Available online at www.sciencedirect.com
ScienceDirect
Journal of Pure and Applied Algebra 209 (2007) 57–69
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

A number-theoretic approach to homotopy exponents of SU(n)

Donald M. Davis[a,*], Zhi-Wei Sun[b]

[a] *Department of Mathematics, Lehigh University, Bethlehem, PA 18015, USA*
[b] *Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China*

Received 1 November 2005; received in revised form 10 April 2006
Available online 12 June 2006
Communicated by C.A. Weibel

Abstract

We use methods of combinatorial number theory to prove that, for each $n \geq 2$ and any prime p, some homotopy group $\pi_i(\mathrm{SU}(n))$ contains an element of order $p^{n-1+\mathrm{ord}_p(\lfloor n/p \rfloor !)}$, where $\mathrm{ord}_p(m)$ denotes the largest integer α such that $p^\alpha \mid m$.

MSC: 55Q52; 57T20; 11A07; 11B65; 11S05

1. Introduction

Let p be a prime number. The *homotopy p-exponent* of a topological space X, denoted by $\exp_p(X)$, is defined to be the largest $e \in \mathbb{N} = \{0, 1, 2, \ldots\}$ such that some homotopy group $\pi_i(X)$ has an element of order p^e. This concept has been studied by various topologists (cf. [12,10,15,3–5,14,18,19]). The most celebrated result about homotopy exponents (proved by Cohen, Moore, and Neisendorfer in [3]) states that $\exp_p(S^{2n+1}) = n$ if $p \neq 2$.

The *special unitary group* SU(n) (of degree n) is the space of all $n \times n$ unitary matrices (the conjugate transpose of such a complex matrix equals its inverse) with determinant one. (See, e.g., [11, p. 68].) It plays a central role in many areas of mathematics and physics. The famous Bott Periodicity Theorem [2] describes $\pi_i(\mathrm{SU}(n))$ with $i < 2n$. In this paper, we provide a strong and elegant lower bound for the homotopy p-exponent of SU(n).

As in number theory, the integral part of a real number c is denoted by $\lfloor c \rfloor$. For a prime p and an integer m, the *p-adic order* of m is given by $\mathrm{ord}_p(m) = \sup\{n \in \mathbb{N} : p^n \mid m\}$ (whence $\mathrm{ord}_p(0) = +\infty$).

Here is our main result.

Theorem 1.1. *For any prime p and $n = 2, 3, \ldots$, some homotopy group $\pi_i(\mathrm{SU}(n))$ contains an element of order $p^{n-1+\mathrm{ord}_p(\lfloor n/p \rfloor !)}$; i.e., we have the inequality*

$$\exp_p(\mathrm{SU}(n)) \geq n - 1 + \mathrm{ord}_p\left(\left\lfloor \frac{n}{p} \right\rfloor !\right).$$

We discuss in Section 2 the extent to which Theorem 1.1 might be sharp.

* Corresponding author.
E-mail addresses: dmd1@lehigh.edu (D.M. Davis), zwsun@nju.edu.cn (Z.-W. Sun).

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2006.05.006

Our reduction from homotopy theory to number theory involves Stirling numbers of the second kind. For $n, k \in \mathbb{N}$ with $n + k \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, the Stirling number $S(n, k)$ of the second kind is the number of partitions of a set of cardinality n into k nonempty subsets; in addition, we define $S(0, 0) = 1$. We will use the following definition.

Definition 1.2. Let p be a prime. For $k, n \in \mathbb{Z}^+$ with $k \geq n$, we define

$$e_p(n, k) = \min_{m \geq n} \mathrm{ord}_p(m!S(k, m)).$$

In Sections 2 and 4 we prove the following standard result.

Proposition 1.3. *Let p be a prime, and let $n \in \mathbb{Z}^+$. Then, for all $k \geq n$, we have $\exp_p(\mathrm{SU}(n)) \geq e_p(n, k)$ unless $p = 2$ and $n \equiv 0 \pmod 2$, in which case $\exp_2(\mathrm{SU}(n)) \geq e_2(n, k) - 1$.*

Our innovation is to extend previous work [16] of the second author in combinatorial number theory to prove the following result, which, together with Proposition 1.3, immediately implies Theorem 1.1 when p or n is odd. In Section 4, we explain the extra ingredient required to deduce Theorem 1.1 from 1.3 and 1.4 when $p = 2$ and n is even.

Theorem 1.4. *Let p be any prime and n be a positive integer.*
(i) *For any $\alpha, h, l, m \in \mathbb{N}$, we have*

$$\mathrm{ord}_p\left(m! \sum_{k=0}^{l} \binom{l}{k} (-1)^k S\left(kh(p-1)p^\alpha + n - 1,\ m\right)\right) \geq \min\left\{l(\alpha+1),\ n - 1 + \mathrm{ord}_p\left(\left\lfloor \frac{m}{p} \right\rfloor !\right)\right\}.$$

(ii) *If we define $N = n - 1 + \lfloor n/(p(p-1)) \rfloor$, then*

$$e_p\left(n, (p-1)p^L + n - 1\right) \geq n - 1 + \mathrm{ord}_p\left(\left\lfloor \frac{n}{p} \right\rfloor !\right) \quad \text{for } L = N, N+1, \ldots.$$

In Section 3, we prove the following broad generalization of Theorem 1.4, and in Section 2, we show that it implies Theorem 1.4.

Theorem 1.5. *Let p be a prime, $\alpha, n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then for any polynomial $f(x) \in \mathbb{Z}[x]$ we have*

$$\mathrm{ord}_p\left(\sum_{k \equiv r \ (\mathrm{mod}\ p^\alpha)} \binom{n}{k} (-1)^k f\left(\frac{k - r}{p^\alpha}\right)\right) \geq \mathrm{ord}_p\left(\left\lfloor \frac{n}{p^\alpha} \right\rfloor !\right).$$

Here we adopt the standard convention that $\binom{n}{k}$ is 0 if k is a negative integer.

In Theorem 5.1, we give a strengthened version of Theorem 1.5, which we conjecture to be optimal in a certain sense. Our application to topology uses the case $r = 0$ of Theorem 1.5; the more technical Theorem 5.1 yields no improvement in this case.

In [5], the first author used totally different, and much more complicated, methods to prove that

$$\exp_p(\mathrm{SU}(n)) \geq n - 1 + \left\lfloor \frac{n + 2p - 3}{p^2} \right\rfloor + \left\lfloor \frac{n + p^2 - p - 1}{p^3} \right\rfloor, \tag{1.6}$$

where p is an odd prime and n is an integer greater than one. Since

$$\mathrm{ord}_p(m!) = \sum_{i=1}^{\infty} \left\lfloor \frac{m}{p^i} \right\rfloor \quad \text{for every } m = 0, 1, 2, \ldots$$

(a well-known fact in number theory), the inequality in Theorem 1.1 can be restated as

$$\exp_p(\mathrm{SU}(n)) \geq n - 1 + \sum_{i=2}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor,$$

a nice improvement of (1.6).

2. Outline of proof

In this section we present the deduction of Theorem 1.1 from Theorem 1.5, which will then be proved in Section 3. We also present some comments regarding the extent to which Theorem 1.1 is sharp.

Let p be any prime. In [8], the first author and Mahowald defined the (p-primary) v_1-periodic homotopy groups $v_1^{-1}\pi_*(X; p)$ of a topological space X and proved that if X is a sphere or compact Lie group, such as $\mathrm{SU}(n)$, each group $v_1^{-1}\pi_i(X; p)$ is a direct summand of some actual homotopy group $\pi_j(X)$. See also [7] for another expository account of v_1-periodic homotopy theory.

In [6, 1.4] and [1, 1.1a], it was proved that if p is odd, or if $p = 2$ and n is odd, then there is an isomorphism

$$v_1^{-1}\pi_{2k}(\mathrm{SU}(n); p) \cong \mathbb{Z}/p^{e_p(n,k)}\mathbb{Z} \tag{2.1}$$

for all $k \ge n$, where $e_p(n, k)$ is as defined in 1.2 and we use $\mathbb{Z}/m\mathbb{Z}$ to denote the additive group of residue classes modulo m. Thus, unless $p = 2$ and n is even, for any integer $k \ge n$, we have

$$\exp_p(\mathrm{SU}(n)) \ge e_p(n, k),$$

establishing Proposition 1.3 in these cases. The situation when $p = 2$ and n is even is somewhat more technical, and will be discussed in Section 4.

Next we show that Theorem 1.5 implies Theorem 1.4.

Proof of Theorem 1.4. (i) By a well-known property of Stirling numbers of the second kind (cf. [13, pp. 125–126]),

$$m!S(kh(p-1)p^\alpha + n - 1, m) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} j^{kh(p-1)p^\alpha + n-1}$$

for any $k \in \mathbb{N}$. Thus

$$(-1)^m m! \sum_{k=0}^{l} \binom{l}{k} (-1)^k S(kh(p-1)p^\alpha + n - 1, m) = \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 = \sum_{k=0}^{l} \binom{l}{k} (-1)^k p^{n-1+kh(p-1)p^\alpha} \sum_{j \equiv 0 \, (\mathrm{mod}\ p)} \binom{m}{j} (-1)^j \left(\frac{j}{p}\right)^{n-1+kh(p-1)p^\alpha}$$

and

$$\begin{aligned}\Sigma_2 &= \sum_{j \not\equiv 0 \, (\mathrm{mod}\ p)} \binom{m}{j} (-1)^j \sum_{k=0}^{l} \binom{l}{k} (-1)^k j^{n-1+kh(p-1)p^\alpha} \\ &= \sum_{j \not\equiv 0 \, (\mathrm{mod}\ p)} \binom{m}{j} (-1)^j j^{n-1} \left(1 - j^{h(p-1)p^\alpha}\right)^l.\end{aligned}$$

Clearly $\mathrm{ord}_p(\Sigma_1) \ge n - 1 + \mathrm{ord}_p(\lfloor m/p \rfloor!)$ by Theorem 1.5, and $\mathrm{ord}_p(\Sigma_2) \ge l(\alpha + 1)$ by Euler's theorem in number theory. Therefore the first part of Theorem 1.4 holds.

(ii) Observe that

$$N + 1 - (n-1) > \frac{n}{p(p-1)} = \sum_{i=2}^{\infty} \frac{n}{p^i} > \sum_{i=2}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor = \mathrm{ord}_p\left(\left\lfloor \frac{n}{p} \right\rfloor !\right).$$

By part (i) in the case $l = h = 1$ and $\alpha = L \ge N$, if $m \ge n$ then

$$\mathrm{ord}_p\left(m!S(n-1, m) - m!S((p-1)p^L + n - 1,\ m)\right) \ge n - 1 + \mathrm{ord}_p\left(\left\lfloor \frac{n}{p} \right\rfloor !\right).$$

Since $S(n-1,m)=0$ for $m \ge n$, we finally have

$$e_p(n,(p-1)p^L+n-1) \ge n-1+\mathrm{ord}_p\left(\left\lfloor \frac{n}{p} \right\rfloor !\right)$$

as required. □

The following proposition, although not needed for our main results, sheds more light on the large exponents N and L which appear in Theorem 1.4(ii), and is useful in our subsequent exposition.

Proposition 2.2. *Let p be a prime and let $n>1$ be an integer. Then there exists an integer $N_0 \ge 0$, effectively computable in terms of p and n, such that $e_p(n,(p-1)p^L+n-1)$ has the same value for all $L \ge N_0$.*

Proof. For integers $m \ge n$ and $L \ge 0$, we write

$$(-1)^m m! S((p-1)p^L+n-1,m)=\sum_{j=0}^{m}\binom{m}{j}(-1)^j j^{(p-1)p^L+n-1}=S_m+S'_{m,L}+S''_{m,L},$$

where

$$S_m=\sum_{j\not\equiv 0 \,(\mathrm{mod}\, p)}\binom{m}{j}(-1)^j j^{n-1},$$

$$S'_{m,L}=\sum_{j\not\equiv 0 \,(\mathrm{mod}\, p)}\binom{m}{j}(-1)^j j^{n-1}(j^{(p-1)p^L}-1),$$

$$S''_{m,L}=\sum_{j\equiv 0 \,(\mathrm{mod}\, p)}\binom{m}{j}(-1)^j j^{(p-1)p^L+n-1}.$$

Note that both $S'_{m,L}$ and $S''_{m,L}$ are divisible by p^{L+1}.

Assume that $S_n, S_{n+1}, \ldots$ are not all zero. (This will be shown later.) Then $L_0=\min_{m\ge n}\mathrm{ord}_p(S_m)$ is finite. Let $m_0 \ge n$ satisfy $\mathrm{ord}_p(S_{m_0})=L_0$. Whenever $L \ge L_0$, we have $\mathrm{ord}_p(S_m+S'_{m,L}+S''_{m,L}) \ge L_0$ for every $m \ge n$, and equality is attained for $m=m_0$. Thus, if $L \ge L_0$ then

$$\begin{aligned} e_p(n,(p-1)p^L+n-1) &= \min_{m\ge n}\mathrm{ord}_p(m!S((p-1)p^L+n-1,m)) \\ &= \min_{m\ge n}\mathrm{ord}_p(S_m+S'_{m,L}+S''_{m,L})=L_0. \end{aligned}$$

Although L_0 is finite, it may not be effectively computable. Instead of L_0 we use the p-adic order N_0 of the first nonzero term in the sequence $S_n, S_{n+1}, \ldots$. This N_0 is computable, also $e_p(n,(p-1)p^L+n-1)=L_0$ for all $L \ge N_0$ since $N_0 \ge L_0$.

To complete the proof, we must show that S_m is nonzero for some $m \ge n$. First note that this is clearly true for $p=2$ since then S_m is a sum of negative terms. If p is odd and $S_m=0$ for all $m \ge n$, then $e_p(n,(p-1)p^L+n-1)=\min_{m\ge n}\mathrm{ord}_p(S'_{m,L}+S''_{m,L}) \ge L+1$ for any $L \ge 0$. By (2.1), this would imply that $v_1^{-1}\pi_*(\mathrm{SU}(n);p)$ has elements of arbitrarily large p-exponent. However, this is not true, for in [6, 5.8], it was shown that the v_1-periodic p-exponent of $\mathrm{SU}(n)$ does not exceed $e := \lfloor (n-1)(1+(p-1)^{-1}+(p-1)^{-2})\rfloor$; i.e., for this e, $p^e v_1^{-1}\pi_*(\mathrm{SU}(n);p)=0$. □

In the remainder of this section and in Section 4, once a prime p and an integer $n>1$ are given, L will refer to any integer not smaller than $\max\{N,N_0\}$, where N and N_0 are described in Theorem 1.4(ii) and the proof of Proposition 2.2 respectively.

We now comment on the extent to which Theorem 1.1 might be sharp. In Table 1, we present, for $p=3$ and a representative set of values of n, three numbers. The first, labelled $\exp_3(v_1^{-1}\mathrm{SU}(n))$, is the largest value of $e_3(n,k)$ over all values of $k \ge n$; thus it is the largest exponent of the 3-primary v_1-periodic homotopy groups of $\mathrm{SU}(n)$. The second number in the table is the exponent of the v_1-periodic homotopy group on which we have been focusing, which, at least in the range of this table, is equal to or just slightly less than the maximal exponent. The third number is the nice estimate for this exponent given by Theorem 1.4(ii).

Table 1
Comparison of exponents when $p = 3$

n	$\exp_3(v_1^{-1}\mathrm{SU}(n))$	$e_3(n, 2\cdot 3^L + n - 1)$	$n - 1 + \mathrm{ord}_3(\lfloor n/3\rfloor!)$
19	21	20	20
20	22	21	21
21	22	22	22
22	25	25	23
23	26	26	24
24	28	28	25
25	29	28	26
26	30	30	27
27	31	31	30
28	32	32	31
29	34	32	32
30	34	33	33
31	34	34	34
32	35	35	35
33	37	37	36
34	38	37	37
35	39	39	38
36	41	41	40
37	42	41	41
38	43	42	42
39	43	43	43
40	45	44	44
41	45	45	45

Note that, for more than half of the values of n in the table, the largest group $v_1^{-1}\pi_{2k}(\mathrm{SU}(n);3)$ occurs when $k = 2\cdot 3^L + n - 1$. In the worst case in the table, $n = 29$, detailed `Maple` calculations suggest that if $k \geq 29$ and $k \equiv 10 \pmod{18}$, then

$$e_3(29, k) = \min\{\mathrm{ord}_3(k - 28 - 8\cdot 3^{20}) + 12,\ 34\}.$$

Shifts (as by $8\cdot 3^{20}$) were already noted in [6, p. 543]. Note also that for more than half of the cases in the table, our estimate for $e_3(n, 2\cdot 3^L + n - 1)$ is sharp, and it never misses by more than 3.

The big question for topologists, though, is whether the v_1-periodic p-exponent agrees (or almost agrees) with the actual homotopy p-exponent. The fact that they agree for S^{2n+1} when p is an odd prime [3,10] leads the first author to conjecture that they also agree for $\mathrm{SU}(n)$ if $p \neq 2$, but we have no idea how to prove this. Theriault [18,19] has made good progress in proving that some of the first author's lower bounds for p-exponents of certain exceptional Lie groups are sharp.

3. Proof of Theorem 1.5

In this section, we prove Theorem 1.5, which we have already shown to imply Theorem 1.1.

Lemma 3.1. *Let p be any prime, and let $\alpha, n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then*

$$\mathrm{ord}_p\left(\sum_{k\equiv r \pmod{p^\alpha}} \binom{n}{k}(-1)^k\right) \geq \mathrm{ord}_p\left(\left\lfloor \frac{n}{p^{\alpha-1}}\right\rfloor!\right) = \left\lfloor \frac{n}{p^\alpha}\right\rfloor + \mathrm{ord}_p\left(\left\lfloor \frac{n}{p^\alpha}\right\rfloor!\right).$$

Proof. The equality is easy, for,

$$\begin{aligned}\mathrm{ord}_p\left(\left\lfloor \frac{n}{p^{\alpha-1}}\right\rfloor!\right) &= \sum_{i=1}^{\infty}\left\lfloor \frac{\lfloor n/p^{\alpha-1}\rfloor}{p^i}\right\rfloor = \sum_{j=\alpha}^{\infty}\left\lfloor \frac{n}{p^j}\right\rfloor \\ &= \left\lfloor \frac{n}{p^\alpha}\right\rfloor + \sum_{i=1}^{\infty}\left\lfloor \frac{\lfloor n/p^\alpha\rfloor}{p^i}\right\rfloor = \left\lfloor \frac{n}{p^\alpha}\right\rfloor + \mathrm{ord}_p\left(\left\lfloor \frac{n}{p^\alpha}\right\rfloor!\right).\end{aligned}$$

When $\alpha = 0$ or $n < p^{\alpha-1}$, the desired inequality is obvious.

Now let $\alpha > 0$ and $m = \lfloor n/p^{\alpha-1} \rfloor \geq 1$. Observe that

$$\operatorname{ord}_p(m!) = \sum_{i=1}^{\infty} \left\lfloor \frac{m}{p^i} \right\rfloor < \sum_{i=1}^{\infty} \frac{m}{p^i} = \frac{m}{p} \sum_{j=0}^{\infty} \frac{1}{p^j} = \frac{m}{p} \cdot \frac{1}{1-p^{-1}} = \frac{m}{p-1}.$$

Thus $(p-1)\operatorname{ord}_p(m!) \leq m-1$, and hence

$$\operatorname{ord}_p(m!) \leq \left\lfloor \frac{m-1}{p-1} \right\rfloor = \left\lfloor \frac{n/p^{\alpha-1}-1}{p-1} \right\rfloor = \left\lfloor \frac{n-p^{\alpha-1}}{\varphi(p^\alpha)} \right\rfloor,$$

where φ is Euler's totient function. By a result of Weisman [21],

$$\operatorname{ord}_p\left(\sum_{k\equiv r \ (\operatorname{mod} p^\alpha)} \binom{n}{k}(-1)^k\right) \geq \left\lfloor \frac{n-p^{\alpha-1}}{\varphi(p^\alpha)} \right\rfloor.$$

(Weisman's proof is complicated, but an easy induction proof appeared in [16].) So we have the desired inequality. □

Now we restate Lemma 2.1 of Sun [16], which will be used later.

Lemma 3.2 *([16]). Let m and n be positive integers, and let $f(x)$ be a function from $\mathbb{Z}$ to a field. Then, for any $r \in \mathbb{Z}$, we have*

$$\sum_{k=0}^{n} \binom{n}{k}(-1)^k f\left(\left\lfloor \frac{k-r}{m} \right\rfloor\right) = \sum_{k\equiv \bar{r} (\operatorname{mod} m)} \binom{n-1}{k}(-1)^{k-1} \Delta f\left(\frac{k-\bar{r}}{m}\right),$$

where $\bar{r} = r-1+m$ and $\Delta f(x) = f(x+1) - f(x)$.

Lemma 3.3. *Let $m, n \in \mathbb{Z}^+$ and $r \in \mathbb{Z}$, and let $f(x)$ be a complex-valued function defined on $\mathbb{Z}$. Then we have*

$$\begin{aligned}&\sum_{k\equiv r \ (\operatorname{mod} m)} \binom{n}{k}(-1)^k f\left(\frac{k-r}{m}\right) - f\left(\left\lfloor \frac{n-r}{m} \right\rfloor\right) \sum_{k\equiv r \ (\operatorname{mod} m)} \binom{n}{k}(-1)^k \\ &\quad = -\sum_{j=0}^{n-1} \binom{n}{j} \sum_{i\equiv r \ (\operatorname{mod} m)} \binom{j}{i}(-1)^i \sum_{k\equiv r_j \ (\operatorname{mod} m)} \binom{n-j-1}{k}(-1)^k \Delta f\left(\frac{k-r_j}{m}\right),\end{aligned}$$

where $r_j = r-j+m-1$.

Proof. Let ζ be a primitive mth root of unity. Clearly

$$\sum_{s=0}^{m-1} \zeta^{(k-r)s} = \begin{cases} \displaystyle\sum_{s=0}^{m-1} 1 = m & \text{if } k \equiv r \ (\operatorname{mod} m), \\ \displaystyle\frac{1-\zeta^{(k-r)m}}{1-\zeta^{k-r}} = 0 & \text{otherwise.} \end{cases}$$

Thus

$$\sum_{k\equiv r \ (\operatorname{mod} m)} \binom{n}{k}(-1)^k f\left(\frac{k-r}{m}\right) = \sum_{k=0}^{n}\left(\frac{1}{m}\sum_{s=0}^{m-1}\zeta^{(k-r)s}\right)\binom{n}{k}(-1)^k f\left(\left\lfloor \frac{k-r}{m} \right\rfloor\right) = \frac{1}{m}\sum_{s=0}^{m-1}\zeta^{-rs} c_s,$$

where

$$c_s = \sum_{k=0}^{n} \binom{n}{k}(-\zeta^s)^k f\left(\left\lfloor \frac{k-r}{m} \right\rfloor\right).$$

Observe that

$$\begin{aligned} c_s &= \sum_{k=0}^{n}\binom{n}{k}((1-\zeta^s)-1)^k f\left(\left\lfloor\frac{k-r}{m}\right\rfloor\right) \\ &= \sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(1-\zeta^s)^j(-1)^{k-j} f\left(\left\lfloor\frac{k-r}{m}\right\rfloor\right) \\ &= \sum_{j=0}^{n}\binom{n}{j}(1-\zeta^s)^j\sum_{k=j}^{n}\binom{n-j}{k-j}(-1)^{k-j} f\left(\left\lfloor\frac{k-r}{m}\right\rfloor\right) \\ &= \sum_{j=0}^{n}\binom{n}{j}(1-\zeta^s)^j\sum_{k=0}^{n-j}\binom{n-j}{k}(-1)^{k} f\left(\left\lfloor\frac{k-(r-j)}{m}\right\rfloor\right). \end{aligned}$$

Applying Lemma 3.2, we find that

$$c_s-(1-\zeta^s)^n f\left(\left\lfloor\frac{n-r}{m}\right\rfloor\right)=\sum_{j=0}^{n-1}\binom{n}{j}(1-\zeta^s)^j\sum_{k\equiv r_j \ (\mathrm{mod}\, m)}\binom{n-j-1}{k}(-1)^{k-1}\Delta f\left(\frac{k-r_j}{m}\right).$$

In view of the above, it suffices to note that

$$\sum_{s=0}^{m-1}\frac{\zeta^{-rs}}{m}(1-\zeta^s)^j=\sum_{i=0}^{j}\binom{j}{i}\frac{(-1)^i}{m}\sum_{s=0}^{m-1}\zeta^{s(i-r)}=\sum_{i\equiv r \ (\mathrm{mod}\, m)}\binom{j}{i}(-1)^i.$$

This concludes the proof. □

With help of Lemmas 3.1 and 3.3, we are able to prove the following equivalent version of Theorem 1.5.

Theorem 3.4. *Let p be a prime, and let $\alpha, l, n \in \mathbb{N}$. Then for any $r \in \mathbb{Z}$ we have*

$$\mathrm{ord}_p\left(\sum_{k\equiv r \ (\mathrm{mod}\ p^\alpha)}\binom{n}{k}(-1)^k\left(\frac{k-r}{p^\alpha}\right)^l\right)\ge \mathrm{ord}_p\left(\left\lfloor\frac{n}{p^\alpha}\right\rfloor!\right).$$

Proof. We use induction on l.

In the case $l=0$, the desired result follows from Lemma 3.1.

Now let $l>0$ and assume the result for smaller values of l. We use induction on n to prove the inequality in Theorem 3.4.

The case $n=0$ is trivial. So we now let $n>0$ and assume that the inequality holds with smaller values of n. Observe that

$$\begin{aligned} &\sum_{k\equiv r \ (\mathrm{mod}\ p^\alpha)}\binom{n}{k}(-1)^k\left(\frac{k-r}{p^\alpha}\right)^l \\ &= \sum_{k\equiv r \ (\mathrm{mod}\ p^\alpha)}\left(\binom{n-1}{k}+\binom{n-1}{k-1}\right)(-1)^k\left(\frac{k-r}{p^\alpha}\right)^l \\ &= \sum_{k\equiv r \ (\mathrm{mod}\ p^\alpha)}\binom{n-1}{k}(-1)^k\left(\frac{k-r}{p^\alpha}\right)^l-\sum_{k'\equiv r-1 \ (\mathrm{mod}\ p^\alpha)}\binom{n-1}{k'}(-1)^{k'}\left(\frac{k'-(r-1)}{p^\alpha}\right)^l. \end{aligned}$$

In view of this, if p^α does not divide n, then, by the induction hypothesis for $n-1$, we have

$$\mathrm{ord}_p\left(\sum_{k\equiv r \ (\mathrm{mod}\ p^\alpha)}\binom{n}{k}(-1)^k\left(\frac{k-r}{p^\alpha}\right)^l\right)\ge \mathrm{ord}_p\left(\left\lfloor\frac{n-1}{p^\alpha}\right\rfloor\right)=\mathrm{ord}_p\left(\left\lfloor\frac{n}{p^\alpha}\right\rfloor\right).$$

Below we let $p^\alpha \mid n$ and set $m=n/p^\alpha$.

Case 1. $r \equiv 0 \pmod{p^\alpha}$. In this case,

$$\begin{aligned}
&\frac{1}{m!}\sum_{k\equiv r \pmod{p^\alpha}}\binom{n}{k}(-1)^k\left(\frac{k}{p^\alpha}-\frac{r}{p^\alpha}\right)\left(\frac{k-r}{p^\alpha}\right)^{l-1}\\
&=\frac{n/p^\alpha}{m!}\sum_{k\equiv r \pmod{p^\alpha}}\binom{n-1}{k-1}(-1)^k\left(\frac{k-r}{p^\alpha}\right)^{l-1}-\frac{r/p^\alpha}{m!}\sum_{k\equiv r \pmod{p^\alpha}}\binom{n}{k}(-1)^k\left(\frac{k-r}{p^\alpha}\right)^{l-1}\\
&=\frac{1}{\lfloor (n-1)/p^\alpha\rfloor!}\sum_{k\equiv r-1 \pmod{p^\alpha}}\binom{n-1}{k}(-1)^{k+1}\left(\frac{k-(r-1)}{p^\alpha}\right)^{l-1}\\
&\quad-\frac{r/p^\alpha}{\lfloor n/p^\alpha\rfloor!}\sum_{k\equiv r \pmod{p^\alpha}}\binom{n}{k}(-1)^k\left(\frac{k-r}{p^\alpha}\right)^{l-1}.
\end{aligned}$$

Thus, by the induction hypothesis for $l-1$,

$$\frac{1}{m!}\sum_{k\equiv r \pmod{p^\alpha}}\binom{n}{k}(-1)^k\left(\frac{k-r}{p^\alpha}\right)^{l}$$

is a p-integer (i.e., its denominator is relatively prime to p) and hence the desired inequality follows.

Case 2. $r \not\equiv 0 \pmod{p^\alpha}$. Note that $\sum_{i\equiv r \pmod{p^\alpha}}\binom{0}{i}(-1)^i = 0$. Also,

$$\mathrm{ord}_p\left(\sum_{k\equiv r \pmod{p^\alpha}}\binom{n}{k}(-1)^k\right)\ge \mathrm{ord}_p\left(\frac{n}{p^{\alpha-1}}!\right)=m+\mathrm{ord}_p(m!)$$

by Lemma 3.1. Thus, in view of Lemma 3.3, it suffices to show that if $0<j<n$ then the p-adic order of

$$\sigma_j=\binom{n}{j}\sum_{i\equiv r \pmod{p^\alpha}}\binom{j}{i}(-1)^i\sum_{k\equiv r_j \pmod{p^\alpha}}\binom{n-j-1}{k}(-1)^k\Delta f\left(\frac{k-r_j}{p^\alpha}\right)$$

is at least $\mathrm{ord}_p(m!)$, where $r_j=r-j+p^\alpha-1$ and $f(x)=x^l$.

Let $0<j\le n-1$ and write $j=p^\alpha s+t$, where $s,t\in\mathbb{N}$ and $t<p^\alpha$. Note that

$$\left\lfloor\frac{j}{p^\alpha}\right\rfloor=s\quad\text{and}\quad\left\lfloor\frac{n-j-1}{p^\alpha}\right\rfloor=\left\lfloor m-s-\frac{t+1}{p^\alpha}\right\rfloor=m-s-1.$$

Since $\Delta f(x)=(x+1)^l-x^l=\sum_{i=0}^{l-1}\binom{l}{i}x^i$, by Lemma 3.1 and the induction hypothesis with respect to l, we have

$$\begin{aligned}
\mathrm{ord}_p(\sigma_j)&=\mathrm{ord}_p\binom{n}{j}+\mathrm{ord}_p\left(\sum_{i\equiv r \pmod{p^\alpha}}\binom{j}{i}(-1)^i\right)\\
&\quad+\mathrm{ord}_p\left(\sum_{k\equiv r_j \pmod{p^\alpha}}\binom{n-j-1}{k}(-1)^k\Delta f\left(\frac{k-r_j}{p^\alpha}\right)\right)\\
&\ge\mathrm{ord}_p\binom{n}{j}+(s+\mathrm{ord}_p(s!))+\mathrm{ord}_p((m-s-1)!)\\
&=\mathrm{ord}_p\binom{n}{j}+s+\mathrm{ord}_p(s!)-\mathrm{ord}_p\left(\prod_{i=0}^{s}(m-i)\right)+\mathrm{ord}_p(m!)\\
&=\mathrm{ord}_p\binom{p^\alpha m}{p^\alpha s+t}-\mathrm{ord}_p\binom{m}{s}+s-\mathrm{ord}_p(m-s)+\mathrm{ord}_p(m!).
\end{aligned}$$

When $t=0$ (i.e., $j=p^\alpha s$) we have the stronger inequality

$$\mathrm{ord}_p(\sigma_j)\ge\mathrm{ord}_p\binom{p^\alpha m}{p^\alpha s}-\mathrm{ord}_p\binom{m}{s}+s+\mathrm{ord}_p(m!),$$

because

$$\begin{aligned}
&\mathrm{ord}_p\left(\sum_{k\equiv r_j \pmod{p^\alpha}}\binom{n-j-1}{k}(-1)^k\Delta f\left(\frac{k-r_j}{p^\alpha}\right)\right)\\
&\quad=\mathrm{ord}_p\left(\sum_{k=0}^{n-j}\binom{n-j}{k}(-1)^k f\left(\left\lfloor\frac{k-(r-j)}{p^\alpha}\right\rfloor\right)\right)\quad(\text{by Lemma 3.2})\\
&\quad=\mathrm{ord}_p\left(\sum_{i=0}^{p^\alpha-1}\sum_{k-(r-j)\equiv i \pmod{p^\alpha}}\binom{n-j}{k}(-1)^k\left(\frac{k-(r-j)-i}{p^\alpha}\right)^l\right)\\
&\quad\geq\mathrm{ord}_p\left(\frac{n-j}{p^\alpha}!\right)=\mathrm{ord}_p((m-s)!)\\
&\qquad(\text{by the induction hypothesis with respect to } n).
\end{aligned}$$

Observe that

$$\begin{aligned}
\mathrm{ord}_p\binom{p^\alpha m}{p^\alpha s}&=\sum_{i=1}^{\alpha}\left(\frac{p^\alpha m}{p^i}-\frac{p^\alpha s}{p^i}-\frac{p^\alpha(m-s)}{p^i}\right)+\sum_{i=\alpha+1}^{\infty}\left(\left\lfloor\frac{p^\alpha m}{p^i}\right\rfloor-\left\lfloor\frac{p^\alpha s}{p^i}\right\rfloor-\left\lfloor\frac{p^\alpha(m-s)}{p^i}\right\rfloor\right)\\
&=\sum_{i=1}^{\infty}\left(\left\lfloor\frac{m}{p^i}\right\rfloor-\left\lfloor\frac{s}{p^i}\right\rfloor-\left\lfloor\frac{m-s}{p^i}\right\rfloor\right)=\mathrm{ord}_p\binom{m}{s}.
\end{aligned}$$

Thus, when $t=0$ we have $\mathrm{ord}_p(\sigma_j)\geq s+\mathrm{ord}_p(m!)\geq\mathrm{ord}_p(m!)$.

Define $\mathrm{ord}_p(a/b)=\mathrm{ord}_p(a)-\mathrm{ord}_p(b)$ if $a,b\in\mathbb{Z}$ and a is not divisible by b. If $t>0$ then

$$\begin{aligned}
\mathrm{ord}_p\binom{p^\alpha m}{p^\alpha s+t}-\mathrm{ord}_p\binom{m}{s}&=\mathrm{ord}_p\frac{\binom{p^\alpha m}{p^\alpha s+t}}{\binom{p^\alpha m}{p^\alpha s}}\\
&=\mathrm{ord}_p\frac{(p^\alpha s)!(p^\alpha(m-s))!}{(p^\alpha s+t)!(p^\alpha(m-s)-t)!}\\
&=\mathrm{ord}_p\frac{p^\alpha(m-s)}{p^\alpha s+t}+\mathrm{ord}_p\prod_{0<i<t}\frac{p^\alpha(m-s)-i}{p^\alpha s+i}.
\end{aligned}$$

For $0<i<p^\alpha$, clearly

$$\mathrm{ord}_p(p^\alpha(m-s)-i)=\mathrm{ord}_p(p^\alpha s+i)=\mathrm{ord}_p(i)<\alpha.$$

Therefore, when $0<t<p^\alpha$ we have

$$\mathrm{ord}_p\binom{p^\alpha m}{p^\alpha s+t}-\mathrm{ord}_p\binom{m}{s}=\mathrm{ord}_p(m-s)+\alpha-\mathrm{ord}_p(p^\alpha s+t)>\mathrm{ord}_p(m-s)$$

and hence $\mathrm{ord}_p(\sigma_j)>s+\mathrm{ord}_p(m!)\geq\mathrm{ord}_p(m!)$. This concludes the analysis of the second case.

The proof of Theorem 3.4 is now complete. □

Note that, in the proof of Theorem 3.4, the technique used to handle the first case is of no use in the second case, and vice versa. Thus, the distinction of the two cases is important.

Corollary 3.5. *Let p be a prime, and let $\alpha,l,n\in\mathbb{N}$ and $r\in\mathbb{Z}$. Then we have*

$$\mathrm{ord}_p\left(\sum_{k\equiv r \pmod{p^\alpha}}\binom{n}{k}(-1)^k\binom{(k-r)/p^\alpha}{l}\right)\geq\mathrm{ord}_p\left(\left\lfloor\frac{n}{p^\alpha}\right\rfloor!\right)-\mathrm{ord}_p(l!).$$

Proof. Simply apply Theorem 1.5 with $f(x)=l!\binom{x}{l}\in\mathbb{Z}[x]$. □

4. Changes when $p = 2$ and n is even

When $p = 2$ and n is even, the relationship between $v_1^{-1}\pi_{2k}(\mathrm{SU}(n);\,p)$ and $e_p(n,k)$ (with $k \geq n$) is not so simple as in (2.1). As described in [1] and [9], there is a spectral sequence converging to $v_1^{-1}\pi_*(\mathrm{SU}(n);\,p)$ and satisfying $E_2^{1,\,2k+1}(\mathrm{SU}(n)) \cong \mathbb{Z}/p^{e_p(n,k)}\mathbb{Z}$. If p or n is odd, the spectral sequence necessarily collapses and $v_1^{-1}\pi_{2k}(\mathrm{SU}(n);\,p) \cong E_2^{1,\,2k+1}$. (Here we begin abbreviating $E_r^{*,\,*}(\mathrm{SU}(n))$ just as $E_r^{*,\,*}$.) If $p = 2$ and n is even, there are two ways in which the corresponding summand of $v_1^{-1}\pi_{2k}(\mathrm{SU}(n);\,2)$ may differ from this.

It is conceivable that there could be an extension in the spectral sequence, which would make the exponent of the homotopy group 1 larger than that of $E_\infty^{1,\,2k+1}$. However, as observed in [9, 6.2(1)], it is easily seen that this does not happen.

It is also conceivable that the differential $d_3 : E_3^{1,\,2k+1} \to E_3^{4,\,2k+3} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ could be nonzero, which would make the exponent of $v_1^{-1}\pi_{2k}(\mathrm{SU}(n);\,2)$ equal to $e_2(n,k) - 1$. This is the reason for the -1 at the end of Proposition 1.3. By [9, 1.6], if $n \equiv 0 \pmod 4$ and $k = 2^L + n - 1$, then $d_3 : E_3^{1,\,2k+1} \to E_3^{4,\,2k+3}$ must be 0.

Now suppose $n \equiv 2 \pmod 4$. If $n = 2$, then $n - 1 + \mathrm{ord}_2(\lfloor n/2\rfloor!) = 1 < \exp_2(\mathrm{SU}(n))$ since $\pi_6(\mathrm{SU}(2)) \cong \mathbb{Z}/12\mathbb{Z}$ (cf. [20]). Below we let $n > 2$, hence $n/2 + 1$ is even and not larger than $n - 1$. As first noted in [1, 1.1] and restated in [9, 6.5], for $k = 2^L + n - 1$, the differential $d_3 : E_3^{1,\,2k+1} \to E_3^{4,\,2k+3}$ is nonzero if and only if

$$e_2(n, 2^L + n - 1) = e_2(n - 1, 2^L + n - 1) + n - 1.$$

We show at the end of the section that

$$e_2(n - 1, 2^L + n - 1) = \mathrm{ord}_2((n - 1)!). \tag{4.1}$$

Thus, if the above d_3 is nonzero, then $e_2(n, 2^L + n - 1) = n - 1 + \mathrm{ord}_2((n - 1)!)$ and hence

$$\exp_2(\mathrm{SU}(n)) \geq e_2(n, 2^L + n - 1) - 1 = n - 1 + \mathrm{ord}_2((n - 1)!) - 1 \geq n - 1 + \mathrm{ord}_2(\lfloor n/2\rfloor!),$$

as claimed in Theorem 1.1.

Proof of (4.1). Putting $p = 2$, $\alpha = L$, $l = h = 1$ and $m = n - 1$ in the first part of Theorem 1.4, we get that

$$\begin{aligned}&\mathrm{ord}_2\Big((n-1)!S(n-1, n-1) - (n-1)!S(2^L + n - 1, n - 1)\Big)\\ &\quad \geq n - 1 + \mathrm{ord}_2\left(\left\lfloor \frac{n-1}{2}\right\rfloor!\right) \geq n - 1 > \mathrm{ord}_2((n-1)!).\end{aligned}$$

Therefore $\mathrm{ord}_2((n-1)!S(2^L + n - 1, n - 1)) = \mathrm{ord}_2((n-1)!)$. On the other hand, by the second part of Theorem 1.4, $\mathrm{ord}_2(m!S(2^L + n - 1, m)) \geq n - 1 + \mathrm{ord}_2(\lfloor n/2\rfloor!)$ for all $m \geq n$. So we have (4.1). □

5. Strengthening and sharpness of Theorem 3.4

In this section, we give an example illustrating the extent to which Theorem 3.4 is sharp when $r = 0$, which is the situation that is used in our application to topology. Then we show in Theorem 5.1 that the lower bound in Theorem 3.4 can sometimes be increased slightly.

We begin with a typical example of Theorem 3.4. Let $p = \alpha = 2$, $r = 0$ and $n = 100$. Then $\lfloor n/p^\alpha\rfloor = 25$ and $\mathrm{ord}_p(\lfloor n/p^\alpha\rfloor!) = 22$. For $l \geq 25$, set

$$\delta(l) = \mathrm{ord}_2\left(\sum_{k\equiv 0 \,(\mathrm{mod}\, 4)} \binom{n}{k}\left(\frac{k}{4}\right)^l\right) - 22.$$

The range $l \geq \lfloor n/p^\alpha\rfloor = 25$ is that in which we feel Theorem 3.4 to be very strong. (See Remark 5.3(2).) Clearly $\delta(l)$ measures the amount by which the actual p-adic order of the sum in Theorem 3.4 exceeds our bound for it. The values of $\delta(l)$ for $25 \leq l \leq 45$ are given in order as

0, 0, 0, 0, 2, 3, 2, 4, 1, 1, 1, 1, 2, 2, 4, 1, 0, 0, 0, 0, 3.

When $r = 0$ and in many other situations, Theorem 3.4 appears to be sharp for infinitely many values of l.

Before presenting our strengthening of Theorem 3.4 we need some notation. For $a \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$, we let $\{a\}_m$ denote the least nonnegative residue of a modulo m. Given a prime p, for any $a, b \in \mathbb{N}$ we let $\tau_p(a, b)$ represent the number of carries occurring in the addition of a and b in base p; actually

$$\tau_p(a,b) = \sum_{i=1}^{\infty} \left(\left\lfloor \frac{a+b}{p^i} \right\rfloor - \left\lfloor \frac{a}{p^i} \right\rfloor - \left\lfloor \frac{b}{p^i} \right\rfloor \right) = \mathrm{ord}_p \binom{a+b}{a}$$

as observed by E. Kummer.

Here is our strengthening of Theorem 3.4. The right-hand side is the amount by which the bound in Theorem 3.4 can be improved. This amount does not exceed α, by the definition of τ_p. In Table 2, we illustrate this amount when $p = 3$ and $\alpha = 2$.

Theorem 5.1. *Let p be a prime, and let $\alpha, l, n \in \mathbb{N}$. Then, for all $r \in \mathbb{Z}$, we have*

$$\begin{aligned} &\mathrm{ord}_p \left(\sum_{k \equiv r \ (\mathrm{mod}\ p^\alpha)} \binom{n}{k} (-1)^k \left(\frac{k-r}{p^\alpha} \right)^l \right) - \mathrm{ord}_p \left(\left\lfloor \frac{n}{p^\alpha} \right\rfloor ! \right) \\ &\quad \geq \tau_p(\{r\}_{p^\alpha}, \{n-r\}_{p^\alpha}) = \mathrm{ord}_p \binom{\{r\}_{p^\alpha} + \{n-r\}_{p^\alpha}}{\{r\}_{p^\alpha}}. \end{aligned}$$

Proof. We use induction on n.

In the case $n = 0$, whether $r \equiv 0 \ (\mathrm{mod}\ p^\alpha)$ or not, the desired result holds trivially.

Now let $n > 0$ and assume the corresponding result for $n - 1$. Suppose that $\tau_p(\{r\}_{p^\alpha}, \{n-r\}_{p^\alpha}) > 0$. Then neither r nor $n - r$ is divisible by p^α.

Set

$$R = \frac{1}{\lfloor n/p^\alpha \rfloor !} \sum_{k \equiv r \ (\mathrm{mod}\ p^\alpha)} \binom{n}{k} (-1)^k \left(\frac{k-r}{p^\alpha} \right)^l$$

and

$$R' = \frac{n/p^\alpha}{\lfloor n/p^\alpha \rfloor !} \sum_{k \equiv r-1 \ (\mathrm{mod}\ p^\alpha)} \binom{n-1}{k} (-1)^k \left(\frac{k-(r-1)}{p^\alpha} \right)^l .$$

Clearly

$$\begin{aligned} R' &= -\frac{n/p^\alpha}{\lfloor n/p^\alpha \rfloor !} \sum_{k \equiv r \ (\mathrm{mod}\ p^\alpha)} \binom{n-1}{k-1} (-1)^k \left(\frac{k-r}{p^\alpha} \right)^l \\ &= -\frac{1}{\lfloor n/p^\alpha \rfloor !} \sum_{k \equiv r \ (\mathrm{mod}\ p^\alpha)} \binom{n}{k} (-1)^k \frac{k}{p^\alpha} \left(\frac{k-r}{p^\alpha} \right)^l , \end{aligned}$$

and thus

$$\frac{r}{p^\alpha} R + R' = -\frac{1}{\lfloor n/p^\alpha \rfloor !} \sum_{k \equiv r \ (\mathrm{mod}\ p^\alpha)} \binom{n}{k} (-1)^k \left(\frac{k-r}{p^\alpha} \right)^{l+1} .$$

This is a p-integer by Theorem 3.4; therefore $\mathrm{ord}_p(rR + p^\alpha R') \geq \alpha$.

Let $\beta = \mathrm{ord}_p(n)$. We consider three cases.

Case 1. $\beta \geq \alpha$. In this case, $\lfloor n/p^\alpha \rfloor !/(n/p^\alpha) = \lfloor (n-1)/p^\alpha \rfloor !$ and hence R' is a p-integer by Theorem 3.4. In view of the inequality $\mathrm{ord}_p(rR + p^\alpha R') \geq \alpha$, we have

$$\mathrm{ord}_p(R) \geq \alpha - \mathrm{ord}_p(r) = \tau_p(\{r\}_{p^\alpha}, \{n-r\}_{p^\alpha}),$$

where the last equality follows from the definition of τ_p and the condition $n \equiv 0 \not\equiv r \ (\mathrm{mod}\ p^\alpha)$.

Case 2. $\mathrm{ord}_p(r) \leq \beta < \alpha$. Since $\lfloor n/p^\alpha \rfloor = \lfloor (n-1)/p^\alpha \rfloor$, the definition of R' implies that

$$\frac{p^\alpha R'}{n} = \frac{1}{\lfloor (n-1)/p^\alpha \rfloor !} \sum_{k \equiv r-1 \ (\mathrm{mod}\ p^\alpha)} \binom{n-1}{k} (-1)^k \left(\frac{k-(r-1)}{p^\alpha} \right)^l .$$

Applying the induction hypothesis, we find that

$$\mathrm{ord}_p(p^\alpha R') - \beta \geq \tau_p(\{r-1\}_{p^\alpha}, \{n-1-(r-1)\}_{p^\alpha}) = \tau_p(\{r-1\}_{p^\alpha}, \{n-r\}_{p^\alpha}).$$

Since $\{r\}_{p^\alpha} + \{n-r\}_{p^\alpha} \equiv n \not\equiv 0 \pmod{p^\alpha}$ and

$$\begin{aligned}\binom{\{r\}_{p^\alpha} + \{n-r\}_{p^\alpha}}{\{r\}_{p^\alpha}} &= \frac{\{r\}_{p^\alpha} + \{n-r\}_{p^\alpha}}{\{r\}_{p^\alpha}} \binom{\{r\}_{p^\alpha} + \{n-r\}_{p^\alpha} - 1}{\{r\}_{p^\alpha} - 1} \\ &= \frac{\{r\}_{p^\alpha} + \{n-r\}_{p^\alpha}}{\{r\}_{p^\alpha}} \binom{\{r-1\}_{p^\alpha} + \{n-r\}_{p^\alpha}}{\{r-1\}_{p^\alpha}},\end{aligned}$$

we have

$$\tau_p(\{r\}_{p^\alpha}, \{n-r\}_{p^\alpha}) = \tau_p(\{r-1\}_{p^\alpha}, \{n-r\}_{p^\alpha}) + \beta - \mathrm{ord}_p(r).$$

Thus

$$\mathrm{ord}_p(p^\alpha R') \geq \mathrm{ord}_p(r) + \tau_p(\{r\}_{p^\alpha}, \{n-r\}_{p^\alpha}).$$

Clearly $\tau_p(\{r\}_{p^\alpha}, \{n-r\}_{p^\alpha}) \leq \alpha - \mathrm{ord}_p(r)$ by the definition of τ_p, so we also have

$$\mathrm{ord}_p(rR + p^\alpha R') \geq \mathrm{ord}_p(r) + \tau_p(\{r\}_{p^\alpha}, \{n-r\}_{p^\alpha}).$$

Therefore

$$\mathrm{ord}_p(R) = \mathrm{ord}_p(rR) - \mathrm{ord}_p(r) \geq \tau_p(\{r\}_{p^\alpha}, \{n-r\}_{p^\alpha}).$$

Case 3. $\beta < \min\{\alpha, \mathrm{ord}_p(r)\}$. In this case, $\mathrm{ord}_p(\bar{r}) = \beta < \alpha$, where $\bar{r} = n - r$. Also,

$$\begin{aligned}\sum_{k \equiv \bar{r} \pmod{p^\alpha}} \binom{n}{k} (-1)^k \left(\frac{k-\bar{r}}{p^\alpha}\right)^l &= \sum_{n-k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k \left(\frac{r-(n-k)}{p^\alpha}\right)^l \\ &= (-1)^{l+n} \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k \left(\frac{k-r}{p^\alpha}\right)^l.\end{aligned}$$

Thus, as in the second case, we have

$$\begin{aligned}\mathrm{ord}_p(R) &= \mathrm{ord}_p\left(\frac{1}{\lfloor n/p^\alpha \rfloor!} \sum_{k \equiv \bar{r} \pmod{p^\alpha}} \binom{n}{k} (-1)^k \left(\frac{k-\bar{r}}{p^\alpha}\right)^l\right) \\ &\geq \tau_p(\{\bar{r}\}_{p^\alpha}, \{n-\bar{r}\}_{p^\alpha}) = \tau_p(\{r\}_{p^\alpha}, \{n-r\}_{p^\alpha}).\end{aligned}$$

The induction proof of Theorem 5.1 is now complete. □

The following conjecture is based on extensive `Maple` calculations.

Conjecture 5.2. *Let p be any prime. And let $\alpha, l \in \mathbb{N}$, $n, r \in \mathbb{Z}$, with $n \geq 2p^\alpha - 1$. Then equality in Theorem 5.1 is attained if $l \geq \lfloor n/p^\alpha \rfloor$ and*

$$l \equiv \left\lfloor \frac{r}{p^\alpha} \right\rfloor + \left\lfloor \frac{n-r}{p^\alpha} \right\rfloor \left(\mathrm{mod}\ (p-1)p^{\lfloor \log_p(n/p^\alpha) \rfloor}\right).$$

Remark 5.3. (1) The conjecture, if proved, would show that Theorem 5.1 would be optimal in the sense that it is sharp for infinitely many values of l.

(2) Note that the conjecture only deals with equality when $l \geq \lfloor n/p^\alpha \rfloor$. For smaller values of l, our inequality is still true, but not so strong. In [17], we obtain a stronger inequality when $l < \lfloor n/p^\alpha \rfloor$.

We close with a table showing the amount by which the bound in Theorem 5.1 improves on that of Theorem 3.4. That is, we tabulate $\tau_p(\{r\}_{p^\alpha}, \{n-r\}_{p^\alpha})$ when $p = 3$ and $\alpha = 2$.

Table 2
Values of $\tau_3(\{r\}_9, \{n-r\}_9)$

		$\{r\}_9$								
		0	1	2	3	4	5	6	7	8
$\{n\}_9$	0	0	2	2	1	2	2	1	2	2
	1	0	0	2	1	1	2	1	1	2
	2	0	0	0	1	1	1	1	1	1
	3	0	1	1	0	2	2	1	2	2
	4	0	0	1	0	0	2	1	1	2
	5	0	0	0	0	0	0	1	1	1
	6	0	1	1	0	1	1	0	2	2
	7	0	0	1	0	0	1	0	0	2
	8	0	0	0	0	0	0	0	0	0

Acknowledgement

The second author is supported by the National Science Fund for Distinguished Young Scholars (no. 10425103) in China.

References

[1] M. Bendersky, D.M. Davis, 2-primary v_1-periodic homotopy groups of SU(n), Amer. J. Math. 114 (1991) 529–544.
[2] R. Bott, The stable homotopy of the classical groups, Ann. of Math. 70 (1959) 313–337.
[3] F.R. Cohen, J.C. Moore, J.A. Neisendorfer, The double suspension and exponents of the homotopy groups of spheres, Ann. of Math. 110 (1979) 549–565.
[4] F.R. Cohen, J.C. Moore, J.A. Neisendorfer, Exponents in homotopy theory, Ann. of Math. Stud. 113 (1987) 3–34.
[5] D.M. Davis, Elements of large order in $\pi_*(\mathrm{SU}(n))$, Topology 37 (1998) 293–327.
[6] D.M. Davis, v_1-periodic homotopy groups of SU(n) at odd primes, Proc. London Math. Soc. 43 (1991) 529–541.
[7] D.M. Davis, Computing v_1-periodic homotopy groups of spheres and certain Lie groups, in: Handbook of Algebraic Topology, Elsevier, 1995, pp. 993–1049.
[8] D.M. Davis, M. Mahowald, Some remarks on v_1-periodic homotopy groups, London Math. Soc. Lecture Note 176 (1992) 55–72.
[9] D.M. Davis, K. Potocka, 2-primary v_1-periodic homotopy groups of SU(n) revisited, Forum Math. (in press). On-line version: http://www.lehigh.edu/~dmd1/sun2long.pdf.
[10] B. Gray, On the sphere of origin of infinite families in the homotopy groups of spheres, Topology 8 (1969) 219–232.
[11] D. Husemoller, Fibre Bundles, 2nd ed., Springer, 1975.
[12] I.M. James, On the suspension sequence, Ann. of Math. 65 (1957) 74–107.
[13] J.H. van Lint, R.M. Wilson, A Course in Combinatorics, 2nd ed., Cambridge Univ. Press, Cambridge, 2001.
[14] J.A. Neisendorfer, A survey of Anick-Gray-Theriault constructions and applications to exponent theory of spheres and Moore spaces, Contemp. Math. Amer. Math. Soc. 265 (2000) 159–174.
[15] P. Selick, Odd-primary torsion in $\pi_k(S^3)$, Topology 17 (1978) 407–412.
[16] Z.W. Sun, Polynomial extension of Fleck's congruence, Acta Arith. 122 (2006) 91–100.
[17] Z.W. Sun, D.M. Davis, Combinatorial congruences modulo prime powers, Trans. Amer. Math. Soc. (in press). On-line version: http://arxiv.org/abs/math.NT/0508087.
[18] S.D. Theriault, 2-primary exponent bounds for Lie groups of low rank, Canad. Math. Bull. 47 (2004) 119–132.
[19] S.D. Theriault, The 5-primary homotopy exponent of the exceptional Lie group E_8, J. Math. Kyoto Univ. 44 (2004) 569–593.
[20] H. Toda, A topological proof of theorems of Bott and Borel-Hirzebruch for homotopy groups of unitary groups, Mem. Coll. Sci. Univ. Kyoto 32 (1959) 103–119.
[21] C.S. Weisman, Some congruences for binomial coefficients, Michigan Math. J. 24 (1977) 141–151.