vec{v} + 2A_x\dfrac{\partial \vec{v}}{\partial x}\right]_1, \quad x \in (0, 1), \\ \dfrac{\partial^2 v_1}{\partial x^2}(0, t) = \dfrac{\partial^2 v_1}{\partial x^2}(1, t) = 0. \end{cases} \tag{13}$$

Note that all terms of the right-hand side and $\partial^2 v_1/\partial x^2$ are bounded. Then, the previous differential equation directly proves

$$\left\| \frac{\partial^4 v_1}{\partial x^4}(x, t) \right\|_{\bar{Q}} \leqslant C\varepsilon_1^{-1}.$$

Taking into account that the right-hand side of problem (13) is parameter uniform bounded, we can apply the argument of [16] on $\{(x, t) \mid x \in \bar{\Omega}\}$ for any $t \in (0, T]$, obtaining the following bound

$$\left\| \frac{\partial^3 v_1}{\partial x^3}(x, t) \right\|_{\bar{Q}} \leqslant C\varepsilon_1^{-1/2}. \qquad \square$$

Below to simplify the notation, we define the function

$$B_\gamma(x) = \mathrm{e}^{-x\sqrt{\alpha/\gamma}} + \mathrm{e}^{-(1-x)\sqrt{\alpha/\gamma}},$$

where α is a positive constant.

Lemma 8. *The singular component satisfies*

$$\left| \frac{\partial^i \vec{w}}{\partial t^i} \right| \leqslant B_{\varepsilon_2}(x)\vec{C} \quad \forall (x, t) \in \bar{Q}, \quad i = 0, 1, 2. \tag{14}$$

Proof. Firstly, we note that from (6) and Lemma 4

$$\left|\frac{\partial^i \vec{w}}{\partial t^i}(x, t)\right| \leqslant \left|\frac{\partial^i \vec{u}}{\partial t^i}(x, t)\right| + \left|\frac{\partial^i \vec{v}}{\partial t^i}(x, t)\right| \leqslant \max_{t \in [0, T]} \left\{|\vec{g}_0^{(i)}(t)|, |\vec{g}_1^{(i)}(t)|\right\} + \left|\frac{\partial^i \vec{v}}{\partial t^i}(x, t)\right| \leqslant \vec{C},$$

for $(x, t) \in \Gamma_1$ and $i = 0, 1, 2$. On Γ_0, using that $\vec{u}(x, 0) = \vec{v}(x, 0) = \vec{0}$, we have that $\vec{w}(x, 0) = \vec{0}$, $x \in \bar{\Omega}$, and by continuity $\vec{w}_t(x, 0) = \vec{w}_{tt}(x, 0) = \vec{0}$, $x \in \bar{\Omega}$.

If $(x, t) \in Q$, it holds

$$L_{\vec{\varepsilon}}\vec{w} = \vec{0}, \quad L_{\vec{\varepsilon}}\vec{w}_t = -A_t\vec{w}, \quad L_{\vec{\varepsilon}}\vec{w}_{tt} = -A_{tt}\vec{w} - 2A_t\vec{w}_t.$$

On the other hand, the function $\vec{\psi}(x, t) = \mathrm{e}^{2\alpha t}B_{\varepsilon_2}(x)\vec{C}_1 \geqslant \vec{0}$, $\forall (x, t) \in \bar{Q}$, with C_1 a sufficiently large positive constant, for $(x, t) \in Q$ satisfies

$$\vec{\psi} \geqslant \vec{C}_1, \ (x, t) \in \Gamma_1, \quad L_{\vec{\varepsilon}}\vec{\psi} \geqslant C_1\alpha\mathrm{e}^{2\alpha t}B_{\varepsilon_2}(x)(2 - \varepsilon_1/\varepsilon_2, 1)^{\mathrm{T}} \geqslant \alpha\mathrm{e}^{2\alpha t}B_{\varepsilon_2}(x)\vec{C}_1,$$

since $\varepsilon_1 \leqslant \varepsilon_2$. Using successively the comparison principle we can deduce that the function $\vec{\psi}(x, t)$ is a barrier function for $\vec{w}$, $\vec{w}_t$ and $\vec{w}_{tt}$. Finally, note that $\vec{\psi}(x, t) = B_{\varepsilon_2}(x)\vec{C}$ with $C = \mathrm{e}^{2\alpha T}C_1$. $\square$

Lemma 9. *The singular component satisfies*

$$|w_1(x)| \leqslant C B_{\varepsilon_2}(x), \quad |w_2(x)| \leqslant C B_{\varepsilon_2}(x), \tag{15}$$

$$\left|\frac{\partial w_1}{\partial x}\right| \leqslant C(\varepsilon_1^{-1/2} B_{\varepsilon_1}(x) + \varepsilon_2^{-1/2} B_{\varepsilon_2}(x)), \quad \left|\frac{\partial w_2}{\partial x}\right| \leqslant C\varepsilon_2^{-1/2} B_{\varepsilon_2}(x), \tag{16}$$

$$\left|\frac{\partial^2 w_1}{\partial x^2}\right| \leqslant C(\varepsilon_1^{-1} B_{\varepsilon_1}(x) + \varepsilon_2^{-1} B_{\varepsilon_2}(x)), \quad \left|\frac{\partial^2 w_2}{\partial x^2}\right| \leqslant C\varepsilon_2^{-1} B_{\varepsilon_2}(x), \tag{17}$$

$$\left|\frac{\partial^3 w_1}{\partial x^3}\right| \leqslant C(\varepsilon_1^{-3/2} B_{\varepsilon_1}(x) + \varepsilon_2^{-3/2} B_{\varepsilon_2}(x)), \tag{18}$$

$$\left|\frac{\partial^3 w_2}{\partial x^3}\right| \leqslant C\varepsilon_2^{-1}(\varepsilon_1^{-1/2} B_{\varepsilon_1}(x) + \varepsilon_2^{-1/2} B_{\varepsilon_2}(x)), \tag{19}$$

$$\left|\frac{\partial^4 w_1}{\partial x^4}\right| \leqslant C(\varepsilon_1^{-2} B_{\varepsilon_1}(x) + \varepsilon_2^{-2} B_{\varepsilon_2}(x)), \tag{20}$$

$$\left|\frac{\partial^4 w_2}{\partial x^4}\right| \leqslant C\varepsilon_2^{-1}(\varepsilon_1^{-1} B_{\varepsilon_1}(x) + \varepsilon_2^{-1} B_{\varepsilon_2}(x)). \tag{21}$$

Proof. We follow the proof given in [11, Lemma 4], showing the differences that appears in the parabolic case.

Bounds (15) are proved in Lemma 8. Taking into account that $|\partial\vec{w}/\partial t| \leqslant B_{\varepsilon_2}(x)\vec{C}$, the same argument as in [11] proves

$$\left|\frac{\partial w_1}{\partial x}(x,t)\right| \leqslant C\varepsilon_1^{-1/2} B_{\varepsilon_2}(x), \quad \left|\frac{\partial^2 w_1}{\partial x^2}(x,t)\right| \leqslant C\varepsilon_1^{-1} B_{\varepsilon_2}(x), \tag{22}$$

$$\left|\frac{\partial w_2}{\partial x}(x,t)\right| \leqslant C\varepsilon_2^{-1/2} B_{\varepsilon_2}(x), \quad \left|\frac{\partial^2 w_2}{\partial x^2}(x,t)\right| \leqslant C\varepsilon_2^{-1} B_{\varepsilon_2}(x). \tag{23}$$

The bounds for the first component are not the required, but from them we deduce the following bounds on the boundary

$$\left|\frac{\partial w_1}{\partial x}(x,t)\right| \leqslant C\varepsilon_1^{-1/2}, \quad \left|\frac{\partial^2 w_1}{\partial x^2}(x,t)\right| \leqslant C\varepsilon_1^{-1}, \quad (x,t) \in \Gamma_1, \tag{24}$$

that we shall use to improve (22).

The function $\psi_1(x,t) = C\mathrm{e}^{2\alpha t}(\varepsilon_1^{-1/2} B_{\varepsilon_1}(x) + \varepsilon_2^{-1/2} B_{\varepsilon_2}(x))$, with C a sufficiently large positive constant, satisfies

$$0 = \frac{\partial w_1}{\partial x}(x,0) \leqslant \psi_1(x,0), \quad x \in \bar{\Omega}, \quad \left|\frac{\partial w_1}{\partial x}(x,t)\right| \leqslant \psi_1(x,t), \quad (x,t) \in \Gamma_1 \tag{25}$$

and

$$\left|L^*_{\varepsilon_1}\frac{\partial w_1}{\partial x}(x,t)\right| \leqslant L^*_{\varepsilon_1}\psi_1(x,t), \quad (x,t) \in Q, \tag{26}$$

where the differential operator $L^*_{\varepsilon_1} := \partial/\partial t - \varepsilon_1\partial^2/\partial x^2 + a_{11}$, with $a_{11} > 0$, satisfies a maximum principle. From (25) and (26), we deduce that ψ_1 is a barrier function for $\partial w_1/\partial x$.

A similar argument proves that $\psi_2(x,t)=C\mathrm{e}^{2\alpha t}(\varepsilon_1^{-1}B_{\varepsilon_1}(x)+\varepsilon_2^{-1}B_{\varepsilon_2}(x))$ is a barrier function for $\partial^2 w_1/\partial x^2$.

Now we consider the third-order partial derivatives. Differentiating w.r.t. t the equation

$$\frac{\partial w_2}{\partial t}-\varepsilon_2\frac{\partial^2 w_2}{\partial x^2}+a_{21}w_1+a_{22}w_2=0, \tag{27}$$

and using bounds of Lemma 8, we have

$$\left|\frac{\partial^3 w_2}{\partial x^2\partial t}\right|\leqslant C\varepsilon_2^{-1}B_{\varepsilon_2}(x). \tag{28}$$

Hence, using a similar technique to that of [11] we have that

$$\left|\frac{\partial^2 w_2}{\partial x\partial t}\right|\leqslant C\varepsilon_2^{-1/2}B_{\varepsilon_2}(x). \tag{29}$$

Differentiating (27) w.r.t. x and using bounds (15), (16) and (29), it follows that

$$\left|\frac{\partial^3 w_2}{\partial x^3}\right|\leqslant C\varepsilon_2^{-1}(\varepsilon_1^{-1/2}B_{\varepsilon_1}(x)+\varepsilon_2^{-1/2}B_{\varepsilon_2}(x)). \tag{30}$$

In a similar way, it is possible to deduce the crude bound $|\partial^3 w_1(x,t)/\partial x^3|\leqslant C\varepsilon_1^{-3/2}$, $(x,t)\in\bar{Q}$, which allows to establish appropriated bounds on the boundary Γ_1. The function $\psi_3(x,t)=C\mathrm{e}^{2\alpha t}(\varepsilon_1^{-3/2}B_{\varepsilon_1}(x)+\varepsilon_2^{-3/2}B_{\varepsilon_2}(x))$, is a barrier function for $\partial^3 w_1/\partial x^3$ which proves (18).

The argument used for the third-order partial derivatives can be extended to the fourth-order partial derivatives. □

Lemma 10. *Suppose that* $\varepsilon_1<\varepsilon_2$. *Then, the singular component* $\vec{w}=(w_1,w_2)$ *can be decomposed as*

$$w_1=w_{1,\varepsilon_1}+w_{1,\varepsilon_2},\quad w_2=w_{2,\varepsilon_1}+w_{2,\varepsilon_2}, \tag{31}$$

where

$$\left|\frac{\partial^2 w_{1,\varepsilon_1}}{\partial x^2}\right|\leqslant C\varepsilon_1^{-1}B_{\varepsilon_1}(x),\quad \left|\frac{\partial^3 w_{1,\varepsilon_2}}{\partial x^3}\right|\leqslant C\varepsilon_2^{-3/2}B_{\varepsilon_2}(x), \tag{32}$$

$$\left|\frac{\partial^2 w_{2,\varepsilon_1}}{\partial x^2}\right|\leqslant C\varepsilon_2^{-1}B_{\varepsilon_1}(x),\quad \left|\frac{\partial^3 w_{2,\varepsilon_2}}{\partial x^3}\right|\leqslant C\varepsilon_2^{-3/2}B_{\varepsilon_2}(x). \tag{33}$$

Also, the singular component $\vec{w}=(w_1,w_2)$ *can be decomposed as*

$$w_1=z_{1,\varepsilon_1}+z_{1,\varepsilon_2},\quad w_2=z_{2,\varepsilon_1}+z_{2,\varepsilon_2}, \tag{34}$$

where

$$\left|\frac{\partial^2 z_{1,\varepsilon_1}}{\partial x^2}\right|\leqslant C\varepsilon_1^{-1}B_{\varepsilon_1}(x),\quad \left|\frac{\partial^4 z_{1,\varepsilon_2}}{\partial x^4}\right|\leqslant C\varepsilon_1^{-1}\varepsilon_2^{-1}B_{\varepsilon_2}(x), \tag{35}$$

$$\left|\frac{\partial^2 z_{2,\varepsilon_1}}{\partial x^2}\right|\leqslant C\varepsilon_2^{-1}B_{\varepsilon_1}(x),\quad \left|\frac{\partial^4 z_{2,\varepsilon_2}}{\partial x^4}\right|\leqslant C\varepsilon_2^{-2}B_{\varepsilon_2}(x). \tag{36}$$

Proof. See [11, Lemma 5], for the decomposition (31) and [10, Lemma 2], for the decomposition (34). □

3. Numerical scheme. Analysis of the uniform convergence

To approximate the solution of (1), we consider the implicit Euler and the central difference schemes to discretize the time and spatial variables, respectively. The numerical solution is defined on the mesh

$$\bar{Q}^N = \bar{\Omega}^N \times \bar{\omega}^N,$$

where, for simplicity, we consider a uniform mesh for the time discretization

$$\bar{\omega}^N = \{k\Delta t,\ 0 \leqslant k \leqslant M,\ \Delta t = T/M\},$$

and for the spatial discretization, $\bar{\Omega}^N$ is a piecewise uniform mesh. Because of the components of the solution in the x-direction can have two boundary layers at both sides $x = 0$ and 1, the mesh $\bar{\Omega}^N$ is defined by means of two transition parameters. We define

$$\tau_{\varepsilon_2} = \min\{1/4, m\sqrt{\varepsilon_2}\ln N\}, \quad \tau_{\varepsilon_1} = \min\{\tau_{\varepsilon_2}/2, m\sqrt{\varepsilon_1}\ln N\}, \tag{37}$$

where m is an arbitrary positive real number. In the subintervals $[0, \tau_{\varepsilon_1}]$, $[\tau_{\varepsilon_1}, \tau_{\varepsilon_2}]$, $[\tau_{\varepsilon_2}, 1-\tau_{\varepsilon_2}]$, $[1-\tau_{\varepsilon_2}, 1-\tau_{\varepsilon_1}]$ and $[1-\tau_{\varepsilon_1}, 1]$ we distribute uniformly $N/8+1$, $N/8+1$, $N/2+1$, $N/8+1$ and $N/8+1$ mesh points, respectively. So the mesh points are given by

$$x_j = \begin{cases} jh_{\varepsilon_1}, & j = 0, \ldots, N/8, \\ x_{N/8} + (j - N/8)h_{\varepsilon_2}, & j = N/8+1, \ldots, N/4, \\ x_{N/4} + (j - N/4)H, & j = N/4+1, \ldots, 3N/4, \\ x_{3N/4} + (j - 3N/4)h_{\varepsilon_2}, & j = 3N/4+1, \ldots, 7N/8, \\ x_{7N/8} + (j - 7N/8)h_{\varepsilon_1}, & j = 7N/8+1, \ldots, N, \end{cases}$$

where

$$h_{\varepsilon_1} = \frac{8\tau_{\varepsilon_1}}{N}, \quad h_{\varepsilon_2} = \frac{8(\tau_{\varepsilon_2} - \tau_{\varepsilon_1})}{N}, \quad H = \frac{2(1 - 2\tau_{\varepsilon_2})}{N}.$$

When $\tau_{\varepsilon_1} \neq \frac{1}{8}$ and $\tau_{\varepsilon_2} = \frac{1}{4}$, it means that only the first component of the exact solution has boundary layers. In this case we will take a new grid with equally distributed points in the subintervals $[0, \tau_{\varepsilon_1}]$, $[\tau_{\varepsilon_1}, 1-\tau_{\varepsilon_1}]$ and $[1-\tau_{\varepsilon_1}, 1]$. Now the mesh points are given by

$$x_j = \begin{cases} jh_{\varepsilon_1}, & j = 0, \ldots, N/8, \\ x_{N/8} + (j - N/8)\hat{H}, & j = N/8+1, \ldots, 7N/8, \\ x_{7N/8} + (j - 7N/8)h_{\varepsilon_1}, & j = 7N/8+1, \ldots, N, \end{cases}$$

where

$$\hat{H} = \frac{4(1 - 2\tau_{\varepsilon_1})}{3N}.$$

On this mesh, we define the following finite difference scheme

$$(I + \Delta t L^N_{x,\vec{\varepsilon}})\vec{U}^{n+1}_j = \vec{U}^n_j + \Delta t \vec{f}^{n+1}_j, \quad 0 < j < N, \ \ n = 0, \ldots, M-1, \tag{38}$$

where

$$L^N_{x,\vec{\varepsilon}} \equiv \begin{pmatrix} -\varepsilon_1\delta^2 & \\ & -\varepsilon_2\delta^2 \end{pmatrix} + A^{n+1}_j, \quad \delta^2 Z_j \equiv \frac{2}{h_j + h_{j+1}} \left(\frac{Z_{j+1} - Z_j}{h_{j+1}} - \frac{Z_j - Z_{j-1}}{h_j} \right),$$

with $h_j = x_j - x_{j-1}$, $j = 1, \ldots, N$, $A^{n+1}_j = (a_{ik}(x_j, t_{n+1}))$, $i = 1, 2$, $k = 1, 2$, $\vec{f}^{n+1}_j = \vec{f}(x_j, t_{n+1})$ and $\vec{U}^n_j$ denotes the approximation of the value $\vec{u}(x_j, t_n)$. So, $\vec{U}^0_j = \vec{0}$, $0 \leqslant j \leqslant N$ and $\vec{U}^{n+1}_0 = \vec{g}_0(t_{n+1})$, $\vec{U}^{n+1}_N = \vec{g}_1(t_{n+1})$.

Lemma 11 (*Discrete maximum principle*). *Let* $(I+\Delta t L^N_{x,\vec{\varepsilon}})$ *be the discrete operator given in* (38) *and we assume that the coefficients of matrix A satisfy the positivity conditions* (2a) *and* (2b). *If* $\vec{Y}$ *is a vectorial mesh function such that* $\vec{Y}_0 \geqslant \vec{0}$, $\vec{Y}_N \geqslant \vec{0}$ *and* $(I+\Delta t L^N_{x,\vec{\varepsilon}})\vec{Y}_j \geqslant \vec{0}$, *for* $j=1,\ldots,N-1$, *then* $\vec{Y}_j \geqslant \vec{0}$ *for* $j=0,\ldots,N$. *Moreover, it is uniformly stable and it holds*

$$\|\vec{Y}\|_{\bar{\Omega}^N} \leqslant \|(I+\Delta t L^N_{x,\vec{\varepsilon}})\vec{Y}\|_{\bar{\Omega}^N}.$$

Proof. We refer to [11, Lemmas 6 and 7]. □

To analyse the uniform convergence of scheme (38), we consider the following decomposition of the discrete solution

$$\vec{U}^{n+1} = \vec{V}^{n+1} + \vec{W}^{n+1}, \quad n=0,\ldots,M-1,$$

where $\vec{V}^{n+1}$ and $\vec{W}^{n+1}$ are solutions of the discrete problems

$$\begin{aligned}&(I+\Delta t L^N_{x,\vec{\varepsilon}})\vec{V}^{n+1}_j = \vec{V}^n_j + \Delta t \vec{f}^{n+1}_j, \ \ 0<j<N, \quad \vec{V}^{n+1}_0 = \vec{v}(0,t_{n+1}), \quad \vec{V}^{n+1}_N = \vec{v}(1,t_{n+1}),\\ &(I+\Delta t L^N_{x,\vec{\varepsilon}})\vec{W}^{n+1}_j = \vec{W}^n_j, \ \ 0<j<N, \quad \vec{W}^{n+1}_0 = \vec{w}(0,t_{n+1}), \ \ \vec{W}^{n+1}_N = \vec{w}(1,t_{n+1}),\end{aligned} \tag{39}$$

with $\vec{V}^0_j = \vec{W}^0_j = \vec{0}$, $0 \leqslant j \leqslant N$.

Theorem 12. *Let* $\vec{u}(x,t)$ *be the solution of* (1) *and* $\{\vec{U}^{n+1}_i\}$ *the solution of* (38). *If the coefficients of matrix A satisfy the positivity conditions* (2a) *and* (2b), *then*

$$\|\vec{u}(x_j,t_{n+1}) - \vec{U}^{n+1}_j\|_{\bar{\Omega}^N} \leqslant C(N^{-2+q}\ln^2 N + \Delta t), \quad 0<q<1, \tag{40}$$

where N, Δt *and* q *are such that* $N^{-q} \leqslant C\Delta t$.

Proof. If the mesh is uniform, it is straightforward to deduce

$$\begin{aligned}|(I+\Delta t L^N_{x,\vec{\varepsilon}})(\vec{u}(x_j,t_{n+1}) - \vec{U}^{n+1}_j)| &\leqslant \left|\vec{u}(x_j,t_{n+1}) - \vec{u}(x_j,t_n) - \Delta t \frac{\partial \vec{u}}{\partial t}(x_j,t_{n+1})\right|\\ &\quad + \Delta t |L^N_{x,\vec{\varepsilon}}\vec{u}(x_j,t_{n+1}) - L_{x,\vec{\varepsilon}}\vec{u}(x_j,t_{n+1})| + |\vec{u}(x_j,t_n) - \vec{U}^n_j|\\ &\leqslant C((\Delta t)^2 + \Delta t N^{-2}\varepsilon_1^{-1}) + |\vec{u}(x_j,t_n) - \vec{U}^n_j|\\ &\leqslant C\Delta t(\Delta t + (N^{-1}\ln N)^2) + |\vec{u}(x_j,t_n) - \vec{U}^n_j|.\end{aligned}$$

The discrete maximum principle proves

$$\|\vec{u}(x_j,t_{n+1}) - \vec{U}^{n+1}_j\|_{\bar{\Omega}^N} \leqslant C\Delta t(\Delta t + (N^{-1}\ln N)^2) + \|\vec{u}(x_j,t_n) - \vec{U}^n_j\|_{\bar{\Omega}^N}.$$

By using recursively this expression, we obtain

$$\|\vec{u}(x_j,t_{n+1}) - \vec{U}^{n+1}_j\|_{\bar{\Omega}^N} \leqslant C(\Delta t + (N^{-1}\ln N)^2).$$

In second place we assume that the spatial mesh is not uniform. Similarly to [20], for the analysis we must distinguish several cases.

If $\varepsilon_2 = 1$ or $\varepsilon_1 = \varepsilon_2$, we calculate the local error associated to the regular and singular components. Taking Taylor expansions and using bounds (7) and (14), we deduce that

$$\begin{aligned}&\left|\vec{v}(x_j,t_{n+1}) - \vec{v}(x_j,t_n) - \Delta t \frac{\partial \vec{v}}{\partial t}(x_j,t_{n+1})\right| \leqslant (\Delta t)^2 \vec{C},\\ &\left|\vec{w}(x_j,t_{n+1}) - \vec{w}(x_j,t_n) - \Delta t \frac{\partial \vec{w}}{\partial t}(x_j,t_{n+1})\right| \leqslant (\Delta t)^2 \vec{C}.\end{aligned} \tag{41}$$

From [8] (case $\varepsilon_1 = \varepsilon_2$) and [13] (case $\varepsilon_2 = 1$), we have that

$$|L^N_{x,\vec{\varepsilon}}\vec{v}(x_j, t_{n+1}) - L_{x,\vec{\varepsilon}}\vec{v}(x_j, t_{n+1})| \leqslant CN^{-1}\begin{pmatrix}\varepsilon_1^{1/2}\\ \varepsilon_2^{1/2}\end{pmatrix},$$
$$|L^N_{x,\vec{\varepsilon}}\vec{w}(x_j, t_{n+1}) - L_{x,\vec{\varepsilon}}\vec{w}(x_j, t_{n+1})| \leqslant (N^{-1}\ln N)^2\vec{C}. \tag{42}$$

Bounds (41) and (42), the discrete maximum principle and a recursive argument prove

$$\|\vec{v}(x_j, t_{n+1}) - \vec{V}_j^{n+1}\|_{\bar{\Omega}^N} \leqslant C(\Delta t + N^{-1}\varepsilon_2^{1/2}),$$
$$\|\vec{w}(x_j, t_{n+1}) - \vec{W}_j^{n+1}\|_{\bar{\Omega}^N} \leqslant C(\Delta t + (N^{-1}\ln N)^2). \tag{43}$$

Hence, it follows

$$\|\vec{u}(x_j, t_{n+1}) - \vec{U}_j^{n+1}\|_{\bar{\Omega}^N} \leqslant C(\Delta t + N^{-1}(\varepsilon_2^{1/2} + N^{-1}\ln^2 N)).$$

Note that the reduction of the order of convergence is due to the regular component. We can sharpen the bound of the error associated to this component by using the barrier functions given in [8] (case $\varepsilon_1 = \varepsilon_2$) and [13] (case $\varepsilon_2 = 1$). These barrier functions are defined by means of a piecewise linear function

$$\varphi_\gamma(x) = \begin{cases} x\tau_\gamma^{-1}, & x \in [0, \tau_\gamma],\\ 1, & x \in [\tau_\gamma, 1-\tau_\gamma],\\ (1-x)\tau_\gamma^{-1}, & x \in [1-\tau_\gamma, 1].\end{cases}$$

Taking

$$\vec{\psi}(x_i) = (N^{-2} + (\Delta t)^2 + N^{-2}\tau_{\varepsilon_1}\varepsilon_1^{-1/2}\varphi_{\varepsilon_1}(x_i))\vec{C} + \|\vec{v}(x_j, t_n) - \vec{V}_j^n\|_{\bar{\Omega}^N}\vec{1},$$

as barrier function if $\varepsilon_1 = \varepsilon_2$, and

$$\vec{\psi}(x_i) = (N^{-2} + (\Delta t)^2)\vec{C} + CN^{-2}\tau_{\varepsilon_1}\varphi_{\varepsilon_1}(x_i)(\varepsilon_1^{-1/2}, 1)^{\mathrm{T}} + \|\vec{v}(x_j, t_n) - \vec{V}_j^n\|_{\bar{\Omega}^N}\vec{1},$$

if $\varepsilon_2 = 1$, the discrete maximum principle proves

$$\|\vec{v}(x_j, t_{n+1}) - \vec{V}_j^{n+1}\|_{\bar{\Omega}^N} \leqslant C((\Delta t)^2 + N^{-2}\ln N) + \|\vec{v}(x_j, t_n) - \vec{V}_j^n\|_{\bar{\Omega}^N}.$$

Using the hypothesis $N^{-q} \leqslant C\Delta t$, since the barrier function does not give the necessary dependence on Δt, we deduce

$$\|\vec{v}(x_j, t_{n+1}) - \vec{V}_j^{n+1}\|_{\bar{\Omega}^N} \leqslant C\sum_{n=1}^{M}\Delta t(\Delta t + N^{-2+q}\ln N) \leqslant C(\Delta t + N^{-2+q}\ln N). \tag{44}$$

Then, the result follows combining (43) and (44).

In the third case (ε_1 and ε_2 are arbitrary) we use Lemmas 7, 9 and 10 to estimate $|L^N_{x,\vec{\varepsilon}}\vec{v}(x_j, t_{n+1}) - L_{x,\vec{\varepsilon}}\vec{v}(x_j, t_{n+1})|$ and $|L^N_{x,\vec{\varepsilon}}\vec{w}(x_j, t_{n+1}) - L_{x,\vec{\varepsilon}}\vec{w}(x_j, t_{n+1})|$, and from them we deduce the same crude bound that in [10, Theorem 1], for $|L^N_{x,\vec{\varepsilon}}\vec{u}(x_j, t_{n+1}) - L_{x,\vec{\varepsilon}}\vec{u}(x_j, t_{n+1})|$. So, we can compare the local error associated to the numerical solution

$$|(I + \Delta t L^N_{x,\vec{\varepsilon}})(\vec{u}(x_j, t_{n+1}) - \vec{U}_j^{n+1})| \leqslant (\Delta t)^2\vec{C} + \Delta t|L^N_{x,\vec{\varepsilon}}\vec{u}(x_j, t_{n+1}) - L_{x,\vec{\varepsilon}}\vec{u}(x_j, t_{n+1})| + |\vec{u}(x_j, t_n) - \vec{U}_j^n|,$$

with the barrier function (see [10])

$$\vec{\psi}(x_i) = ((\Delta t)^2 + N^{-2}\ln^2 N(1 + \varphi_{\varepsilon_1}(x_i) + \varphi_{\varepsilon_2}(x_i)))\vec{C} + \|\vec{u}(x_j, t_n) - \vec{U}_j^n\|_{\bar{\Omega}^N}\vec{1},$$

and the comparison principle proves

$$\|\vec{u}(x_j, t_{n+1}) - \vec{U}_j^{n+1}\|_{\bar{\Omega}^N} \leqslant C((\Delta t)^2 + (N^{-1}\ln N)^2) + \|\vec{u}(x_j, t_n) - \vec{U}_j^n\|_{\bar{\Omega}^N}.$$

The same argument given above finishes the proof. □

4. Numerical results

In this section we show some numerical results obtained with the numerical scheme (38) to approximate the solution of the following problem

$$\left.\begin{aligned}\frac{\partial u_1}{\partial t}-\varepsilon_1\frac{\partial^2 u_1}{\partial x^2}+\kappa(u_1-u_2)=1,\\ \frac{\partial u_2}{\partial t}-\varepsilon_2\frac{\partial^2 u_2}{\partial x^2}+\kappa(u_2-u_1)=1,\end{aligned}\right\}\quad (x,t)\in(0,1)\times(0,1],$$
$$u(0,t)=u(1,t)=0,\quad t\in[0,T],\qquad u(x,0)=0,\quad x\in[0,1]. \tag{45}$$

This coupled system is used to model the flow in fractured porous media. The first equation of the system models the flow in the fracture system and the second equation models the flow in the porous matrix structure (see [1]). In these equations u_i are the fluid pressures, ε_i the permeabilities and κ is the coefficient that control the exchange of fluid between the pores and the fractures. In the numerical experiments we take $\kappa=1$.

We use a variant of the double mesh principle to estimate the pointwise errors $|\vec{U}_i^n-\vec{u}(x_i,t_n)|$ in the mesh points $\{(x_i,t_n)\}$. We calculate a new approximation $\{\hat{\vec{U}}_i^n\}$ on the mesh $\{(\hat{x}_i,\hat{t}_n)\}$ that contains the mesh points of the original mesh and their midpoints, i.e.,

$$\hat{x}_{2i}=x_i,\quad i=0,\dots,N,\quad \hat{x}_{2i+1}=(x_i+x_{i+1})/2,\quad i=0,\dots,N-1,$$
$$\hat{t}_{2n}=t_n,\quad n=0,\dots,M,\quad \hat{t}_{2n+1}=(t_n+t_{n+1})/2,\quad n=0,\dots,M-1.$$

At the mesh points of the coarse mesh we calculate the maximum errors and the uniform errors by

$$\vec{d}_{\vec{\varepsilon},N,\Delta t}=\max_{0\leqslant n\leqslant M}\max_{0\leqslant i\leqslant N}|\vec{U}_i^n-\hat{\vec{U}}_{2i}^{2n}|,\qquad \vec{d}_{N,\Delta t}=\max_{S} d_{\vec{\varepsilon},N,\Delta t}, \tag{46}$$

where the singular perturbation parameters take values on the set $S=\{(\varepsilon_1,\varepsilon_2)|\varepsilon_2=2^0,2^{-2},\dots,2^{-30},\ \varepsilon_1=\varepsilon_2,\ 2^{-2}\varepsilon_2,\dots,2^{-58},2^{-60}\}$ in order to permit that the maximum errors stabilize. The extreme values of ε_1 taken are very small, but in the numerical calculations we are working with numbers $2\varepsilon_j/(h_i(h_i+h_{i+1}))$ and $2\varepsilon_j/(h_{i+1}(h_i+h_{i+1}))$. These quantities are not beyond standard machine precision inside the boundary layers and have any relevant influence outside them.

From the values giving by (46) we can obtain, in a standard way, the corresponding orders of convergence and the uniform orders of convergence

$$\vec{p}=\frac{\log(\vec{d}_{\vec{\varepsilon},N,\Delta t}/\vec{d}_{\vec{\varepsilon},2N,\Delta t/4})}{\log 2},\qquad \vec{p}_{\mathrm{uni}}=\frac{\log(\vec{d}_{N,\Delta t}/\vec{d}_{2N,\Delta t/4})}{\log 2}.$$

In Tables 1–4 the spatial discretization parameter takes the values N=64, 128, 256, 512, 1024 and the time discretization parameter $\Delta t=0.5, 0.5/4, 0.5/4^2, 0.5/4^3, 0.5/4^4$. We have divided the step sizes into a different ratio in order to corroborate at the same time first order and second order of convergence in time and space, respectively. The first time step is sufficiently large relative to the spatial step in order that the last values of Δt will not become very small.

Table 1
Uniform errors $\vec{d}_{N,\Delta t}$ and uniform orders of convergence $\vec{p}_{\mathrm{uni}}$ for problem (45) on a uniform mesh

$\varepsilon_1,\varepsilon_2\in S$	$N=64$ $\Delta t=0.5$	$N=128$ $\Delta t=0.5/4$	$N=256$ $\Delta t=0.5/4^2$	$N=512$ $\Delta t=0.5/4^3$	$N=1024$ $\Delta t=0.5/4^4$
$[\vec{d}_{N,\Delta t}]_1$	0.519E − 1	0.382E − 1	0.354E − 1	0.347E − 1	0.345E − 1
$[\vec{p}_{\mathrm{uni}}]_1$	0.443	0.111	0.030	0.008	
$[\vec{d}_{N,\Delta t}]_2$	0.480E − 1	0.382E − 1	0.354E − 1	0.347E − 1	0.345E − 1
$[\vec{p}_{\mathrm{uni}}]_2$	0.330	0.111	0.030	0.008	

Table 2
Uniform errors $\vec{d}_{N,\Delta t}$ and uniform orders of convergence $\vec{p}_{\text{uni}}$ for problem (45)

$\varepsilon_1, \varepsilon_2 \in S$	$N=64$ $\Delta t=0.5$	$N=128$ $\Delta t=0.5/4$	$N=256$ $\Delta t=0.5/4^2$	$N=512$ $\Delta t=0.5/4^3$	$N=1024$ $\Delta t=0.5/4^4$
$m=1$					
$[\vec{d}_{N,\Delta t}]_1$	0.466E − 1	0.150E − 1	0.405E − 2	0.103E − 2	0.260E − 3
$[\vec{p}_{\text{uni}}]_1$	1.631	1.892	1.972	1.993	
$[\vec{d}_{N,\Delta t}]_2$	0.399E − 1	0.122E − 1	0.348E − 2	0.903E − 3	0.228E − 3
$[\vec{p}_{\text{uni}}]_2$	1.713	1.805	1.947	1.987	
$m=2$					
$[\vec{d}_{N,\Delta t}]_1$	0.466E − 1	0.150E − 1	0.420E − 2	0.118E − 2	0.326E − 3
$[\vec{p}_{\text{uni}}]_1$	1.631	1.840	1.834	1.856	
$[\vec{d}_{N,\Delta t}]_2$	0.399E − 1	0.122E − 1	0.374E − 2	0.111E − 2	0.325E − 3
$[\vec{p}_{\text{uni}}]_2$	1.713	1.704	1.752	1.771	
$m=4$					
$[\vec{d}_{N,\Delta t}]_1$	0.519E − 1	0.238E − 1	0.823E − 2	0.266E − 2	0.809E − 3
$[\vec{p}_{\text{uni}}]_1$	1.126	1.531	1.629	1.719	
$[\vec{d}_{N,\Delta t}]_2$	0.601E − 1	0.337E − 1	0.128E − 1	0.434E − 2	0.139E − 2
$[\vec{p}_{\text{uni}}]_2$	0.833	1.396	1.561	1.640	

Table 3
Problem (45): ε_1-uniform errors, uniform errors and orders of convergence associated to the component u_1

ε_2	$N=64$ $\Delta t=0.5$	$N=128$ $\Delta t=0.5/4$	$N=256$ $\Delta t=0.5/4^2$	$N=512$ $\Delta t=0.5/4^3$	$N=1024$ $\Delta t=0.5/4^4$
$\varepsilon_2=2^0$	0.410E − 1	0.136E − 1	0.372E − 2	0.966E − 3	0.245E − 3
	1.598	1.864	1.947	1.981	
$\varepsilon_2=2^{-2}$	0.466E − 1	0.150E − 1	0.405E − 2	0.103E − 2	0.260E − 3
	1.631	1.892	1.972	1.993	
$\varepsilon_2=2^{-4}$	0.399E − 1	0.120E − 1	0.327E − 2	0.855E − 3	0.220E − 3
	1.727	1.879	1.937	1.955	
$\varepsilon_2=2^{-6}$	0.371E − 1	0.119E − 1	0.327E − 2	0.855E − 3	0.220E − 3
	1.642	1.860	1.937	1.955	
$\varepsilon_2=2^{-8}$	0.371E − 1	0.119E − 1	0.327E − 2	0.855E − 3	0.220E − 3
	1.642	1.860	1.937	1.955	
$\varepsilon_2=2^{-10}$	0.371E − 1	0.119E − 1	0.327E − 2	0.855E − 3	0.220E − 3
	1.642	1.860	1.937	1.955	
$\varepsilon_2=2^{-12}$	0.371E − 1	0.119E − 1	0.327E − 2	0.855E − 3	0.220E − 3
	1.642	1.860	1.937	1.955	
$\varepsilon_2=2^{-14}$	0.371E − 1	0.119E − 1	0.327E − 2	0.855E − 3	0.220E − 3
	1.642	1.860	1.937	1.955	
$\varepsilon_2=2^{-16}$	0.371E − 1	0.119E − 1	0.327E − 2	0.855E − 3	0.220E − 3
	1.642	1.860	1.937	1.955	
$\varepsilon_2=2^{-18}$	0.371E − 1	0.119E − 1	0.327E − 2	0.855E − 3	0.220E − 3
	1.642	1.860	1.937	1.955	
$\varepsilon_2=2^{-20}$	0.371E − 1	0.119E − 1	0.327E − 2	0.855E − 3	0.220E − 3
	1.642	1.860	1.937	1.955	
⋮	⋮	⋮	⋮	⋮	⋮
$\varepsilon_2=2^{-30}$	0.371E − 1	0.119E − 1	0.327E − 2	0.855E − 3	0.220E − 3
	1.642	1.860	1.937	1.955	
$[\vec{d}_{N,\Delta t}]_1$	0.466E − 1	0.150E − 1	0.405E − 2	0.103E − 2	0.260E − 3
$[\vec{p}_{\text{uni}}]_1$	1.631	1.892	1.972	1.993	

Table 4
Problem (45): ε_1-uniform errors, uniform errors and orders of convergence associated to the component u_2

ε_2	$N=64$ $\Delta t=0.5$	$N=128$ $\Delta t=0.5/4$	$N=256$ $\Delta t=0.5/4^2$	$N=512$ $\Delta t=0.5/4^3$	$N=1024$ $\Delta t=0.5/4^4$
$\varepsilon_2=2^0$	0.125E − 1	0.815E − 2	0.304E − 2	0.870E − 3	0.226E − 3
	0.623	1.422	1.805	1.947	
$\varepsilon_2=2^{-2}$	0.326E − 1	0.122E − 1	0.348E − 2	0.903E − 3	0.228E − 3
	1.422	1.805	1.947	1.987	
$\varepsilon_2=2^{-4}$	0.399E − 1	0.120E − 1	0.318E − 2	0.807E − 3	0.203E − 3
	1.727	1.920	1.979	1.995	
$\varepsilon_2=2^{-6}$	0.244E − 1	0.712E − 2	0.185E − 2	0.470E − 3	0.118E − 3
	1.777	1.941	1.981	1.991	
$\varepsilon_2=2^{-8}$	0.246E − 1	0.724E − 2	0.189E − 2	0.477E − 3	0.120E − 3
	1.767	1.939	1.985	1.996	
$\varepsilon_2=2^{-10}$	0.252E − 1	0.749E − 2	0.197E − 2	0.524E − 3	0.137E − 3
	1.751	1.928	1.910	1.930	
$\varepsilon_2=2^{-12}$	0.252E − 1	0.749E − 2	0.198E − 2	0.562E − 3	0.155E − 3
	1.749	1.922	1.813	1.855	
$\varepsilon_2=2^{-14}$	0.251E − 1	0.749E − 2	0.198E − 2	0.562E − 3	0.155E − 3
	1.747	1.922	1.813	1.855	
$\varepsilon_2=2^{-16}$	0.251E − 1	0.749E − 2	0.198E − 2	0.562E − 3	0.155E − 3
	1.745	1.922	1.813	1.855	
$\varepsilon_2=2^{-18}$	0.251E − 1	0.749E − 2	0.198E − 2	0.562E − 3	0.155E − 3
	1.744	1.922	1.814	1.855	
$\varepsilon_2=2^{-20}$	0.251E − 1	0.749E − 2	0.198E − 2	0.562E − 3	0.155E − 3
	1.744	1.922	1.814	1.855	
⋮	⋮	⋮	⋮	⋮	⋮
$\varepsilon_2=2^{-30}$	0.251E − 1	0.749E − 2	0.198E − 2	0.562E − 3	0.155E − 3
	1.743	1.922	1.814	1.855	
$[\vec{d}_{N,\Delta t}]_2$	0.399E − 1	0.122E − 1	0.348E − 2	0.903E − 3	0.228E − 3
$[\vec{p}_{\text{uni}}]_2$	1.712	1.805	1.947	1.987	

In Fig. 1 we show for $\varepsilon_1=2^{-30}$ and $\varepsilon_2=2^{-15}$ the numerical solutions corresponding to $T=1$. The time and spatial discretization parameters are $\Delta t=0.5$ and $N=128$, respectively. Clearly, from this figure we observe two boundary layers at $x=0$. Note that the values of the solution outside of the boundary layers are close to the solution of the reduced problem

$$\left.\begin{aligned}\frac{\partial v_1}{\partial t}+(v_1-v_2)=1,\\ \frac{\partial v_2}{\partial t}+(v_2-v_1)=1,\end{aligned}\right\}\quad (x,t)\in(0,1)\times(0,1],$$

$$v(x,0)=0,\quad x\in[0,1], \tag{47}$$

given by $v_1(x,t)=v_2(x,t)=t$. In the boundary layers the solutions are different. The first component is greater and varies faster than the second one.

It is clear that we cannot use a uniform mesh to approximate the solution of this problem (see Table 1). In Table 2, we show the uniform errors and the orders of convergence for several values of the constant involved in the definition of the Shishkin mesh (37). In both tables the values of the singular perturbation parameters belong to the set S and the results for the components u_1 and u_2 appear in the first and second rows, respectively. Table 2 has a double purpose, first from it we observe second order of uniform convergence proved theoretically for $m=1,2,4$; second we deduce that the error constant depends on m similarly to steady equations in the scalar case (see [5]).

Tables 3 and 4 display the numerical results for the components u_1 and u_2, respectively, taking $m=1$. At each row we show the maximum errors and the orders of convergence for a fix value of ε_2 and $\varepsilon_1\in\{\varepsilon_2,2^{-2}\varepsilon_2,\ldots,2^{-60}\}$. The uniform errors and the uniform orders of convergence appear at the last row.

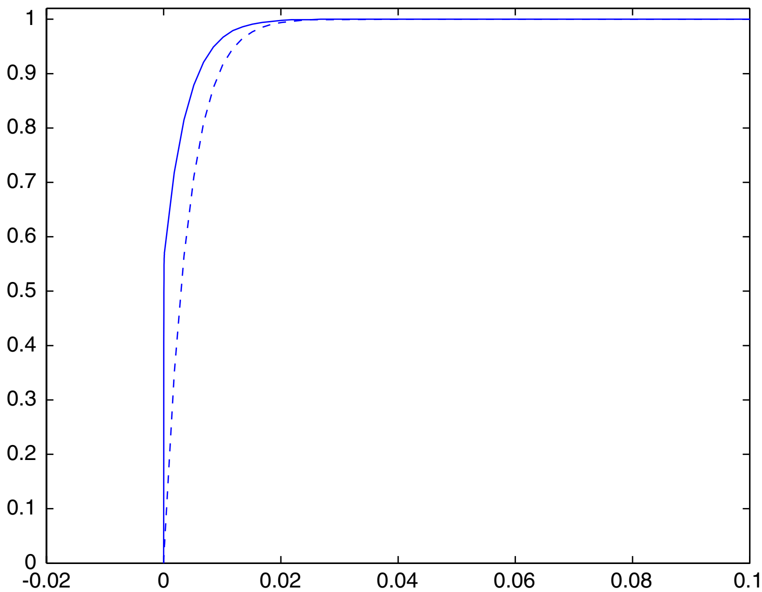

Fig. 1. Numerical solutions U_1 (solid line) and U_2 (dashed line) for $T = 1$ generated by the numerical method (38) applied to problem (45) with $N = 128$ and $\varepsilon_1 = 2^{-30}$, $\varepsilon_2 = 2^{-15}$.

Table 5
Uniform errors $\vec{d}_{N,\Delta t}$ and uniform orders of convergence $\vec{p}_{\text{uni}}$ for problem (45)

$\varepsilon_1, \varepsilon_2 \in S$	$N = 64$ $\Delta t = 0.01$	$N = 128$ $\Delta t = 0.01/4$	$N = 256$ $\Delta t = 0.01/4^2$	$N = 512$ $\Delta t = 0.01/4^3$
$[\vec{d}_{N,\Delta t}]_1$	0.422E − 2	0.147E − 2	0.478E − 3	0.151E − 3
$[\vec{p}_{\text{uni}}]_1$	1.521	1.620	1.665	
$[\vec{d}_{N,\Delta t}]_2$	0.609E − 2	0.221E − 2	0.768E − 3	0.254E − 3
$[\vec{p}_{\text{uni}}]_2$	1.466	1.522	1.597	

From all these tables we see that the numerical results are in agreement with Theorem 12, even though we do not have sufficiently compatibility conditions (in the corners $(0, 0)$ and $(1, 0)$ only the condition of order zero holds). Finally, we wish to note that the condition $N^{-q} \leqslant \Delta t,\ 0 < q < 1$, that we have imposed in Theorem 12, it is not necessary in the different numerical experiments that we have performed. We think that it is only a theoretical restriction in the proof by recurrence given in Theorem 12. To support it, in Table 5 we present the numerical result for problem (45) taking $m = 1$ and different starting values for the discretization parameters such that $\Delta t \approx N^{-1}$, observing any anomalous behaviour.

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