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The regular algebra of a quiver ☆

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Abstract

Let K be a fixed field. We attach to each column-finite quiver E a von Neumann regular K-algebra $Q(E)$ in a functorial way. The algebra $Q(E)$ is a universal localization of the usual path algebra $P(E)$ associated with E. The monoid of isomorphism classes of finitely generated projective right $Q(E)$-modules is explicitly computed.

Keywords: Von Neumann regular ring; Path algebra; Leavitt path algebra; Universal localization

Introduction

In a series of recent papers [1,2,6–8] different aspects of the algebraic structure of the so-called Leavitt path algebras $L_K(E)$ have been analyzed. These algebras, defined with coefficients in an arbitrary field K, are the purely algebraic analogues of the important class of Cuntz–Krieger graph C^*-algebras. See the book of Raeburn [24] for a recent account about Cuntz–Krieger algebras.

Fix a field K. For a column-finite quiver E, denote by $P(E)$ the usual path K-algebra associated to E and by $L(E)$ the Leavitt path K-algebra of E. In this paper, we show that the algebra $L(E)$ can be embedded in a von Neumann regular algebra $Q(E)$ in such a way that the embed-

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ding preserves the monoid of isomorphism classes of finitely generated projective right modules, that is the inclusion $L(E) \to Q(E)$ induces a monoid isomorphism $\mathcal{V}(L(E)) \cong \mathcal{V}(Q(E))$. Moreover the von Neumann regular algebra $Q(E)$ is obtained from $P(E)$ and also from $L(E)$ by universal localization, so that it is a (generalized) ring of fractions of both algebras. Using this we prove that the construction of $Q(E)$ is functorial with respect to *complete* graph homomorphisms (see Section 4 for the definition). This enables us to extend many results from the case of a finite quiver to the case of an arbitrary column-finite quiver.

Our construction is relevant to the following realization problem for von Neumann regular rings, which is a variant of a problem posed by Goodearl in [19, Fundamental Open Problem]. Recall that an abelian monoid M is conical in case $x + y = 0$ implies $x = y = 0$ and that it is a *refinement monoid* in case any equality $x_1 + x_2 = y_1 + y_2$ admits a refinement, that is, there are x_{ij}, $1 \leqslant i, j \leqslant 2$ such that $x_i = x_{i1} + x_{i2}$ and $y_j = x_{1j} + x_{2j}$ for all i, j, see e.g. [6].

Realization problem for von Neumann regular rings. Let M be a countable refinement conical abelian monoid. Is there a von Neumann regular ring R such that $\mathcal{V}(R) \cong M$?

For every von Neumann regular ring R, it is known that $\mathcal{V}(R)$ is a conical refinement abelian monoid. Indeed this is the case for the larger class of exchange rings, by [5, Corollary 1.3].

The countability condition is important here. Fred Wehrung [28] proved that there are (even cancellative) refinement conical abelian monoids of size $\aleph_2$ such that cannot be realized as $\mathcal{V}(R)$ for any von Neumann regular ring R. Note that, by the results in [29], an affirmative answer to the above realization problem would give a negative answer to the fundamental separativity problem for von Neumann regular rings [5].

Since the monoids $\mathcal{V}(L(E))$ corresponding to column-finite quivers E were neatly computed in [6], and their properties are fairly well understood (see [6, Section 6]), the results in the present paper represent a significant contribution to the realization problem. In particular, let M_E be the monoid corresponding to the quiver E (definition given prior to Theorem 3.1). We demonstrate in Theorem 4.2 that there exists a unital von Neumann regular hereditary ring $Q(E)$ such that $\mathcal{V}(Q(E)) \cong M_E$. Furthermore, if E is a column-finite quiver, then there exists a (not necessarily unital) von Neumann regular ring $Q(E)$ such that $\mathcal{V}(Q(E)) \cong M_E$ (Theorem 4.4).

The only systematic approaches to the realization problem known to the authors are the well-known realization theorem for dimension monoids [18, Theorem 15.24(b)], and the realization theorem given in [4, Theorem 8.4], saying that every countable abelian group G can be obtained as $K_0(R)$ for some purely infinite simple regular ring R. Since $\mathcal{V}(R) = \{0\} \sqcup K_0(R)$ for every purely infinite simple ring R, we get that all the monoids of the form $\{0\} \sqcup G$, for G a countable abelian group, can be realized as monoids associated to a purely infinite simple regular ring. It can be shown that these monoids are of the form M_E for suitable quivers E, so the main result in the present paper incorporates the above-mentioned ones. Indeed, taking into account [6, Theorems 3.5 and 7.1], we see that the case of dimension monoids follows from [24, Proposition 2.12] and the case of monoids of the form $\{0\} \sqcup G$, with G a countable abelian group, follows from [27, Theorem 1.2].

Our results are a generalization of the ones obtained in [4], where a von Neumann regular envelope of the Leavitt algebra of type $(1, n)$ was constructed. The Leavitt algebra of type $(1, n)$ can be seen as the Leavitt path algebra associated with a quiver with just one vertex and $n + 1$ arrows. The corresponding path algebra is the free associative algebra with $n + 1$ generators, and the construction in [6] uses the algebra of rational formal power series on these generators

[12,14]. A large part of the present paper is devoted to generalize properties of the (rational) formal power series algebra to the more general setting of quiver algebras.

We now summarize the content of the paper. In Section 1 we review some basic concepts and we generalize some results known for the free algebra to the context of path algebras. In particular, we study the algebra of formal power series on the quiver E and the algebra $P_{\mathrm{rat}}(E)$ of rational series over E, which is defined as the division closure of $P(E)$ in $P(\!(E)\!)$.

Section 2 contains the basic constructions of our algebras of Leavitt type, associated with an algebra R which is a subalgebra of the algebra $P(\!(E)\!)$ of formal power series on a finite quiver E containing the path algebra $P(E)$. We show that if the algebra R is closed under inversion in $P(\!(E)\!)$ then the resulting ring of Leavitt type is von Neumann regular. When we take $R = P(E)$ (which is not closed under inversion in $P(\!(E)\!)$ in general) we recover the usual Leavitt path algebra $L(E)$ (which is not von Neumann regular in general). When we take $R = P_{\mathrm{rat}}(E)$, the algebra of rational power series on E (which is closed under inversion in $P(\!(E)\!)$), we get what we call the *regular algebra* $Q(E)$ associated with E, which, by Theorem 4.2, is both von Neumann regular and hereditary.

Section 3 contains the computation of the monoid of finitely generated projective modules over the von Neumann regular algebras T of Leavitt type constructed in Section 2. In particular, we get that the inclusion $L(E) \to Q(E)$ induces a monoid isomorphism $\mathcal{V}(L(E)) \cong \mathcal{V}(Q(E))$. Section 4 gives the functoriality of the construction with respect to complete graph homomorphisms, which enables us to extend the construction and the computations from finite to column-finite quivers.

1. A universal localization of the path algebra of a quiver

Let R be a ring. We will use the notation R^n (respectively, ${}^nR$) for the left (respectively, right) R-module of n-rows (respectively, n-columns) with coefficients in R. We will use $M_{m\times n}(R)$ for the space of matrices of size $m \times n$ over R and $M_n(R)$ for the ring of $n \times n$ matrices over R.

In the following, K will denote a fixed field and $E = (E^0, E^1, r, s)$ a finite quiver (oriented graph) with $E^0 = \{1, \dots, d\}$. Here $s(e)$ is the *source vertex* of the arrow e, and $r(e)$ is the *range vertex* of e. A *path* in E is either an ordered sequence of arrows $\alpha = e_1 \cdots e_n$ with $r(e_t) = s(e_{t+1})$ for $1 \leqslant t < n$, or a path of length 0 corresponding to a vertex $i \in E^0$, which will be denoted by p_i. The paths p_i are called trivial paths, and we have $r(p_i) = s(p_i) = i$. A non-trivial path $\alpha = e_1 \cdots e_n$ has length n and we define $s(\alpha) = s(e_1)$ and $r(\alpha) = r(e_n)$. We will denote the length of a path α by $|\alpha|$, the set of all paths of length n by E^n, for $n > 1$, and the set of all paths by E^*.

Let $P(E)$ be the K-vector space with basis E^*. It is easy to see that $P(E)$ has a structure of K-algebra (see for example [9, Section III.1]), which is called the *path algebra*. Indeed, $P(E)$ is the K-algebra given by free generators $\{p_i \mid i \in E^0\} \cup E^1$ and relations:

(i) $p_i p_j = \delta_{ij} p_i$ for all $i, j \in E^0$.
(ii) $p_{s(e)} e = e p_{r(e)} = e$ for all $e \in E^1$.

Observe that $A = \bigoplus_{i\in E^0} K p_i \subseteq P(E)$ is a subring isomorphic to K^d. In general we will identify $A \subseteq P(E)$ with K^d. An element in $P(E)$ can be written in a unique way as a finite sum $\sum_{\gamma\in E^*} \lambda_\gamma \gamma$ with $\lambda_\gamma \in K$. We will denote by ε the augmentation homomorphism, which is the ring homomorphism $\varepsilon : P(E) \to K^d \subseteq P(E)$ defined by $\varepsilon(\sum_{\gamma\in E^*} \lambda_\gamma \gamma) = \sum_{\gamma\in E^0} \lambda_\gamma \gamma$.

Definition 1.1. Let $I = \ker(\varepsilon)$ be the augmentation ideal of $P(E)$. Then the *K-algebra of formal power series* over the quiver E, denoted by $P((E))$, is the I-adic completion of $P(E)$, that is $P((E)) \cong \varprojlim P(E)/I^n$.

An element of $P((E))$ can be written in a unique way as a possibly infinite sum $\sum_{\gamma\in E^*} \lambda_\gamma \gamma$ with $\lambda_\gamma \in K$. We will also denote by ε the augmentation homomorphism on $P((E))$.

Set $R = P(E)$ or $P((E))$. Observe that the elements in $M_n(R)$ (or in R^n) can also be uniquely written as possibly infinite sums $\sum_{\gamma\in E^*} \lambda_\gamma \gamma$ with $\lambda_\gamma \in M_n(K)$ (respectively with $\lambda_\gamma \in K^n$) and so we can also define over them the augmentation homomorphisms, which will be denoted also by ε.

Given an element $r = \sum_{\gamma\in E^*} \lambda_\gamma \gamma$ in R, $M_n(R)$ or R^n we define its support as $\mathrm{supp}(r) = \{\gamma \in E^* \mid \lambda_\gamma \neq 0\}$ and we define its order $o(r)$ as the minimum length of the paths in $\mathrm{supp}(r)$.

Define, for $e \in E^1$, the following additive mappings:

$$\delta_e : R \to R,$$
$$\sum_{\alpha\in E^*} \lambda_\alpha \alpha \mapsto \sum_{\substack{\alpha\in E^*\\ r(\alpha)=s(e)}} \lambda_{\alpha e}\alpha.$$

We will write δ_e on the right of its argument. We will sometimes refer to the maps δ_e as the (left) transductions. There are corresponding maps on $M_n(R)$ and on R^n, defined componentwise, denoted also by δ_e.

The right transductions $\tilde{\delta}_e : R \to R$ are defined similarly by

$$\tilde{\delta}_e\biggl(\sum_{\alpha\in E^*} \lambda_\alpha \alpha\biggr) = \sum_{\substack{\alpha\in E^*\\ s(\alpha)=r(e)}} \lambda_{e\alpha}\alpha.$$

The following well-known result follows easily from [10, Theorem 5.3].

Proposition 1.2. *We have an isomorphism $\mathcal{V}(P(E)) \overset{\mathcal{V}(\varepsilon)}{\cong} \mathcal{V}(K^d) \cong (\mathbb{Z}^+)^d$ induced by the augmentation homomorphism $\varepsilon : P(E) \to K^d$.*

For a f.g. projective $P(E)$-module M, we write $\mathrm{rank}_{P(E)}(M) \in (\mathbb{Z}^+)^d$ for the image of $[M]$ under the isomorphism $\mathcal{V}(P(E)) \cong (\mathbb{Z}^+)^d$ of Proposition 1.2. Note that $\mathrm{rank}_{P(E)}(M) = (r_1, \dots, r_d)$ if and only if $M \cong (p_1 P(E))^{r_1} \oplus \cdots \oplus (p_d P(E))^{r_d}$. Similarly, we use the notation $\mathrm{rank}_{K^d}(M) \in (\mathbb{Z}^+)^d$ for a f.g. (projective) K^d-module M.

Definition 1.3. Let R be a subalgebra of $P((E))$ closed under the left transductions δ_e and let B be a right R-submodule of ${}^nR$. We say that B is *regular* if for every $b \in B$ with $o(b) > 0$, we have $(b)\delta_e \in B$ for all arrows $e \in E^1$.

The regularity passes to direct summands:

Lemma 1.4. *Given a regular module $B = B_1 \oplus B_2$ we have that B_1, B_2 are also regular modules.*

Proof. Given $b \in B_1$ with $o(b) > 0$, we have $(b)\delta_e \in B$ for each arrow $e \in E^1$ by regularity of B. So $(b)\delta_e = b_1^e + b_2^e$ for some unique $b_i^e \in B_i$ with $b_i^e p_{s(e)} = b_i^e$. Thus $b = \sum_e (b\delta_e) \cdot e = \sum_e b_1^e e + \sum_e b_2^e e$ and we get that $\sum_e b_2^e e = 0$. It follows that $b_2^e = 0$ for every $e \in E^1$ and that $(b)\delta_e = b_1^e \in B_1$. $\square$

The following result provides a generalization of [12, Theorem VII.3.1].

Theorem 1.5 *(Characterization of regular modules). Let $B \subseteq {}^n P(E)$ be a right $P(E)$-module. Then B is regular if and only if $B = \bigoplus_{i=1}^t B_i$ where each B_i is a cyclic projective $P(E)$-module, $t \leqslant n$, and* $\operatorname{rank}_{P(E)}(B) = \operatorname{rank}_{K^d}(\varepsilon(B))$.

Proof. Denote by ε_j the following composition ${}^n P(E) \to {}^n(K^d) \cong ({}^n K)^d \to {}^n K$, where the last homomorphism is the projection onto the jth component of $({}^n K)^d$.

We first show that there are $t_1, \dots, t_d \in \mathbb{N}$ and $v_1^{(j)}, \dots, v_{t_j}^{(j)} \in Bp_j$, where $j = 1, 2, \dots, d$ such that

(i) $\deg(v_1^{(j)}) \leqslant \cdots \leqslant \deg(v_{t_j}^{(j)})$.
(ii) The vectors $\{\varepsilon_j(v_i^{(j)})\}_{i=1,\dots,t_j}$ form a K-basis of $\varepsilon_j(Bp_j)$.
(iii) If $v \in Bp_j$ and $\deg(v) < \deg(v_i^{(j)})$, then $\varepsilon_j(v)$ is a linear combination of $\varepsilon_j(v_1^{(j)}), \dots, \varepsilon_j(v_{i-1}^{(j)})$.

For fixed j and $r = -1, 0, 1, 2, \dots$ write

$$F(r) = \{\varepsilon_j(v) \in {}^n K \mid v \in Bp_j,\ \deg(v) \leqslant r\}.$$

Observe that $F(r)$ are K-vector spaces. We have inclusions $0 = F(-1) \subseteq F(0) \subseteq \cdots \subseteq F(r) \subseteq \cdots$. Take integer numbers $0 \leqslant r_1 < \cdots < r_q$ such that $F(r_i - 1) \subset F(r_i)$ and that, for all r with $F(r) \neq 0$, we have $F(r) = F(r_i)$ for some i. In particular, $F(r_q) = \varepsilon_j(Bp_j)$.

Choose now a K-basis $\mathcal{B}_1$ of $F(r_1)$, a K-basis $\mathcal{B}_2$ of $F(r_2)$ modulo $F(r_1)$ and in general a K-basis $\mathcal{B}_i$ of $F(r_i)$ modulo $F(r_{i-1})$. By definition of the F's, for each $i = 1, \dots, q$ we can find elements $w_{i,1}, \dots, w_{i,k_i} \in B$ of degree less than or equal to r_i such that $\{\varepsilon_j(w_{i,1}), \dots, \varepsilon_j(w_{i,k_i})\} = \mathcal{B}_i$. Indeed, the degree of each $w_{i,j}$ is exactly r_i, otherwise we would have that $\varepsilon_j(w_{i,j})$ would belong to $F(r_i - 1) = F(r_{i-1})$, contradicting that $\mathcal{B}_i$ is a basis modulo $F(r_{i-1})$. Now put $t_j = k_1 + k_2 + \cdots + k_q$ and define $v_1^{(j)}, \dots, v_{t_j}^{(j)}$ by

$$(v_1^{(j)}, \dots, v_{t_j}^{(j)}) = (w_{1,1}, \dots, w_{1,k_1}, w_{2,1}, \dots, w_{2,k_2}, \dots, w_{q,k_q}).$$

It is clear from this definition that condition (i) is satisfied. Moreover, since $F(r_q) = \varepsilon_j(Bp_j)$, we see that condition (ii) is also satisfied. Let $v \in Bp_j$ with $\deg(v) < \deg(v_i^{(j)})$. Then $v_i^{(j)} = w_{\ell,m}$ for some ℓ, m, so that $\deg(v) < r_\ell = \deg(w_{\ell,m})$, and we conclude that $\varepsilon_j(v) \in F(r_\ell - 1) = F(r_{\ell-1})$ and that $\varepsilon_j(v)$ is a K-linear combination of $\varepsilon(w_{1,1}), \dots, \varepsilon(w_{\ell-1,k_{\ell-1}})$, and so of $\varepsilon(v_1^{(j)}), \dots, \varepsilon(v_{i-1}^{(j)})$. This shows (iii).

Set $t = \max\{t_1, \dots, t_d\}$. Since $t_j = \dim_K \varepsilon_j(Bp_j) \leqslant n$ we have that $t \leqslant n$. For $i = 1, \dots, t$, take the following submodules of B

$$B_i = v_i^{(1)} P(E) + v_i^{(2)} P(E) + \cdots + v_i^{(d)} P(E) \subseteq B$$

where $v_i^{(j)} = 0$ if $i > t_j$. We will now see that B_i are cyclic projective modules.

For $i = 1, \dots, t$ and $j = 1, \dots, d$ set

$$a_{ij} = \begin{cases} 0 & \text{if } i > t_j, \\ 1 & \text{if } i \leqslant t_j. \end{cases}$$

With this notation, for $i = 1, \dots, t$, we have isomorphisms $B_i \cong a_{i1}(p_1 P(E)) \oplus a_{i2}(p_2 P(E)) \oplus \cdots \oplus a_{id}(p_d P(E))$. Indeed, define

$$\varphi_i : a_{i1}\big(p_1 P(E)\big) \oplus \cdots \oplus a_{id}\big(p_d P(E)\big) \to B_i,$$
$$(a_{i1} p_1 a_1, \dots, a_{id} p_d a_d) \mapsto \sum_{j=1}^{d} v_i^{(j)} p_j a_j.$$

It is clear that the maps φ_i are surjective since $a_{ij} = 0$ if and only if $v_i^{(j)} = 0$. We claim that

$$\sum_{j=1}^{d} \sum_{i=1}^{t_j} v_i^{(j)} (p_j b_{ij}) = 0 \tag{1.1}$$

implies $p_j b_{ij} = 0$ for all i, j. This will show at once that each φ_i is injective and that the sum $B_1 + \cdots + B_t$ is a direct sum.

To prove the claim, we proceed by way of contradiction, so let

$$\sum_{j=1}^{d} \sum_{i=1}^{t_j} v_i^{(j)} (p_j b_{ij}) = 0 \quad \text{with } 0 \leqslant \max_{i,j}\big\{\deg(p_j b_{ij})\big\} \text{ smallest possible.} \tag{1.2}$$

For all j we have $\sum_{i=1}^{t_j} \varepsilon_j(v_i^{(j)}) \varepsilon_j(p_j b_{ij}) = 0$ and so we deduce from (ii) that, for all i and j, $\varepsilon_j(p_j b_{ij}) = 0$ so that $\varepsilon(p_j b_{ij}) = 0$, since $\varepsilon_k(p_j) = 0$ if $k \neq j$. Since these elements have zero augmentation we can apply the transductions so reducing the degree:

$$\sum_{j=1}^{d} \sum_{i=1}^{t_j} v_i^{(j)} \cdot \big((p_j b_{ij})\delta_e\big) = 0.$$

Observe that $(p_j b_{ij})\delta_e = p_j((p_j b_{ij})\delta_e)$ for all e and that $(p_j b_{ij})\delta_e \neq 0$ for some i, j and some $e \in E^1$. This leads to a contradiction to the minimality of the degree in (1.2).

Now set $B' = \bigoplus_{i=1}^{t} B_i \subseteq B$. We want to see that $B' = B$. Suppose there is $v \in B \setminus B'$, which we take of minimal degree. We can write $v = vp_1 + \cdots + vp_d$. Denote $\deg(v)$ by d_v. Let j be an integer such that $\deg(vp_j) = d_v$ and let i be the smallest integer such that $\deg(vp_j) < \deg(v_i^{(j)})$ ($i = t_j + 1$ if no such integer exists). By (ii) or (iii), depending on the case, we have that $\varepsilon_j(vp_j) = \sum_{k=1}^{i-1} \lambda_k \varepsilon_j(v_k^{(j)})$ and thus $v' = vp_j - \sum_{k=1}^{i-1} \lambda_k v_k^{(j)} \in B$ satisfies that $\deg(v') \leqslant \deg(v)$ and $v' \in Bp_j$ with $\varepsilon_j(v') = 0$. It follows that $\varepsilon(v') = 0$. Since B is a regular module we have $(v')\delta_e \in B$ for all $e \in E^1$. By the minimality of the degree we have that $(v')\delta_e \in B'$ and then

$v' = \sum_{e \in E^1} ((v')\delta_e)e \in B'$. We get from this that $vp_j \in B'$ for all j such that $\deg(vp_j) = d_v$. Since

$$v - \sum_{\deg(vp_j)=d_v} vp_j = \sum_{\deg(vp_k)<d_v} vp_k,$$

we have that

$$\deg\left(v - \sum_{\deg(vp_j)=d_v} vp_j\right) < d_v$$

and, again by the minimality of the degree, we get that $v \in B'$.

We now prove the converse. Suppose that $B = \bigoplus_{i=1}^t B_i$ where each B_i is a cyclic projective module, $t \leqslant n$ and $\operatorname{rank}_{P(E)}(B) = \operatorname{rank}_{K^d}(\varepsilon(B))$. Observe that we have

$$B = \bigoplus_{j=1}^{d} \bigoplus_{i=1}^{t_j} x_i^{(j)} P(E),$$

with $x_i^{(j)} = x_i^{(j)} p_j$ for all i, j, where $\operatorname{rank}_{P(E)}(B) = (t_1, \dots, t_d) = \operatorname{rank}_{K^d}(\varepsilon(B))$. It follows from the latter equality that $\{\varepsilon_j(x_i^{(j)}) \mid i = 1, \dots, t_j\}$ is a linearly independent family of vectors in ${}^nK$.

Let $v \in B$ such that $\varepsilon(v) = 0$. Then for all j we have

$$0 = \varepsilon_j(v) = \sum_{i=1}^{t_j} \varepsilon_j\bigl(x_i^{(j)}\bigr)\varepsilon_j\bigl(z_j^i\bigr),$$

where $v = \sum_{i,j} x_i^{(j)} z_j^i$, with $z_j^i \in p_j P(E)$. Since $\{\varepsilon_j(x_i^{(j)}) \mid i = 1, \dots, t_j\}$ are K-linearly independent, we get $\varepsilon_j(z_j^i) = 0$ for all i, j, but $z_j^i = p_j z_j^i$ so $\varepsilon(z_j^i) = 0$ and thus $v\delta_e = \sum_{i,j} x_i^{(j)} \cdot ((z_j^i)\delta_e) \in B$. $\square$

Corollary 1.6. *If $B \subseteq {}^nP(E)$ is a regular module, then there exists $u \in M_n(P(E))$ such that $B = u\,{}^nP(E)$.*

The proof of next lemma is standard, see for example [15, pp. 284–285].

Lemma 1.7 *(Higman's trick). Given a matrix $M \in M_{n\times m}(P(E))$, there exist $\ell \in \mathbb{N}$, $P \in GL_{n+\ell}(P(E))$ and $Q \in GL_{m+\ell}(P(E))$ such that $P\left(\begin{smallmatrix} M & 0 \\ 0 & \mathbf{1}_\ell \end{smallmatrix}\right)Q$ is a linear matrix, that is, a matrix whose entries have degree $\leqslant 1$.*

Lemma 1.8. *We have $GL_n(P((E))) = \{M \in M_n(P((E))) \mid \varepsilon(M) \in GL_n(K^d)\}$.*

Proof. If $M \in GL_n(P((E)))$ then clearly $\varepsilon(M) \in GL_n(K^d)$. Let now $M \in M_n(P((E)))$ such that $\varepsilon(M) \in GL_n(K^d)$. We can write $M = \varepsilon(M) - D$ for some $D \in M_n(P((E)))$ with $o(D) > 0$. We have that

$$M^{-1} = \varepsilon(M)^{-1}\bigl(\mathbf{1}_n + D\varepsilon(M)^{-1} + \bigl(D\varepsilon(M)^{-1}\bigr)^2 + \cdots\bigr). \qquad \square$$

Lemma 1.9. *Let $u_0 = p - D \in M_n(P(E))$, where $p \in \mathrm{Idem}(M_n(K^d))$ and D is homogeneous of degree 1. Suppose further that $B = u_0\,{}^nP(E)$ is a regular $P(E)$-module. Then there exist $u \in M_n(P(E))$ and $v \in M_n(P((E)))$ such that $uvu = u$, $vuv = v$ with $B = u\,{}^nP(E)$ and $vu \in M_n(P(E))$.*

Proof. If the matrix $u_0(1-p) = -D(1-p)$ is nonzero then it is a homogeneous matrix of degree 1. The columns of this matrix are elements of B, that is, denoting the elements of the canonical basis of ${}^nP(E)$ by E_i, we have $u_0(1-p)E_i \in B$ for all $i = 1, \dots, n$. These elements are of positive order, so they decompose as follows:

$$u_0(1-p)E_i = \sum_{e \in E^1} u^i_{0e} e \quad \text{with unique } u^i_{0e} \in {}^n\bigl(P(E)p_{s(e)}\bigr),$$

and indeed, being B a regular module, we get $u^i_{0e} \in B$. We also have that $\deg(u^i_{0e}) < 1$, that is, they are elements of $\varepsilon(B) \cap B$. Since $\varepsilon(B) = p\,{}^n(K^d)$ we have $pu^i_{0e} = u^i_{0e}$ for all i and e. In particular, $(1-p)D(1-p) = 0$. Consider the K^d-submodule V_1 of $\varepsilon(B)$ generated by $\{u^i_{0e} \mid e \in E^1, i = 1, \dots, n\}$. Since K^d is a semisimple ring, there exists $q_1 \in \mathrm{Idem}(M_n(K^d))$, $q_1 \leqslant p$, such that $V_1 = q_1\,{}^n(K^d)$. Therefore by the above we have that $q_1\,{}^nP(E) \subseteq B$ and, in conclusion we have seen that

$$u_0 = p - pDp - q_1D(1-p) - (1-p)Dp.$$

Setting $u_1 = (1-q_1)u_0 = (p-q_1) - (p-q_1)Dp - (1-p)Dp$, we obtain from the modular law:

$$B = q_1\,{}^nP(E) \oplus u_1\,{}^nP(E).$$

By Lemma 1.4, $u_1\,{}^nP(E)$ is again a regular $P(E)$-module, so that we can repeat the above process with u_1. We have that $u_1q_1 = -(p-q_1)Dq_1 - (1-p)Dq_1$. As before, for $i = 1, \dots, n$, we get $u_1q_1E_i = \sum_{e \in E^1} u^i_{1e} e$ with $u^i_{1e} \in B$ and since we know by degree considerations that $u^i_{1e} \in \varepsilon(B)$ we obtain that $(1-p)Dq_1 = 0$.

Consider the K^d-submodule V_2 of $\varepsilon(B)$ generated by $\{u^i_{1e} \mid e \in E^1,\ i = 1, \dots, n\}$. There exists an idempotent matrix $q_2 \in M_n(P(E))$, $q_2 \leqslant (p - q_1)$, such that $V_2 = q_2\,{}^n(K^d) \leqslant \varepsilon(B)$. Similarly to the above argument we get $q_2\,{}^nP(E) \subseteq B$ since $u^i_{1e} \in B$. Putting

$$\begin{aligned} u_2 &= \bigl(1-(q_1+q_2)\bigr)u_1 = \bigl(1-(q_1+q_2)\bigr)u_0 \\ &= \bigl(p-(q_1+q_2)\bigr) - \bigl(p-(q_1+q_2)\bigr)D(p-q_1) - (1-p)D(p-q_1) \end{aligned}$$

we have that $B = (q_1+q_2)\,{}^nP(E) \oplus u_2\,{}^nP(E)$.

Iterating this process we get a sequence of idempotent, pairwise orthogonal matrices $q_1, \dots, q_\ell \in \mathrm{Idem}(M_n(K^d))$, with $q_i \leqslant p$ for all i in such a way that $q_i\,{}^nP(E) \subseteq B$; we also have independent K^d-modules $V_i = q_i\,{}^n(K^d)$ and $u_1, \dots, u_\ell \in M_n(P(E))$,

$$\begin{aligned} u_i = {} & \bigl(p-(q_1+\dots+q_i)\bigr) - \bigl(p-(q_1+\dots+q_i)\bigr)D\bigl(p-(q_1+\dots+q_{i-1})\bigr) \\ & - (1-p)D\bigl(p-(q_1+\dots+q_{i-1})\bigr) \end{aligned}$$

such that $B = (q_1+\dots+q_i)\,{}^nP(E) \oplus u_i\,{}^nP(E)$ for all i.

Since we have an ascending chain of submodules of a Noetherian module:

$$V_1 \subseteq V_1 \oplus V_2 \subseteq \cdots \subseteq V_1 \oplus \cdots \oplus V_i \subseteq \cdots \subseteq \varepsilon(B)$$

we see that the above process will stop in a finite number of steps, say after ℓ steps. Write $q = q_1 + \cdots + q_\ell$. In principle we have

$$u_\ell = (p-q) - (p-q)D\big(p - (q_1 + \cdots + q_{\ell-1})\big) - (1-p)D\big(p - (q_1 + \cdots + q_{\ell-1})\big),$$

but since this process stops in the step ℓ necessarily we have $u_\ell q_\ell = 0$, that is,

$$u_\ell = (p-q) - (p-q)D(p-q) - (1-p)D(p-q)$$

with $B = q\,{}^nP(E) \oplus u_\ell\,{}^nP(E)$. Now set

$$u = q + u_\ell = p - (p-q)D(p-q) - (1-p)D(p-q),$$

and note that $B = u\,{}^nP(E)$. If we set

$$v = p + (p-q)D(p-q) + \big((p-q)D(p-q)\big)^2 + \cdots \in M_n\big(P((E))\big)$$

then $vu = p \in M_n(P(E))$, $uvu = u$ and $vuv = v$. $\square$

The following result is well known for free algebras; see [12, Corollary VII.3.4] and [16, Appendix].

Theorem 1.10 *(Stable inertia). Let ${}_{P(E)}A \subseteq P((E))^n$ and $B_{P(E)} \subseteq {}^nP((E))$ be left and right $P(E)$-submodules, respectively, such that for every $a \in A$ and for every $b \in B$ we have $ab \in P(E)$. Then there exist $m \in \mathbb{N}$ and $u, v \in M_{n+m}(P((E)))$ such that for all $a \in A \oplus P(E)^m$ and all $b \in B \oplus {}^mP(E)$ we have*

$$au \in P(E)^{n+m}, \quad vb \in {}^{n+m}P(E) \quad \text{and} \quad ab = (au)(vb).$$

Proof. We will show first the case where $B \subseteq {}^nP(E)$. We can assume that

$$B = \big\{b \in {}^nP(E) \,\big|\, \forall a \in A,\ ab \in P(E)\big\}.$$

Under this assumption we have that B is a regular right $P(E)$-module. Indeed, if we take $b \in B$ with $o(b) > 0$ we know that $b = \sum_{e \in E^1} b_e e$ for some unique elements $b_e \in {}^n(P(E)p_{s(e)})$. For every $a \in A$ we have that

$$ab = a\Big(\sum_{e \in E^1} b_e e\Big) = \sum_{e \in E^1} (ab_e)e \in P(E),$$

so that $ab_e \in P(E)$ for all $e \in E^1$ and for all $a \in A$, which tells us that $b_e \in B$ for all $e \in E^1$.

By Corollary 1.6, there exists $u_0 \in M_n(P(E))$ such that $B = u_0\,{}^nP(E)$. We get $\varepsilon(B) = \varepsilon(u_0)\,{}^n(K^d)$.

By Higman's trick (Lemma 1.7) there exist $m \in \mathbb{N}$ and $P,\ Q \in GL_{n+m}(P(E))$ such that the matrix

$$u_1 = P \begin{pmatrix} u_0 & 0 \\ 0 & \mathbf{1}_m \end{pmatrix} Q$$

is a linear matrix. Now we consider the left $P(E)$-submodule $A' = (A \oplus P(E)^m)P^{-1}$ of $P((E))^{n+m}$ and the right $P(E)$-submodule $B' = u_1{}^{n+m}P(E) = P(B \oplus {}^mP(E))$ of ${}^{n+m}P(E)$, and observe that B' is regular and that the generating matrix u_1 of B' is linear.

Since $M_{n+m}(K^d)$ is unit-regular there exists $x \in GL_{n+m}(K^d)$ such that $\varepsilon(u_1)x\varepsilon(u_1) = \varepsilon(u_1)$. Thus the idempotent matrix $p := \varepsilon(u_1)x \in \mathrm{Idem}(M_{n+m}(K^d))$ satisfies $\varepsilon(B) = p^{n+m}(K^d)$. Using u_1x instead of u_1 as a generator of B', we can assume that $u_1 = p - D$ with $p \in \mathrm{Idem}(M_{n+m}(K^d))$ and $D \in M_{n+m}(P(E))$ homogeneous of degree 1.

We are now in the hypothesis of Lemma 1.9, so that there exist $u_2 \in M_{n+m}(P(E))$ and $v_0 \in M_{n+m}(P((E)))$ such that $u_2v_0u_2 = u_2$, $v_0u_2v_0 = v_0$ with $B' = u_2{}^{n+m}P(E)$ and $v_0u_2 \in M_{n+m}(P(E))$.

Set $u = P^{-1}u_2$ and $v = v_0P$. Given $a \in A \oplus P(E)^m$ and $b \in B \oplus {}^mP(E)$ we have that $aP^{-1} \in A'$ and $Pb \in B'$. Thus $Pb = u_2b'$ for some $b' \in {}^{n+m}P(E)$. The following identities hold:

$$\begin{aligned}(au)(vb) &= \big((aP^{-1})u_2\big)\big(v_0(Pb)\big) = \big(aP^{-1}\big)\big((u_2v_0u_2)b'\big) \\ &= \big(aP^{-1}\big)(u_2b') = \big(aP^{-1}\big)(Pb) = ab, \\ vb &= v_0(Pb) = (v_0u_2)b' \in {}^{n+m}P(E).\end{aligned}$$

Moreover, using that $u_2 = u_1y$ for some $y \in M_{n+m}(P(E))$ we have that

$$au = \big(aP^{-1}\big)u_2 = \big(aP^{-1}\big)u_1y = a \begin{pmatrix} u_0 & 0 \\ 0 & \mathbf{1}_m \end{pmatrix} Qy \in P(E)^{n+m}.$$

We have shown the result in the case where $B \subseteq {}^nP(E)$.

Now we shall see that the general case can be reduced to the case considered before. Taking, if needed, a bigger subset A, we can assume that

$$A = \big\{a \in P((E))^n \;\big|\; \forall b \in B,\ ab \in P(E)\big\}$$

so that A is a regular left $P(E)$-module.

For $i \in \{1, \dots, d\}$, put

$$J_i = \big\{j \in \{1, \dots, n\} \;\big|\; \forall b = (b_1, \dots, b_n)^t \in B,\ p_ib_j \in P(E)\big\}.$$

These sets give us a measure of how far we are from the preceding situation. Assume that for all i with $1 \leqslant i \leqslant d$, for all $a = (a_1, \dots, a_n) \in A$ and for all $j \notin J_i$ we have $a_jp_i = 0$. In this case, we consider the following diagonal matrix:

$$q = \mathrm{diag}(d_1, \dots, d_n) \quad \text{where } d_j = \sum_{i \in \{i \mid j \in J_i\}} p_i.$$

For all $a=(a_1,\dots,a_n)\in A$ and all $b=(b_1,\dots,b_n)^t\in B$ we have

$$ab=\sum_i a_ib_i=\sum_i a_id_ib_i+\sum_i a_i(1-d_i)b_i=\sum_i a_id_ib_i=a(qb)$$

with $qb\in {}^nP(E)$ so that considering qB instead of B we can reduce ourselves to the above case.

Otherwise, suppose that there exist $i_0\in\{1,\dots,d\}$, $a=(a_1,\dots,a_n)\in A$ and $j_0\notin J_{i_0}$ such that $a_{j_0}p_{i_0}\neq 0$. In particular, we can take $w\in \operatorname{supp}(a_{j_0})$ such that $wp_{i_0}\neq 0$. Write $w=e_1\cdots e_m$ for some arrows $e_1,\dots,e_m\in E^1$ with $r(e_m)=i_0$. Consider the following set:

$$S_a:=\bigcup_{i=1}^d\biggl(\biggl(\bigcup_{j\notin J_i}\operatorname{supp}\bigl(\varepsilon(a_j)\bigr)\biggr)\cap\{p_i\}\biggr).$$

Observe that, if we denote the elements of the canonical basis of $P(E)^n$ by E_j, by the definition of J_i we have that $p_iE_j\in A$ if and only if $j\in J_i$. Thus in case that $S_a=\emptyset$ we get that $\varepsilon(a)\in A$ and we can assume that $o(a)>0$. Consider the subword $w_1=e_1\cdots e_{m_1}$ of w, where $m_1=o(a)$. By the regularity of A we have that $a'=\tilde\delta_{e_{m_1}}\cdots\tilde\delta_{e_2}\tilde\delta_{e_1}(a)$ is a nonzero element in A with $w'=e_{m_1+1}\cdots e_m\in\operatorname{supp}(a'_{j_0})$.

Now we can repeat the same argument with a' and w'. Since $wp_{i_0}\neq 0$ and the above process decrease the length of the word, we will arrive at some $a^{(k)}\in A$ such that $S_{a^{(k)}}\neq\emptyset$ and therefore we will always be able to reduce to this case.

If $S_a\neq\emptyset$ we can take $p_\ell\in S_a$ in such a way that for some $j_0\notin J_\ell$ we have $p_\ell\in\operatorname{supp}(\varepsilon(a_{j_0}))$. Set

$$a''=(p_\ell a_1,\dots,p_\ell a_{j_0-1},p_\ell a_{j_0}+p_1+\dots+p_{\ell-1}+p_{\ell+1}+\dots+p_r,p_\ell a_{j_0+1},\dots,p_\ell a_n).$$

By Lemma 1.8 we have that the j_0th component of a'' is invertible in $P((E))$, so that we can consider the following invertible matrix:

$$M=\begin{pmatrix}1&0&\dots&\dots&0\\ &\ddots&&&\\ a''_1&\dots&a''_{j_0}&\dots&a''_n\\ &&&\ddots&\\ 0&\dots&\dots&0&1\end{pmatrix}\in GL_n\bigl(P((E))\bigr)$$

obtained by substituting the j_0th row of the identity matrix by a''.

For all $b=(b_1,\dots,b_n)^t\in B$ we have that

$$Mb=\bigl(b_1,\dots,b_{j_0-1},b'_{j_0},b_{j_0+1},\dots,b_n\bigr)^t,$$

where $b'_{j_0}=p_\ell ab+(p_1+\dots+p_{j_0-1}+p_{j_0+1}+\dots+p_n)b_{j_0}$. Hence, if we substitute A by AM^{-1} and B by MB, this will not change the sets J_i for $i\neq\ell$ but will increase $|J_\ell|$ by one (we now have $j_0\in J_\ell$). Repeating the above process a finite number of times, we will arrive at the case where $B\subseteq {}^nP(E)$. $\square$

Let R be a ring. Recall that an R-module P is *stably free* if $P\oplus R^m\cong R^n$ for some $m,n\in\mathbb{N}$.

Definition 1.11. [14, p. 15] A ring R is called a *Hermite ring* if it has IBN (invariant basis number) and the stably free modules are free.

Recall that a *full matrix* over a ring R is a square matrix A over R, of size $n \times n$ say, such that A cannot be written as a product $A = BC$, where $B \in M_{n\times(n-1)}(R)$ and $C \in M_{(n-1)\times n}(R)$, see [14, p. 159]. We need the following lemma:

Lemma 1.12. *If A is a full matrix over $P(E)$ then*

$$A \oplus \mathbf{1}_m := \begin{pmatrix} A & 0 \\ 0 & \mathbf{1}_m \end{pmatrix}$$

is also full, for every $m \geqslant 0$.

Proof. It follows from Proposition 1.2 that $P(E)$ is a Hermite ring. Now the result is a consequence of [14, Proposition 5.6.2]. $\square$

Definition 1.13. [14, p. 250] A ring homomorphism $f : R \to S$ is *honest* if it sends full matrices over R to full matrices over S.

Corollary 1.14. *The inclusion $P(E) \hookrightarrow P((E))$ is a honest inclusion.*

Proof. Let $C \in M_n(P(E))$ be a matrix such that C is not a full matrix over $P((E))$. Then we can write $C = AB$ where $A \in M_{n\times\ell}(P((E)))$ and $B \in M_{\ell\times n}(P((E)))$ with $\ell < n$. By Theorem 1.10 there exist $m \in \mathbb{N}$ and $u, v \in M_{\ell+m}(P((E)))$ such that

$$\begin{pmatrix} C & 0 \\ 0 & \mathbf{1}_m \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \mathbf{1}_m \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & \mathbf{1}_m \end{pmatrix} = \left(\begin{pmatrix} A & 0 \\ 0 & \mathbf{1}_m \end{pmatrix} u\right)\left(v \begin{pmatrix} B & 0 \\ 0 & \mathbf{1}_m \end{pmatrix}\right),$$

so that we obtain a decomposition of the matrix $C \oplus \mathbf{1}_m$ in $P(E)$, showing that it is not full over $P(E)$. By Lemma 1.12 the matrix C is not full over $P(E)$, as required. $\square$

We recall the following notation and definitions; see for example [22, 10.2.2].

Notation 1.15. Given a ring S and a subring $R \subseteq S$, we denote by $T(R \subseteq S)$ the set of all the elements of R which are invertible in S, and we denote by $\Sigma(R \subseteq S)$ the set of all square matrices over R which are invertible over S.

Definition 1.16. Let S be a ring.

(i) A subring $R \subseteq S$ is *closed under inversion* in S if $T(R \subseteq S) = GL_1(R)$, that is, if $GL_1(R) = R \cap GL_1(S)$.
(ii) A subring $R \subseteq S$ is *rationally closed* in S if $\Sigma(R \subseteq S) = GL(R)$, that is, if $GL(R) = M(R) \cap GL(S)$.
(iii) Given a subring $R \subseteq S$ the *division closure* of R in S, denoted by $\mathcal{D}(R \subseteq S)$, is the smallest subring of S which is closed under inversion and contains R.
(iv) Given a subring $R \subseteq S$ the *rational closure* of R in S, denoted by $\mathcal{R}(R \subseteq S)$, is the smallest subring of S which is rationally closed and contains R.

Note that the division closure and the rational closure always exist, because the relevant properties are inherited by intersections.

Definition 1.17. The *K-algebra of rational series over the quiver E*, denoted by $P_{\mathrm{rat}}(E)$, is the division closure of $P(E)$ in $P((E))$.

Observation 1.18. We note that $P_{\mathrm{rat}}(E)$ is also the rational closure of $P(E)$. Indeed if M is a matrix over $P_{\mathrm{rat}}(E)$ that becomes invertible in $P((E))$, then $\varepsilon(M)$ is invertible over K^d, and replacing M with $\varepsilon(M)^{-1}M$, we may assume that $\varepsilon(M)$ is an identity matrix. Hence the diagonal entries of M are invertible in $P((E))$ by Lemma 1.8 and so are invertible in $P_{\mathrm{rat}}(E)$. By applying to M a suitable sequence of elementary row transformations, we may further assume that M is diagonal. It follows that M is invertible over $P_{\mathrm{rat}}(E)$, as claimed.

Observation 1.19. The same argument as above shows that if R is a subalgebra of $P((E))$ closed under inversion and containing K^d then a matrix M over R is invertible over R if and only if $\varepsilon(M)$ is invertible over K^d.

Set $\Sigma := \Sigma(P(E) \subseteq P((E)))$, and let $\iota : P(E) \to \Sigma^{-1}P(E)$ be the *universal localization* of $P(E)$ with respect to Σ, cf. [14,25]. By the universal property, we get a unique K-algebra homomorphism $f : \Sigma^{-1}P(E) \to P_{\mathrm{rat}}(E)$ such that $\phi = f \circ \iota$, where we denote by $\phi : P(E) \to P_{\mathrm{rat}}(E)$ the natural inclusion. It follows from a well-known general fact (see for instance [22, Lemma 10.35(3)]) that the map f is surjective.

Theorem 1.20. *Let $\Sigma = \Sigma(P(E) \subseteq P((E)))$. Then $P_{\mathrm{rat}}(E)$ coincides with the universal localization of $P(E)$ with respect to Σ.*

Proof. Consider $\iota : P(E) \to \Sigma^{-1}P(E)$, the universal localization of the path algebra with respect to the set Σ. As observed above, we have a surjective K-algebra homomorphism $f : \Sigma^{-1}P(E) \to P_{\mathrm{rat}}(E)$, and we want to see that it is injective.

By Corollary 1.14 we have that the inclusion $P(E) \hookrightarrow P((E))$ is honest. Moreover, it is Σ-inverting and it is easily seen that Σ is multiplicative and factor-closed (see [14, Chapter 7] for the definitions of these concepts). It follows from [14, Proposition 7.5.7(ii)] that f is injective. $\square$

2. Construction of the algebras

In this section, we will give the basic construction of the algebras associated with a finite quiver.

Definition 2.1. Given a quiver $E = (E^0, E^1, r, s)$, consider the sets $\overline{E}^0 = E^0$, $\overline{E}^1 = \{\bar{e} \mid e \in E^1\}$ and the maps $\bar{r}, \bar{s} : \overline{E}^1 \to \overline{E}^0$ defined via $\bar{r}(\bar{e}) = s(e)$ and $\bar{s}(\bar{e}) = r(e)$. Define the *inverse quiver* of E as the quiver $\overline{E} = (\overline{E}^0, \overline{E}^1, \bar{r}, \bar{s})$.

Notation 2.2. Given a path $\alpha = e_1 \cdots e_n \in E^*$ denote by $\bar{\alpha} = \bar{e}_n \cdots \bar{e}_1$ the corresponding path in the inverse quiver. Of course, if $i \in E^0$, then $\overline{p}_i = p_i$.

Set $R = P(E)$ or $P((E))$. For $e \in E^1$ we define the following K-algebra endomorphism,

$$\begin{aligned} \tau_e : R &\to R, \\ p_{s(e)} &\mapsto p_{r(e)}, \\ p_{r(e)} &\mapsto p_{s(e)}, \\ p_i &\mapsto p_i, \quad i \neq s(e),\ r(e), \\ f &\mapsto 0, \quad \forall f \in E^1. \end{aligned}$$

It is clear that they are K-algebra endomorphisms, since they are defined by the composition of the augmentation with an automorphism of $\varepsilon(R)$ and the inclusion of $\varepsilon(R)$ in R. We will write τ_e on the right of its argument (and compositions will act accordingly).

Definition 2.3. Let R be a ring and $\tau : R \to R$ a ring endomorphism. A left τ-*derivation* is an additive mapping $\delta : R \to R$ satisfying $(rs)\delta = (r\delta) \cdot (s\tau) + r \cdot (s\delta)$ for all $r, s \in R$.

Lemma 2.4. *For every $e \in E^1$, δ_e is a left τ_e-derivation.*

Proof. Set $r = \sum_{\alpha \in E^*} \lambda_\alpha \alpha$ and $s = \sum_{\beta \in E^*} \mu_\beta \beta$. Its product is $rs = \sum_{\gamma \in E^*} \nu_\gamma \gamma$ where $\nu_\gamma = \sum_{\gamma = \alpha\beta} \lambda_\alpha \mu_\beta$. On the one hand, we have that, if $s(e) \neq r(e)$,

$$\begin{aligned} (r\delta_e) \cdot (s\tau_e) &= \Bigg(\sum_{\substack{\alpha \in E^* \\ r(\alpha) = s(e)}} \lambda_{\alpha e} \alpha \Bigg) \Bigg(\mu_{r(e)} p_{s(e)} + \mu_{s(e)} p_{r(e)} + \sum_{\substack{i \in E^0 \\ i \neq r(e), s(e)}} \mu_i p_i \Bigg) \\ &= \sum_{\substack{\alpha \in E^* \\ r(\alpha) = s(e)}} (\lambda_{\alpha e} \mu_{r(e)}) \alpha \end{aligned}$$

and note that, in case $s(e) = r(e)$, we get indeed the same expression. Also,

$$r \cdot (s\delta_e) = \Bigg(\sum_{\alpha \in E^*} \lambda_\alpha \alpha \Bigg) \Bigg(\sum_{\substack{\beta \in E^* \\ r(\beta) = s(e)}} \mu_{\beta e} \beta \Bigg) = \sum_{\substack{\gamma \in E^* \\ r(\gamma) = s(e)}} \Bigg(\sum_{\gamma = \alpha\beta} \lambda_\alpha \mu_{\beta e} \Bigg) \gamma.$$

On the other hand, we see that

$$\begin{aligned} (rs)\delta_e &= \Bigg(\sum_{\gamma \in E^*} \nu_\gamma \gamma \Bigg) \delta_e = \sum_{\substack{\gamma \in E^* \\ r(\gamma) = s(e)}} \nu_{\gamma e} \gamma = \sum_{\substack{\gamma \in E^* \\ r(\gamma) = s(e)}} \Bigg(\sum_{\gamma e = \alpha\beta} \lambda_\alpha \mu_\beta \Bigg) \gamma \\ &= \sum_{\substack{\gamma \in E^* \\ r(\gamma) = s(e)}} \Bigg(\sum_{\gamma = \alpha\beta} \lambda_\alpha \mu_{\beta e} \Bigg) \gamma + \sum_{\substack{\gamma \in E^* \\ r(\gamma) = s(e)}} (\lambda_{\gamma e} \mu_{r(e)}) \gamma. \end{aligned}$$

Therefore, $(rs)\delta_e = (r\delta_e) \cdot (s\tau_e) + r \cdot (s\delta_e)$. $\square$

In the rest of this section, E will denote a finite quiver with $E^0 = \{1, \ldots, d\}$.

Proposition 2.5. *Given a quiver E and a K-subalgebra R of $P((E))$, containing $P(E)$ and closed under all the left transductions δ_e, there exists a ring S such that*:

(i) *There are embeddings*

$$\begin{aligned} \mathcal{R}: R &\to S, \\ r &\mapsto \mathcal{R}_r, \end{aligned} \quad \text{and} \quad \begin{aligned} z: P(\overline{E}) &\to S, \\ \bar{\alpha} &\mapsto z_{\bar{\alpha}}, \end{aligned}$$

such that $z_{p_i} = \mathcal{R}_{p_i}$ for all i and

$$\mathcal{R}_r \cdot z_{\bar{e}} = z_{\bar{e}} \cdot \mathcal{R}_{(r\tau_e)} + \mathcal{R}_{(r\delta_e)} \tag{2.1}$$

for all $e \in E^1$ and all $r \in R$.

(ii) *S is projective as a right R-module. Indeed, $S = \bigoplus_{\gamma \in E^*} S_\gamma$ with $S_\gamma \cong p_{s(\gamma)} R$ as R-modules. Moreover, every element of S can be uniquely written as a finite sum $\sum_{\gamma \in E^*} z_{\bar{\gamma}} \mathcal{R}_{a_\gamma}$, where $a_\gamma \in p_{s(\gamma)} R$ for all $\gamma \in E^*$.*

Proof. Set $T = \mathrm{End}_K(R)$. The elements of T will act on the right of their arguments. For $r \in R$ denote by $\mathcal{R}_r$ the operator in T given by right multiplication by r. The map $\mathcal{R}: R \to T$ is clearly an injective K-algebra morphism.

For each $\bar{e} \in \overline{E}^1$ consider the elements $z_{\bar{e}} \in T$ defined by

$$(r) z_{\bar{e}} = (r)\delta_e.$$

Let S be the subring of T generated by R and by all the elements $z_{\bar{e}}$ defined above. For $\bar{e} \in \overline{E}^1$, we have

$$z_{\bar{e}} = z_{\bar{e}} \mathcal{R}_{p_{s(e)}} = \mathcal{R}_{p_{r(e)}} z_{\bar{e}},$$

so that there exists a unique K-algebra morphism $z: P(\overline{E}) \to S$ such that $z(\bar{e}) = z_{\bar{e}}$ for all $e \in E^1$ and $z(p_i) = \mathcal{R}_{p_i}$ for all $i \in E^0$.

For $r, s \in R$ and $e \in E^1$, we have

$$\begin{aligned} (s)(\mathcal{R}_r z_{\bar{e}}) &= (sr) z_{\bar{e}} = (sr)\delta_e = (s\delta_e) \cdot (r\tau_e) + s \cdot (r\delta_e) \\ &= (s)[z_{\bar{e}} \mathcal{R}_{(r\tau_e)} + \mathcal{R}_{(r\delta_e)}]. \end{aligned}$$

Hence,

$$\mathcal{R}_r \cdot z_{\bar{e}} = z_{\bar{e}} \cdot \mathcal{R}_{(r\tau_e)} + \mathcal{R}_{(r\delta_e)}$$

for all $\bar{e} \in \overline{E}^1$ and all $r \in R$, which shows the formula (2.1). From this we conclude that S is generated as a right R-module by monomials $z_{\bar{\gamma}}$ where $\gamma \in E^*$, and thus every element in S can be written as a finite sum $\sum_{\gamma \in E^*} z_{\bar{\gamma}} \mathcal{R}_{a_\gamma}$, where $a_\gamma \in p_{s(\gamma)} R$ for all $\gamma \in E^*$. It remains to check uniqueness of the expression. For, assume that we have $\sum_{\gamma \in E^*} z_{\bar{\gamma}} \mathcal{R}_{a_\gamma} = 0$, where $a_\gamma \in p_{s(\gamma)} R$

are not all 0. Let $\gamma_0 \in E^*$ be a path of minimal length in the support of this expression, so that $a_{\gamma_0} \neq 0$. Observe that

$$0 = (\gamma_0)\left(\sum_{\gamma \in E^*} z_{\overline{\gamma}} \mathcal{R}_{a_\gamma}\right) = p_{s(\gamma_0)} a_{\gamma_0} = a_{\gamma_0},$$

which gives a contradiction. It follows that every element in S can be uniquely written as a finite sum $\sum_{\gamma \in E^*} z_{\overline{\gamma}} \mathcal{R}_{a_\gamma}$, where $a_\gamma \in p_{s(\gamma)} R$ for all $\gamma \in E^*$, which gives (ii) and also gives the injectivity of the map $z : P(\overline{E}) \to S$. This completes the proof. $\square$

Notation 2.6. We will denote the ring S of Proposition 2.5 by $R\langle \overline{E}; \tau, \delta\rangle$ where τ and δ stand for $(\tau_e)_{e \in E^1}$ and $(\delta_e)_{e \in E^1}$, respectively. Moreover, since the maps $\mathcal{R} : R \to S$ and $z : P(\overline{E}) \to S$ are injective, we will identify the elements of R and of $P(\overline{E})$ with their images in S under these maps. Note that the fundamental relation (2.1) in Proposition 2.5 becomes $r\bar{e} = \bar{e}(r\tau_e) + (r\delta_e)$ for all $e \in E^1$. In particular, $f\bar{e} = \delta_{e,f} p_{s(e)}$ for $e, f \in E^1$. With this notation, Proposition 2.5(ii) says that each element $s \in R\langle \overline{E}; \tau, \delta\rangle$ can be uniquely written as a finite sum

$$s = \sum_{\gamma \in E^*} \overline{\gamma} a_\gamma, \tag{2.2}$$

where $a_\gamma \in p_{s(\gamma)} R$ for all $\gamma \in E^*$.

The algebra $R\langle \overline{E}; \tau, \delta\rangle$ is characterized by the following universal property:

Proposition 2.7. *Let $\phi : R \to B$ be a homomorphism of K-algebras and assume that for each $e \in E^1$ there exists $t_{\bar{e}} \in (p_{r(e)}\phi) B (p_{s(e)}\phi)$ such that $(r\phi) t_{\bar{e}} = t_{\bar{e}}(r\tau_e\phi) + (r\delta_e\phi)$ for all $r \in R$ and all $e \in E^1$. Then ϕ can be uniquely extended to a K-algebra homomorphism $\overline{\phi} : R\langle \overline{E}; \tau, \delta\rangle \to B$ such that $\bar{e}\overline{\phi} = t_{\bar{e}}$ for all $e \in E^1$.*

Proof. The proof is similar to the one of [4, Proposition 3.3]. Set $S = R\langle \overline{E}; \tau, \delta\rangle$. It is enough to build a K-algebra with the universal property and to show that it is isomorphic with S. Let $F := R *_{K^d} P(\overline{E})$ be the coproduct of R and $P(\overline{E})$ over K^d, with canonical maps $\psi_1 : R \to F$ and $\psi_2 : P(\overline{E}) \to F$. Let S_1 be the K-algebra obtained by imposing the relations $(r\psi_1)(\bar{e}\psi_2) = (\bar{e}\psi_2)(r\tau_e\psi_1) + (r\delta_e\psi_1)$ for all $e \in E^1$ and all $r \in R$. Then clearly S_1 satisfies the required universal property. In particular, we get a K-algebra homomorphism $S_1 \to S$ extending the canonical maps $R \to S$ and $P(\overline{E}) \to S$. Using the defining relations, we see that every element in S_1 can be written in the form $\sum_{\gamma \in E^*} (\overline{\gamma}\psi_2)((p_{s(\gamma)} a_\gamma)\psi_1)$, where $a_\gamma \in R$. Now it follows from Proposition 2.5 that the map $S_1 \to S$ is an isomorphism. $\square$

In the following R will denote a K-subalgebra of $P((E))$ containing $P(E)$, closed under inversion (in $P((E))$) and closed under all the left transductions δ_e. Examples include the power series algebra $P((E))$ and the algebra $P_{\mathrm{rat}}(E)$ of rational series. Indeed this follows from the fact that elements $a \in \Sigma^{-1} P(E)$ can be written as

$$a = bA^{-1}c,$$

where $A \in \Sigma$ and b and c are a row vector and a column vector of suitable size. This implies that elements in $P_{\mathrm{rat}}(E)$ also have this expression (see Theorem 1.20). Observe that, for $A \in \Sigma$ we have $(A^{-1})\delta_e = -A^{-1} \cdot (A\delta_e) \cdot (A\tau_e)^{-1} \in M_n(P_{\mathrm{rat}}(E))$ and $(A^{-1})\tau_e = (A\tau_e)^{-1} \in M_n(P_{\mathrm{rat}}(E))$ for some $n \geqslant 1$, so the result follows. Of course a similar argument shows that $P_{\mathrm{rat}}(E)$ is closed under all the right transductions $\tilde{\delta}_e$.

Let $X \subseteq E^0$ be the set of vertices which are not sources. Given a vertex $i \in X$, consider the following element:

$$q_i = p_i - \sum_{e \in r^{-1}(i)} \bar{e}e \in R\langle \overline{E}; \tau, \delta\rangle.$$

Lemma 2.8. *The elements q_i defined above are pairwise orthogonal, nonzero idempotents and $q_i \leqslant p_i$ for all $i \in X$.*

Proof. Using the relations $e\bar{f} = \delta_{e,f} p_{s(e)}$ and the relations in $P(E)$ and $P(\overline{E})$ we have that

$$q_i^2 = p_i^2 - \sum_{e \in r^{-1}(i)} \bar{e}ep_i - \sum_{e \in r^{-1}(i)} p_i\bar{e}e + \left(\sum_{e \in r^{-1}(i)} \bar{e}e\right)^2 = q_i,$$

moreover, $q_i p_i = p_i q_i = q_i$ so that q_i are idempotent elements and $q_i \leqslant p_i$. Since the p_i's are pairwise orthogonal, it is clear that the q_i's are also orthogonal. It follows from Proposition 2.5(ii) that $q_i \neq 0$ for all $i \in X$. □

Notation 2.9. We write $q = \sum_{i \in X} q_i = \sum_{i \in X} p_i - \sum_{e \in E^1} \bar{e}e$ which is, by the above lemma, an idempotent.

Lemma 2.10. *Let R be a K-subalgebra of $P((E))$ containing $P(E)$, closed under inversion and closed under all the left transductions δ_e. Set $S = R\langle \overline{E}; \tau, \delta\rangle$ and $I = SqS$, the two-sided ideal generated by the idempotent q. Then the following properties hold*:

(1) *If $r \in R \setminus \{0\}$ then there exists $y \in S$ such that $p_i r y = p_i$ for some $i \in E^0$.*
(2) *If $s \in S \setminus I$ then there are $s_1, s_2 \in S$ such that $s_1 s s_2 = p_i$ for some $i \in E^0$.*
(3) *If $s \in I \setminus \{0\}$ then there exist $s_1, s_2 \in S$ such that $s_1 s s_2 = q_i$ for some $i \in X$.*

Proof. (1) Take $r \in R \setminus \{0\}$, with order k. Let $w \in E^*$ be a path of length k in the support of r and put $i = s(w)$. Then $r\overline{w} = \lambda p_i + r'$ where $\lambda \in K \setminus \{0\}$ and r' is an element in R of order different from 0. Thus it follows that $r\overline{w} + (1 - p_i)$ is an invertible element in R. Let t be the inverse of $r\overline{w} + (1 - p_i)$, and observe that

$$p_i r(\overline{w}t) = p_i,$$

as wanted.

(2) Let $s \in S \setminus I$. By Proposition 2.5 we know that s can be written as a (finite) right R-linear combination $s = \sum_{\gamma \in E^*} \overline{\gamma} a_\gamma$, where $a_\gamma \in p_{s(\gamma)}R$. Observe that $p_j\bar{e} = 0$ for all $e \in E^1$ and all $j \in E^0 \setminus X$, so that $p_j s \in R$ for all $j \in E^0 \setminus X$. Therefore if there exists $j \in E^0 \setminus X$ such that $p_j s \neq 0$ then the result follows from part (1).

So we can assume that $s = p_X s$, where $p_X = \sum_{i\in X} p_i$. By an obvious induction, it is enough to show that there is $e \in E^1$ such that $es \notin I$. If $es \in I$ for all $e \in E^1$ then we have

$$s = qs + (p_X - q)s = qs + \sum_{e\in E^1} \bar{e}es \in I,$$

a contradiction with our hypothesis. This shows the result.

(3) Since for all $i, j \in X$, all $e \in E^1$ and all $r \in R$, we have $q_i\bar{e} = 0$ and $rq_j = \varepsilon_j(r)q_j \in Kq_j$, we see that $q_iSq_j = \delta_{i,j}q_iK \cong K$. In particular, we see that $I = \sum_{i\in X} Sq_iS$ and indeed, using the above relations and Proposition 2.5, we get that every element $s \in I$ can be uniquely written as a finite sum

$$s = \sum_{i\in X} \sum_{\{\gamma\in E^*\mid s(\gamma)=i\}} \overline{\gamma} q_i a_\gamma, \tag{2.3}$$

where $a_\gamma \in p_{s(\gamma)}R$.

Now if $a_\gamma = 0$ for every $\gamma \in E^*$ of positive length, then the result follows from (1). Assume that $a_\gamma \neq 0$ for some $\gamma \in E^*$ of positive length. By induction, it suffices to show there is $e \in E^1$ such that $es \neq 0$. Let $w \in E^*$ be a path of maximum length in the support of s (with respect to the above expression) and let e be the final arrow in w, in such a way that $\overline{w} = \bar{e}\overline{w'}$ for some w'. Then

$$es = \sum_{i\in X} \sum_{\{\gamma\in E^*\mid s(\gamma)=i\}} \overline{(\gamma\delta_e)} \cdot q_i \cdot a_\gamma$$

is a nonzero element in I. □

Proposition 2.11. *Let R be a K-subalgebra of $P((E))$ containing $P(E)$, closed under inversion and closed under all the left transductions δ_e. The ring $S = R\langle \overline{E}; \tau, \delta\rangle$ is a semiprime ring and SqS is a direct summand of* $\mathrm{Soc}(S)$. *Moreover, SqS and* $\mathrm{Soc}(S)$ *are both von Neumann regular ideals of S.*

Proof. It will be a convenient notation in this proof to set $q_i = p_i$ for $i \in E^0 \setminus X$. Given $s \in S \setminus \{0\}$, we have by Lemma 2.10 that $q_i \in SsS$ for some $i \in E^0$. Since q_i are nonzero idempotents, we get that $(SsS)^2 \neq \{0\}$, which shows that S is a semiprime ring.

As observed in the proof of Lemma 2.10, we have $q_iSq_j = \delta_{i,j}q_iK \cong \delta_{i,j}K$ for all i, j. In particular, q_iSq_i is a division ring (indeed a field) and so by [20, Proposition 21.16(2)] the right ideals q_iS are minimal. Hence we get $Sq_iS \subseteq \mathrm{Soc}(S)$ for all $i \in E^0$.

Now we show that $\mathrm{Soc}(S) \subseteq \bigoplus_{i\in E^0} Sq_iS$. By definition $\mathrm{Soc}(S)$ is the sum of all the minimal right (or left) ideals of S (see [20, p. 186]). Since S is semiprime, every minimal right ideal of S is of the form eS, where e is a (nonzero) idempotent in S (see e.g. [20, Corollary 10.23]). If e is an idempotent such that eS is a minimal right ideal, then by Lemma 2.10 there exist $s_1, s_2 \in S$ such that $s_1es_2 = q_i$ for some $i \in E^0$. Since eS is a minimal right ideal, we have that $(es_2)S = eS$ and, so there exists $s_3 \in S$ such that $es_2s_3 = e$. Moreover by [20, Lemma 11.9] we have that $S(es_2)$ is a minimal left ideal and, since $q_i \in S(es_2)$, we get $S(es_2) = Sq_i$ so that there exists $s_4 \in S$ such that $s_4q_i = es_2$. Finally, $e = es_2s_3 = s_4q_is_3 \in Sq_iS$ which proves the desired inclusion. We also see now that SqS is a direct summand of $\mathrm{Soc}(S)$.

Now we show that $\mathrm{Soc}(S)$ and SqS are (von Neumann) regular ideals. Observe that SqS is the orthogonal sum of the ideals Sq_iS, $i=1,\dots,d$, and that these ideals are simple rings (possibly without unit) and contain a minimal one-sided ideal. Thus the result follows from Litoff's theorem (see [17]). $\square$

We recall the following result from [4], which will be very useful later on.

Lemma 2.12. [4, Lemma 5.3] *Let A be a left semihereditary ring and let $B=\Sigma^{-1}A$ be a universal localization of A. Suppose that for every finitely presented right A-module M such that* $\mathrm{Hom}_A(M,A)=0$*, we have that $M\otimes_A B=0$. Then B is a von Neumann regular ring and every finitely generated projective B-module is induced by a finitely generated projective A-module.*

Let R be a K-subalgebra of $P((E))$ containing $P(E)$. As before, let $X=E^0\setminus \mathrm{Sour}(E)$ be the set of vertices which are not sources in E. For $i\in X$ put $r^{-1}(i)=\{e_1^i,\dots,e_{n_i}^i\}$ and consider the right R-module homomorphisms

$$\mu_i : p_iR \to \bigoplus_{j=1}^{n_i} p_{s(e_j^i)}R,$$
$$r \mapsto \left(e_1^i r,\dots,e_{n_i}^i r\right).$$

Write $\Sigma_1=\{\mu_i \mid i\in X\}$. Observe that the elements of Σ_1 are homomorphisms between finitely generated projective right R-modules, so that we can consider the universal localization $\Sigma_1^{-1}R$.

Proposition 2.13. *Let R be a K-subalgebra of $P((E))$ containing $P(E)$ and closed under the left transductions δ_e. Set $S=R\langle \overline{E};\tau,\delta\rangle$, let I be the ideal of S generated by q and let Σ_1 be as above. Then $\Sigma_1^{-1}R\cong S/I$.*

Proof. Set $T:=S/I$. Given $s\in S$ we will denote by $\tilde{s}$ its class in T. As before, we will identify R with a subring of S. Let $f:R\to T$ be the composition of the inclusion $\iota:R\hookrightarrow S$ with the canonical projection $\pi:S\to T$. By the above identification, we can write $f(r)=\tilde{r}$.

We want to see that f is a universal Σ_1-inverting homomorphism. Define, for $i\in X$, the following right T-module homomorphisms

$$\tilde{\mu}_i : \left(\bigoplus_{j=1}^{n_i} p_{s(e_j^i)}R\right)\otimes_R T \to p_iR\otimes_R T,$$
$$(r_1,\dots,r_{n_i})\otimes t \mapsto p_i\otimes\left(\sum_{j=1}^{n_i}\widetilde{\overline{e_j^i}}\, r_j t\right).$$

Now, the relations $\tilde{p}_i=\sum_{j=1}^{n_i}\widetilde{\overline{e_j^i}e_j^i}$ and $\widetilde{e\overline{f}}=\delta_{e,f}\tilde{p}_{s(e)}$ in T give that $\tilde{\mu}_i=(\mu_i\otimes \mathbf{1}_T)^{-1}$. Therefore f is Σ_1-inverting.

To show that f is universal Σ_1-inverting, consider a K-algebra A and a Σ_1-inverting algebra homomorphism $g : R \to A$. For $i \in X$ the following diagram is commutative:

$$\begin{array}{ccc} (\bigoplus_{j=1}^{n_i} p_{s(e^i_j)} R) \otimes_R A & \xrightarrow{(\mu_i \otimes \mathbf{1}_A)^{-1}} & p_i R \otimes_R A \\ \cong \big\downarrow & & \big\downarrow \cong \\ \bigoplus_{j=1}^{n_i} g(p_{s(e^i_j)}) A & \xrightarrow{(a^i_1, \ldots, a^i_{n_i}) \cdot} & g(p_i) A \end{array}$$

for some $a^i_j \in g(p_i) A g(p_{s(e^i_j)})$. From this we conclude that the compositions

$$\left(a^i_1, \ldots, a^i_{n_i}\right) \begin{pmatrix} g(e^i_1) \\ \vdots \\ g(e^i_{n_i}) \end{pmatrix} = g(p_i) \tag{2.4}$$

and

$$\begin{pmatrix} g(e^i_1) \\ \vdots \\ g(e^i_{n_i}) \end{pmatrix} \left(a^i_1, \ldots, a^i_{n_i}\right) = \operatorname{diag}\left(g(p_{s(e^i_1)}), \ldots, g(p_{s(e^i_{n_i})})\right), \tag{2.5}$$

give the identities on $g(p_i)A$ and on $\bigoplus_{j=1}^{n_i} g(p_{s(e^i_j)})A$, respectively.

Take $e \in E^1$; we have that $e = e^i_j$ for some $e^i_j \in r(i)^{-1}$, where $i = r(e)$. Putting $t_{\bar{e}} = a^i_j$ we conclude from (2.5) that $g(e)t_{\bar{e}} = g(p_{s(e)})$ and $g(e^i_k)t_{\bar{e}} = 0$ for $k \neq j$. Moreover, if we take $f \in E^1$ such that $r(f) \neq i$ we have that $g(f)t_{\bar{e}} = g(f)g(p_{r(f)})g(p_i)t_{\bar{e}} = 0$. We are thus in the hypothesis of Proposition 2.7 and there exists a unique algebra homomorphism $\bar{g} : S \to A$ extending g and such that $\bar{g}(\bar{e}) = t_{\bar{e}}$ for all $e \in E^1$. From (2.4), we get that $g(p_i) = \sum_{j=1}^{n_i} a^i_j g(e^i_j)$ in A, which entails that $p_i - \sum_{j=1}^{n_i} \overline{e^i_j} e^i_j \in \ker(\bar{g})$. Hence $\bar{g}$ factorizes uniquely through T and we have $h : T \to A$ such that $h \circ \pi = \bar{g}$. Now, $g = \bar{g} \circ \iota = h \circ \pi \circ \iota = h \circ f$ and h is unique by uniqueness of inverses and the fact that T is generated by R and $\overline{E}^1$. We have seen that f is universal Σ_1-inverting. Therefore $\Sigma_1^{-1} R \cong T$. □

Remark 2.14. The uniqueness of the expression in (2.2) for elements in $S = R\langle \overline{E}; \tau, \delta \rangle$ and the uniqueness of the expression in (2.3) for elements in $I = SqS$ give that the natural maps $R \to T$ and $P(\overline{E}) \to T$ are both injective, where $T = S/I = \Sigma_1^{-1} R$. They also give that, for two K-algebras R_1 and R_2 such that $P(E) \subseteq R_1 \subseteq R_2 \subseteq P((E))$ and R_1 and R_2 are closed under all the transductions δ_e, we have that $I_1 = I_2 \cap S_1$, where $S_i = R_i\langle \overline{E}; \tau, \delta \rangle$ and $I_i = S_i q S_i$ for $i = 1, 2$. It follows that the natural map $T_1 = S_1/I_1 \to T_2 = S_2/I_2$ is injective.

Proposition 2.15. *Let R be a K-subalgebra of $P((E))$ containing $P(E)$ and closed under inversion and under all the right transductions $\tilde{\delta}_e$. Then R is left semihereditary.*

Proof. By [26] (see also [21, Proposition 7.63]), it is enough to show that for every $n \geqslant 1$, $M_n(R)$ is left Rickart, that is $\ell.\mathrm{ann}_{M_n(R)}(A)$ is generated by an idempotent for all $A \in M_n(R)$.

Let $A \in M_n(R)$. We will use the following notation for the annihilator: $I_A = \ell.\mathrm{ann}_{M_n(R)}(A)$. Given $U \in GL_n(R)$ we have that $I_{UA} = \{XU^{-1} \mid X \in I_A\}$.

Observe that I_A is a regular submodule of $M_n(R)$ in the sense of Section 1, that is if $X \in I_A$ and $o(X) > 0$ then $\tilde{\delta}_e(X) \in I_A$ for every $e \in E^1$. This will be used later in the proof.

Since $\varepsilon(I_A)$ is a left ideal of $M_n(K^d)$, which is semisimple, there exists an idempotent $D \in \mathrm{Idem}(M_n(K^d))$ such that $\varepsilon(I_A) = M_n(K^d)D$. We can therefore take $B \in I_A$ such that $\varepsilon(B) = D$. Thus we have that $B = D - B'$ for some $B' \in M_n(R)$ such that $o(B') > 0$. Moreover, we can assume that $DB' = B'$. If we set $U = \mathbf{1}_n - B'$, since R is closed under inversion, we get from Observation 1.19 that $U \in GL_n(R)$. Now, we see that $BU^{-1} = D$, so that $D \in I_{UA}$. Since $\varepsilon(U) = \mathbf{1}_n$ we also have that $\varepsilon(I_{UA}) = M_n(K^d)D$.

We now show that $I_{UA} = M_n(R)D$. We have shown before that $M_n(R)D \subseteq I_{UA}$. Suppose that there is $X \in I_{UA} \setminus M_n(R)D$. Substituting X by $X - XD$ we can assume that $X = X(\mathbf{1}_n - D)$. Writing $X = \sum_{\alpha \in E^*} \alpha\lambda_\alpha$ for some $\lambda_\alpha \in p_{r(\alpha)}M_n(K^d)$, we get $\lambda_\alpha = \lambda_\alpha(\mathbf{1}_n - D)$. On the other hand, if $o(X) = m$ we have that $X = \sum_{\alpha \in E^m} \alpha \cdot \tilde{\delta}_\alpha(X)$. Since I_{UA} is a regular submodule of $M_n(R)$, for every $\alpha \in E^m$ we have that $\tilde{\delta}_\alpha(X) \in I_{UA}$. Take $\alpha \in E^m$ such that $\lambda_\alpha \neq 0$. We have that $\varepsilon(\tilde{\delta}_\alpha(X)) = \lambda_\alpha$ but, since $\lambda_\alpha(\mathbf{1}_n - D) = \lambda_\alpha$, this leads us to a contradiction with $\varepsilon(I_{UA}) = M_n(K^d)D$. We have shown that $I_{UA} = M_n(R)D$ and we deduce that $I_A = M_n(R)H$ where $H = U^{-1}DU \in \mathrm{Idem}(M_n(R))$. It follows that $M_n(R)$ is left Rickart, as desired. □

Theorem 2.16. *Let E be a finite quiver and let R be a K-subalgebra of $P((E))$ containing $P(E)$ and closed under inversion and under all the transductions δ_e and $\tilde{\delta}_e$. Set $S = R\langle \overline{E}; \tau, \delta\rangle$, $I = SqS$ and $T = S/I$. Then T and S are von Neumann regular.*

Proof. By Proposition 2.13 we have that T is a universal localization of R and, moreover, by Proposition 2.15 R is left semihereditary. Let M be a finitely presented right R-module such that $\mathrm{Hom}_R(M, R) = 0$. We want to show that $M \otimes_R T = 0$. Consider the following presentation of M:

$$^{s}R \xrightarrow{\mathcal{L}_A} {}^{t}R \to M \to 0, \tag{2.6}$$

where $A \in M_{t\times s}(R)$. Adding some zero columns to A, we can assume that $t \leqslant s$. Applying the functor $\mathrm{Hom}_R(-, R)$ to (2.6) we obtain the exact sequence:

$$0 \to \mathrm{Hom}_R(M, R) \to R^t \xrightarrow{\mathcal{R}_A} R^s$$

and, since $\mathrm{Hom}_R(M, R) = 0$, we have that $\mathcal{R}_A$ is a monomorphism. By the right exactness of the functor $- \otimes_R T$, applied to (2.6), we get the exact sequence:

$$^{s}T \xrightarrow{\mathcal{L}_A} {}^{t}T \to M \otimes_R T \to 0.$$

We want to see that $A\,{}^{s}T = {}^{t}T$, that is, that the columns of A generate ${}^{t}T$ as a right T-module.

By a standard argument of linear algebra we know that there are matrices $P \in GL_t(K^d)$ and $Q \in GL_s(K^d)$ such that $P\varepsilon(A)Q = D$, where

$$D = \sum_{i=1}^{d} p_i \begin{pmatrix} \mathbf{1}_{r_i} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with } r_1, \dots, r_d \leqslant t. \tag{2.7}$$

Hence, $PAQ = D - X$, where $X \in M_{t\times s}(R)$ with $o(X) > 0$. Since $t \leqslant s$, we have that $D = (D'0)$ with $D' \in M_t(R)$. Observe that, in the case where $D' = \mathbf{1}_t$, it follows from Observation 1.19 that PAQ is right invertible over R so that $(PAQ)T^s = T^t$ and we are done.

Otherwise, consider the matrix $\binom{X}{0} \in M_s(R)$. By Observation 1.19 we know that $Q' = \mathbf{1}_s - \binom{X}{0} \in GL_s(R)$. Therefore,

$$PAQ(Q')^{-1} = (D-X)\left(\mathbf{1}_s + \begin{pmatrix} X \\ 0 \end{pmatrix} + \begin{pmatrix} X \\ 0 \end{pmatrix}^2 + \cdots\right) = D - (X_1 X_2),$$

where $X_1 \in M_t(R)$ satisfies that $(\mathbf{1}_t - D')X_1 = X_1$.

Again by Observation 1.19 we have that $\mathbf{1}_t - X_1 \in GL_t(R)$. Now, setting

$$A' = (\mathbf{1}_t - X_1)^{-1} PAQ(Q')^{-1} = D - (X_3 X_4),$$

we have that $X_3(\mathbf{1}_t - D') = (\mathbf{1}_t - D')X_3 = X_3$.

We distinguish two cases, depending on whether X_3 is zero or not. If $X_3 \neq 0$ we take α of minimal length amongst the monomials in the support of the entries of X_3. Suppose that α belongs to the ith column of X_3. Consider the column $v = (0,\dots,0,\bar\alpha,0,\dots,0)^t \in {}^sS$, where $\bar\alpha$ is in the ith position. From the condition $X_3(\mathbf{1}_t - D') = X_3$ we deduce that $Dv = 0$ and thus $A'v \in {}^tR$. In addition we have $(\mathbf{1}_t - D')A'v = A'v$ and $p_{s(\alpha)} \in \operatorname{supp}(A'v)$.

If $X_3 = 0$ then by multiplying A' on the right by the matrix $\left(\begin{smallmatrix} \mathbf{1}_t & -X_4 \\ 0 & \mathbf{1}_{s-t} \end{smallmatrix}\right) \in GL_s(R)$ we can assume that X_4 satisfies that $(\mathbf{1}_t - D')X_4 = X_4$. Since $\mathcal{R}_A$ is injective we have that $X_4 \neq 0$. As before, we take α of minimal length amongst all the monomials in the support of the entries of X_4 and we get a column $v = (0,\dots,0,\bar\alpha,0,\dots,0)^t \in {}^sS$ satisfying that $A'v \in {}^tR$, $(\mathbf{1}_t - D')A'v = A'v$ and $p_{s(\alpha)} \in \operatorname{supp}(A'v)$.

In each of the above two cases, we consider the matrix $A'' = (A' \;\; A'v) \in M_{t\times(s+1)}(R)$. We have

$$\varepsilon(A'') = \begin{pmatrix} D' & 0 & \varepsilon(A'v) \end{pmatrix}$$

so from the conditions $(\mathbf{1}_t - D')A'v = A'v$ and $p_{s(\alpha)} \in \operatorname{supp}(A'v)$, we infer that $\operatorname{rank}_{K^d}(\varepsilon(A'')) > \operatorname{rank}_{K^d}(\varepsilon(A))$. We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
{}^sR \arrow[r, "\mathcal{L}_A"] \arrow[d, "\cong"'] & {}^tR \arrow[r] \arrow[d, "\cong"] & M \arrow[r] \arrow[d, equal] & 0 \\
{}^sR \arrow[r, "\mathcal{L}_{A'}"] \arrow[d, "i"'] & {}^tR \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "f"] & 0 \\
{}^sR \oplus R \arrow[r, "\mathcal{L}_{A''}"] & {}^tR \arrow[r] & M' \arrow[r] & 0
\end{tikzcd}$$

where i denotes the inclusion in the first component and f exists by the universal property of the cokernel. Applying the functor $-\otimes_R T$ to the above diagram we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
{}^sT \arrow[r, "\mathcal{L}_A"] \arrow[d, "\cong"'] & {}^tT \arrow[r] \arrow[d, "\cong"] & M\otimes_R T \arrow[r] \arrow[d, equal] & 0 \\
{}^sT \arrow[r, "\mathcal{L}_{A'}"] \arrow[d] & {}^tT \arrow[r] \arrow[d, equal] & M\otimes_R T \arrow[r] \arrow[d] & 0 \\
{}^{s+1}T \arrow[r, "\mathcal{L}_{A''}"] & {}^tT \arrow[r] & M'\otimes_R T \arrow[r] & 0.
\end{tikzcd}$$

Since $A'\,{}^sT = A''\,{}^{s+1}T$ we have that $M\otimes_R T\cong M'\otimes_R T$. Moreover it is clear that $\mathcal{R}_{A''}: R^t\to R^{s+1}$ is injective, so that we can repeat the above argument. After a finite number of steps we will arrive at a matrix $B\in M_{t\times(s+\ell)}(R)$ such that $\varepsilon(B)=(\mathbf{1}_t\,0)$ and we see that $M\otimes_R T\cong \operatorname{coker}\mathcal{L}_B=0$.

Now it follows from Lemma 2.12 that $T\cong S/I\cong \Sigma_1^{-1}R$ is a von Neumann regular ring. Since, by Proposition 2.11, I is a von Neumann regular ideal of S, we get from [18, Lemma 1.3] that S is von Neumann regular too. □

3. The structure of finitely generated projective modules

Let E be a finite quiver and let R be a K-subalgebra of $P((E))$ containing $P(E)$ and closed under inversion and under all the transductions δ_e and $\tilde{\delta}_e$. Set $S=R\langle\overline{E};\tau,\delta\rangle$, $I=SqS$ and $T=S/I\cong\Sigma_1^{-1}R$ (see Section 2).

We have a commutative diagram of inclusion maps

$$\begin{tikzcd}
K^d \arrow[r] \arrow[d] & P(E) \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & P((E)) \arrow[d] \\
P(\overline{E}) \arrow[r] & L(E) \arrow[r] & T \arrow[r] & U
\end{tikzcd}$$

where $U=\Sigma_1^{-1}P((E))$, $T=\Sigma_1^{-1}R$ and $L(E)=\Sigma_1^{-1}P(E)$, being Σ_1 the set of homomorphisms between finitely generated projective modules defined in the previous section. In this section we will compute the structure of the monoid $\mathcal{V}(T)$, indeed we will show that the maps in the bottom row of the above diagram induce isomorphisms $\mathcal{V}(L(E))\cong\mathcal{V}(T)\cong\mathcal{V}(U)$. The algebras $L(E)$ are the Leavitt path algebras of [1,2,6].

The monoid $\mathcal{V}(L(E))$ has been computed in [6] as follows. Let F_E be the free abelian monoid on the set E^0. The nonzero elements of F_E can be written in a unique form up to permutation as $\sum_{i=1}^n v_i$, where $v_i\in E^0$. For $v\in E^0$ such that $r^{-1}(v)\neq\emptyset$, write

$$\mathbf{s}(v):=\sum_{\{e\in E^1\mid r(e)=v\}} s(e)\in F_E.$$

We define the monoid M_E as $M_E=F_E/{\sim}$, where $\sim$ is the congruence on F_E generated by all pairs $(v,\mathbf{s}(v))$ with $r^{-1}(v)\neq\emptyset$. By [6, Theorem 3.5], there is a canonical monoid isomorphism $M_E\cong\mathcal{V}(L(E))$ sending each generator $\bar{v}$ of M_E to the class in $\mathcal{V}(L(E))$ of the projective module

$p_v L(E)$. We shall now prove that the canonical map $\mathcal{V}(L(E)) \to \mathcal{V}(T)$ is an isomorphism for an algebra T as above.

Theorem 3.1. *There is a canonical isomorphism $M_E \cong \mathcal{V}(T)$.*

Proof. We have a natural monoid homomorphism $\varphi : M_E \cong \mathcal{V}(L(E)) \to \mathcal{V}(T)$. We have seen in the proof of Theorem 2.16 and in Proposition 2.15 that the ring R satisfies the hypothesis of Lemma 2.12, so we get that all the finitely generated projective T-modules are induced from finitely generated projective R-modules. Observe that $J(R) = \ker \varepsilon$, so that $R/J(R) \cong K^d$ and R is a semiperfect ring. This follows from the well-known characterization of $J(R)$ as the set of elements x in R such that $1 - xy$ is invertible for all $y \in R$. Since R is inversion-closed in $P((E))$ all the elements of the form $1 - x$, with $x \in \ker \varepsilon$ will be invertible in R. It follows that we have an isomorphism

$$\mathcal{V}(R) \cong \mathcal{V}\big(K^d\big).$$

We conclude that all the finitely generated projective T-modules are isomorphic to finite direct sums of modules of the form $p_i T$, where p_i are the basic idempotents in K^d. It follows that the map $\varphi : M_E \to \mathcal{V}(T)$ is surjective.

Now we will show injectivity of φ. Assume that $\bigoplus_{i=1}^n p_{v(i)} T \cong \bigoplus_{j=1}^m p_{w(j)} T$, where $v(i), w(j) \in E^0$. We want to show that $\sum_{i=1}^n v(i) \sim \sum_{j=1}^m w(j)$ in F_E. Since T is von Neumann regular, the refinement property for f.g. projective modules holds [18, Theorem 2.8], so that we can reduce ourselves to the case where $n = 1$. Let $\alpha : p_v T \to \bigoplus_{j=1}^m p_{v(j)} T$ be an isomorphism. Write $\alpha = (\alpha_1, \dots, \alpha_m)$ with $\alpha_j \in p_{v(j)} T p_v$. Each α_j can be written as $\alpha_j = \sum_k w_{jk} \gamma_{jk}$, where $w_{jk} \in \overline{E}^*$ and $\gamma_{jk} \in R$, all j, k, and since $\alpha_j \in p_{v(j)} T$, we can assume that $\bar{s}(w_{jk}) = v(j)$ for all j, k.

We proceed by induction on the maximum of the lengths of the paths w_{jk} appearing in these decompositions. If the maximum is 0 then the map α is induced from a map $p_v R \to \bigoplus_{j=1}^m p_{v(j)} R$ and so the result follows from Lemma 3.2. Assume that the maximum N_0 of the lengths of the paths w_{jk} is strictly greater than 0. Take any path $w_{j_0 k_0}$ of length N_0. Since $\bar{s}(w_{j_0 k_0}) = v(j_0)$, we see that $v(j_0)$ is not a source in E (i.e. is not a sink in $\overline{E}$), and so we may consider the right T-module isomorphism

$$\big((e)_{e \in r^{-1}(v(j_0))}\big)^t : p_{v(j_0)} T \to \bigoplus_{e \in r^{-1}(v(j_0))} p_{s(e)} T, \tag{3.1}$$

which is given by left multiplication by the column $((e)_{e \in r^{-1}(v(j_0))})^t$, with inverse given by left multiplication by the row $(\bar{e})_{e \in r^{-1}(v(j_0))}$. (The maps $(e)_{e \in r^{-1}(v(j_0))}$ are the maps $\mu_{v(j_0)} \otimes \mathbf{1}_T$ considered in Proposition 2.13.) Composing the isomorphism $\alpha : p_v T \to \bigoplus_{j=1}^m p_{v(j)} T$ with the isomorphism obtained by applying the above canonical isomorphisms to all the modules $p_{v(j)} T$ such that there is a path w_{jk} of length N_0 in the corresponding representation of α_j, we obtain a new isomorphism $\alpha' : p_v T \to \bigoplus_{j=1}^{m'} p_{w(j)} T$ such that the maximum length of the paths in $\overline{E}$ appearing in the representations of the elements $\alpha'_j \in T$ is less than N_0. Each of the isomorphisms (3.1) contributes to a basic transformation $\sum_{i=1}^{n} v(i) \to_1 \sum_{i \neq j} v(i) + \mathbf{s}(v(j))$ (see [6]), and therefore we have that $\sum_{j=1}^m v(j) \sim \sum_{j=1}^{m'} w(j)$ in F_E. By induction we have that $v \sim \sum_{j=1}^{m'} w(j)$ and thus $v \sim \sum_{j=1}^m v(j)$, which completes the proof. $\square$

The following lemma completes the proof of Theorem 3.1, its proof follows the lines of the one of [4, Lemma 5.5].

Lemma 3.2. *Let $p = p_v$ for a fixed $v \in E^0$ and $\alpha : pR \to \bigoplus_{i=1}^{s} p_{v(i)}R$ such that α becomes invertible over T. Then $v \sim \sum_{i=1}^{s} v_i$ in F_E.*

Proof. Write $\alpha = (\alpha_1, \dots, \alpha_s)^t$, where each $\alpha_i \in p_{v(i)}Rp$. We will construct, by induction on i, paths $w_i \in \overline{E}^*$ and invertible elements $g_i \in p_{v(i)}Rp_{v(i)}$ such that the following statements hold:

(A_i) There exists an invertible map $\alpha^{(i)} : pT \to \bigoplus_{i=1}^{s} p_{v(i)}T$ satisfying the following properties:

(1) $\alpha_{i+1}^{(i)}, \dots, \alpha_s^{(i)} \in R$.

(2) The inverse of $\alpha^{(i)}$ is the row $(w_1g_1, \dots, w_ig_i, \beta_{i+1}, \dots, \beta_s)$ for some elements $\beta_\ell \in pTp_{v(\ell)}$, $\ell = i+1, \dots, s$.

The statement is obvious for $i = 0$. Assume that $0 \leqslant i < s$ and that (A_i) holds. We will prove (A_{i+1}). Without loss of generality, we can assume that the order of the series $\alpha_{i+1}^{(i)}$ is less than or equal to the order of $\alpha_{i+t}^{(i)}$ for all $t \geqslant 2$. Choose a path $w_{i+1} \in \overline{E}^*$ with length equal to the order of $\alpha_{i+1}^{(i)}$ such that $w_{i+1} = pw_{i+1}p_{v(i+1)}$ and such that $\alpha_{i+1}^{(i)}w_{i+1}$ is invertible in $p_{v(i+1)}Rp_{v(i+1)}$. Let $g_{i+1} \in p_{v(i+1)}Rp_{v(i+1)}$ be the inverse of $\alpha_{i+1}^{(i)}w_{i+1}$ and note that

$$p_{v(i+1)} = \alpha_{i+1}^{(i)}w_{i+1}g_{i+1} = \alpha_{i+1}^{(i)}\beta_{i+1}.$$

It follows that

$$u := \beta_{i+1}\alpha_{i+1}^{(i)} + \big(p - w_{i+1}g_{i+1}\alpha_{i+1}^{(i)}\big)$$

is invertible in pTp with inverse

$$u^{-1} = w_{i+1}g_{i+1}\alpha_{i+1}^{(i)} + \big(p - \beta_{i+1}\alpha_{i+1}^{(i)}\big).$$

Therefore, $\alpha^{(i+1)} := \alpha^{(i)}u$ is invertible with inverse

$$u^{-1}(w_1g_1, \dots, w_ig_i, \beta_{i+1}, \dots, \beta_s).$$

Note that, for $t > 1$, we have

$$\alpha_{i+t}^{(i)}u = \alpha_{i+t}^{(i)}\big(p - w_{i+1}g_{i+1}\alpha_{i+1}^{(i)}\big).$$

Since the order of $\alpha_{i+t}^{(i)}$ is greater than or equal to the length of w_{i+1}, we conclude that $\alpha_{i+t}^{(i+1)} \in R$, and condition (1) of (A_{i+1}) holds. On the other hand, for $m \leqslant i$ we have $\alpha_{i+1}^{(i)}w_m = 0$ and so $u^{-1}w_mg_m = w_mg_m$. We also have

$$u^{-1}\beta_{i+1} = w_{i+1}g_{i+1}\alpha_{i+1}^{(i)}\beta_{i+1} + \big(p - \beta_{i+1}\alpha_{i+1}^{(i)}\big)\beta_{i+1} = w_{i+1}g_{i+1},$$

and so condition (2) of (A_{i+1}) is also satisfied. Therefore, the induction works.

Take $h_i = g_i \alpha_i^{(s)} \in p_{v(i)} T p$ for $i = 1, \dots, s$. Then

$$\sum_{i=1}^{s} w_i h_i = p, \tag{3.2}$$

$h_i w_i \neq 0$ for all i, and $h_i w_j = 0$ for $i \neq j$. We claim that these conditions imply $v \sim \sum_{i=1}^{s} v(i)$ in F_E. We proceed by induction on the maximum of the lengths of the w_i. If this maximum is 0 then $s = 1$ and $h_1 = p$. So assume that either $s > 1$ or $s = 1$ and the length of w_1 is $\geqslant 1$. In either case, all w_i are different from p. Note that $w_i = \overline{\gamma_i}$ for a path γ_i in E of length $\geqslant 1$ such that $s(\gamma_i) = v(i)$ and $r(\gamma_i) = v$. Let $e(i) \in E^1$ be the ending arrow of the path γ_i, so that $r(e(i)) = v$. For each $e \in E^1$ such that $r(e) = v$, define

$$A_e := \big\{ i \in \{1, \dots, s\} \bigm| e(i) = e \big\} = \big\{ i \in \{1, \dots, s\} \bigm| e w_i \neq 0 \big\}.$$

Then the set $\{1, \dots, s\}$ is the disjoint union of the sets A_e, for $e \in r^{-1}(v)$.

Fix an arrow $e \in E^1$ such that $r(e) = v$. Left multiplying (3.2) by e and right multiplying it by $\bar{e}$, we get

$$\sum_{i \in A_e} (e w_i)(h_i \bar{e}) = e p \bar{e} = p_{s(e)}.$$

Observe also that for $i, j \in A_e$ we have $(h_i \bar{e})(e w_j) = h_i w_j$. So this term is 0 if $i \neq j$ and nonzero if $i = j$. By induction, $p_{s(e)} \sim \sum_{i \in A_e} p_{v(i)}$. Therefore

$$p = p_v \sim \sum_{e \in r^{-1}(v)} p_{s(e)} \sim \sum_{e \in r^{-1}(v)} \sum_{i \in A_e} p_{v(i)} = \sum_{i=1}^{s} p_{v(i)}.$$

This shows the result. $\square$

4. The regular algebra of a quiver

In this section we will observe that our construction can be made functorial in E if we choose a suitable class of morphisms between quivers. This also enables us to extend the construction to all the column-finite quivers.

Definition 4.1. For a finite quiver E and a field K, we define *the regular algebra* of E as the algebra $Q(E)$ obtained by using the construction in Section 2 taking as coefficients $R = P_{\mathrm{rat}}(E)$. So we have

$$Q(E) = S/SqS = (\Sigma_1)^{-1}\big(P_{\mathrm{rat}}(E)\big),$$

where $S = (P_{\mathrm{rat}}(E))\langle \overline{E}; \tau, \delta \rangle$.

The algebra $Q(E)$ fits into a commutative diagram of injective algebra morphisms:

$$\begin{array}{ccccccc}
K^d & \longrightarrow & P(E) & \xrightarrow{\iota_\Sigma} & P_{\mathrm{rat}}(E) & \longrightarrow & P(\!(E)\!) \\
\big\downarrow & & {\scriptstyle \iota_{\Sigma_1}}\big\downarrow & & {\scriptstyle \iota_{\Sigma_1}}\big\downarrow & & {\scriptstyle \iota_{\Sigma_1}}\big\downarrow \\
P(\overline{E}) & \longrightarrow & L(E) & \xrightarrow{\iota_\Sigma} & Q(E) & \longrightarrow & U(E)
\end{array}$$

where $U(E) = \Sigma_1^{-1} P(\!(E)\!)$, $Q(E) = \Sigma_1^{-1} P_{\mathrm{rat}}(E)$ and $L(E) = \Sigma_1^{-1} P(E)$, being Σ_1 the set of homomorphisms between finitely generated projective modules defined in Section 2. We summarize the properties of the algebra $Q(E)$ in the following theorem:

Theorem 4.2. *Let E be a finite quiver. Then the regular algebra $Q(E)$ of E is a unital von Neumann regular hereditary ring, and $Q(E) = (\Sigma \cup \Sigma_1)^{-1} P(E)$ is a universal localization of the path algebra $P(E)$. Moreover we have $\mathcal{V}(Q(E)) \cong M_E$ canonically.*

Proof. By Theorem 2.16 we have that $Q(E)$ is von Neumann regular, and by Theorem 3.1 we have that $\mathcal{V}(Q(E)) \cong M_E$ canonically. Using Theorem 1.20, we get $Q(E) = (\Sigma \cup \Sigma_1)^{-1} P(E)$, and, since $P(E)$ is a hereditary ring, a result of Bergman and Dicks [11] gives that $Q(E)$ is hereditary too. $\square$

We remark that the algebra $U(E)$ is also unital and von Neumann regular, with $\mathcal{V}(U(E)) \cong M_E$ (same proof as in Theorem 4.2). However, it is unlikely that $U(E)$ is hereditary in general, because, for instance, the algebra $K\langle\langle X \rangle\rangle$ of formal power series with $|X| > 1$ is not hereditary (see [14, Proposition 5.10.9]).

Examples 4.3. As we mentioned in the introduction, when we take the quiver with one vertex and n arrows, we recover the constructions in [4] (see also [3]). We consider now a different example, which also recovers some known rings. Let E be the quiver with $E^0 = \{1, 2\}$ and $E^1 = \{e, f\}$ such that $s(e) = r(e) = 1$ and $s(f) = 2$, $r(f) = 1$. Then the Leavitt path algebra is exactly the universal non-directly finite algebra:

$$L(E) \cong K\langle x, y \mid xy = 1 \rangle.$$

The isomorphism is obtained by sending x to $e + f$ and y to $\bar{e} + \bar{f}$. As is well known, $L(E)$ has a two-sided ideal $I \cong M_\infty(K)$, the ring of countably infinite matrices with only finitely many nonzero entries, and $L(E)/I \cong K[x, x^{-1}]$, the algebra of Laurent polynomials. Now the algebra $U(E)$ associated with the power series algebra $P(\!(E)\!)$ has a two-sided ideal M isomorphic with the ideal of finite rank transformations on a K-vector space of countable dimension, and $U(E)/M \cong K(\!(x)\!)$, the field of Laurent power series. The regular ring $U(E)$ is isomorphic to the ring constructed by Menal and Moncasi in [23, Example 1]. Finally the regular algebra $Q(E)$ of E has a two-sided ideal J which is simple as a ring and satisfies $J = \mathrm{Soc}(Q(E))$, such that $Q(E)/J \cong K(x)$, the field of rational functions. A regular ring with these properties was constructed (for K countable) in [13].

A quiver E is said to be *column-finite* in case each vertex in E receives only a finite number of arrows, that is $r_E^{-1}(v)$ is finite for all $v \in E^0$. For a column-finite quiver E, one can define the Leavitt path algebra $L(E)$ [1,6] and also the quiver monoid M_E just as in Section 3 (see [6]).

Note that the Leavitt path algebra $L(E)$ is unital if and only if the quiver E is finite. As shown in [6, Section 2] these constructions are functorial with respect to complete graph homomorphisms, defined below.

Let $f = (f^0, f^1): E \to F$ be a graph homomorphism. Then f is said to be *complete* if f^0 and f^1 are injective and f^1 restricts to a bijection between $r_E^{-1}(v)$ and $r_F^{-1}(f^0(v))$ for every vertex $v \in E^0$ that receives arrows.

If $f: E \to F$ is a complete graph homomorphism between finite quivers E and F, then f induces a non-unital algebra homomorphism $P(f): P(E) \to P(F)$ between the corresponding path algebras and a non-unital homomorphism $L(f): L(E) \to L(F)$ between the corresponding Leavitt path algebras. Note that the image of the identity under these homomorphisms is the idempotent

$$p_E := \sum_{v \in E^0} p_{f^0(v)} \in P(F).$$

We get a morphism $P(E) \to P(F) \to L(F) \to Q(F)$ such that every map in $\Sigma_1(E)$ becomes invertible over $Q(F)$.

Observe that we have a commutative diagram

$$\begin{array}{ccc} P(E) & \longrightarrow & P(F) \\ {\scriptstyle \varepsilon_E}\big\downarrow & & {\scriptstyle \varepsilon_F}\big\downarrow \\ K^{d_E} & \longrightarrow & K^{d_F} \end{array}$$

so a matrix $A \in M_n(P(E))$ such that $\varepsilon_E(A)$ is invertible is sent to a matrix $P(f)(A) \in M_n(p_E P(F) p_E)$ such that $\varepsilon_F(P(f)(A))$ is invertible over $p_E K^{d_F} p_E$. It follows that the unital algebra homomorphism $P(E) \to p_E Q(F) p_E$ factorizes uniquely through $Q(E) = (\Sigma \cup \Sigma_1)^{-1} P(E)$, so that we get a unital algebra homomorphism $Q(E) \to p_E Q(F) p_E$, thus a non-unital algebra homomorphism $Q(f): Q(E) \to Q(F)$ such that $Q(f)(1) = p_E$.

This gives the functoriality property of the regular algebra of a quiver, for finite quivers. By [6, Lemma 3.1] every column-finite quiver E is the direct limit, in the category of quivers with complete graph homomorphisms, of the directed family $\{E_\lambda\}$ of its complete finite subquivers. Thus we get a directed system $\{Q(E_\lambda)\}$ of von Neumann regular algebras and (non-unital) algebra morphisms, and we define the regular algebra of E as:

$$Q(E) = \varinjlim Q(E_\lambda).$$

Since the functor $\mathcal{V}$ commutes with direct limits and $M_E \cong \varinjlim M_{E_\lambda}$ [6, Lemma 3.4] we get:

Theorem 4.4. *Let E be any column-finite quiver. Then there is a (possibly non-unital) von Neumann regular algebra $Q(E)$ such that*

$$\mathcal{V}(Q(E)) \cong M_E.$$

This solves the realization problem for the monoids associated to column-finite quivers. Clearly the functoriality of Q extends to the category of column-finite quivers and complete graph homomorphisms.

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