Let $F$ be a field of characteristic $p > 0$ and $A$ a commutative (associative) nilpotent finite-dimensional algebra over $F$. Let $A^{(p)}$ be the subalgebra generated by the set \{\(a^p\mid a \in A\)\}. N. Eggert \[4\] (1971) conjectured that

$$p \cdot \dim A^{(p)} \leq \dim A.$$

This conjecture gives an answer to the problem, when a finite abelian group is isomorphic to the adjoint group of some finite commutative nilpotent $F$-algebra. Recall that the adjoint group of $A$ is the set $A$ with the operation $x \circ y = x + y + xy$ for every $x, y \in A$. Besides, the validity of Eggert's conjecture would give an upper bound of the rank of the product of groups (see \[2\]).

It is also good to point out that, since the Jacobson's radical of an artinian commutative ring is nilpotent and, on the other hand, every commutative nilpotent algebra is equal to the Jacobson's radical in some local ring, we get through this conjecture additional information about the structure of commutative rings (with unit), especially of the finite ones.
N. Eggert proved his conjecture only when \( \dim A(p) \leq 2 \). Five years later, R. Bautista [3] (1976) proved it when \( \dim A(p) = 3 \). C. Stack confirmed this results in [10,11] (1996, 1998), but provided shorter proofs. Finally, B. Amberg and L.S. Kazarin [1] (2001) proved the conjecture for the case \( \dim A(p) \leq 4 \).

Another type of results presented K.R. McLean in [8,9] (2004, 2006). He showed that this conjecture is true if the algebra \( A \) is either radical of a group algebra of a finite abelian group or \( A \) is graded and at least one of the following conditions is fulfilled:

(i) \( p = 2 \) and \( (A(p))^4 = 0 \).
(ii) \( A(p) \) is 2-generated.
(iii) \( (A(p))^3 = 0 \).
(iv) \( n < 3p \) and \( 3 \leq s - 1 \leq p \), where \( n \) is the number of generators of \( A(p) \) and \( s \) is the index of nilpotence of \( A(p) \).

We also should mention the result of V.O. Gorlov [5] (1995). He proved the conjecture for nilpotent algebras \( A \) with a metacyclic adjoint group.

One paper concerning Eggert’s conjecture appeared in 2002 and the author L. Hammoudi [6] claimed he proved it. But, as B. Amberg and L. Kazarin [2] have shown, his proof was incorrect. A similar counterexample to Hammoudi’s method provided also K.R. McLean [9].

In this paper we show that Eggert’s conjecture is true if the subalgebra \( A(p) \) has at most two generators. Our result needs no limitation on the dimension of \( A(p) \), no assumption on grading of \( A \) and approaches Eggert’s conjecture from a different point of view. Our method will be more combinatorial than algebraic. We will use the theory of standard bases (a generalization of the well-known Gröbner bases).

1. Introduction

Throughout this paper, all algebras are assumed to be commutative (and associative, of course). Henceforth, the word ‘algebra’ will always mean a commutative one.

We will denote by \( \mathbb{N} (\mathbb{N}_0, \text{resp.}) \) the set of positive (non-negative, resp.) integers. For \( r \in \mathbb{Q} \) let \( \lfloor r \rfloor \) (\( \lceil r \rceil \), resp.) be the lower (upper, resp.) integral part of \( r \).

Let \( A \) be an algebra over \( F \) and \( X \subseteq A \) a subset. We denote by \( \langle X \rangle \) (\( \{X\}, \text{resp.} \) the algebra (vector space, resp.) generated by \( X \).

An algebra \( A \) is called nilpotent if \( A^m = 0 \) for some \( m \in \mathbb{N} \).

Through this paper let always \( F \) be a field of characteristic \( p > 0 \) and \( R = F[x, y] \) be the ring of polynomials over the variables \( x, y \) and the field \( F \).

First we recall some basic properties of nilpotent \( F \)-algebras.

**Lemma 1.1.** Let \( A \) be an \( F \)-algebra, \( \text{char } F = p > 0 \). Then:

(i) \( (a + b)^p = a^p + b^p \) for all \( a, b \in A \).
(ii) If \( A = \langle a_1, \ldots, a_n \rangle \), then \( A(p) = \langle a_1^p, \ldots, a_n^p \rangle \).

**Lemma 1.2.** Let \( A \) be an \( F \)-algebra.

(i) If \( A \) is nilpotent, then \( \dim A < \infty \) if and only if \( A \) is finitely generated.
(ii) If \( A \) is generated by \( a_1, \ldots, a_n \in A \), then \( A \) is nilpotent if and only if all the elements \( a_i, i = 1, \ldots, n \), are nilpotent.

**Lemma 1.3.** (See [7, 1.3.8].) Let \( S \) be an \( F \)-subalgebra of a nilpotent \( F \)-algebra \( A \), such that \( A = S + A^2 \), then \( A = S \).
As an immediate consequence we get the following:

**Corollary 1.4.** Let $A$ be a nilpotent $F$-algebra such that $A = \langle a_1, \ldots, a_n \rangle = \langle b_1, \ldots, b_m \rangle$, $m \leq n$. Then there are $i_1, \ldots, i_m \in \mathbb{N}$ such that $A = \langle a_{i_1}, \ldots, a_{i_m} \rangle$. In particular, minimal sets of generators of $A$ have the same cardinality.

To prove our main claim we can restrict our consideration, using the next two assertions, only on 2-generated algebras which arise as factors of polynomials.

**Lemma 1.5.** Suppose that Eggert’s conjecture holds for every nilpotent 2-generated $F$-algebra. Then it also holds for every nilpotent $F$-algebra $A$ such that $A(p)$ is a 2-generated $F$-algebra.

**Proof.** Let $A$ be a nilpotent $F$-algebra of finite dimension and let $A(p)$ be 2-generated. By 1.2, we have $A = \langle a_1, \ldots, a_n \rangle$ for some $a_1, \ldots, a_n \in A$. By 1.1, $A(p) = \langle a_1^p, \ldots, a_n^p \rangle$ and, by 1.4, we get (without loss of generality) that $A(p) = \langle a_1^p, a_2^p \rangle$. Consider now the subalgebra $B = \langle a_1, a_2 \rangle$. Then $A(p) = \langle a_1^p, a_2^p \rangle = B(p)$. Hence, by assumption, we get $p \dim A(p) = p \dim B(p) \leq \dim B \leq \dim A$. □

**Lemma 1.6.** Let $A$ be a nilpotent $F$-algebra generated by $a_1, a_2 \in A$. Set $I = \{ f \in Rx + Ry \mid f(a_1, a_2) = 0 \}$. Then $I$ is an ideal of $R$ and there is $k \in \mathbb{N}$ such that $x^k, y^k \in I$. The map $\varphi : Rx + Ry/I \to A$, $\varphi(f + I) = f(a_1, a_2)$ is an isomorphism of $F$-algebras.

On the other hand, let $J \subseteq Rx + Ry$ be an ideal of $R$ such that $x^k, y^k \in J$ for some $k \in \mathbb{N}$. Then $Rx + Ry/J$ is a nilpotent $F$-algebra generated by $x + J, y + J$.

Our aim in the rest of the paper will be to prove the following:

**Theorem.** Let $A$ be a nilpotent 2-generated $F$-algebra, $\text{char } F = p > 0$. Then $p \cdot \dim A(p) \leq \dim A$.

And as an immediate consequence (using 1.5) we get

**Theorem.** Let $A$ be a nilpotent $F$-algebra, $\text{char } F = p > 0$, such that $A(p)$ is 2-generated. Then $p \cdot \dim A(p) \leq \dim A$.

### 2. Orderings and polynomials

In this and the following sections we will use the well-known concept of monomial orderings.

**Definition 2.1.** Define the lexicographical ordering $\leq$ on $\mathbb{N}_0^2$ such that

$$(i, j) \leq (i', j') \iff i < i' \lor (i = i' \land j \leq j').$$

Define the component-wise ordering $\leq_{\Pi}$ on $\mathbb{N}_0^2$ such that

$$(i, j) \leq_{\Pi} (i', j') \iff i \leq i' \land j \leq j'.$$

Consider $(\mathbb{N}_0^2, +)$ to be a semigroup with operation $+$ defined component-wise. For $\alpha = (i, j) \in \mathbb{N}_0^2$ put

$$x^\alpha = x^i y^j \in F[x, y].$$
It is well known that $\leq$ is a total order on $\mathbb{N}_0^2$ with the following properties:

(i) $\alpha \leq \Pi \beta \Rightarrow \alpha \leq \beta$;
(ii) $\alpha \leq \Pi \beta \Rightarrow \alpha + \gamma \leq \Pi \beta + \gamma$;
(iii) $\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$

for every $\alpha, \beta, \gamma \in \mathbb{N}_0^2$.

Definition 2.2. For $0 \neq f = \sum \lambda_{\alpha} x^\alpha \in F[x, y]$ put

$$m(f) = \min \{ \alpha \in \mathbb{N}_0^2 | \lambda_{\alpha} \neq 0 \}.$$ 

$f$ will be called normal iff $\lambda_m(f) = 1$ and $m(f) < \Pi \alpha$ implies $\lambda_{\alpha} = 0$ for every $\alpha \in \mathbb{N}_0^2$.

Fig. 1 illustrates the difference between a general polynomial $f$ with $m(f) = (6, 6)$ (the gray area) and a general normal polynomial $g$ with $m(g) = (2, 4)$ (the hatched area). The marked areas are the most common sets of $\alpha \in \mathbb{N}_0^2$ such that the monomial $x^\alpha$ can occur in the given polynomial.

First, we recall some basic properties of $m$ and $\leq$. The following lemmas are easy to prove.

Lemma 2.3. Let $0 \neq f, 0 \neq g \in F[x, y]$. Then:

(i) $m(fg) = m(f) + m(g)$.
(ii) $m(f + g) \geq \min \{ m(f), m(g) \}$, if $f + g \neq 0$. Moreover, $m(f + g) = m(f)$ if $m(f) < m(g)$.
(iii) $m(f(x^p, y^p)) = p m(f)$.

Lemma 2.4. Let $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in R$ be non-zero polynomials. Let $h_1, \ldots, h_n \in R$. If $\sum_i h_i f_i \neq 0$, then $m(\sum_i h_i f_i) \geq \min \{ m(f_i) | i = 1, \ldots, n \}$.

Lemma 2.5. Let $0 \neq f \in R$, $m(f) = (i, j) \in \mathbb{N}_0^2$. Then $f$ is normal if and only if there are $g_0, \ldots, g_{j-1} \in F[x]$ such that $f = x^i y^j + x^{i+1} h$, where $h = \sum_{k=0}^{j-1} g_k y^k \in R$. In particular, if $j = 0$ then $f = x^i y^j$. 

**Fig. 1.** Common polynomials vs. normal polynomials.
Definition 2.6. Let $X \subseteq \mathbb{N}_0^2$. A set $X$ will be called upper (lower, resp.) if $\alpha \in X$ and $\alpha \leq \Pi \beta$ ($\beta \leq \Pi \alpha$, resp.) implies $\beta \in X$ for every $\alpha, \beta \in \mathbb{N}_0^2$.

Clearly, a set $X \subseteq \mathbb{N}_0^2$ is lower if and only if $\mathbb{N}_0^2 \setminus X$ is an upper set.

Definition 2.7. Let $\emptyset \neq B \subseteq \mathbb{N}_0^2$ be a finite lower set. Put

$$n_0(B) = |B \cap (\{0\} \times \mathbb{N}_0)| - 1,$$
$$d_i(B) = |B \cap (\{i\} \times \mathbb{N}_0)|$$

for $i \in \mathbb{N}_0$.

As we see on Fig. 2, the finite lower set $B$ is “echelon-shaped”. The number $n_0(B) + 1$ is just the width of the base of $B$ and the number $d_i(B)$ is the height of the $i$-th column in $B$. The following lemma is now easy to see.

Lemma 2.8. Let $\emptyset \neq B \subseteq \mathbb{N}_0^2$ be a finite lower set and $C = \mathbb{N}_0^2 \setminus B$. Let $n_0 = n_0(B)$. Then:

(i) $d_0(B) \geq d_1(B) \geq \ldots \geq d_{n_0}(B) > 0 = d_{n_0+1}(B) = \ldots$.

(ii) $d_0(B) + \ldots + d_{n_0}(B) = |B|$.

(iii) $B = \{\alpha \in \mathbb{N}_0^2 \mid (\exists 0 \leq i \leq n_0) (i, 0) \leq \alpha < (i, d_i(B))\}$.

(iv) $C = \{\alpha \in \mathbb{N}_0^2 \mid (\exists 0 \leq i \leq n_0 + 1) (i, d_i(B)) \leq \Pi \alpha\}$.

(v) $(i, j) \in C$ if and only if $d_i(B) \leq j$.

3. Bases of nilpotent algebras

Through this section let $A$ be a (commutative) nilpotent $F$-algebra generated by $a_1, a_2 \in A$. We will now naturally apply previous results about polynomials and lower sets to the case of 2-generated nilpotent $F$-algebras.
Definition 3.1. Set $a^\alpha = a^\alpha_1 a^\alpha_2 \in A$ for $(0, 0) \neq \alpha = (i, j) \in \mathbb{N}^2_0$.

Put

$$C_A(a_1, a_2) = \{ \alpha \in \mathbb{N}^2_0 \mid (\exists f \in Rx + Ry) m(f) = \alpha \wedge f(a_1, a_2) = 0 \}$$

and

$$B_A(a_1, a_2) = \mathbb{N}^2_0 \setminus C_A(a_1, a_2).$$

Theorem 3.2.

(i) $C_A(a_1, a_2)$ is an upper set.

(ii) $B_A(a_1, a_2)$ is a lower set and $(0, 0) \in B_A(a_1, a_2)$.

(iii) The set $\{a^\alpha \mid (0, 0) \neq \alpha \in B_A(a_1, a_2)\}$ is a basis of $A$. In particular, $B_A(a_1, a_2)$ is finite.

Proof. (i) Let $f \in Rx + Ry$ be such that $m(f) = \alpha \in \mathbb{N}^2_0$ and $\alpha \leq_{\Pi} \beta$. Then $m(f x^{\beta-\alpha}) = m(f) + m(x^{\beta-\alpha}) = \alpha + (\beta - \alpha) = \beta$.

(ii) From the definition of $C_A(a_1, a_2)$ we have that $C_A(a_1, a_2) \subseteq \mathbb{N}^2_0 \setminus \{(0, 0)\}$. The rest follows from (i).

(iii) First, we show that $B = \{a^\alpha \mid (0, 0) \neq \alpha \in B_A(a_1, a_2)\}$ generates $A$ as a vector space. Since $A$ is nilpotent, there is $k \in \mathbb{N}$ such that $a^k_1 = a^k_2 = 0$. Denote $I = \{\alpha \in \mathbb{N}^2_0 \mid (0, 0) \neq \alpha \leq_{\Pi} (k, k)\}$. Obviously $A = \{a^\alpha \mid \alpha \in I\}$, since $a^\alpha = 0$ for $\alpha \notin_{\Pi} (k, k)$. It remains to show, that $a^\alpha \in [B]$ for every $\alpha \in I$.

Let, on contrary, $\alpha$ be the greatest element in $I$ with respect to $\leq$, such that $a^\alpha \notin [B]$. Then $\alpha \in C_A(a_1, a_2)$ and hence, by 3.1, we have $a^\alpha = \sum_{\beta < \alpha} \lambda_\beta a^\beta$, where $\lambda_\beta \in F$. For $\beta > \alpha$ we have either $\beta \notin I$, and then $a^\beta = 0 \in [B]$, or $\beta \in I$, and then $a^\beta \in [B]$ by choice of $\alpha$. Hence $\sum_{\alpha < \beta} \lambda_\beta a^\beta \in [B]$, a contradiction.

Now we show that $B$ is linearly independent. Suppose, on contrary, there is a non-trivial polynomial $0 \neq f = \sum_{\alpha \in B_A(a_1, a_2)} \lambda_\alpha a^\alpha$ such that $\sum_{\alpha \in B_A(a_1, a_2)} \lambda_\alpha a^\alpha = 0$. Then $m(f) \in C_A(a_1, a_2)$, a contradiction. □

Lemma 3.3. $a^\alpha \in \{[a^\beta \mid \beta \in B_A(a_1, a_2) \wedge \alpha < \beta]\}$ for every $\alpha \in C_A(a_1, a_2)$.

Proof. Let $M$ be the set of all $\alpha \in C_A(a_1, a_2)$ such that $a^\alpha \notin \{[a^\beta \mid \beta \in B_A(a_1, a_2) \wedge \alpha < \beta]\}$. Since $A$ is nilpotent, there are only finitely many $\alpha \in \mathbb{N}^2_0$ such that $a^\alpha \notin 0$. Hence $M$ is finite. Suppose, for contrary, that $M$ is not empty. Let $\alpha_0$ be the greatest element of $M$. Then, by 3.1, $a^{\alpha_0} = \sum_{\alpha < \beta} \lambda_\beta a^\beta$ for some $\lambda_\beta \in F$. But for every $\beta > \alpha_0$ is either $\beta \in B_A(a_1, a_2)$ or $\beta \in C_A(a_1, a_2)$ and $a^\beta \in \{[a^\gamma \mid \gamma \in B_A(a_1, a_2) \wedge \beta < \gamma]\}$. Hence $a^{\alpha_0} \in \{[a^\beta \mid \beta \in B_A(a_1, a_2) \wedge \alpha_0 < \beta]\}$, a contradiction. □

Corollary 3.4. $C_A(a_1, a_2) = \{\alpha \in \mathbb{N}^2_0 \mid (\exists f \in Rx + Ry) m(f) = \alpha \wedge f(a_1, a_2) = 0 \wedge f$ is normal}.

Proof. Let $\alpha \in C_A(a_1, a_2)$. Put $K = B_A(a_1, a_2) \cap \{\beta \in \mathbb{N}^2_0 \mid \alpha < \beta\}$. By 3.3, $a^\alpha = \sum_{\beta \in K} \lambda_\beta a^\beta$, where $\lambda_\beta \in F$. Now, set $f = x^\alpha - \sum_{\beta \in K} \lambda_\beta x^\beta$. Then $m(f) = \alpha$, $f(a_1, a_2) = 0$ and, by 3.2, $f$ is normal. □

This corollary says that for every $0 \neq f \in Rx_1 + Rx_2$ such that $f(a_1, a_2) = 0$, there is $0 \neq g \in Rx_1 + Rx_2$ such that $g(a_1, a_2) = 0$, $m(g) = m(f)$ and $g$ is normal. Hence we can always work only with normal polynomials.

Since the subalgebra $A(p)$ is also nilpotent and, by 1.1, generated by $a^p_1, a^p_2 \in A$, we can use the results on 2-generated nilpotent $F$-algebras for it. By 3.2(iii), the set $\{a^\alpha \mid (0, 0) \neq \alpha \in B_A(a_1, a_2)\}$ is a basis of $A$ and similarly the set $\{a^\alpha \mid (0, 0) \neq \alpha \in B_A(p)(a^p_1, a^p_2)\}$ is a basis of $A(p)$ (see 3.9(V)). One example of these two bases for a certain $A$ and $p = 2$ is shown in Fig. 3 (for more details see 5.10(1)).
The set \( B_A(a_1, a_2) \) corresponds to the gray area and the elements of the set \( p \cdot B_{A(p)}(a_1^p, a_2^p) \) are marked with crosses.

In the following part we apply the results of the section 2 for the sets \( B_A(a_1, a_2) \) and \( B_{A(p)}(a_1^p, a_2^p) \).

**Definition 3.5.** Denote

\[
\begin{align*}
n_0^A(a_1, a_2) &= n_0(B_A(a_1, a_2)), \\
d_i^A(a_1, a_2) &= d_i(B_A(a_1, a_2)), \\
\bar{n}_0^A(a_1, a_2) &= n_0(B_{A(p)}(a_1^p, a_2^p)), \\
\bar{d}_i^A(a_1, a_2) &= d_i(B_{A(p)}(a_1^p, a_2^p))
\end{align*}
\]

and

\[
D_i^A(a_1, a_2) = \sum_{k=pi}^{pi+p-1} d_k^A(a_1, a_2)
\]

for \( i \in \mathbb{N}_0 \).

**Lemma 3.6.** Let \( n_0 = n_0^A(a_1, a_2) \) and \( d_i = d_i^A(a_1, a_2) \). Then:

(i) \( d_0 \geq d_1 \geq \cdots \geq d_{n_0} > 0 = d_{n_0+1} = \cdots \).

(ii) \( d_0 + \cdots + d_{n_0} = |B_A(a_1, a_2)| = 1 + \dim A \).

(iii) \( B_A(a_1, a_2) = \{ \alpha \in \mathbb{N}_0^2 \mid (30 \leq i \leq n_0) \ (i, 0) \leq \alpha < (i, d_i) \} \).

(iv) \( C_A(a_1, a_2) = \{ \alpha \in \mathbb{N}_0^2 \mid (30 \leq i \leq n_0 + 1) \ (i, d_i) \leq \alpha \leq (i, d_i) \} \).

(v) \( (i, j) \in C_A(a_1, a_2) \) if and only if \( d_i \leq j \).

**Proof.** Follows from 2.8 and 3.2. \( \square \)

**Definition 3.7.** For \( a \in A \) put \( \text{nil}(a) = \min \{ k \in \mathbb{N} \mid a^k = 0 \} \).
Lemma 3.9.

Let \( \mathfrak{m}_0 = \mathfrak{n}_0^A(a_1, a_2) \) and \( D_i = D_i^A(a_1, a_2) \). Then:

(i) \( \tilde{d}_0 > \tilde{d}_1 > \cdots > \tilde{d}_{\mathfrak{m}_0} > 0 = \tilde{d}_{\mathfrak{m}_0+1} > \cdots \).

(ii) \( D_0 + \cdots + D_{\mathfrak{m}_0} = |\mathcal{B}_A(a_1, a_2)| = 1 + \dim A^p \).

(iii) \( D_i = 0 \) if and only if \( \tilde{d}_i = 0 \).

(iv) \( \alpha \in A^p(a_1^p, a_2^p) \) if and only if there is \( 0 \neq f \in Rx + Ry \) such that \( \mathbf{m}(f) = \alpha \) and \( f(a_1^p, a_2^p) = 0 \).

(v) \( p \cdot C_{A^p}(a_1^p, a_2^p) \subseteq C_A(a_1, a_2) \).

Proof. Put \( A' = A^p \), \( a'_1 = a_1^p \) and \( a'_2 = a_2^p \).

(i) Use (i). (iv) Use 3.8 and 3.6 for \( A' = \langle a'_1, a'_2 \rangle \).

(ii) From \( 0 < n_0/p < |n_0/p| + 1 \) follows that \( d_k = 0 \) for \( k = p(|n_0/p| + 1) \), by 3.6(i). Hence \( D_i = 0 \) for \( i > |n_0/p| + 1 = \mathfrak{m}_0 + 1 \), by 3.8(iii).

(v) Use 3.2 for \( A' = \langle a'_1, a'_2 \rangle \).

(vi) See 3.1 for \( A' = \langle a'_1, a'_2 \rangle \).

(vii) Let \( (i, j) \in C_{A^p}(a_1^p, a_2^p) \). Then there is \( f \in R \) such that \( f(a_1^p, a_2^p) = 0 \) and \( \mathbf{m}(f) = (i, j) \). Hence \( (p, p) \in C_A(a_1, a_2) \). \( \square \)

The next lemma is the first step to upper estimation of the numbers \( \tilde{d}_i^A(a_1, a_2) \) and hence of the dimension of the subalgebra \( A^p \). Since in this estimation plays an important role the upper integral part, we will later need a slightly different form of the polynomials – instead of \( a_1^m f(a_1^p, a_2^p) = 0 \) we use \( a_1^m f(a_1^p, a_2^p) = \lambda w_A(a_1, a_2) \), \( \lambda \in F \) (see 5.1 and 5.8).

Lemma 3.10. Let \( 0 \neq f \in Rx + Ry \) and \( m \in \mathbb{N}_0 \) be such that \( a_1^m f(a_1^p, a_2^p) = 0 \). If \( \mathbf{m}(f) = (i, j) \in \mathbb{N}_0^2 \) then:

(i) \( \mathbf{m}(y^m f(x^p, y^p)) = (pi, pj + m) \).

(ii) \( \tilde{d}^A_i(a_1, a_2) \leq [(pj + m)/p] \).

Proof. (i) Use 2.3.

(ii) Let \( m = pk + r \), where \( k, r \in \mathbb{N}_0 \), \( 0 \leq r < p \). First, let \( r = 0 \). Put \( g = y^k f \). Then \( g(x^p, y^p) = y^m f(x^p, y^p) \), \( g(a_1^p, a_2^p) = 0 \) and \( \mathbf{m}(g) = (i, j + k) \), by (i). Hence, by 3.9(vi), \( (i, j + k) \in C_{A^p}(a_1^p, a_2^p) \). Thus, by 3.9(iv), \( \tilde{d}^A_i(a_1, a_2) \leq j + k = [(pj + m)/p] \).
Now, suppose that $r > 0$. Put $h = y^{r+1}f$. Then $h(x^p, y^p) = y^{m+p-r}f(x^p, y^p)$, $h(a_p^i, a_p^j) = 0$ and $m(h) = (i, j + k + 1)$. Hence, by 3.9(vi), $(i, j + k + 1) \in C_{A(P)}(a_p^i, a_p^j)$. Thus, by 3.9(iv), $\bar{d}_t^A(a_1, a_2) \leq j + k + 1 = [(p + m)/p]$. □

**Proposition 3.11.** $[d_{pi}^A(a_1, a_2)/p] \leq \bar{d}_t^A(a_1, a_2) \leq d_{pi}^A(a_1, a_2)$ for every $i \in \mathbb{N}_0$.

**Proof.** Let $i \in \mathbb{N}_0$. Let $j \in \mathbb{N}_0$ be such that $pj < d_{pi}^A(a_1, a_2)$. Then $(pi, pj) \in B_A(a_1, a_2)$, by 3.6. Hence $(i, j) \in B_{A(P)}(a_p^i, a_p^j)$ (otherwise $(pi, pj) \in pC_{A(P)}(a_p^i, a_p^j) \subseteq C_A(a_1, a_2)$, by 3.9(vii), a contradiction). It follows that $[d_{pi}^A(a_1, a_2)/p] \leq \bar{d}_t^A(a_1, a_2)$.

Further, put $d = d_{pi}^A(a_1, a_2)$. By 3.6(iv), $(pi, d) \in C_A(a_1, a_2)$. By 3.4 and 2.5, there is $f \in R$ such that $f(a_1, a_2) = 0$ and $f = h^d(y^d + xh)$ for some $h \in R$. Clearly, there is $h' \in R$ such that $h(x, y)^p = h'(x^p, y^p)$. Hence $0 = d_{pi}^A(a_p^i + a_1h(a_1, a_2))^p = d_{pi}^A(a_p^i + a_p^j h'(a_p^i, a_p^j))$. Thus $(i, d) \in C_{A(P)}(a_p^i, a_p^j)$. By 3.9(iv), $\bar{d}_t^A(a_1, a_2) \leq d = d_{pi}^A(a_1, a_2)$. □

4. Polynomial presentation of 2-generated nilpotent algebras

In this section we pay our attention only on the nilpotent algebras of the form $Rx + Ry/I$ for some ideal $I$ of $R$.

Through this (and the next) section let $I \subseteq Rx + Ry$ be an ideal in $R$ such that $A = Rx + Ry/I$ is a non-zero nilpotent $F$-algebra (i.e. $x^k, y^k \in I$ for some $k \in \mathbb{N}$, by 1.6). The congruence of $R$ corresponding to $I$ will be denoted by $\equiv I$ or just $\equiv$.

We have $A = (x + I, y + I)$, by 1.6, and $A^{(p)} = (x^p + I, y^p + I)$, by 1.1.

For shorter expressions we write: $C_A = C_A(x + I, y + I)$, $B_A = B_A(x + I, y + I)$, $C_{A(P)} = C_{A(P)}(x^p + I, y^p + I)$, $B_{A(P)} = B_{A(P)}(x^p + I, y^p + I)$, $n_0 = n_0^0(x + I, y + I)$, $n_0 = n_0^0(x + I, y + I)$, $d_i = d_i(x + I, y + I)$ and $D_i = D_i(x + I, y + I)$.

By 3.6(iv), $(i, d_i) \in C_A$ for $0 \leq i \leq n_0 + 1$. Now, by 3.4, choose $f_i \in Rx + Ry$, $0 \leq i \leq n_0 + 1$, such that $m(f_i) = (i, d_i)$, $f_i \equiv 0$ and $f_i$ are normal. Hence, by 2.5, there are $h_i \in Rx + Ry$ such that $f_i = x^{d_i} - x^{i+1}h_i$ and $h_{n_0+1} = 0$.

**Lemma 4.1.** Let $J \subseteq Rx + Ry$ be an ideal of $R$. Let $n, m \in \mathbb{N}$ and $h \in R$ be such that $y^m - xh, x^n \in J$. Then $Rx + Ry/J$ is a nilpotent $F$-algebra.

**Proof.** Denote $\equiv J$ the congruence corresponding to $J$. We have $x^d \equiv J 0$ and $y^{mn} \equiv J x^n h^n \equiv J 0$. Hence $x + J$ and $y + J$ are nilpotent elements and $Rx + Ry/J$ is nilpotent, by 1.2(ii). □

**Proposition 4.2.** Let $M \subseteq I$ be such that every $0 \neq h \in M$ is normal and $C_A = \{\alpha \in \mathbb{N}_0^2 \mid (\exists h \in M) m(h) \leq_I \alpha\}$. Then $I = \sum_{h \in M} Rh$.

**Proof.** Put $\bar{T} = \sum_{h \in M} Rh$. By 1.6, there is $k \in \mathbb{N}$ such that $x^k, y^k \in I$. Hence $(k, 0), (0, k) \in C_A$. Thus there are $h_1, h_2 \in M$ such that $(i_1, j_1) = m(h_1) \leq_I (k, 0)$ and $(i_2, j_2) = m(h_2) \leq_I (0, k)$. Since $h_1$ and $h_2$ are normal, we have, by 2.5, that $h_1 = x_1$ and $h_2 = y_2 - xg$, where $g \in R$. By 4.1, $A = Rx + Ry/\bar{T}$ is a nilpotent algebra.

Clearly, $\bar{T} \subseteq I$. Consider the natural projection $\pi : \bar{A} \rightarrow A$, $\pi(f + \bar{T}) = f + I$. Since $m(h) \in C_A(x + I, y + I)$ for every $h \in M$, we get, by assumption, $C_A(x + I, y + I) \subseteq C_A(x + \bar{T}, y + \bar{T})$. Hence $B_2(x + \bar{T}, y + \bar{T}) \subseteq B_A(x + I, y + I)$ and $\dim \bar{A} \leq \dim A$, by 3.2(iii). Since $\pi$ is an epimorphism, it follows that $\pi$ is an isomorphism. Hence $0 = \ker(\pi) = I/\bar{T}$ and $I = \bar{T}$. □

**Lemma 4.3.** Let $0 \neq f \in R$, $f \equiv 0$ and $m(f) \geq (i, 0)$, where $i \in \mathbb{N}_0$. Then:

(i) $f \in RF_i + \cdots + RF_{n_0+1}$ for $0 \leq i \leq n_0 + 1$.

(ii) $f \in RF_{n_0+1} x^{i-(n_0+1)}$ for $n_0 + 1 \leq i$. 

Proof. (i) Let $0 \leq i < n_0 + 1$. Put $g = f/x^i$ and $g_k = f_k/x^i$ for $k = i, \ldots, n_0 + 1$. By 2.3(i), $m(g_k) = (k - i, d_k)$. Set $\overline{I} = Rg + Rg_i + \cdots + Rg_{n_0 + 1}$. Since $g_i = y^{d_i} - xh_i$ and $g_{n_0 + 1} = x^{n_0 + 1 - i}$, we get, by 4.1, that $\overline{A} = Rx + Ry/\overline{I}$ is a nilpotent $F$-algebra.

Now, put $C = (\alpha \in \mathbb{N}_0^g \mid (\exists k \in \{i, \ldots, n_0 + 1\}) m(g_k) \leq \alpha_I$. We show that $C = C_\overline{A}((x + \overline{I}, y + \overline{I})$.

Obviously $m(g_k) \in C_\overline{A}(x + \overline{I}, y + \overline{I})$ for every $k = i, \ldots, n_0 + 1$. Thus $C \subseteq C_\overline{A}(x + \overline{I}, y + \overline{I})$.

Let $\alpha = (i', j') \in C_\overline{A}(x + \overline{I}, y + \overline{I})$. Then there is $h \in R$ such that $m(h) = \alpha$ and $h \in \overline{I}$. Hence $x'h \in x'\overline{I} \subseteq I$ and $m(x'h) = \alpha + (i, 0)$. Thus $\alpha + (i, 0) = (i' + i, j') \in C_\overline{A}$.

Now, put $I = Rf_i + \cdots + Rf_{n_0 + 1}$, where $f_i = g_i = y^{d_i} - xh_i$, and $x'f_i = f_i$ for $i = 1, \ldots, n_0$. By 4.2, finally, $f = x^0g$ for some $g \in R$. Hence $f = g f_{n_0 + 1} x^{i - (n_0 + 1)}$.

The previous lemma says that the ideal $\{[x^0 + I \mid (i, 0) \leq \alpha]\}$ of the $F$-algebra $A$ is determined only by the polynomials $f_1, \ldots, f_{n_0 + 1}$.

Lemma 4.4.

(i) $I = Rf_0 + \cdots + Rf_{n_0 + 1}$.

(ii) $xf_i \in Rf_{i+1} + \cdots + Rf_{n_0 + 1}$ for $0 \leq i \leq n_0$.

(iii) $y^{d_i-1-d_i} f_i - x f_{i-1} \in Rf_{i+1} + \cdots + Rf_{n_0 + 1}$ for $1 \leq i \leq n_0$.

(iv) $y^{d_0} f_{n_0 + 1} - x f_n \in Rx f_{n_0 + 1}$.

Proof. Use 4.3 and the following:

(i) $m(f) \geq (0, 0)$ for every $f \in I$.

(ii) $m(xf_i) = (i + 1, d_i) \geq (i + 1, 0)$.

(iii), (iv) $y^{d_i-1-d_i} f_i - x f_{i-1} = y^{d_i-1-d_i} (x y^{d_i} - x^{i+1} h_i) - x (x^{i-1} y^{d_i} - x^i h_i) - x = x y^{d_i-1} - x^{i+1} y^{d_i-1-d_i} h_i - x^i y^{d_i} - x^{i+1} h_i - x^{i+1} h_{i-1} = x^{i+1} (h_{i-1} - y^{d_i-1-d_i} h_i)$. Hence $m(y^{d_i-1-d_i} f_i - x f_{i-1}) \geq (i + 1, 0)$, if $y^{d_i-1-d_i} f_i - x f_{i-1} \neq 0$.

Remark 4.5. Our choice of polynomials $f_i$ with combination of previous lemmas gave us the following conditions:

(1) $f_0 = y^{d_0} - x h_0$, where $h_0 \in R$, and $f_{n_0 + 1} = x^{n_0 + 1}$.

(2) $m(f_i) = (i, d_i)$ for $i = 1, \ldots, n_0$.

(3) $xf_i \in Rf_{i+1} + \cdots + Rf_{n_0 + 1}$ for $i = 0, \ldots, n_0$ (by 4.4(ii)).

In the next part we show that this can also be reversed (see 4.7).

Lemma 4.6. Let $m \in \mathbb{N}_0$, and $f_0, \ldots, f_m \in R$ be non-zero polynomials such that $m(f_k) = (k, c_k)$ for some $c_k \in \mathbb{N}_0$ and $xf_k \in Rf_{k+1} + \cdots + Rf_m$ for $k = 0, \ldots, m - 1$.

If $0 \neq f \in Rf_0 + \cdots + Rf_m$ is such that $m(f) \geq (i, 0)$, where $0 \leq i \leq m$, then $f \in Rf_1 + \cdots + Rf_m$.

Proof. Let $j \in \{0, \ldots, m\}$ be the greatest integer such that $f \in Rf_j + \cdots + Rf_m$. Suppose, for contrary, that $j < i$. Since $xf_j \in Rf_{j+1} + \cdots + Rf_m$, we have $f = g f_j + \sum_{k=j+1}^m h_k f_k$, where $h_k \in F[x, y]$, $g \in F[y]$. By choice of $j$ is $g \neq 0$. Let $m(g) = (0, 0)$. If $\sum_{k=j+1}^m h_k f_k \neq 0$, then $m(\sum_{k=j+1}^m h_k f_k) \geq (j + 1, 0)$, by 2.4. Since $m(g f_j) = (j, c_j + l) < (j + 1, 0)$, we get $(i, 0) \leq m(f) = m(g f_j) < (j + 1, 0)$, by 2.3. Hence $i < j + 1 \leq i$, a contradiction. □
Theorem 4.7. Let $\emptyset \neq B \subseteq \mathbb{N}_0^2$ be a finite lower set. Let $n_0 = n_0(B)$, $d_i = d_i(B)$ and $f_i \in R$, $i = 0, \ldots, n_0 + 1$ be such that:

1. $f_0 = y^{d_0} - x_{\emptyset}$, where $h_0 \in R$, and $f_{n_0 + 1} = x^{n_0 + 1}$.
2. $m(f_i) = (i, d_i)$ for $i = 1, \ldots, n_0$.
3. $x f_i \in Rf_{i+1} + \cdots + Rf_{n_0 + 1}$ for $i = 0, \ldots, n_0$.

Put $I = Rf_0 + \cdots + Rf_{n_0 + 1}$. Then:

(i) $A = Rx + Ry / I$ is a nilpotent $F$-algebra.
(ii) $B_A(x + I, y + I) = B$.
(iii) $n_0 = n_0(x + I, y + I)$.
(iv) $d_i(x + I, y + I) = d_i$ for $i \in \mathbb{N}_0$.

In particular, the set $\{ x^\alpha + I \mid (0, 0) \neq \alpha \in B \}$ is a basis of $A$.

Proof. (i) Since $x^{n_0 + 1}, y^{d_0} - x_{\emptyset} \in I$, we have that $A$ is nilpotent, by 4.1.
(ii) Put $C = \mathbb{N}_0^2 \setminus B$. By 2.8, $C = \{ \alpha \in \mathbb{N}_0^2 \mid \exists 0 \leq i \leq n_0 + 1 \} \{ i, d_i \} \leq n_0 \alpha$. Since $m(f_i) = (i, d_i)$ for $i = 0, \ldots, n_0 + 1$, we have $(i, d_i) \in A(x + I, y + I)$. Hence $C \subseteq C_A(x + I, y + I)$.

Let be now $\alpha = (i, j) \in C_A(x + I, y + I)$. Then there is $0 \neq f \in I$ such that $m(f) = \alpha$. If $n_0 + 1 \leq i$, then $(n_0 + 1, d_{n_0 + 1}) = (n_0 + 1, 0) \leq (i, j) \in \alpha$, by 2.8(i). Hence $\alpha \in C$.

Suppose $i \leq n_0$. By 4.6, $f = \sum_{k=1}^{n_0 + 1} h_k f_k$ for some $h_k \in R$. By 2.4, $(i, j) = m(f) = m(\sum_{k=1}^{n_0 + 1} h_k f_k) = \min_{\leq \alpha} (m(f_k)) = \leq k \leq n_0 + 1 = (i, d_i)$. Hence $d_k \leq j$ and $(i, d_i) \leq (i, j) \in \alpha$. Thus $\alpha \in C$.

We have shown that $C = C_A(x + I, y + I)$.

(iii), (iv) follow from (ii).

Remark 4.5 and Theorem 4.7 give us the complete description of 2-generated nilpotent $F$-algebras as the factors of polynomials. Especially 4.7 will be quite useful for constructing of certain examples of these algebras (see 5.10 and 7.5). For all that we still do not know how to find all the sets of the polynomials appropriate for a given finite lower set $B$ in 5.10. The final answer gives us the following Proposition 4.8. The idea is simply to construct the polynomials inductively starting from the last one of them (i.e. with the polynomial with highest index).

Proposition 4.8. Let $\emptyset \neq B \subseteq \mathbb{N}_0^2$ be a finite lower set. Let $n_0 = n_0(B)$ and $d_i = d_i(B)$.

Let $f_i \in R$, $i = 0, \ldots, n_0 + 1$. The following are equivalent:

(i) $f_i - x^m(f_i) \in Rx^{i+1}$ for every $i = 0, \ldots, n_0 + 1$ and $f_i$ fulfill the conditions (1)–(3) in 4.7.

(ii) $f_{n_0 + 1} = x^{n_0 + 1}$ and there are $h_{i,j} \in R$, $0 \leq i < j \leq n_0 + 1$ such that

\[ f_i = (y^{d_i - d_{i+1}} + x h_{i,i+1}) f_{i+1} / x + \sum_{j=i+2}^{n_0 + 1} h_{i,j} f_j / x \]

for every $i = 0, \ldots, n_0$.

Proof. ($\Rightarrow$) Let $0 \leq i \leq n_0$. By (3), there are $h_{i,j} \in R$, $i < j \leq n_0 + 1$ and $g_i \in R$ such that $x f_i = g_i f_{i+1} + \sum_{j=i+2}^{n_0 + 1} h_{i,j} f_j$. By (1), (2) and 2.4, we have $m(\sum_{j=i+2}^{n_0 + 1} h_{i,j} f_j) \geq i + 2, d_{i+2})$ provided $\sum_{j=i+2}^{n_0 + 1} h_{i,j} f_j \neq 0$. Since $m(x f_i) = (i + 1, d_i)$, must be $g_i f_{i+1} \neq 0$. By 2.3(ii), we get $(i + 1, d_i) = m(x f_i) = m(g_i f_{i+1}) = m(g_i) + (i + 1, d_{i+1})$. Thus $m(g_i) = (0, d_i - d_{i+1})$. By assumption, $f_j = x^{d_j} + x^{j+1} h_j$, $0 \leq j \leq n_0 + 1$. Now, by comparing the monomials in the equality $x f_i = g_i f_{i+1} + \sum_{j=i+2}^{n_0 + 1} h_{i,j} f_j$ we get, that $g_i = y^{d_i - d_{i+1}} + x h_{i,i+1}$ for some $h_{i,i+1} \in R$. 


We need to show that \( m(f_i) = (i, d_i) \) and \( f_i - x^{m(f_i)} \in R \) for every \( i = 0, \ldots, n_0 \). Since \( f_{n_0+1} = x^{n_0+1} \), we can assume that this is true for every \( j = i + 1, \ldots, n_0 + 1 \). Now, \( f_i = (y^{d_i-d_{i+1}} + x^{h_{i,i+1}}) f_{i+1} / x + \sum_{j=i+2}^{n_0+1} h_{i,j} f_j / x \) and \( m(f_{j+1}) < (i+2, 0) \leq m(f_j) \) for \( j > i + 1 \). By 2.3(ii) and 2.4, we have \( m(f_j) = (0, d_i - d_{i+1}) + m(f_{i+1}/x) = (i, d_i) \). By assumption, \( f_j = x^j y^{d_j} + x^{j+1} h_j \), where \( h_j \in R \), for every \( j > i \). Hence we get \( f_i = x^i y^{d_i} + x^{i+1} h_i \) for some \( h_i \in R \). \( \square \)

5. Estimation of the dimension

As in the previous section, let \( I \subseteq Rx + Ry \) be an ideal in \( R \) such that \( A = Rx + Ry/I \) is a non-zero nilpotent \( F \)-algebra. The notation remains the same.

This part and the estimation will be rather technical, but the main idea can be quite good viewed as shifting of polynomials.

To get the right estimation of the dimension of the subalgebra \( A^{(p)} \) we will need to consider polynomials that are “almost” contained in the ideal \( I \). The measure for this will be the greatest element in the canonical basis \( \{ x^\alpha + I \mid (0, 0) \neq \alpha \in B_A \} \), denoted by \( W_A + I \) (see 5.1).

**Definition 5.1.** Let \( \alpha_0 \) be the greatest element of \( B_A \) with respect to \( \leq \). Denote

\[
W_A = x^{\alpha_0}.
\]

**Lemma 5.2.**

(i) \( W_A = x^{n_0} y^{d_{n_0} - 1} \in Rx + Ry \).

(ii) \( xW_A \equiv yW_A \equiv 0 \). Hence \( [W_A + I] \) is an ideal in \( A \).

(iii) If \( 1 \leq n_0 \) then \( y^{d_{n_0} - 1}(W_A/x) + I \in [W_A + I] \).

**Proof.** (i) Since \( \dim A \neq 0 \), we have, by 3.2, that \( B_A \neq \{(0, 0)\} \). Hence \( m(W_A) \neq (0, 0) \). By 3.6(iii), we have \( m(W_A) = (n_0, d_{n_0} - 1) \).

(ii) Since \( 0 \equiv f_{n_0} = x^{n_0}(y^{d_{n_0} - xh_{n_0}}) \) and \( 0 \equiv f_{n_0+1} = x^{n_0+1} \), it follows, by (i), that \( yW_A = y^{d_{n_0} x^{n_0}} \equiv x^{n_0+1} h_{n_0} \equiv 0 \) and \( xW_A = x^{n_0+1} y^{d_{n_0} - 1} \equiv 0 \).

(iii) We have \( 0 \equiv f_{n_0-1} = x^{n_0-1}(y^{d_{n_0-1} - xh_{n_0-1}}) \) and \( W_A = x^{n_0} y^{d_{n_0} - 1} \), by (i). Hence \( y^{d_{n_0} - 1}(W_A/x) = y^{d_{n_0} - 1}(x^{n_0-1} y^{d_{n_0-1} - (x^{n_0} h_{n_0-1})} \equiv W_A h_{n_0-1}. \) By (ii), \( y^{d_{n_0} - 1}(W_A/x) + I \in [W_A + I] \). \( \square \)

The next lemma is crucial for the estimation of \( \dim A^{(p)} \). It says that we can divide in some sense in a nilpotent algebra (under some special condition of course!). Namely, we can divide a polynomial \( xf \in I \) by \( x \) if we, in the same moment, multiply it by a suitable \( y^{m} \) such that the result \( y^{m} f \) will be contained again in \( I \).

**Lemma 5.3.** Let \( 1 \leq i \leq n_0 + 1 \) and \( 0 \neq f \in I \) be such that \( m(f) \geq (i, 0) \). Then \( y^{d_{i-1} - 1}(f/x) + I \in [W_A + I] \).

**Proof.** Let \( 1 \leq i \leq n_0 + 1 \) and \( f \in I \) be such that \( m(f) \geq (i, 0) \). First, let \( i = n_0 + 1 \). Then \( f = x^{n_0+1} g \) for some \( g \in R \). Hence \( y^{d_{n_0} - 1}(f/x) = y^{d_{n_0} - 1} x^{n_0} f = f W_A \), by 5.2(i), and \( y^{d_{n_0} - 1}(f/x) + I \in [W_A + I] \) by 5.2(ii).

Now, let \( 1 \leq i < n_0 + 1 \) and suppose that for every \( k \in \{i + 1, \ldots, n_0 + 1 \} \) and every \( g \in I \) such that \( m(g) \geq (k, 0) \) holds \( y^{d_{k-1} - 1}(g/x) + I \in [W_A + I] \).

By 4.3, we have \( f = \sum_{k=i+1}^{n_0+1} g_k f_k \), where \( g_k \in R \). By 4.4(iii), \( y^{d_{i-1} - d_i}(f_i/x) = \sum_{k=i+1}^{n_0+1} h_k (f_k/x) \), for some \( h_k \in R \). Hence
Finally, we get
\[ y_{d_i-1}(f/x) = y_{d_i-1} \sum_{k=i}^{n_0+1} g_k(f_k/x) = g_1 y_{d_i-1} y_{d_i-1}(f_i/x) \]
\[ + y_{d_i-1} \sum_{k=i+1}^{n_0+1} g_k(f_k/x) \equiv g_1 y_{d_i-1} \sum_{k=i+1}^{n_0+1} h_k(f_k/x) + y_{d_i-1} \sum_{k=i+1}^{n_0+1} g_k(f_k/x) \]
\[ = \sum_{k=i+1}^{n_0+1} (g_k h_k + y_{d_i-1} y_{d_i-1}(f_k/x)). \]
\[ (* \quad \text{(*)}) \]

Now, \( y_{d_k-1}(f_k/x) + l \in [A + l] \), by assumption. Since \( d_i \geq d_k - 1 \) for every \( k \in \{i + 1, \ldots, n_0 + 1\} \), we get \( y_{d_i-1}(f_k/x) + l \in [A + l] \), by 5.2(ii). By (**) and 5.2(ii), we have \( y_{d_i-1}(f_i/x) + l \in [A + l]. \) □

In the following Lemmas 5.4 and 5.7 we can see how the dimension of the original algebra \( A \) (namely, the sums of the numbers \( d_i \)) appears in the estimation of the dimension of the subalgebra \( A^p \) (see 5.7 and 3.10).

**Lemma 5.4.** Let \( 0 \leq j < i \leq n_0 + 1 \) and \( 0 \neq f \in l \) be such that \( m(f) \geq (i, 0). \)

(i) Put \( l = (\sum_{k=j}^{i-1} d_k) - 1. \) Then \( y^f(f/x^{i-j}) + l \in [A + l]. \)

(ii) Put \( l' = \sum_{k=j}^{i-1} d_k - 1. \) Let \( d_i - 1 \geq d_{n_0} + 1, \) then \( y^f(f/x^{i-j}) + l \in [A + l]. \)

**Proof.** (i) We will proceed by induction on \( m = i - j. \) By 5.3, the statement is true for \( m = 1. \)

Now, let \( 0 < j < i \leq n_0 + 1 \) and \( f \in l \) be such that \( m(f) \geq (i, 0). \) Suppose that \( y^f(f/x^{i-j}) + l \in [A + l], \) where \( l = (\sum_{k=j}^{i-1} d_k) - 1. \) Put \( g = y^f(f/x^{i-j}). \) We have \( yg \equiv 0, \) by 5.2(ii), and \( m(yg) = m(y^{i+1}(f/x^{i-j})) = (0, l + 1) + m(f) \geq (i, 0) \) (since \( m(f) \geq (i, 0) \)).

Hence, 5.3, we get \( y^{g_i+1}(f/x^{i-j}) + l = y^{g_i+1}(f/x^{i-j}) + l \in [A + l]. \)

(ii) Again, we proceed by induction on \( m = i - j. \) By 5.3, the statement is true for \( m = 1. \)

Let \( 0 < j < i \leq n_0 + 1, \) \( d_i - 1 > d_{n_0} + 1, \) and \( f \in l \) be such that \( m(f) \geq (i, 0). \) Suppose that \( y^f(f/x^{i-j}) + l \in [A + l], \) where \( l' = \sum_{k=j}^{i-1} d_k - 1. \) Put \( g' = y^f(f/x^{i-j}). \) Then we have \( m(g') \geq (j, 0) \) and \( g' - \lambda w_A \equiv 0 \) for some \( \lambda \in F. \)

We show that \( y^{g_i+1}(g'/(x/y)) \equiv \lambda y^{g_i+1}(w_A/x) + \mu w_A \) for some \( \mu \in F. \) For \( g' = \lambda w_A \) it is clear. Suppose that \( g' - \lambda w_A \neq 0, \) then, by 2.3(ii), \( m(g' - \lambda w_A) \geq \min_{\lambda \in F} \{m(g'), m(-\lambda w_A)\} \geq \min_{\lambda \in F}((j, 0), (n_0, d_{n_0} - 1)) = (j, 0). \) Hence, by 5.3, we get \( y^{g_i+1}(g' - \lambda w_A)/x \equiv \mu w_A \) for some \( \mu \in F. \)

Now, since \( d_i - 1 - 1 \geq d_{i-1} - 1 \geq d_{n_0} - 1, \) we have \( y^{d_i-1}(w_A/x) \equiv \lambda' w_A, \) where \( \lambda' \in F, \) by 5.2(ii).

(iii) Finally, we get \( y^{d_i-1}(f/x^{i-j+1}) = y^{d_i-1}(g'/(x/y)) \equiv \lambda y^{d_i-1}(w_A/x) + \mu w_A \equiv \lambda' w_A + \mu w_A = (\lambda' + \mu)w_A. \) □

For shorter expressions let us define another two auxiliary numbers \( m_i \) and \( l_i. \)

**Definition 5.5.** For \( 0 \leq i \leq n_0 \) denote

\[ m_i \in \mathbb{N}_0 \]

the least integer such that \( pi \leq m_i \leq pi + p - 1 \) and \( d_{pi} \geq \cdots \geq d_{m_i} = d_{m_i+1} = \cdots = d_{pi+p-1}. \) Put

\[ l_i = \left( \sum_{k=pi}^{m_i-1} (dk - 1) \right) - (p-1)d_{m_i}. \]
Lemma 5.6. Let $0 \leq i < \bar{n}_0$, $p_i < m_i$ and $l \in \mathbb{N}_0$. The following are equivalent:

(i) $l + (p_i + p - 1 - j)d_{m_i} \geq \sum_{k=j}^{m_i-1} (d_k - 1)$ for every $j \in \{p_i, \ldots, m_i - 1\}$ ($>,$ resp.).

(ii) $l \geq l_i$ ($l > l_i$, resp.).

Proof. It’s easy to see that (i) is equivalent to

$$l \geq \left( \sum_{k=j}^{m_i-1} (d_k - d_{m_i} - 1) \right) - (p_i + p - 1 - m_i)d_{m_i} \quad (>,$$ resp.) \hfill (*)

for $j \in \{p_i, \ldots, m_i - 1\}$. We have $l_i = \left( \sum_{k=p_i}^{m_i-1} (d_k - 1) \right) - (p - 1)d_{m_i} = \left( \sum_{k=p_i}^{m_i-1} (d_k - d_{m_i} - 1) \right) - (p_i + p - 1 - m_i)d_{m_i}$. The right-hand side of (*) is a decreasing function in $j$, since $d_k \geq d_{m_i} + 1$ for $k \in \{p_i, \ldots, m_i - 1\}$. Hence (i) holds iff $l \geq l_i$ ($l > l_i$, resp.). \hfill \square

To estimate the number $\tilde{d}_i$ we need, by Definitions 3.5 and 3.1, a suitable polynomial $f \in R$ such that $f(x^p, y^p) \in I$ and $\text{m}(f) = (i, j)$ for some $j$. But how to find such a polynomial? The idea is to take some $f_k$ and change it a little bit (see 5.7). The way how to find such a polynomial is the crucial point of the whole construction. (But, surprisingly, the only thing we need for that purpose will be a suitable usage of the binomial formula.)

Proposition 5.7.

(i) If $0 \leq i < \bar{n}_0$ and $l_i \geq 0$, then $y^i x^{p_i} (f_{m_i}/x^{m_i})^p + I \subseteq [w_A + I]$.

(ii) If $0 \leq i < \bar{n}_0$ and $l_i < 0$, then $x^{p_i} (f_{m_i}/x^{m_i})^p \equiv 0$.

(iii) If $i = \bar{n}_0$, then $y^{D_i-1} x^{p_i} + I \subseteq [w_A + I]$.

Proof. (i), (ii) Let $0 \leq i < \bar{n}_0$ and $l \in \mathbb{N}_0$. Since $0 \equiv f_{m_i} = x^{m_i} (y^{d_{m_i}} - xh_{m_i})$, we get, using the binomial formula, that

$$y^{i} x^{p_i} (f_{m_i}/x^{m_i})^p = y^{i} x^{p_i} (f_{m_i}/x^{m_i})(f_{m_i}/x^{m_i})^{p-1} = y^{i} (f_{m_i}/x^{m_i-p_i})(y^{d_{m_i}} - xh_{m_i})^{p-1} = y^{i} (f_{m_i}/x^{m_i-p_i}) \sum_{j=0}^{p-1} \binom{p-1}{j} (-xh_{m_i})^j y^{d_{m_i}(p-1-j)} = \sum_{j=0}^{p-1} \binom{p-1}{j} (-h_{m_i})^j y^{l+d_{m_i}(p-1-j)} (f_{m_i}/x^{m_i-p_i+j}) = \sum_{j=p_i}^{p_i+p-1} \binom{p-1}{j-p_i} (-h_{m_i})^j y^{l+d_{m_i}(p-1-j)} (f_{m_i}/x^{m_i-j}) \equiv \sum_{j=p_i}^{m_i-1} \binom{p-1}{j-p_i} (-h_{m_i})^j y^{l+d_{m_i}(p-1-j)} (f_{m_i}/x^{m_i-j}). \quad (\ast)$$

(In the last step we used that $f_{m_i}/x^{m_i-j} \equiv 0$ for $j > m_i$.)

Hence, if $m_i = p_i$, our claim is true.
Let be $m_i > pi$ and $j \in \{pi, \ldots, m_i - 1\}$. By 3.8(iii), $m_i < pi < p[n_0/p] \leq n_0$. By choice of $m_i$, we have $d_{m_i-1} \geq d_{m_i} + 1 \geq m_{n_0-1} + 1$.

First, suppose that $l_i \geq 0$. By 5.6, $l_i + d_{m_i}(pi + p - 1 - j) \geq \sum_{k=j}^{m_i-1}(d_k - 1)$. Hence, by 5.4(ii) and 5.2(ii), $y^{l_i+d_{m_i}(pi+p-1-j)}(f_{m_i}/x^{m_i-1}) \in [W_A + I]$. Thus $y^{l_i}x^{p\hat{I}}(f_{m_i}/x^{m_i})^p \in [W_A + I]$, by (*)& 5.2(ii).

Now, let $l_i < 0 = l$. By 5.6, $l_i + d_{m_i}(pi + p - 1 - j) > \sum_{k=j}^{m_i-1}(d_k - 1)$. Hence, by 5.4(ii) and 5.2(ii), $y^{l_i+d_{m_i}(pi+p-1-j)}(f_{m_i}/x^{m_i-1}) \not\equiv 0$. Thus $x^{p\hat{I}}(f_{m_i}/x^{m_i})^p \not\equiv 0$, by (*)& 5.2(ii).

(iii) Let $i = \hat{\alpha}_0$. By 3.8(iii), we have $\hat{\alpha}_0 = [n_0/p]$, hence $0 \leq pi = p[n_0/p] \leq n_0 < p([n_0/p] + 1) = p(i + 1)$. Thus $l = (\sum_{k=p_i}^{p_0}d_k) - 1 = D_i - 1 \geq 0$, by 3.6(i). Now, put $f = x^{n_0} + j$, $j = pi$ and $i^\prime = n_0 + 1$. Then $0 \neq f \in I$, $m(f) \geq (i^\prime, 0)$, $0 \leq j^\prime \leq n_0 + 1$ and $l = (\sum_{k=j^\prime}^{i^\prime-1}d_k) - 1$. Using 5.4(i) we get $y^{D_i-1}x^{p\hat{I}} + I = y^l(f/x^{x^j-j'}) \not\equiv 0$. □

In the proofs of Lemma 5.8 and Proposition 5.9 we finally see, why we needed to take in consideration the element $w_A$.

**Lemma 5.8.** Let be $0 \leq i \leq \hat{\alpha}_0$. One of the following four cases takes place:

(i) $p\hat{d}_i \leq D_i$.

(ii) There are $k \in \mathbb{N}_0$, $f \in Rx + R$ and $0 \neq \lambda \in F$ such that $f(x^p, y^p) \equiv \lambda w_A$, $m(f) = (i, k)$, $D_i = pk + p - 1$ and $p\hat{d}_i \leq D_i + 1$.

(iii) $i = \hat{\alpha}_0$ and there are $k \in \mathbb{N}_0$ and $0 \neq \lambda \in F$ such that $(i, k) \neq (0, 0)$, $x^{pi}y^{pk} \equiv \lambda w_A$, $D_i = pk + 1$ and $p\hat{d}_i \leq D_i + p - 1$.

(iv) $i = \hat{\alpha}_0$ and $p\hat{d}_i \leq D_i + p - 2$.

**Proof.** We divide our proof into three cases (a), (b) and (c).

(a) Let $i < \hat{\alpha}_0$ and $l_i \geq 0$. By 3.8(iii), $pi + p - 1 < p\hat{\alpha}_0 = p[n_0/p] \leq n_0$. Hence $d_k > 1$ for $k \in \{pi, \ldots, pi + p - 1\}$. Since $l_i \geq 0$, we get $pi + 1 \leq m_i$. We have

\[
pd_{m_i} + l_i = pd_{m_i} + \left(\sum_{k=pi}^{m_i-1}(d_k - 1)\right) - (p - 1)d_{m_i} = d_{m_i} + \sum_{k=pi}^{m_i-1}(d_k - 1) \leq d_{m_i} + \sum_{k=pi}^{m_i-1}(d_k - 1) + \sum_{k=m_i+1}^{pi+p-1}(d_k - 1) = D_i - (p - 1). \tag{*}
\]

By 5.7(i) there is $\mu \in F$ such that $y^{l_i}x^{pi}(f_{m_i}/x^{m_i})^p \equiv \mu w_A$. Clearly, there is $0 \neq g \in R$ such that $g(x^p, y^p) = x^{pi}(f_{m_i}/x^{m_i})^p$. Thus $y^{l_i} g(x^p, y^p) \equiv \mu w_A$ and $m(g) = (i, d_{m_i})$, since $pm(g) = m(x^{pi}(f_{m_i}/x^{m_i})^p) = (pi, 0) + p(0, d_{m_i})$, by 2.3.

Suppose now, that:

(a1) $D_i = pk + r$ for some $k \in \mathbb{N}_0$, $0 \leq r \leq p - 2$. By 5.2(ii), $y^{l_i+1}g(x^p, y^p) \equiv 0$. By 3.10 and (*), we get

\[
\hat{d}_i \leq \left\lfloor \frac{(pd_{m_i} + l_i + 1)/p}{(D_i - (p - 1) + 1)/p} \right\rfloor = \left\lfloor \frac{(pk + r + 2 - p)/p}{k + (r + 2 - p)/p} \right\rfloor \leq k + r/p = D_i/p.
\]

(We obtained case (i).)
(a2) $D_i = pk + p - 1$ for some $k \in \mathbb{N}_0$. By (\star), $l = D_i - (p - 1) - (pd_{m_i} + l_i) \geq 0$. Then $y^{l+1}g(x^p, y^p) \equiv \lambda w_A$ for some $\lambda \in F$. If $\lambda = 0$ then, by 3.10, we have

$$\bar{d}_i \leq \left\lfloor (pd_{m_i} + l_i + l)/p \right\rfloor = \left\lfloor (D_i - p + 1)/p \right\rfloor = k \leq D_i/p.$$  
(We obtained case (i).)

Let $\lambda \neq 0$. Since $l \geq 0$ and $l_i \geq 0$, we have $k - d_{m_i} \geq 0$. Put $f = y^{k-d_{m_i}}g$. We have $f(x^p, y^p) = y^{p(k-d_i)}g(x^p, y^p) = y^{l+1}g(x^p, y^p) \equiv \lambda w_A$ and $m(f) = m(y^{k-d_{m_i}}g) = (0, k - d_{m_i}) + (i, d_{m_i}) = (i, k)$. By 5.2(ii), $y^{l+1+1}g(x^p, y^p) \equiv 0$. By 3.10, we get

$$\bar{d}_i \leq \left\lfloor (pd_{m_i} + l_i + l + 1)/p \right\rfloor = \left\lfloor (pk + 1)/p \right\rfloor = k + 1 = (D_i + 1)/p.$$  
(We obtained case (ii).)

(b) Let $i < \overline{m_0}$ and $l_i < 0$. By 5.7(ii), we have $x^{pi}(f_{m_i}/x^{m_i})^p \equiv 0$. Clearly, there is $g(x, y) \in R$ such that $g(x^p, y^p) = x^{pi}(f_{m_i}/x^{m_i})^p$. Thus $m(g) = (i, d_{m_i})$. By choice of $m_i$ we have that $d_{m_i} \leq d_k$ for every $k \in \{pi, \ldots, pi + p - 1\}$. By 3.10, we get

$$pd_i \leq pd_{m_i} \leq \sum_{k=pi}^{pi+p-1} d_k = D_i.$$  
(We obtained case (ii).)

(c) Let $i = \overline{m_0}$. By 5.7(iii), $D_i \geq 1$ and there is $\lambda \in F$ such that $y^{D_i-1}x^{pi} \equiv \lambda w_A$. By 5.2(ii), is $y^{D_i}x^{pi} \equiv 0$.

Suppose now, that:

(c1) $D_i = pk$ for some $k \in \mathbb{N}$. By 3.10, we have

$$\bar{d}_i \leq \left\lfloor D_i/p \right\rfloor = D_i/p.$$  
(We obtained case (i).)

(c2) $D_i = pk + 1$ for some $k \in \mathbb{N}_0$. If $\lambda = 0$ then, by 3.10, is

$$\bar{d}_i \leq \left\lfloor (D_i - 1)/p \right\rfloor = k \leq D_i/p.$$  
(We obtained case (i).)

Let be $\lambda \neq 0$. By 3.10, we have that

$$\bar{d}_i \leq \left\lfloor D_i/p \right\rfloor = \left\lfloor (pk + 1)/p \right\rfloor = k + 1 = (D_i + p - 1)/p.$$  
(We obtained case (ii).)

(c3) $D_i = pk + r$ for some $k \in \mathbb{N}_0$, $2 \leq r < p$. By 3.10, we have that

$$\bar{d}_i \leq \left\lfloor D_i/p \right\rfloor = \left\lfloor (pk + r)/p \right\rfloor = k + 1 \leq (D_i + p - 2)/p.$$  
(We obtained case (iii).)

(We obtained case (iv).) □

The next proposition says that the inequality “$pd_i \leq D_i$” holds for almost every $i$. 
Proposition 5.9. One of the following cases takes place:

(i) \( p \overline{d}_{i0} \leq D_{i0} + p - 2 \) and \( p \overline{d}_i \leq D_i + 1 \) for every \( 0 \leq i < \overline{n}_0 \). Moreover, \( p \overline{d}_{i0} = D_{i0} + 1 \) for at most one \( 0 \leq i_0 < \overline{n}_0 \).

(ii) \( p \overline{d}_{i0} \leq D_{i0} + p - 1 \) and \( p \overline{d}_i \leq D_i + 1 \) for every \( 0 \leq i < \overline{n}_0 \).

Proof. By 5.8, we have \( p \overline{d}_{i0} \leq D_{i0} + p - 1 \) and \( p \overline{d}_i \leq D_i + 1 \) for every \( 0 \leq i < \overline{n}_0 \).

For contrary, assume that our claim is not true. Then \( M = \{ i \mid 0 \leq i < \overline{n}_0 \} \) and \( p \overline{d}_{i0} = D_{i0} + 1 \) and either \( |M| \geq 2 \) or \( |M| = 1 \) and \( p \overline{d}_{i0} = D_{i0} + p - 1 \).

Let \( i_1 \) be the least element of \( M \). By 5.8(ii), there is \( k \in \mathbb{N}_0 \), \( f \in R_x + R_y \) and \( 0 \neq \lambda \in F \) such that \( f(x^p, y^p) \equiv \lambda w_A \), \( \text{m}(f) = (i_1, k) \), \( D_{i1} = pk + p - 1 \) and \( p \overline{d}_{i1} = D_{i1} + 1 \).

Further, let \( i_2 \) be the greatest element of \( M \), if \( |M| \geq 2 \), and \( i_2 = \overline{n}_0 \), if \( |M| = 1 \). By 5.8(ii), (iii) there are \( l \in \mathbb{N}_0 \), \( g \in R_x + R_y \) and \( 0 \neq \mu \in F \) such that \( g(x^p, y^p) \equiv \mu w_A \) and \( \text{m}(g) = (i_2, l) \).

Now, put \( h = f - \frac{k}{n} g \). Since \( i_1 < i_2 \), we have \( \text{m}(h) = \text{m}(f) = (i_1, k) \), by 2.3(ii). Clearly, \( h(x^p, y^p) \equiv 0 \) and, by 3.10, we get \( \overline{d}_{i1} \leq \lfloor pk/p \rfloor = k = (D_{i1} - p + 1)/p \leq D_{i1}/p \), a contradiction to \( i_1 \in M \). \( \square \)

To see that the inequalities in 5.9 are not overestimated, we construct the following two examples (see 5.10). The first one is for the case (i) and the second is for the case (ii) in 5.9.

Example 5.10. (i) Let char \( F = 2 \). Put \( f_0 = y^6 + yx^2 + x^2 + x^3y \), \( f_1 = x^2f_0 \), \( f_2 = x^2y^3 + x^3y + x + x^3y^4 \), \( f_3 = x^2y^2 + x^4 \), \( f_4 = x^2f_3 \) and \( f_5 = x^3f_3 \). Let \( I \) be an ideal generated by \( f_i \), \( i = 0, \ldots, 5 \). It is easy to verify conditions (1), (2), (3) in 4.7. Thus \( A = R_x + R_y/I \) is a nilpotent \( F \)-algebra, \( d_0 = d_1 = 6 \), \( d_2 = 3 \), \( d_3 = d_4 = 2 \) and \( \dim A = 18 \) (see Fig. 3). By 3.11, \( \overline{d}_0 \leq d_0 = 6 \), \( \overline{d}_1 \leq d_2 = 3 \), \( \overline{d}_2 = 1 \) and \( \overline{d}_3 = 0 \).

First, we show that \( \overline{d}_1 = 3 \). Suppose, on contrary, that \( \overline{d}_1 \leq 2 \). By 3.9, (1.2) \( C_{A(2)} \) and there is a normal polynomial \( f \in R \) such that \( f(x^2, y^2) \not\in I \) and \( \text{m}(f) = (1, 2) \). This means that \( x^2y^4 \equiv \lambda x^4 \) for some \( \lambda \in F \), since \( x^4y^2 \equiv x^5 \equiv 0 \), by 5.2. On the other hand, from \( f_2 \equiv 0 \equiv f_3 \) follows that \( \lambda x^4 \equiv x^2y^3 \equiv x^2y^2 + x^4y + x^4y^2 \equiv x^4 + x^4y \). But this is a contradiction, since \( x^4 + 1 \) and \( x^4y + 1 \) are elements of the basis of \( A \).

Now, we show that \( \overline{d}_0 = 6 \). Using \( f_0 \equiv 0 \), \( f_1 \equiv 0 \) and \( f_3 \equiv 0 \) we get \( y^{10} \equiv xy^6 + x^2y^4 + x^3y^5 \equiv (x^2y^2 + x^2y^4 + x^4y^2 + x^4y^3 + x^4 + x^4y^5) \). We already know that \( \{ \alpha \in B_{A(2)} \mid (1, 0) \leq \alpha \} = \{(1, 0), (1, 1), (1, 2), (2, 0) \} \). Suppose now, for contrary, that \( \overline{d}_0 \leq 5 \). Then \( (0, 5) \in C_{A(2)} \). Using 3.3 for \( A(2) \) and previous equations, we have that \( y^{10} + 1 \in [x^2, x^2y^2, x^2y^4, x^4] + I = [x^2, x^2y^2, x^4y, x^4] + I \).

Hence \( x^3 + I = y^{10} + x^2y^2 + x^2 + I \in [x^2, x^2y^2, x^4y, x^4] + I \), a contradiction.

We conclude with \( 2\overline{d}_0 = 12 = D_0 \), \( 2\overline{d}_1 = 6 = D_1 + 1 \) and \( 2\overline{d}_2 = 2 = D_2 \).

(ii) Let \( B = \{ \alpha \in B_{A(2)} \mid (p, p) \} \). Let \( f_0 = y^{p+1} - xh \), where \( \text{m}(h) = (0, 0) \), \( f_1 = x^hf_0 \) for \( 1 \leq i \leq p \) and \( f_{p+1} \equiv x^{p+1} \). Again, it is easy to verify, that the conditions (1), (2), (3) in 4.7 are fulfilled. Hence \( A = R_x + R_y/I \) is a nilpotent \( F \)-algebra, where \( I = Rf_0 + Rf_{p+1} \) and \( \dim A = |B| - 1 = p(p + 2) \).

By 3.8, \( \overline{n}_0 = |n_0/p| = 1 \). By 3.11, we get \( 2 = [(p + 1)/p] = [d_{p}/p] \leq \overline{d}_1 \). Since \( x^py^{p+1} \equiv 0 \), by 5.2, we have \( \overline{d}_1 = 2 \) and \( p\overline{d}_1 = 2p = D_1 + 1 + 1 \). Hence the case (ii) in 5.9 takes place.

Now, we show that \( p\overline{d}_0 = D_0 \). Since \( y^{p+1} - xh \equiv 0 \) and \( \text{m}(h) = (0, 0) \), we have, by 1.3, that \( A \) is generated by \( y + J \). Using the proof of 7.4, we get that \( \dim A^{(p)} = |\dim A/p| = p + 2 \). Now, \( D_0 = \dim A + 1 - D_1 = p(p + 1) \) and \( \overline{d}_0 = \dim A^{(p)} + 1 - \overline{d}_1 = p + 1 \). Hence \( p\overline{d}_0 = p(p + 1) = D_0 \).

6. Eggert's conjecture for 2-generated algebras

In this section we finally prove Eggert's conjecture for \( F \)-algebra \( A \) with at most \( 2 \)-generated subalgebra \( A^{(p)} \).

Theorem 6.1. Let \( A \) be a nilpotent \( 2 \)-generated \( F \)-algebra, char \( F = p > 0 \). Then \( p \cdot \dim A^{(p)} \leq \dim A \).

Proof. For \( \dim A = 0 \) is it clear. Let \( \dim A > 0 \).
We can assume, by 1.6, without loss of generality that $A = Rx + Ry/I$, where $I \subseteq Rx + Ry$ is an ideal such that $x^k, y^k \in I$ for some $k \in \mathbb{N}$.

Let $\bar{m}_0 = \bar{m}_0 A(x + I, y + I)$. By 3.9, we have $\sum_{i=0}^{\bar{m}} D_i = 1 + \dim A$ and $\sum_{i=0}^{\bar{m}} \bar{d}_i = 1 + \dim A^{(p)}$.

If $\bar{m}_0 = 0$ then, by 5.8, we have $\bar{p} \bar{d}_0 \leq D_0 + p - 1$. Hence $p \dim A^{(p)} = \bar{p} \bar{d}_0 - p \leq D_0 - 1 = \dim A$.

Let be now $i_1 = \bar{m}_0 > 1$. By 5.9, we get the following two cases:

(i) $\bar{p} \bar{d}_i \leq D_i + p - 2$ and there is $0 \leq i_0 < i_1$ such that $\bar{p} \bar{d}_i < D_i + 1$ and $\bar{d}_i \leq D_i$ for every $0 \leq i < i_1$, $i \neq i_0$. Then $p \dim A^{(p)} = \bar{p} \bar{d}_i + \bar{p} \bar{d}_i + p \left( \sum_{i \neq i_0, i_1} \bar{d}_i \right) - p \leq (D_i + 1) + (D_i + p - 2) + (\sum_{i \neq i_0, i_1} D_i) - p = (\sum_i D_i) - 1 = \dim A$.

(ii) $\bar{p} \bar{d}_i \leq D_i + p - 1$ and $\bar{d}_i \leq D_i$ for every $0 \leq i < i_1$. Then $p \dim A^{(p)} = \bar{p} \bar{d}_i + (p \sum_{i \neq i_1} \bar{d}_i) - p \leq (D_i + p - 1) + (\sum_{i \neq i_1} D_i) - p = (\sum_i D_i) - 1 = \dim A$. □

**Theorem 6.2.** Let $A$ be a nilpotent $F$-algebra, char $F = p > 0$, such that $A^{(p)}$ is 2-generated. Then $p \cdot \dim A^{(p)} \leq \dim A$.

**Proof.** Use 6.1 and 1.5. □

7. Stronger version of Eggert’s conjecture

In the last section we reformulate the Eggert’s conjecture and slightly generalize it.

**Definition 7.1.** Let $A$ be an $F$-algebra, char $F = p > 0$. Put

$$A_p = \{ a \in A \mid a^p = 0 \}.$$

**Proposition 7.2.** Let $A$ be an $F$-algebra, char $F = p > 0$. Then:

(i) $\varphi: A \to A^{(p)}$, $\varphi(a) = a^p$, is a ring homomorphism and $\ker(\varphi) = A_p$. Hence $A_p$ is an ideal in $A$.

(ii) $\dim A^{(p)} \leq \dim A / A_p$. Moreover, if $F$ is a perfect field, then $\dim A^{(p)} = \dim A / A_p$.

**Proof.** (i) Use 1.1.

(ii) Since $A_p$ is a vector subspace of $A$, there is a basis $\{ e_i \}_{i \in X \cup Y}$ of $A$, where $X \cap Y = \emptyset$ and $\{ e_i \}_{i \in X}$ is a basis of $A_p$. By 1.1, we have $A^{(p)} = \{ [a^p] \mid a \in A \} = \{ [e_i^p] \mid i \in X \cup Y \} = \{ [e_i^p] \mid i \in Y \}$. Hence $\dim A^{(p)} \leq |Y| = \dim A / A_p$.

Suppose now, that $F$ is a perfect field. It is enough to show, that $\{ e_i^p \}_{i \in Y}$ is linearly independent. Let $\sum_{i \in K} \lambda_i e_i^p = 0$ for some finite set $K \subseteq Y$ and $\lambda_i \in F$. Since $F$ is perfect, there are $\mu_i \in F$ such that $\lambda_i = \mu_i^p$. Hence $0 = \sum_{i \in K} \mu_i^p e_i = (\sum_{i \in K} \mu_i e_i)^p$ and $\sum_{i \in K} \mu_i e_i \in A_p$. Hence $\sum_{i \in K} \mu_i e_i = 0$ and $\mu_i = 0$ for every $i \in K$. Thus $\lambda_i = 0$ for every $i \in K$. It follows $\dim A^{(p)} = |Y| = \dim A / A_p$. □

**Remark 7.3.** Let $A$ be a nilpotent $F$-algebra generated by $a \in A$. Put $a_1 = a_2 = a$. Then $A = \langle a_1, a_2 \rangle$. Consider $f = y - x$. Then $f(a_1, a_2) = 0$, hence $m(f) = (0, 1) \in C_A(a_1, a_2)$. Thus $d_0^\infty(a_1, a_2) = 1$, by 3.6(i), (v). It follows, that $B_A(a_1, a_2) = \{ (i, 0) \mid i \leq n_0 \} \subseteq \{ n_0 \}$. Finally, by 3.8, $\text{nil}(a_1) = n_0 \in \mathbb{N}$. Hence, by 3.2(iii), we get that $a, a^2, \ldots, a^n$ is a basis of $A$ and $a^{n+1} = 0$, where $n + 1 = \text{nil}(a)$.

**Lemma 7.4.** Let $A$ be a 1-generated nilpotent $F$-algebra, char $F = p > 0$. Then $\dim A^{(p)} = \dim A / A_p$.

**Proof.** Let $A$ be generated by $a \in A$. By 7.3, $a, \ldots, a^n$ is a basis of $A$, where $n + 1 = \text{nil}(a)$. By 11, $A^{(p)} = \langle a^p \rangle$. Thus, by 7.3 and 3.8, we have that $a^p, \ldots, a^{pm}$ is a basis of $A^{(p)}$, where $m = \lfloor n/p \rfloor$. 
Further, \( x = \sum_{i=1}^{n} \lambda_i a^i \in A_p \) if and only if \( 0 = x^p = \sum_{i=1}^{n} \lambda_i^p a^i \) and this is equivalent to \( \lambda_i = 0 \) for \( i \leq n/p \). Hence \( \dim A_p = n - \lfloor n/p \rfloor \). Finally, \( \dim A^{(p)} = \dim A - \dim A_p = \dim A/A_p \). □

We have seen that the equality \( \dim A^{(p)} = \dim A/A_p \) holds for a nilpotent \( F \)-algebra \( A \) whenever \( F \) is a perfect field or \( A \) is 1-generated. In the next Example 7.5 we show that this is not true for a 2-generated \( F \)-algebra in general.

**Example 7.5.** Let \( F \) be a field, \( \text{char } F = p > 0 \), that is not perfect. We construct a 2-generated nilpotent \( F \)-algebra, such that \( \dim A^{(p)} < \dim A/A_p \).

Let \( \lambda \in F \) be such that \( \lambda \neq \mu^p \) for any \( \mu \in F \). Let \( k, n \in \mathbb{N} \) be such that \( pk \leq n \) and \( B = \{ \alpha \in \mathbb{N}^{\mathbb{N}} \mid \alpha \leq n, (n, p - 1) \} \). Put \( f_0 = y^p - \lambda x^{pk} \), \( f_{n+1} = x^{n+1} \) and \( f_i = x^if_0 \) for \( i = 1, \ldots, n \). Clearly, \( I = Rf_0 + Rf_{n+1} = Rf_0 + \cdots + Rf_{n+1} \).

Now, it is easy to check that the conditions (i), (2) and (3) in 4.7 are fulfilled. Hence, by 4.7 and 3.2(iii), \( A = Rx + Ry/I \) is a nilpotent \( F \)-algebra with the basis \( B = \{ x^\alpha + I \mid (0, 0) \neq \alpha \leq n, (n, p - 1) \} \).

(i) Since \( x^{k+1} + I, y + I \in B \), we get that \( \dim [x^{k+1}, y + I] = 2 \).

(ii) We show that \([x^{k+1}, y + I] \cap A_p = 0 \). Let \( \mu_1, \mu_2 \in F \) be such that \((\mu_1 x^{k+1} + \mu_2 y) + I \in A_p \). Then \( 0 \equiv_{I} (\mu_1 x^{k+1} + \mu_2 y)^p = \mu_1^p x^{pk} + \mu_2^p y^p + I \equiv_{I} (\mu_1^p + \mu_2^p) x^{pk} \). Since \( x^{pk} + I \in B \), we get \( \mu_1^p + \mu_2^p = 0 \). Thus \( \mu_1 = \mu_2 = 0 \) and \([x^{k+1}, y + I] \cap A_p = 0 \).

(ii) Let \( a_1, \ldots, a_m \) be a basis of \( A_p \). By (i) and (ii), there are \( b_1, \ldots, b_l \in A \) such that \( a_1, \ldots, a_m, x^{k+1}, y + I, b_1, \ldots, b_l \) is a basis of \( A \). Hence \( A^{(p)} = \{ a_1^p, \ldots, a_m^p, x^{pk} + I, y^p + I, b_1^p, \ldots, b_l^p \} = \{ x^{pk} + I, b_1^p, \ldots, b_l^p \} \), since \( y^p + I = \lambda(x^{pk} + I) \) and \( a_i^p = 0 \) for \( i = 1, \ldots, m \). We get \( \dim A^{(p)} < l + 2 = \dim A - \dim A_p = \dim A/A_p \).

Let \( A \) be a finitely-dimensional \( F \)-algebra. By 7.2, we have that \( p \dim A^{(p)} \leq \dim A \) is equivalent to \( \frac{p-1}{p} \dim A \leq \dim A_p \), provided that \( F \) is a perfect field. By 7.2, we also see that the inequality \( p \dim A^{(p)} \leq \dim A \) follows from \( \frac{p-1}{p} \dim A \leq \dim A_p \) for any field \( F \) with \( \text{char } F = p > 0 \). Hence we can think about the following:

**Stronger version of Eggert’s conjecture.** Let \( A \) be a finitely-dimensional nilpotent \( F \)-algebra, \( \text{char } F = p > 0 \). Let \( A_p = \{ a \in A \mid a^p = 0 \} \). Then

\[
\frac{p-1}{p} \dim A \leq \dim A_p.
\]

**References**


