JOURNAL OF ALGEBRA 170, 916–928 (1994)

Regularity Index of Fat Points in the Projective Plane

GIULIANA FATABBI

Department of Mathematics, Universitá di Perugia, Via Vanvitelli 1, 06100 Perugia, Italy

Communicated by D. A. Buchsbaum

Received April 1, 1993

In this paper we determine a new upper bound for the regularity index of fat points of P^2, without requiring any geometric condition on the points. This bound is intermediate between Segre's bound, that holds for points in the general position, and the more general bound, that is attained when the points are collinear: in fact, both of these bounds can be recovered as particular cases. Furthermore, our bound cannot, in general, be sharpened: in fact, it is attained if there are either many collinear points or collinear points with high multiplicities.

INTRODUCTION

Let $\{P_1, \ldots, P_s\}$ be a set of s distinct points in the projective plane, P^2, over an algebraically closed field k of characteristic 0.

If we assign to each P_i a "multiplicity," that is, a positive integer m_i, we obtain a set of "fat points" $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ of P^2, the study of which consists in examining the plane curves passing through each P_i with multiplicity at least m_i.

In this paper, extracted from the author's thesis [Fa], we study the regularity index of a set of fat points, that is, the least integer τ such that the points with their multiplicities impose independent conditions to the curves of degree greater than or equal to τ; or equivalently, the minimum integer τ such that the Hilbert function of X reaches the degree of X, $\delta(X) = \sum_{i=1}^{s}(m_i(m_i + 1)/2)$.

This problem, though classical (see, for instance, [C], [Sg], [S]), has been recently dealt with by many authors (M. V. Catalisano in [C1–C3], E. Davis and A. V. Geramita in [DG], A. Gimigliano in [G1, G2], S. Greco in [G], and P. Maroscia in [Ma]). Nevertheless, τ can be precisely determined only in some particular cases (for collinear points, see [DG]; for fat points on an irreducible conic, see [C3]). In more general cases only an upper bound for τ can be given.

0021-8693/94 $6.00

An upper bound as general as possible is given by

$$\tau \leq \sum_{i=1}^{s} m_i - 1$$

(see, for instance, [DG]).

However, this upper bound characterizes collinear points, so it can be sharpened as soon as the points are not all on a line.

B. Segre, in [Sg], indicated an upper bound for τ in the case of fat points in the general position, that is, never three on a line:

$$\tau \leq \max\left\{m_1 + m_2 - 1, \left[\frac{\sum_{i=1}^{s} m_i}{2}\right]\right\} \qquad \text{with } m_1 \geq \cdots \geq m_s.$$

The bound is sharp if the points lie on an irreducible conic.

The bound for τ determined in this paper is intermediate between Segre's bound and the most general one, in the sense that no condition on the position of the points is required, and that it coincides with the more general bound if the points are collinear and with Segre's bound if they are in the general position.

In fact, in Theorem 3.3, we show that if $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ is a set of fat points of P^2 and h is the maximum among the sums of multiplicities taken over all subsets of collinear points, then

$$\tau \leq \max\left\{h - 1, \left[\frac{\sum_{i=1}^{s} m_i}{2}\right]\right\}.$$

Furthermore, if $\max\{h - 1, [\sum_{i=1}^{s} m_i/2]\} = h - 1$, then we can determine τ exactly (in fact in this case we have $\tau = h - 1$, see Corollary 3.6). This also shows that the bound determined in this paper is, in general, sharp.

The proof of the above theorem is based upon induction on the sum of the multiplicities: thus it is first necessary to show the result when all the multiplicities are equal to 1, and this is taken care of in Section 2.

In Section 3 we give two different proofs of our main theorem (Theorem 3.3) in its full generality: a more direct one and another one based on Corollary 3.2 of [C2], coupled with our Lemma 3.2. We show how to recover the two other bounds mentioned and, finally we precisely determine τ in the case in which there are "many" collinear points or collinear points with high multiplicities, or when τ is "sufficiently" large (Corollary 3.6). In the latter case X necessarily contains a subset of collinear points, the sum of whose multiplicities is exactly $\tau + 1$. When all the multiplicities are equal to 1 this obviously means that X must contain $\tau + 1$ collinear points.

Many thanks are due to A. Lorenzini and S. Greco for introducing the author to this type of problem and for the many constructive discussions about this subject.

1. Preliminaries

Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of fat points of P^2 and let $p_1, \ldots, p_s$ be the (homogeneous) prime ideals of $R = k[X_0, X_1, X_2]$ corresponding to the points P_i. We denote by $A = \oplus_{i \geq 0} A_i$ the homogeneous coordinate ring of X: $A = R/I$, where $I = p_1^{m_1} \cap \cdots \cap p_s^{m_s}$.

It is well known that A is a graded Cohen–Macaulay ring of (Krull) dimension 1, and its multiplicity is $\delta(A) = \sum_{i=1}^{s} \binom{m_i + 1}{2}$, which is by definition the degree of X, $\delta(X)$. Moreover, we may assume that X_0 is not a 0-divisor *modulo* I (i.e., its image in A is not a 0-divisor).

It is also known that the Hilbert function of X (or of A), which is defined by $H(X, t) = \dim_k A_t$, strictly increases until it reaches the degree of X, and keeps constant thereafter (see, for instance, [GM]). Thus the regularity index can be viewed as the least integer τ such that $H(X, \tau) = \delta(X)$.

Then it is clear that the difference of the Hilbert function of X, $\Delta H(X, t) = H(X, t) - H(X, t-1)$, becomes 0 from $\sigma = \tau + 1$ on.

Now, let X and X' be two sets of fat points such that $X' \subset X$. Then there is a relation between $\Delta H(X, -)$ and $\Delta H(X' -)$, as the following lemma shows:

Lemma 1.1. $\Delta H(X', -) \leq \Delta H(X, -)$.

For a proof, see for instance [R] or [D2].

Other properties of the difference function which are known, whose proofs can be recovered from [D1, D3] (see also [Fa, Proposition 1.5.1 and Proposition 1.5.2]), are the following:

(1) $\Delta H(X, t) \leq t + 1$ $\forall t \geq 0$; $\Delta H(X, t) = t + 1 \Leftrightarrow t \leq \alpha - 1$, where $\alpha = \min\{t \mid I_t \neq (0)\}$;

(2) If $t \geq \alpha$, then $\Delta H(X, t) \leq \Delta H(X, t-1)$;

(3) If $\beta \leq t \leq \sigma$, then $\Delta H(X, t) < \Delta H(X, t-1)$, where $\beta = \min\{t \mid \gcd(I_t) = 1\}$;

(4) $\sum_{i=0}^{t} \Delta H(X, i) \leq \delta(X)$ $\forall t \geq 0$; moreover $\sum_{i=0}^{t} \Delta H(X, i) = \delta(X) \Leftrightarrow t \geq \tau$.

A case of interest is when there exists an integer t ($\alpha \leq t < \beta$, by property (3)), such that $\Delta H(X, t) = \Delta H(X, t-1) = d$: in fact this implies the existence of a "fixed component" of degree d for I_t. In other words $I_t = DW$, where deg $D = d$ and $W \subseteq R_{t-d}$.

The existence of such a component gives some information about the difference function:

LEMMA 1.2. *Let I and J be two ideals of $R = k[X_0, X_1, X_2]$ and D a form of degree d. Suppose there exists $k > d$ such that, for every t, $d \leq t \leq k$, we have $I_t = DJ_{t-d}$. Then $\Delta H(R/I, t) = \Delta H(R/J, t-d) + d$, for every t, $d \leq t \leq k$.*

For a proof see for instance [D2, Proposition 2.7].

Finally, if we denote by Γ the curve corresponding to the fixed component D, then we can determine the degree of the (ideal-theoretic) intersection between X and Γ:

$$\delta(X \cap \Gamma) = \sum_{t=0}^{\infty} \min\{d, \Delta H(X, t)\}$$

(a complete proof of this can be found in [D2] or in [Fa]).

2. REGULARITY INDEX OF DISTINCT POINTS IN P^2

In this section we determine an upper bound for the regularity index of s distinct points of P^2. In this case $m_i = 1\ \forall i$ and $\delta(X) = s$.

To do this we need two lemmas which hold more generally for fat points.

LEMMA 2.1. *If $X \subset P^2$ is a set of fat points of degree δ, then there exists an integer $k > 0$ such that*

$$k \leq \begin{cases} \dfrac{\delta}{2} & \text{if } \delta \text{ is even} \\[2ex] \dfrac{\delta + 1}{2} & \text{if } \delta \text{ is odd} \end{cases}$$

for which we have $\Delta H(X, k) \leq 1$. In particular, $k \geq \alpha$.

Proof. If the points are collinear and $m_i = 1$ for every i, then the inequality holds for every k. Otherwise $\Delta H(X, 1) = 2$.

Let δ be even and let us suppose that $\Delta H(X,t) \geq 2$ for every t, $1 \leq t \leq \delta/2$, if we add up (recalling that $\Delta H(X,0) = 1$), we obtain

$$\sum_{t=0}^{\delta/2} \Delta H(X,t) = 1 + \sum_{t=1}^{\delta/2} \Delta H(X,t) \geq 1 + 2\frac{\delta}{2} = \delta + 1$$

and this contradicts property (4) above.

Now let δ be odd and let us suppose that

$$\sum_{t=0}^{(\delta+1)/2} \Delta H(X,t) = 1 + \sum_{t=1}^{(\delta+1)/2} \Delta H(X,t) \geq 1 + 2\frac{\delta+1}{2} = \delta + 2.$$

Also in this case we have a contradiction. ∎

LEMMA 2.2. *Let $X \subset P^2$ be a set of fat points of degree δ, such that $\delta(X \cap L) \leq h$ for every line L.*

Let us suppose that $h > \alpha$. If $\Delta H(X, h-1) \leq 1$, then we necessarily have $\sigma \leq h$.

Proof. Since $h - 1 \geq \alpha$, by property (2) of Section 1, we have

$$\Delta H(X,h) \leq \Delta H(X,h-1) \leq 1.$$

Thus to prove the result, it suffices to show that $\Delta H(X,h)$ can't equal 1. Let us suppose that

$$\Delta H(X,h) = \Delta H(X,h-1) = 1.$$

Then, for what we said at the end of Section 1, there would exist a line L such that $\delta(X \cap L) = h + 1$, contradicting the hypothesis. ∎

THEOREM 2.3. *Let $X = \{P_1, \ldots, P_s\} \subset P^2$ be a set of s distinct points and let h be the maximum number of collinear point. Then*

$$\sigma \leq \max\left\{h - 1, \left[\frac{s}{2}\right]\right\} + 1.$$

Proof. Let $\rho = \max\{h-1, [s/2]\}$ and let us first suppose that $\rho = [s/2] \geq h - 1$.

If s is even, then, by Lemma 2.1, there exists $0 < k \leq s/2$ such that $\Delta H(X,k) \leq 1$. As $k \geq \alpha$, we have $\Delta H(X, s/2 + 1) \leq \Delta H(X, s/2)$. By Lemma 2.2 it must be $\sigma \leq s/2 + 1$.

Now let s be odd: in this case we have $[s/2] = (s-1)/2$. By Lemma 2.1 there exists $k \leq (s+1)/2$ such that $\Delta H(X,k) \leq 1$, and so $\Delta H(X, (s+1)/2) \leq 1$.

If $k \leq (s-1)/2$ then, by Lemma 2.2, we have $\sigma \leq (s-1)/2 + 1$.

If $k = (s+1)/2$ we have $\Delta H(X,t) \geq 2$ for every t, $1 \leq t \leq (s-1)/2$. By adding up we obtain

$$\sum_{t=0}^{(s-1)/2} \Delta H(X,t) \geq 1 + 2\frac{s-1}{2} = s = \delta(X).$$

Also in this case we have $\sigma \leq (s-1)/2 + 1$.

Now let $\rho = h - 1 \geq [s/2] + 1$.

If s is even, then, by Lemma 2.1, there exists $k \leq s/2$ such that $\Delta H(X,k) \leq 1$. Since $h - 1 \geq [s/2] + 1 \geq \alpha$, then we have $\Delta H(X, h - 1) \leq 1$ and so by Lemma 2.2, $\sigma \leq h$.

Now let s be odd. By Lemma 2.1 there exists $\alpha \leq k \leq (s+1)/2$ such that $\Delta H(X,k) \leq 1$. Since $h - 1 \geq (s+1)/2$, then $\Delta H(X, h-1) \leq 1$ and so, by Lemma 2.2, we have $\sigma \leq h$. ∎

3. Regularity Index of Fat Points in P^2

If $X = \{(P,m)\}$ is a fat point of P^2, not only the regularity index is known, but even the Hilbert function:

$$\Delta H(X,t) = \begin{cases} t+1, & 0 \leq t \leq m-1 \\ 0, & t \geq m \end{cases}$$

(whence $\tau = m - 1$).

From now on when we talk about a set $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ of fat points we shall always assume $s \geq 2$, and $m_i \geq 0$ ($m_i = 0$ simply means P_i imposes no conditions).

Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of fat points of P^2. It is known (see for instance [DG, Corollary 2.3]) that if the points are collinear then $\sigma = \sum_{i=1}^{s} m_i$; if the points are not collinear look at the maximum among the sums of multiplicities with respect to the all possible subsets of X of collinear points and denote it by h:

$$h = \max\left\{ \sum_{j=1}^{k} m_{i_j} | P_{i_1}, \ldots, P_{i_k} \in X \text{ are collinear} \right\}.$$

Observe that $h \leq \sigma$. In fact if $X' \subset X$ is a subset of collinear points with $\delta(X') = h$, then $\sigma(X') = h$. On the other hand, it follows from Lemma 1.1 that $h = \sigma(X') \leq \sigma(X) = \sigma$. Obviously if the points are in the general position and $m_1 \geq \cdots \geq m_s$, then $h = \max\{m_i + m_j | 1 \leq i \neq j \leq s\} = m_1 + m_2$.

As we shall use induction on the sum of the multiplicities to prove Theorem 3.3, we need some information about σ when we lower the m_i's.

LEMMA 3.1. *Let $I = p_1^{m_1} \cap p_2^{m_2} \cap \cdots \cap p_s^{m_s}$ be the ideal of X and let $J = p_1^{m_1-1} \cap p_2^{m_2} \cap \cdots \cap p_s^{m_s}$. Then $\sigma(R/I) \leq \sigma(R/J) + 1$.*

A proof of this lemma can be found in [DG, Proposition 2.1].

LEMMA 3.2. *Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of fat points of P^2, and let Y_1 and Y_2 be two distinct subsets of X, each consisting of collinear points.*

Let us suppose that each of the sums of the multiplicities with respect to the points of Y_1 and Y_2 is greater than $[\sum_{i=1}^s m_i/2]$. Then $Y_1 \cap Y_2 \neq \emptyset$.

Proof. Without loss of generality we can assume that the points of Y_1 are exactly the first k points of X and that $Y_2 = \{(P_{k+1}, m_{k+1}), \ldots, (P_r, m_r)\}$, $r \leq s$.

By adding up we get

$$\sum_{i=1}^{k} m_i + \sum_{i=k+1}^{r} m_i \geq 2\left(\left[\frac{\sum_{i=1}^{s} m_i}{2}\right] + 1\right) = \begin{cases} \sum_{i=1}^{s} m_i + 2 & \text{if } \sum_{i=1}^{s} m_i \text{ is even} \\ \sum_{i=1}^{s} m_i + 1 & \text{if } \sum_{i=1}^{s} m_i \text{ is odd.} \end{cases}$$

In both cases we have $\sum_{i=1}^k m_i + \sum_{i=k+1}^r m_i > \sum_{i=1}^s m_i$, which is a contradiction, because our assumptions yield $\sum_{i=1}^k m_i + \sum_{i=k+1}^r m_i \leq \sum_{i=1}^s m_i$. ∎

THEOREM 3.3. *Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of fat points of P^2. Then*

$$h \leq \sigma \leq \max\left\{h - 1, \left[\frac{\sum_{i=1}^s m_i}{2}\right]\right\} + 1,$$

where $h = \max\{\sum_{j=1}^k m_{i_j} | P_{i_1}, \ldots, P_{i_k} \subset X \text{ are collinear}\}$.

Proof. First of all let us fix the notation that will be used throughout the proof.

Let $\rho = \max\{h - 1, [\sum_{i=1}^s m_i/2]\}$. Denote by L the line containing a subset of points $\{P_{i_1}, \ldots, P_{i_k}\}$ such that $\sum_{j=1}^k m_{i_j} = h$. Without loss of generality we may assume that such points are $P_1, \ldots, P_k$. Let I be the ideal of X, let $J = p_1^{m_1-1} \cap \cdots \cap p_k^{m_k-1} \cap p_{k+1}^{m_{k+1}} \cap \cdots \cap p_s^{m_s}$, and let $\overline{X}$ be the set of points corresponding to J, i.e.,

$$\overline{X} = \{(P_i, \overline{m}_i) | 1 \leq i \leq s\},$$

where

$$\overline{m}_i = \begin{cases} m_i - 1 & \text{if } 1 \le i \le k \\ m_i & \text{if } i \ge k+1. \end{cases}$$

Moreover, denote $\overline{h} = \max\{\Sigma_{j=1}^{k}\overline{m}_{i_j} | P_{i_1}, \dots, P_{i_t} \subset \overline{X}$ are collinear$\}$ and $\overline{\rho} = \max\{\overline{h} - 1, [\Sigma_{i=1}^{s}\overline{m}_i/2]\}$. Then it is easy to see that the following relations hold:

$$\overline{h} \le h, \tag{1}$$

$$\left[\frac{\sum^{s}\overline{m}_i}{2}\right] = \left[\frac{\sum^{s} m_i - k}{2}\right] \le \left[\frac{\sum^{s} m_i}{2}\right] - 1 \le \rho - 1, \tag{2}$$

$$\delta(R/I) = \delta(R/J) + h. \tag{3}$$

Now we proceed by induction on $\Sigma^s m_i$. If $\Sigma^s m_i = s$ then the result follows from Theorem 2.3. Otherwise we distinguish two cases.

Case 1. $\rho = h - 1 \ge [\Sigma^s m_i/2]$.

For every $t = 1, \dots, h-1$, we have $I_t = LJ_{t-1}$. This equality follows from Bezout's theorem, since by virtue of this theorem every curve of degree $t \le h-1$ passing through each P_i with multiplicity at least m_i has the line L as a component. By Lemma 1.2 this implies that

$$\Delta H(X,t) = \Delta H(\overline{X}, t-1) + 1 \qquad \text{if } 1 \le t \le h-1. \tag{4}$$

Obviously $\Delta H(X,0) = 1$, and so by adding up we have

$$\sum_{t=0}^{h-1} \Delta H(X,t) = \sum_{t=0}^{h-2} \Delta H(\overline{X},t) + h. \tag{5}$$

Now we show that $\Sigma_{t=0}^{h-2}\Delta H(\overline{X},t) = \delta(R/J)$, so that the equality (5) will become

$$\sum_{t=0}^{h-1} \Delta H(X,t) = \delta(R/J) + h$$

hence, by (3) we shall have

$$\sum_{t=0}^{h-1} \Delta H(X,t) = \delta(R/I).$$

From this we shall be able to conclude that $\sigma \le h = \rho + 1$.

By property (4) of Section 1 it suffices to show that $\sigma(R/J) \le h-1$. Since by induction we have $\sigma(R/J) \le \bar{\rho}+1$, it will be enough to prove that $\bar{\rho} \le h-2 = \rho - 1$.

By (2) we have $[\Sigma^s \overline{m}_i/2] \le \rho - 1 = h-2$. To show that also $\bar{h}-1 \le h-2$, consider the sets

$$Y_1 = \{(P_1, m_1), \dots, (P_k, m_k)\} \subset X$$
$$\overline{Y}_2 = \left\{(P_{i_1}, \overline{m}_{i_1}), \dots, (P_{i_r}, \overline{m}_{i_r})\right\} \subset \overline{X}$$
$$Y_2 = \left\{(P_{i_1}, m_{i_1}), \dots, (P_{i_r}, m_{i_r})\right\} \subset X,$$

where $\overline{Y}_2$ is a subset of $\overline{X}$ containing points for which $\Sigma_{j=1}^r \overline{m}_{i_j} = \bar{h}$, while Y_2 is the corresponding subset of X. Clearly $\Sigma_{j=1}^r m_{i_j} \le h$.

If $\Sigma_{j=1}^r m_{i_j} \le h-1$, then $\bar{h} = \Sigma_{j=1}^r \overline{m}_{i_j} \le \Sigma_{j=1}^k m_{i_j} \le h-1$.

If $\Sigma_{j=1}^r m_{i_j} = h = \Sigma_{j=1}^k m_i$ then by the previous lemma, $Y_1 \cap Y_2 \ne \varnothing$ and so there exist j and i, with $1 \le j \le r$ and $1 \le i \le k$ such that $m_{i_j} = m_i$. Thus $\overline{m}_{i_j} = \overline{m}_i = m_i - 1$, and so $\bar{h} = \Sigma_{j=1}^r \overline{m}_{i_j} \le \Sigma_{j=1}^r m_{i_j} - 1 = h-1$.

Case 2. $\rho = [\Sigma_{i=1}^s m_i/2] \ge h$.

Let us consider the set $X' = X \cup \{Q_1, \dots, Q_{\rho-h+1}\}$ where the Q_i's are distinct points on L, which are also distinct from the P_i's. Let I' be the ideal corresponding to X'. By Bezout's theorem we have $I'_\rho = LJ_{\rho-1}$.

If we consider the vector spaces $I_\rho, I'_\rho, J_{\rho-1}$, we have the relations

$$\dim_k I_\rho \le \dim_k I'_\rho + \rho - h + 1 = \dim_k J_{\rho-1} + \rho - h + 1.$$

By induction we have $\sigma(R/J) \le \bar{\rho}+1$. Since by (1) and (2) we have $\bar{\rho} \le \rho - 1$, it follows that $\sigma(R/J) \le \rho$.

This yields

$$\dim_k J_{\rho-1} = \binom{\rho+1}{2} - \delta(R/J).$$

If we consider the Hilbert functions we have

$$H(X,\rho) = \binom{\rho+2}{2} - \dim_k I_\rho \ge \binom{\rho+2}{2} - \dim_k J_{\rho-1} + h - (\rho+1)$$
$$= \binom{\rho+1}{2} - \dim_k J_{\rho-1} + h = \delta(R/J) + h = \delta(X).$$

Therefore $H(X,\rho) = \delta(X)$ and so $\sigma \le \rho + 1$. ∎

Another proof of the theorem above can be obtained by coupling our Lemma 3.2 and Corollary 3.2 in [C2].

Second Proof of Theorem 3.3. The notation is the same as that used in the other proof.

We proceed by induction on $\sum_{i=1}^{s} m_i$. It is obvious if $\sum_{i=1}^{s} m_i \leq 2$ or $s = 2$. Let us suppose $\sum_{i=1}^{s} m_i \geq 3$ and $s > 2$.

Now we show that the hypotheses of Corollary 3.2 in [C2] are verified (assuming $C = L$, whence $d = 1$, $g = 0$). In fact, let $t \geq \rho$, then:

(1) $t \geq 1$, because $s > 2$.

(2) $t - \sum_{i=1}^{k} m_i \geq 1$, because $t \geq \rho \geq h - 1 = \sum_{i=1}^{k} m_i - 1$.

(3) $\sigma(\bar{X}) \leq t$; in fact:

(a) If $\bar{\rho} < \rho$, then, by induction $\sigma(\bar{X}) \leq \bar{\rho} + 1 \leq \rho \leq t$.

(b) If $\bar{\rho} = \rho = [\sum_{i=1}^{s} m_i/2]$ then $\bar{\rho} = \bar{h} - 1 = [\sum_{i=1}^{s} m_i/2]$, and so $h - 1 \geq \bar{h} - 1 = [\sum_{i=1}^{s} m_i/2]$. By Lemma 3.2 it follows that $\bar{h} < h$ and this implies $\rho = \bar{\rho} = \bar{h} - 1 < h - 1 \leq [\sum_{i=1}^{s} m_i/2]$, which is a contradiction.

(c) If $\bar{\rho} = \rho = h - 1$ then $h - 1 \geq [\sum_{i=1}^{s} m_i/2]$. Also in this case, by Lemma 3.2, it follows that $\bar{h} < h$, and this is a contradiction as above.

So by Corollary 3.2 in [C2] we have $\sigma(X) \leq t + 1$, in particular, if $t = \rho$ then $\sigma(X) \leq t + 1$. ∎

Now we show that both the general bound and Segre's bound for τ can be recovered from the previous theorem.

COROLLARY 3.4. *Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of fat points of P^2. Then*:

(1) $\sigma \leq \sum_{i=1}^{s} m_i$.

(2) $\sigma = \sum_{i=1}^{s} m_i$ *if and only if $P_1, \ldots, P_s$ are collinear.*

Proof. The first result is a consequence of the inequalities

$$\sigma \leq \max\left\{h - 1, \left[\frac{\sum^{s} m_i}{2}\right]\right\} + 1 \leq \sum^{s} m_i.$$

The first inequality holds by Theorem 3.3 while the second one easily follows the definition of h.

A proof of (2) can be found, for instance, in [DG, Corollary 2.3]. Note that the necessary part also follows from Theorem 3.3. In fact by that theorem we have

$$\sum^{s} m_i = \sigma \leq \max\left\{h - 1, \left[\frac{\sum^{s} m_i}{2}\right]\right\} + 1.$$

Since $[\Sigma^s m_i/2] + 1 \leq \Sigma^s m_i$ then necessarily $\Sigma^s m_i = h$ and so, from the definition of h, the points are collinear. ∎

COROLLARY 3.5. *Let $X = \{(P_1, m_1), \dots, (P_s, m_s)\}$ be a set of fat points of P^2, in general position, with $m_1 \geq \cdots \geq m_s$. Then*

$$\sigma \leq \max\left\{m_1 + m_2 - 1, \left[\frac{\sum^{s} m_i}{2}\right]\right\} + 1.$$

Proof. Since the points are in general position, as we already noticed, we have

$$h = \max\{m_i + m_j \mid 1 \leq i \neq j \leq s\} = m_1 + m_2.$$

Thus the result immediately follows from Theorem 3.3. ∎

The following corollary is an easy consequence of Theorem 3.3 and the definition of h.

COROLLARY 3.6. *If $X \subset P^2$ is a set of fat points, then the following assertions are equivalent*:

(1) $\sigma = h > [\Sigma_{i=1}^{s} m_i/2]$
(2) $\sigma > [\Sigma_{i=1}^{s} m_i/2]$
(3) $h > [\Sigma_{i=1}^{s} m_i/2]$
(4) *There exists collinear points whose multiplicities add up to σ.*

Remark. It is well known that if $X \subset C$, where X is a set of fat points and C an integral curve of degree d, then

$$\sigma \leq \left[\frac{\sum_{i=1}^{s} m_i}{d}\right] + d - 1 \tag{6}$$

(see, for instance, [G2]).

We can observe that, in this case, we necessarily have $h \le d$. This observation together with Theorem 3.3 implies that

$$\sigma \le \max\left\{d - 1, \left[\frac{\sum_{i=1}^{s} m_i}{2}\right]\right\} + 1, \qquad (7)$$

and so (7) might give some hope of sharpening bound (6). Unfortunately, this is not the case. In fact, if $[\sum_{i=1}^{s} m_i/2] \le d \le \sum_{i=1}^{s} m_i$, then the two bounds coincide, otherwise (7) gives even a larger bound as the following example shows:

Example. Let $X = \{(P_1, 2), (P_2, 1), \ldots, (P_7, 1)\}$ be a set of fat points on an integral cubic curve. In this case we have

$$d = h = 3,$$

$$\left[\frac{\sum_{i=1} m_i}{2}\right],$$

and so, (6) gives $\sigma \le 3 = d$, and (7) yields $\sigma \le 5$.

References

[C] G. Castelnuovo, Ricerche generali sopra i sistemi lineari di curve piane, Memorie delle Reale Accademia delle Scienze di Torino XLII, 1891, *in* "Memorie Scelte," Zanichelli, Bologna, 1937.

[C1] M. V. Catalisano, Regolarità e irriducibilità dei sistemi lineari di curve piane passanti per punti di P^2 in posizione generale, con assegnata molteplicità, *Atti Accad. Sci. Torino* **123** (1989), 51–62.

[C2] M. V. Catalisano, Linear systems of plane curves through fixed "fat" points of P^2, *J. Algebra* **142** (1991), 81–100.

[C3] M. V. Catalisano, "Fat" points on a conic, *Comm. Algebra* **19**, No. 8 (1991), 2153–2168.

[CG] M. V. Catalisano and S. Greco, Linear systems: Developments of some results by F. Enriques and B. Segre, *in* "Lecture Notes in Pure and Appl. Math.," Vol. 132, pp. 41–57, Dekker, New York, 1991.

[D1] E. Davis, Complete intersections of codimension 2 in P^r: The Bezout–Jacobi–Segre Theorem revisited, *Rend. Sem. Mat. Torino* **43** (1985), 333–353.

[D2] E. Davis, 0-dimensional subschemes of P^2: New application of Castelnuovo function, *Ann. Univ. Ferrara* **32** (1986), 93–107.

[D3] E. Davis, Complements to a paper of Paul Dubreil, *Ricerche Mat.* **37** (1988), 347–357.

[DG] E. Davis and A. V. Geramita, The Hilbert function of a special class of 1-dimensional Cohen–Macaulay graded algebras, *in* "The Curves Seminar at Queen's, Vol.

III," pp. 1H–29H, Queen's Papers in Pure and Appl. Math., Vol. 67, Queen's Univ., Kingston, Ontario, 1984.

[Fa] G. FATABBI, Sistemi lineari di ipersuperfici per punti base di molteplicità assegnata, Tesi di Laurea, 1992.

[F] W. FULTON, "Algebraic Curves," Benjamin, New York, 1969.

[G1] A. GIMIGLIANO, Regularity of linear systems of plane curves, *J. Algebra* **124** (1989), 343–351.

[G2] A. GIMIGLIANO, Our thin knowledge about fat points, *in* "The Curves Seminar at Queen's, Vol. VI," pp. 1B–50B, Queen's Papers in Pure and Appl. Math., Vol. 83, Queen's Univ., Kingston, Ontario, 1989.

[G] S. GRECO, Remarks on the postulation of zero-dimensional subschemes of projective space, *Math. Ann.* **284** (1989), 343–351.

[GM] A. V. GERAMITA AND P. MAROSCIA, The ideals of forms vanishing at a finite set of points in P^n, *J. Algebra* **90** (1984), 528–555.

[Ma] P. MAROSCIA, Some remarks on the Hilbert function of a zero-cycle in P^2, preprint 6/90, Dipartimento di Matematica, Università della Basilicata, 1990.

[M] H. MATSUMURA, "Commutative Ring Theory," Cambridge Univ. Press, London/New York, 1986.

[P] G. PAXIA, On flat families of fat points, *Proc. Amer. Math. Soc.* **112** (1991), 19–23.

[R] G. RACITI, Sulla funzione di Hilbert di un sottoschema 0-dimensionale di P^3, *Ann. Univ. Ferrara* **35** (1989), 99–112.

[Sg] B. SEGRE, Alcune questioni su insiemi finiti di punti in Geometria Algebrica, *in* "Atti del Convegno Internazionale di Geometria Algebrica, Torino, 1961," pp. 15–33.

[S] F. SEVERI, "Trattato di Geometria Algebrica," Vol. I, Parte I, Zanichelli, Bologna, 1926.