Kernel sections and uniform attractors of multi-valued semiprocesses✩

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Abstract

We present the existence of kernel sections (which are all compact, invariant and pullback attracting) of an infinite-dimensional general multi-valued process constructed by the set-valued backward extension of multi-valued semiprocesses. Moreover, the structure of the uniform attractors of a family of multi-valued semiprocesses and the uniform forward attraction of kernel sections of a family of general multi-valued processes are investigated. Finally, we explain our abstract results by considering the mixed wave systems with supercritical exponent and ordinary differential equations.

Keywords: Multi-valued semiprocess; Kernel section; Uniform attractor; Uniform forward attraction; Nonautonomous wave system with supercritical exponent

1. Introduction

Autonomous set-valued dynamical systems and their attractors have been extensively studied in mathematical literature, especially in the recent years (see, for example, [1,3,6,7,12,14,16,18,21,22,30] and the references cited therein). However, the nonautonomous multi-valued dynamical systems [4,23], in particular, the nonautonomous multi-valued semidynamical systems are less well understood. In this present work, we are mainly concerned with the asymptotical behavior of multi-valued semiprocesses.

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First, we give the sufficient conditions for the existence of kernel sections of an infinite-dimensional general multi-valued process which is generated by the set-valued backward extension of multi-valued semiprocesses. The existence of kernel sections of (single-valued) processes are well known. Related results can be found in [8–12,15,20,24,31,33,34], etc. As far as we know not many results in this line are available in the literature in case of nonautonomous semidynamical systems, even in single-valued case. An important reason, from the point of view of mathematics, may be that the backward extension of the semiprocesses is set-valued in the general sense. Let \( \{ T(h) \mid h \in \mathbb{R}^+ \} \) be a continuous semigroup on a Banach space \( \Sigma \) with norm \( \| \cdot \|_\Sigma \), and let \( \{ U_\sigma(t, \tau) \mid t, \tau \in \mathbb{R}^+ \}, \sigma \in \Sigma \), be a family of multi-valued semiprocesses (MVSP) on a state space \( X \) satisfying the following translation identity:

\[
U_{T(h)\sigma}(t, \tau) = U_{\sigma}(t+h, \tau+h), \quad \forall h \geq 0, \quad t \geq \tau, \quad \tau \geq 0.
\]  

(1.1)

In particular, we suppose that \( X \) is a Banach space with norm \( \| \cdot \|_X \) and \( U_\sigma(t, \tau)x \) is jointly norm-to-weak upper-semicontinuous in \( \sigma \) and \( x \) for any fixed \( t \geq \tau \), \( \tau \in \mathbb{R}^+ \), i.e., if \( \sigma_n \to \sigma \) and \( x_n \to x \), then for any \( y_n \in U_{\sigma_n}(t, \tau)x_n \), there exists a \( y \in U_{\sigma}(t, \tau)x \), such that \( y_n \rightharpoonup y \) (weak convergence), and that the semigroup \( \{ T(h) \} \) is continuous invariant on a subset \( \Xi \) of \( \Sigma \). We will construct a general multi-valued process (GMVP) (see Section 3 for formal definitions) \( \{ P_\sigma(t, \tau) \mid t \geq \tau, \quad \tau \in \mathbb{R} \} \) for each \( \sigma \in \Xi \) by the following formula

\[
P_\sigma(t, \tau) = \begin{cases} 
U_\sigma(t, \tau), & \tau \geq 0, \\
\tilde{U}_\sigma(t, \tau), & \tau < 0,
\end{cases}
\]

(1.2)

where \( \tilde{U}_\sigma(t, \tau) := \{ U_\sigma'(t-\tau, 0) \mid T(|\tau|)\sigma' = \sigma \} \) in \( X \). Let \( K_\sigma \) be the kernel of the GMVP \( \{ P_\sigma(t, \tau) \} \) with \( \sigma \in \Sigma \). The kernel \( K_\sigma \) consists of all bounded complete trajectories of the general multi-valued process, i.e.,

\[
K_\sigma = \left\{ u(\cdot) \mid \sup_{t \in (-\infty, +\infty)} \| u(t) \|_X \leq C_u, \quad u(t) \in P_\sigma(t, \tau)u(\tau), \quad \forall t \geq \tau, \quad \tau \in \mathbb{R} \right\}.
\]

\( K_\sigma(s) \) denotes the kernel section at a time moment \( s \in \mathbb{R} \):

\[
K_\sigma(s) = \left\{ u(s) \mid u(\cdot) \in K_\sigma \right\}, \quad K_\sigma(s) \subseteq X.
\]

Here using the technique of Kuratowski measure of noncompactness, we will show under the uniform dissipativeness and the uniform \( \omega \)-limit compactness of the family of MVSPs \( \{ U_\sigma(t, \tau) \}, \sigma \in \Sigma \), that the GMVP \( \{ P_\sigma(t, \tau) \} \) with \( \sigma \in \Xi \), which is norm-to-weak upper-semicontinuous on \( X \), has a nonempty kernel; moreover, the kernel sections \( K_\sigma(t) \) are all compact, invariant \((P_\sigma(t, \tau)K_\sigma(\tau) = K_\sigma(t)) \) for all \( t \geq \tau \) and all \( \tau \in \mathbb{R} \) and pullback attracting, i.e., for any fixed \( t \in \mathbb{R} \) and every bounded set \( B \subset X \),

\[
\lim_{s \to +\infty} H_X^+(P_\sigma(t, t-s)B, K_\sigma(t)) = 0,
\]

where \( H_X^+ \) denotes the Hausdorff semidistance in \( X \). Furthermore, we give simple and sufficient methods for verifying the uniform \( \omega \)-limit compactness.

Secondly, we are interested in the structure of the uniform attractors and the uniform forward attraction of the inflated kernel sections. In [12], the structure of the uniform attractors (for the
family of single-valued semiprocesses \( \{U_\sigma(t,\tau)\}, \sigma \in \Sigma \), satisfying the translation identity (1.1) and the semigroup \( \{T(h)\} \) satisfying the backward unique property) are described. Whereas, in general, the semigroup satisfies set-valued backward extension property; see [19]. In this case, the general multi-valued processes defined in (1.2) only satisfy the following translation property

\[
P_{T(h)\sigma}(t,\tau) \supset P_\sigma(t+h,\tau+h), \quad \forall h \geq 0, t \geq \tau, \tau < 0.
\]

This will lead to a more complicated description of the structure of the uniform attractors. Here we will prove that the multi-valued skew product flow \( \{F(t)\}_{t \in \mathbb{R}^+} \)

\[
F(t)(x,\sigma) = (U_\sigma(t,0)x, T(t)\sigma), \quad \forall t \geq 0, (x,\sigma) \in X \times \Sigma,
\]

which is norm-to-weak upper-semicontinuous on \( X \times \Sigma \), corresponding to the family of MVSPs \( \{U_\sigma(t,\tau)\}, \sigma \in \Sigma \), possesses a unique compact attractor \( \mathcal{A} \) which is invariant \( (F(t)\mathcal{A} = \mathcal{A} \text{ for all } t \geq 0) \), and that

1. \( \Pi_1 \mathcal{A} = \mathcal{A}_\Sigma \) is the uniform (w.r.t. \( \sigma \in \Sigma \)) attractor of the family of MVSPs \( \{U_\sigma(t,\tau)\}, \sigma \in \Sigma \),
2. \( \Pi_2 \mathcal{A} = \omega(\Sigma) \) is the attractor of the semigroup \( \{T(t)\} \) acting on \( \Sigma: T(t)\omega(\Sigma) = \omega(\Sigma) \) for all \( t \geq 0 \),
3. the global attractor \( \mathcal{A} \) satisfies

\[
\mathcal{A} = \bigcup_{\sigma \in \omega(\Sigma)} \mathcal{K}_\sigma(0) \times \{\sigma\};
\]
4. the uniform attractor satisfies

\[
\Pi_1 \mathcal{A} = \mathcal{A}_\Sigma = \bigcup_{\sigma \in \omega(\Sigma)} \mathcal{K}_\sigma(0),
\]

where \( \Sigma \) is a bounded Banach space with norm \( || \cdot ||_\Sigma \), and \( \mathcal{K}_\sigma(0) \) is the section at \( t = 0 \) of the kernel \( \mathcal{K}_\sigma \) of the GMVP \( \{P_\sigma(t,\tau)\} \) with \( \sigma \in \omega(\Sigma) \). Furthermore, we consider the family of inflated kernel sections \( \{\mathcal{K}_\sigma^{(\varepsilon)}(0)\}, \sigma \in \omega(\Sigma) \), for any fixed \( \varepsilon_0 > 0 \), with component sets defined by

\[
\mathcal{K}_\sigma^{(\varepsilon_0)}(0) = \bigcup_{||\sigma' - \sigma||_\Sigma \leq \varepsilon_0} \mathcal{K}_{\sigma'}(0).
\]

We will show that, for any fixed \( \varepsilon_0 > 0 \), the family of inflated kernel sections \( \{\mathcal{K}_\sigma^{(\varepsilon_0)}(0)\}, \sigma \in \omega(\Sigma) \), uniformly (w.r.t. \( \sigma \in \omega(\Sigma) \)) forward attracts each bounded subset \( B \) of \( X \), i.e., for any \( \varepsilon > 0 \), there is a \( T_1 = T_1(B,\varepsilon) > 0 \) independent of \( \sigma \in \omega(\Sigma) \) such that

\[
H^*_X(P_{\sigma}(t,0)B, \mathcal{K}_{T(t)\sigma}^{(\varepsilon_0)}(0)) < \varepsilon, \quad \forall \sigma \in \omega(\Sigma), \ t \geq T_1.
\]

As an important case, we consider the family of semiprocesses \( \{U_\sigma(t,\tau)\}, \sigma \in \Sigma \), which is jointly upper-semicontinuous on \( \Sigma \times X \) for any fixed \( t \geq \tau, \tau \in \mathbb{R}^+ \) (i.e., if \( x_n \to x \) in \( X \) and
\( \sigma_n \to \sigma \) in \( \Sigma \), then for any \( y_n \in U_{\sigma_n}(t, \tau)x_n \), there exists a \( y \in U_{\sigma}(t, \tau)x \), such that \( y_n \to y \) in \( X \) as \( n \to \infty \). We will prove that the uniform attractor satisfies
\[
\Pi_1 A = A_{\Sigma} = A_{\omega(\Sigma)} = \bigcup_{\sigma \in \omega(\Sigma)} K_\sigma(0),
\]
where \( A_{\omega(\Sigma)} \) is the uniform attractor of the family of general multi-valued processes \( \{ P_\sigma(t, \tau) \} \), \( \sigma \in \omega(\Sigma) \).

Finally, we apply our abstract results to investigate the mixed wave systems with supercritical exponent and ordinary differential equations. For autonomous wave systems, we refer the reader to [3,12,16,17,27–29]. However, there is little reference on nonautonomous wave systems without uniqueness of solutions. As an application of our abstract theory as above, we discuss the mixed wave systems with supercritical exponent and ordinary differential equations, and obtain the existence of kernel sections which is compact, invariant and pullback attracting in \( H^1_0(D) \times L^2(D) \), where \( D \) is a smooth bounded domain in \( \mathbb{R}^3 \).

This paper is organized as follows. In Section 2 we give some preliminary results and definitions and then in Sections 3–5 we state and prove our main results. Finally, in Section 6 we illustrate our main results in Sections 3–5 by studying the mixed wave systems with supercritical exponent and ordinary differential equations.

2. Preliminaries

Let \( X \) be a Banach space with norm \( \| \cdot \|_X \), let \( 2^X \) be the set of all subsets of \( X \), and let \( \Sigma \) be a Banach space with norm \( \| \cdot \|_\Sigma \). Denote by \( H^*_Z(\cdot, \cdot) \) and \( H_Z(\cdot, \cdot) \), respectively, the Hausdorff semi-distance and Hausdorff distance between two nonempty subsets of a Banach space \( (Z, \| \cdot \|_Z) \), which are defined by
\[
H^*_Z(A, B) = \sup_{a \in A} \text{dist}_Z(a, B),
\]
where \( \text{dist}_Z(a, B) = \inf_{b \in B} \| a - b \|_Z \), and
\[
H_Z(A, B) = \max\left\{ H^*_Z(A, B), H^*_Z(B, A) \right\}.
\]
Finally, denote by \( N(A, r) \) the open neighborhood \( \{ y \in Z \mid \text{dist}_Z(y, A) < r \} \) of radius \( r > 0 \) of a subset \( A \) of a Banach space \( Z \).

**Definition 2.1.** A family of mappings \( F(t) : X \to 2^X, t \in \mathbb{R}^+ \), is said to be a (autonomous) multi-valued semidynamical system (MVSS in short) if the following axioms hold:

1. \( F(0)x = \{ x \}, \forall x \in X; \)
2. \( F(s)F(t)x = F(s + t)x, \forall s, t \in \mathbb{R}^+, x \in X; \)
3. \( F(t)x \) is norm-to-weak upper-semicontinuous in \( x \) for fixed \( t \in \mathbb{R}^+ \) (i.e., if \( x_n \to x \) in \( X \), then for any \( y_n \in F(t)x_n \), there exists a \( y \in F(t)x \), such that \( y_n \to y \) (weak convergence)).

In evolution equation, this type of multi-valued semidynamical system corresponds to the solution that only satisfies weaker stability, and generally, it is neither upper-semicontinuous (i.e., norm-to-norm) nor weak upper-semicontinuous (i.e., weak-to-weak). But obviously, upper-semicontinuous MVSS and the weak upper-semicontinuous MVSS are both norm-to-weak
upper-semicontinuous MVSS. As we know, for some concrete problems, it is difficult to verify that the MVSS is either upper-semicontinuous or weak upper-semicontinuous in stronger normed spaces. However, it follows from the following result that one can easily show that the MVSS is norm-to-weak upper-semicontinuous in stronger normed space.

Let \( X, Y \) be two Banach spaces, and \( X^*, Y^* \) be their dual spaces, respectively. We also assume that \( X \) is a dense subspace of \( Y \), the injection \( i: X \hookrightarrow Y \) is continuous and its adjoint \( i^*: Y^* \hookrightarrow X^* \) is densely injective. Under these assumptions, we have the following result:

**Theorem 2.2.** Let \( X, Y \) be two Banach spaces satisfy the assumptions just above, \( \{F(t)\} \) be a MVSS on \( X \) and \( Y \), respectively. Assume that \( \{F(t)\} \) is upper-semicontinuous or weak upper-semicontinuous on \( Y \). If for fixed \( t \in \mathbb{R}^+ \), \( F(t) \) maps compact subsets of \( X \) into bounded subsets of \( 2^X \), then \( F(t) \) is norm-to-weak upper-semicontinuous on \( X \).

**Proof.** Let \( x_n \to x \) in \( X \) as \( n \to \infty \). We need to check that for any \( y_n \in F(t)x_n \), there exists a \( y \in F(t)x \) such that for any given \( x^* \in X^* \),

\[
\left\langle x^*, y_n - y \right\rangle_{X^*} \to 0 \quad \text{as} \quad n \to \infty.
\]  

(2.1)

Since \( i^*: Y^* \hookrightarrow X^* \) is dense, for any \( \varepsilon > 0 \) and any \( x^* \in X^* \), there exists \( y^*_\varepsilon \in Y^* \) such that

\[
\| i^*(y^*_\varepsilon) - x^* \|_{X^*} < \frac{\varepsilon}{2K},
\]  

(2.2)

where \( K \) is a constant which satisfies

\[
\| F(t)x_n \|_X + \| F(t)x \|_X \leq K, \quad \forall n \in \mathbb{N}.
\]  

(2.3)

Note that the MVSS \( \{F(t)\} \) is upper-semicontinuous or weak upper-semicontinuous in \( Y \), for the \( y^*_\varepsilon \) given above and any \( y_n \in F(t)x_n \), there exist \( y \in F(t)x \) and \( N''_0 > 0 \) such that for any \( n \geq N''_0 \),

\[
\left\langle [y^*_\varepsilon, i(y_n - y)]_{Y^*} \right\rangle < \frac{\varepsilon}{2}.
\]  

(2.4)

Combining (2.2)–(2.4) together, we can conclude that for any \( n \geq N''_0 \),

\[
\left\langle [x^*, y_n - y]_{X^*} \right\rangle \leq \left\| [i^*(y^*_\varepsilon) - x^*, y_n - y]_{X^*} \right\| + \left\| [i^*(y^*_\varepsilon), y_n - y]_{X^*} \right\|
\]

\[
\leq \| i^*(y^*_\varepsilon) - x^* \|_{X^*} \| y_n - y \|_X + \left\langle [y^*_\varepsilon, i(y_n - y)]_{Y^*} \right\rangle < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\]

which means that (2.1) holds true. The proof is finished. \( \square \)

**Remark 2.3.** In concrete problems, we can choose \( Y \) to be a larger and weaker topology space, in which the upper semicontinuity of the MVSS can be obtained easily.

**Definition 2.4.** Let \( \{F(t)\} \) be a multi-valued semidynamical system on \( X \). We say that \( \{F(t)\} \) is

1. dissipative, if there exists a bounded subset \( \mathcal{U} \) of \( X \) so that for any bounded set \( B \subset X \), there exists a \( T_0 = T_0(B) \in \mathbb{R}^+ \), such that

\[
F(t)B \subset \mathcal{U}, \quad \forall t \geq T_0;
\]

2. asymptotically compact, if the set \( \{F(t)x \} \) is relatively compact in \( X \) for any \( x \in X \), and there exists a sequence \( \{T_n\} \to \infty \) such that

\[
\lim_{n \to \infty} \sup_{t \geq T_n} \| F(t)B \|_X = 0.
\]

3. global attractor, if there exists a closed set \( A \) such that for any bounded set \( B \subset X \), there exists a \( t_0 \) such that

\[
F(t)B \subset A, \quad \forall t \geq t_0;
\]

4. limit absorbing, if for any bounded set \( B \subset X \), there exists a \( t_0 \) such that

\[
F(t)B \cap \mathcal{U} = \emptyset, \quad \forall t \geq t_0,
\]

where \( \mathcal{U} \) is a bounded set in \( X \).
(2) $\omega$-limit compact, if for any bounded subset $B$ of $X$ and $\varepsilon > 0$, there exists a $T_1 = T_1(B, \varepsilon) \in \mathbb{R}^+$, such that

$$k\left(\bigcup_{t \geq T_1} F(t)B\right) \leq \varepsilon,$$

where $k$ is the Kuratowski measure of noncompactness.

**Definition 2.5.** A nonempty compact subset $A$ of $X$ is called to be a global attractor for the multi-valued semidynamical system $\{F(t)\}$, if it satisfies

1. $A$ is an invariant set, i.e.,
   $$F(t)A = A, \quad \forall t \in \mathbb{R}^+;$$
2. $A$ attracts each bounded subset $B$ of $X$, i.e.,
   $$\lim_{t \to +\infty} H^\omega_X(F(t)B, A) = 0.$$

**Definition 2.6.** A family of mappings $U_\sigma(t, \tau): X \to 2^X, t \geq \tau, \tau \in \mathbb{R}^+$, is said to be a multi-valued semiprocess (MVSP in short) with $\sigma \in \Sigma$ if it satisfies:

1. $U_\sigma(\tau, \tau)x = \{x\}, \forall \tau \in \mathbb{R}^+, x \in X$;
2. $U_\sigma(t, s)U_\sigma(s, \tau)x = U_\sigma(t, \tau)x, \forall t \geq s \geq \tau, \tau \in \mathbb{R}^+, x \in X$;
3. $U_\sigma(t, \tau)x$ is norm-to-weak upper-semicontinuous in $x$ for fixed $t \geq \tau, \tau \in \mathbb{R}^+$ (i.e., if $x_n \to x$ in $X$, then for any $y_n \in U_\sigma(t, \tau)x_n$, there exists a $y \in U_\sigma(t, \tau)x$ such that $y_n \rightharpoonup y$ (weak convergence)).

Fully analogous to the proof of Theorem 2.2, we have:

**Theorem 2.7.** Let $X, Y$ be two Banach spaces satisfy the assumptions just above, $\{U_\sigma(t, \tau)\}$ with $\sigma \in \Sigma$ be a multi-valued semiprocess on $X$ and $Y$, respectively. Suppose that $\{U_\sigma(t, \tau)\}$ is upper-semicontinuous or weak upper-semicontinuous on $Y$. If for fixed $t \geq \tau, \tau \in \mathbb{R}^+$, $U_\sigma(t, \tau)$ maps compact subsets of $X$ into bounded subsets of $2^X$, then $U_\sigma(t, \tau)$ is norm-to-weak upper-semicontinuous on $X$.

**Definition 2.8.** Let $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$, be a family of multi-valued semiprocesses on $X$. We say that the family of multi-valued semiprocesses $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$, is

1. uniformly dissipative, if for any fixed $\tau \in \mathbb{R}^+$, there exists a bounded subset $\mathcal{V}$ of $X$ so that for any bounded set $B \subset X$, there exists $\tilde{\tau} = \tilde{\tau}(B) \in \mathbb{R}^+$ independent of $\sigma \in \Sigma$, such that
   $$U_\sigma(t + \tau, \tau)B \subset \mathcal{V}, \quad \forall \sigma \in \Sigma, \ t \geq \tilde{\tau};$$
(2) uniformly $\omega$-limit compact, if for any fixed $\tau \in \mathbb{R}^+$, every bounded subset $B$ of $X$ and any $\varepsilon > 0$, there exists a $t_1 = t_1(B, \tau, \varepsilon) \in \mathbb{R}^+$ which is independent of $\sigma \in \Sigma$, such that

$$k\left( \bigcup_{t \geq t_1} \bigcup_{\sigma \in \Sigma} U_\sigma(t + \tau, \tau)B \right) \leq \varepsilon.$$ 

**Definition 2.9.** A compact set $\mu \subset X$ is called to be a uniform attractor of the family of multivalued semiprocesses $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, if it satisfies

(1) $\mu$ uniformly attracts every bounded subset $B$ of $X$, i.e., for any fixed $\tau \in \mathbb{R}^+$,

$$\lim_{t \to +\infty} \sup_{\sigma \in \Sigma} H^\sigma_X(U_\sigma(t + \tau, \tau)B, \mu) = 0.$$ 

(2) If there is another compact set $A'$ satisfying (1), then $\mu \subset A'$.

Let $k(A)$ be the Kuratowski measure of noncompactness of $A$, which is defined by

$$k(A) = \inf\{\delta > 0 \mid A \text{ admits a finite cover by sets whose diameter } \leq \delta\}.$$ 

**Lemma 2.10.** [13,25,35] The Kuratowski measure of noncompactness $k(A)$ on a complete metric space $M$ satisfies the following properties:

(1) $k(A) = 0$ if and only if $\overline{A}$ is compact, where $\overline{A}$ is the closure of $A$;
(2) if $A_1 \subset A_2$, then $k(A_1) \leq k(A_2)$;
(3) $k(A_1 \cup A_2) \leq \max\{k(A_1), k(A_2)\}$;
(4) $k(\overline{A}) = k(A)$;
(5) if $A_t$ is a family of nonempty, closed, bounded sets defined for $t > r$ that satisfy $A_t \subset A_s$, whenever $s \leq t$, and $k(A_t) \to 0$, as $t \to \infty$, then $\bigcap_{t > r} A_t$ is a nonempty, compact set in $M$.

If, in addition, $M$ is a Banach space, then the following statements are valid:

(6) $k(A_1 + A_2) \leq k(A_1) + k(A_2)$;
(7) $k(\text{co } A) = k(A)$, where $\text{co } A$ is the closed convex hull of $A$;
(8) let $M$ has the following decomposition:

$$M = M_1 \oplus M_2, \quad \text{with } \dim M_1 < \infty,$$

$$P : M \to M_1, \quad Q : M \to M_2$$

be the canonical projectors, and $A$ be a bounded subset of $M$. If the diameter of $QA$ is less than $\varepsilon$, then $k(A) < \varepsilon$.

**Definition 2.11.** Let $A$ be a subset of Banach space $M$. The weakly sequential closure $\overline{A}^{\text{WS}}$ of $A$ is defined by

$$\overline{A}^{\text{WS}} = \{ x \in X \mid \exists \{x_n\} \subset A, \text{ s.t. } x_n \rightharpoonup x, \text{ that is, } x_n \text{ converges weakly to } x \}.$$
In general topology space, $\overline{A}^{WS}$ is different from $\overline{A}$ or the weak closure $\overline{A}^W$ of $A$. But if $A$ is a convex subset of some Banach space, then we know that $\overline{A} = \overline{A}^W = \overline{A}^{WS}$.

**Lemma 2.12.** [35] Let $M$ be a Banach space and $k$ be the Kuratowski measure of noncompactness. Then for any subset $A$ of $M$, we have

$$k(A) = k(\overline{A}^{WS}).$$

3. Set-valued backward extension of multi-valued semiprocesses

In this section, we construct general multi-valued processes by the set-valued backward extension of multi-valued semiprocesses.

Let $\Xi$ be a subset of Banach space $\Sigma$ and let $\{T(h) \mid h \in \mathbb{R}^+\}$ be a continuous invariant semigroup on $\Xi$. Let $\{U_{\sigma}(t, \tau) \mid t \geq \tau, \tau \in \mathbb{R}^+\}, \sigma \in \Sigma$, be a family of multi-valued semiprocesses on $X$. In addition, the translation identity (1.1) is valid for the family of multi-valued semiprocesses $\{U_{\sigma}(t, \tau)\}, \sigma \in \Sigma$, and for the semigroup $\{T(h)\}$.

We first consider the set-valued backward extension of the semidynamical system $\{T(t) \mid t \in \mathbb{R}^+\}$. Note that $\{T(t) \mid t \in \mathbb{R}^+\}$ be a invariant semidynamical system on $\Xi$, i.e., $T(t)\Xi = \Xi$ for all $t \in \mathbb{R}^+$. So, it is backwards extendable on $\Xi$ but is not necessarily a dynamical system on $\Xi$ because it may not be uniquely backwards extendable. However, we can say that the backward extension defines a set-valued or general semidynamical system backwards in time on the space $\Xi$.

**Definition 3.1.** A set-valued mapping $G : \mathbb{R}^+ \times \Xi \to 2^\Xi$ is said to be a general semidynamical system if it satisfies:

1. $G(t, \sigma)$ is a nonempty closed subset of $\Xi$ for all $t \geq 0$ and all $\sigma \in \Xi$;
2. $G(0, \sigma) = \{\sigma\}, \forall \sigma \in \Xi$;
3. $G(s + t, \sigma) = G(s, G(t, \sigma)), \forall s, t \geq 0, \sigma \in \Xi$, where $2^\Xi$ be the set of nonempty subsets of $\Xi$.

We define the backward extension of the semidynamical system $T(t)$ by

$$T(t)\sigma := \{\sigma' \in \Xi \mid T(|t|)\sigma' = \sigma\}$$

for each $t \leq 0$.

**Theorem 3.2.** [19] $T = \{T(t), t \leq 0\}$ is a general semidynamical system backwards in time on the space $\Xi$, i.e., the set-valued mapping $G$ defined by $G(t, \sigma) := T(-t)\sigma \in 2^\Xi$ for $(t, \sigma) \in \mathbb{R}^+ \times \Xi$ is a general semidynamical system on $\Xi$.

We observe that $T$ usually cannot be concatenated with $T$ to form a “natural” topological extension of the semidynamical system $T$ on $\Xi$ to give a dynamical system on $2^\Xi$. To see this, we first define $T(t)K := \bigcup_{\sigma \in K} T(t)\sigma$ for each $t \leq 0$ and $K \in 2^\Xi$. Secondly, we notice that, in general,

$$T(t)T(-t)\sigma = T(0)\sigma = \{\sigma\} \subset T(-t)T(t)\sigma, \forall t \geq 0,$$
with a possible strict set inclusion, since there may exist points \( \sigma' \in \Xi \) with \( \sigma' \neq \sigma \) for which \( T(t)\sigma' = T(t)\sigma \). More specifically, we have \( T(t)T(-s-t)\sigma = T(-s)\sigma \) and \( T(-s-t)T(t)\sigma \supseteq T(-s)\sigma \) for \( s, t \geq 0 \), with a possible strict set inclusion.

**Definition 3.3.** A family of mappings \( P_\sigma(t, \tau) : X \to 2^X, t \geq \tau, \tau \in \mathbb{R} \), is called a general multivalued process (GMVP in short) with \( \sigma \in \Xi \) if it satisfies:

1. \( P_\sigma(\tau, \tau)x = \{x\}, \forall \tau \in \mathbb{R}, x \in X \);
2. \( P_\sigma(t, s)P_\sigma(s, \tau)x = P_\sigma(t, \tau)x, \forall t \geq s \geq \tau, \tau \in \mathbb{R}, x \in X \).

**Theorem 3.4.** Let \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \), be a family of multi-valued semiprocesses on \( X \). Then the family of mappings \( \{P_\sigma(t, \tau) | t \geq \tau, \tau \in \mathbb{R}\} \) defined in (1.2) forms a general multi-valued process on \( X \).

**Proof.** Only axiom (2) in Definition 3.3 needs to be checked. Three cases may occur.

**Case 1.** \( s \geq 0, \tau < 0 \). By (1.1) and (1.2), we have

\[
P_\sigma(t, s)P_\sigma(s, \tau) = U_\sigma(t, s)\tilde{U}_\sigma(s, \tau) = U_\sigma(t, s) \bigcup_{\sigma' \in T(\tau)\sigma} U_{\sigma'}(s - \tau, 0)
\]

\[
= \bigcup_{\sigma' \in T(\tau)\sigma} U_\sigma(t, s)U_{\sigma'}(s - \tau, 0) = \bigcup_{\sigma' \in T(\tau)\sigma} U_{T(-\tau)\sigma'}(t, s)U_{\sigma'}(s - \tau, 0)
\]

\[
= \bigcup_{\sigma' \in T(\tau)\sigma} U_{\sigma'}(t - \tau, s - \tau)U_{\sigma'}(s - \tau, 0) = \bigcup_{\sigma' \in T(\tau)\sigma} U_{\sigma'}(t - \tau, 0)
\]

\[
= \tilde{U}_\sigma(t, \tau) = P_\sigma(t, \tau).
\]

**Case 2.** \( s < 0, \tau < 0 \). Similarly, we can deduce from \( T(s - \tau)T(\tau)\sigma = T(s)\sigma \) that

\[
P_\sigma(t, s)P_\sigma(s, \tau) = \tilde{U}_\sigma(t, s)\tilde{U}_\sigma(s, \tau) = \bigcup_{\sigma' \in T(s)\sigma} U_{\sigma'}(t - s, 0)\tilde{U}_\sigma(s, \tau)
\]

\[
= \bigcup_{\sigma'' \in T(\tau)\sigma} U_{T(s - \tau)\sigma''}(t - s, 0)U_{\sigma''}(s - \tau, 0)
\]

\[
= \bigcup_{\sigma'' \in T(\tau)\sigma} U_{\sigma''}(t - \tau, s - \tau)U_{\sigma''}(s - \tau, 0)
\]

\[
= \bigcup_{\sigma'' \in T(\tau)\sigma} U_{\sigma''}(t - \tau, 0) = \tilde{U}_\sigma(t, \tau) = P_\sigma(t, \tau).
\]

**Case 3.** \( s \geq 0, \tau \geq 0 \). In view of (1.2), we easily see that

\[
P_\sigma(t, s)P_\sigma(s, \tau) = U_\sigma(t, s)U_\sigma(s, \tau) = U_\sigma(t, \tau) = P_\sigma(t, \tau).
\]

The proof is complete. \( \Box \)
Theorem 3.5. Assume that the family of MVSPs \( \{U_\sigma(t, \tau)\} \), \( \sigma \in \Sigma \), is jointly norm-to-weak upper-semicontinuous on \( X \times \Sigma \), and that \( T(t)\sigma \) is a nonempty compact subset of \( \Xi \) for each \( t < 0 \) and \( \sigma \in \Xi \). Then the GMVP \( \{P_\sigma(t, \tau)\} \) with \( \sigma \in \Xi \) is norm-to-weak upper-semicontinuous on \( X \), i.e., for any fixed \( t \geq \tau \), \( \tau \in \mathbb{R} \), if \( x_n \to x \) in \( X \), then for any \( y_n \in P_\sigma(t, \tau)x_n \), there exist a subsequence \( y_{nk} \) of \( y_n \) and a \( y \in P_\sigma(t, \tau)x \), such that \( y_{nk} \rightharpoonup y \) (weak convergence).

Proof. The conclusion can be easily obtained by assumption, thus the detailed arguments are omitted here. \( \square \)

Remark 3.6. A simple and important case is that the semigroup \( \{T(t)\} \) satisfies the backward uniqueness property, i.e., if \( T(t)\sigma' = T(t)\sigma \) for some \( t \geq 0 \) implies that \( \sigma' = \sigma \). In such a situation, \( T(t)\sigma \) is a singleton for each \( t < 0 \).

Remark 3.7. If \( \Xi \) is an attractor of the semigroup \( \{T(t)\} \), then the compactness of \( \Xi \) and the closed property of \( T(t)\sigma \) implies that \( T(t)\sigma \) is compact for each \( t < 0 \).

Theorem 3.8. \( P_{T(h)\sigma}(t, \tau) \supseteq P_\sigma(t + h, \tau + h) \), \( \forall h \geq 0, t \geq \tau, \tau < 0 \).

Proof. Two cases may occur.

Case 1. \( \tau < 0, \tau + h < 0 \). Observing that \( T(\tau)T(h)\sigma \supseteq T(\tau + h)\sigma \) for all \( \sigma \in \Xi \). Hence
\[
P_{T(h)\sigma}(t, \tau) = \bigcup_{\sigma' \in T(\tau)T(h)\sigma} U_{\sigma'}(t - \tau, 0) \supseteq \bigcup_{\sigma' \in T(\tau + h)\sigma} U_{\sigma'}(t - \tau, 0) = \tilde{U}_{\sigma}(t + h, \tau + h) = P_\sigma(t + h, \tau + h).
\]

Case 2. \( \tau < 0, \tau + h \geq 0 \). In the similar way, we can conclude that
\[
P_{T(h)\sigma}(t, \tau) = \bigcup_{\sigma' \in T(\tau)T(h)\sigma} U_{\sigma'}(t - \tau, 0) \supseteq \bigcup_{\sigma' \in T(\tau + h)\sigma} U_{\sigma'}(t - \tau, 0) = U_{\sigma}(t + h, \tau + h) = P_\sigma(t + h, \tau + h).
\]

The proof of Theorem 3.8 is finished. \( \square \)

Remark 3.9. It is worth noticing that, in general, the general multi-valued processes defined in (1.2) do not satisfy the translation identity as in (1.1). However, if the semigroup \( \{T(t)\} \) satisfies the backward uniqueness property, then the translation identity holds true.

4. Kernel sections of general multi-valued processes

In this section, we give the sufficient conditions for the existence of kernel sections (which are all compact, invariant and pullback attracting) of an infinite-dimensional general multi-valued process, using the Kuratowski measure of noncompactness.

Let \( \{U_\sigma(t, \tau) \mid t \geq \tau, \tau \in \mathbb{R}^+\}, \sigma \in \Sigma \), be a family of multi-valued semiproceses on \( X \) and let \( \{T(h) \mid h \in \mathbb{R}^+\} \) be a continuous invariant semigroup on a set \( \Xi \subset \Sigma \). As above, for each \( \sigma \in \Xi \), we can construct a general multi-valued process \( \{P_\sigma(t, \tau) \mid t \geq \tau, \tau \in \mathbb{R}\}. \) Let \( \mathcal{K}_\sigma \) be the...
kernel of the GMVP \( \{ P_\sigma (t, \tau) \} \). The kernel \( K_\sigma \) consists of all bounded complete trajectories of the general multi-valued process \( \{ P_\sigma (t, \tau) \} \), i.e.,

\[
K_\sigma = \left\{ u(\cdot) \mid \sup_{t \in (-\infty, +\infty)} \| u(t) \|_X \leq C_u, \ u(t) \in P_\sigma (t, \tau)u(\tau), \forall t \geq \tau, \ \tau \in \mathbb{R} \right\}.
\]

As usual, \( K_\sigma (s) \) denotes the kernel section at a time moment \( s \in \mathbb{R} \):

\[
K_\sigma (s) = \left\{ u(s) \mid u(\cdot) \in K_\sigma \right\}, \ K_\sigma (s) \subset X.
\]

Evidently, the following assertion holds:

**Proposition 4.1.** Let \( K_\sigma \) be the kernel of the general multi-valued process \( \{ P_\sigma (t, \tau) \} \). Then

\[
K_\sigma (t) \subseteq P_\sigma (t, \tau)K_\sigma (\tau), \ \forall t \geq \tau, \ \tau \in \mathbb{R}.
\]

**Definition 4.2.** Let \( \{ P_\sigma (t, \tau) \} \) be a general multi-valued process on \( X \) with \( \sigma \in \Sigma \). For every nonempty subset \( B \) of \( X \) and any \( t \in \mathbb{R} \), the pullback \( \omega \)-limit set \( \omega t, \sigma (B) \) defined by

\[
\omega t, \sigma (B) = \bigcap_{T \geq \max\{0, t\}} \bigcup_{s \geq T} P_\sigma (t, t-s)B^{\text{WS}} = \bigcap_{T \geq \max\{0, t\}} \bigcup_{s \geq T} \bigcup_{\sigma' \in \mathcal{T}(t-s)\sigma} U_{\sigma'}(s, 0)B^{\text{WS}}.
\]

**Theorem 4.3.** Let \( X \) be a Banach space and let \( \{ U_\sigma (t, \tau) \mid t \geq \tau, \ \tau \in \mathbb{R}^+ \} \), \( \sigma \in \Sigma \), be a family of multi-valued semiprocesses on \( X \) possessing jointly norm-to-weak upper semicontinuity on \( X \times \Sigma \). Also let \( \mathcal{T}(t)\sigma \) is a nonempty compact subset of \( \Sigma \) for each \( t < 0 \) and \( \sigma \in \Sigma \) and let \( \{ T(t) \} \) be a continuous invariant \( (T(t)\Sigma = \Sigma \) for all \( t \in \mathbb{R}^+ \) \) semigroup on \( \Sigma \) satisfying the translation identity (1.1). Assume that \( \{ U_\sigma (t, \tau) \}, \ \sigma \in \Sigma \), is

1. uniformly dissipative, i.e., for any fixed \( \tau \in \mathbb{R}^+ \), there exists a bounded subset \( V \) of \( X \) so that for any bounded set \( B \subset X \), there exists \( \tilde{\tau} = \tilde{\tau}(B) \in \mathbb{R}^+ \) independent of \( \sigma \in \Sigma \), such that

\[
U_\sigma (t + \tau, \tau)B \subset V, \ \forall \sigma \in \Sigma, \ t \geq \tilde{\tau}; \quad (4.1)
\]

2. uniformly \( \omega \)-limit compact.

Then the kernel \( K_\sigma \) of the general multi-valued process \( \{ P_\sigma (t, \tau) \mid t \geq \tau, \ \tau \in \mathbb{R} \} \) with \( \sigma \in \Sigma \) is nonempty, the kernel sections \( K_\sigma (t) = \omega t, \sigma (V) \) are all compact, invariant \( (P_\sigma (t, \tau)K_\sigma (\tau) = K_\sigma (t) \) for all \( t \geq \tau \) and all \( \tau \in \mathbb{R} \) \) and pullback attract every bounded set \( B \subset X \), i.e., for any fixed \( t \in \mathbb{R} \),

\[
\lim_{s \to +\infty} H^X_\Sigma (P_\sigma (t, t-s)B, K_\sigma (t)) = \lim_{s \to +\infty} \sup_{\sigma' \in \mathcal{T}(t-s)\sigma} H^X_\Sigma (U_{\sigma'}(s, 0)B, K_\sigma (t)) = 0.
\]

**Proof.** Since the family of MVSPs \( \{ U_\sigma (t, \tau) \}, \ \sigma \in \Sigma \), is uniformly \( \omega \)-limit compact and \( V \) is bounded, for any \( \varepsilon > 0 \), there exists a \( t_\varepsilon = t_\varepsilon (V, \varepsilon) > 0 \) independent of \( \sigma \in \Sigma \) such that

\[
k \left( \bigcup_{t \geq t_\varepsilon} \bigcup_{\sigma \in \Sigma} U_\sigma (t, 0) V \right) \leq \varepsilon.
\]
Take $\varepsilon = \frac{1}{n}, n = 1, 2, \ldots$. We can find a sequence $\{t_n\}, \max\{0, t\} < t_1 < t_2 < \cdots < t_n < \cdots$ such that
\[
\frac{1}{n}, n = 1, 2, \ldots.
\]

By Lemma 2.12, we have
\[
k\left(\bigcup_{t \geq t_n} U_\sigma(t, 0)\right) = k\left(\bigcup_{t \geq t_n} U_\sigma(t, 0)\right).
\]

Thanks to property (5) in Lemma 2.10, noticing that the set $\bigcup_{t \geq t_n} U_\sigma(t, 0)$ is closed in $X$, we know that $\bigcap_{n=1}^{\infty} \bigcup_{s \geq t_n} U_\sigma(s, 0)$ is a nonempty compact set. In view of
\[
\bigcup_{s \geq t_n} U_\sigma(s, 0) \subset \bigcup_{t \geq t_n} U_\sigma(t, 0)
\]
and the closed property of the set $\bigcap_{n=1}^{\infty} \bigcup_{s \geq t_n} U_\sigma(s, 0)$, we can conclude that
\[
A_\sigma(t) = \omega_{t, \sigma}(\mathcal{V}) = \bigcap_{T \geq \max\{0, t\}} \bigcup_{s \geq T} P_\sigma(t, t - s)\mathcal{V}^{WS}
\]
(4.2)
is compact. Observing that $\mathcal{V}$ is a uniformly absorbing set, therefore there exists a $\bar{\tau} = \bar{\tau}(\mathcal{V}) \in \mathbb{R}^+$ independent of $\sigma \in \Sigma$, such that
\[
U_\sigma(t, 0) \subset \mathcal{V}, \quad \forall t \geq \bar{\tau}, \sigma \in \Sigma.
\]

It follows from (4.2) and (4.3) that
\[
A_\sigma(t) = \omega_{t, \sigma}(\mathcal{V}) \subset \mathcal{V}, \quad \forall t \in \mathbb{R}.
\]

Let us show that
\[
A_\sigma(t) = \mathcal{K}_\sigma(t), \quad \forall t \in \mathbb{R},
\]
where $\mathcal{K}_\sigma(t)$ is the section of the kernel $\mathcal{K}_\sigma$ of the GMVP $\{P_\sigma(t, \tau)\}$ with $\sigma \in \Sigma$ at time $t$. We consider an arbitrary bounded complete trajectory $u(s)$ of the GMVP $\{P_\sigma(t, \tau)\}$. Then, according to (4.1), $u(t) \subset \mathcal{V}$ for all $t \in \mathbb{R}$. Indeed, $u(t) \in P_\sigma(t, t - s)u(t - s)$ for all $s \geq 0$. The set $B = \{u(t - s), s \geq 0\}$ is bounded in $X$. Then (4.1) implies that for $s$ sufficiently large,
\[
P_\sigma(t, t - s)B = \bigcup_{\sigma' \in T(t - s)} U_{\sigma'}(s, 0)B \subset \mathcal{V}.
\]
Hence

$$u(t) \in P_{\sigma}(t, t-s)B \subset V.$$  

On the other hand, it follows from $u(t-T) \in V$ for all $t \in \mathbb{R}$ and all $T \in \mathbb{R}^+$ that

$$u(t) \in P_{\sigma}(t, t-T)u(t-T) \subset P_{\sigma}(t, t-T)V \subset \bigcup_{s \geq T} P_{\sigma}(t, t-s)V^{WS}.$$  

Therefore, $u(t) \in \bigcup_{s \geq T} P_{\sigma}(t, t-s)V^{WS}$ for all $T \geq 0$. So,

$$u(t) \in A_{\sigma}(t) = \bigcap_{T \geq \max\{0, t\}} \bigcup_{s \geq T} P_{\sigma}(t, t-s)V^{WS}.$$  

Thus we have established that

$$K_{\sigma}(t) \subset A_{\sigma}(t), \quad \forall t \in \mathbb{R}. \quad (4.4)$$  

To prove the reverse inclusion we need the following lemmas.

**Lemma 4.4.** $y \in \omega_{t, \sigma}(B) \iff$ there exist sequences $x_n \in B$, $s_n \geq \max\{0, t\}$, $s_n \to +\infty$ ($n \to \infty$), $\sigma_n' \in T(t - s_n)\sigma$, $y_n \in U_{\sigma_n'}(s_n, 0)x_n$, such that $y_n \rightharpoonup y$ as $n \to \infty$.

**Proof.** $\Leftarrow$ According to the definition of weakly sequential closure, we know that if there exist sequences $x_n \in B$, $s_n \geq \max\{0, t\}$, $s_n \to +\infty$ ($n \to \infty$), $\sigma_n' \in T(t - s_n)\sigma$, $y_n \in U_{\sigma_n'}(s_n, 0)x_n$ such that $y_n \rightharpoonup y$ as $n \to \infty$, then we can conclude that

$$y \in \bigcup_{s \geq T} \bigcup_{\sigma' \in T(t-s)\sigma} U_{\sigma'}(s, 0)B^{WS}$$

for any $T \geq \max\{0, t\}$, which means that

$$y \in \bigcap_{T \geq \max\{0, t\}} \bigcup_{s \geq T} \bigcup_{\sigma' \in T(t-s)\sigma} U_{\sigma'}(s, 0)B^{WS}.$$  

$\Rightarrow$ Assume that $y \in \omega_{t, \sigma}(B)$. Then for any $n \in \mathbb{N}$ with $n \geq t$, we have

$$y \in \bigcup_{s \geq n} \bigcup_{\sigma' \in T(t-s)\sigma} U_{\sigma'}(s, 0)B^{WS}.$$  

It follows from the definition of weakly sequential closure that there exist sequences $x_n^k \in B$, $s_n^k \geq n$, $\sigma_{n,k}' \in T(t - s_n^k)\sigma$, $y_n^k \in U_{\sigma_{n,k}'}(s_n^k, 0)x_n^k$ such that

$$y_n^k \rightharpoonup y \quad \text{as} \quad k \to \infty.$$  

Since the family of MVSPs $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, is uniformly $\omega$-limit compact, we know that the weakly sequential closure $K^{WS}$ of $K = \{y_n^k \mid k, n = 1, 2, \ldots\}$ is weakly compact. So $K^{WS}$ is
metrizable, and the metric generates the weak topology of $\bar{K}^{WS}$. Suppose that the equivalent metric is $d$, i.e., for $\{y_n\} \subset \bar{K}^{WS}$ and $y \in \bar{K}^{WS}$, $y_n \rightharpoonup y$ if and only if $d(y_n, y) \to 0$ as $n \to \infty$. Using this equivalent metric of $\bar{K}^{WS}$, for each $n \in \mathbb{N}$, we can extract an element $y_n^{k_n}$ which belong to $\{y_n\}_{k=1}^{\infty}$ such that

$$d(y_n^{k_n}, y) < \frac{1}{n}.$$  

Then $d(y_n^{k_n}, y) \to 0$ as $n \to \infty$, which implies that

$$y_n^{k_n} \rightharpoonup y \quad \text{as} \quad n \to \infty,$$

i.e., we can find sequences $x_n^{k_n} \in B$, $s_n^{k_n} \geq n$, $s_n \to +\infty$ ($n \to \infty$), $\sigma'_n, k_n \in T(t-s_n^{k_n})$, $y_n^{k_n} \in U_{\sigma'_n, k_n}(s_n^{k_n}, 0) x_n^{k_n}$ such that $y_n^{k_n} \rightharpoonup y$ as $n \to \infty$. The proof of the lemma is complete. \qed

**Lemma 4.5.** $A_{\sigma}(t) \subset P_{\sigma}(t, \tau) A_{\sigma}(\tau)$, $\forall t \geq \tau, \tau \in \mathbb{R}$.

**Proof.** Let $y \in A_{\sigma}(t) := \omega_{t, \sigma}(\mathcal{V})$. By Lemma 4.4, there exist sequences $x_n \in \mathcal{V}$, $s_n \geq \max\{0, t\}$, $s_n \to +\infty$ ($n \to \infty$), $\sigma'_n \in T(t-s_n)$, $y_n \in U_{\sigma'_n}(s_n, 0) x_n$ such that

$$y_n \rightharpoonup y \quad \text{as} \quad n \to \infty. \quad (4.5)$$

We observe that for $n$ sufficiently large,

$$y_n \in U_{\sigma'_n}(s_n, 0) x_n \subset \bigcup_{\sigma_n' \in T(t-s_n)} U_{\sigma'_n}(s_n, 0) x_n = P_{\sigma}(t-t-s_n) x_n = P_{\sigma}(t, \tau) P_{\sigma}(\tau, t-s_n) x_n.$$

Hence there exists a sequence $\tilde{x}_n \in P_{\sigma}(\tau, t-s_n) x_n$ such that $y_n \in P_{\sigma}(t, \tau) \tilde{x}_n$.

We need to prove that $\{\tilde{x}_n\}$ has a subsequence which converges in $X$. Note that for any $\varepsilon > 0$, there exists a $t_\varepsilon > 0$ such that

$$k\left(\bigcup_{t \geq t_\varepsilon} \bigcup_{\sigma \in \Sigma} U_{\sigma}(t, 0) \mathcal{V}\right) \leq \varepsilon,$$

and that there exists an $N_0$ such that $\tau + s_n - t \geq t_\varepsilon$ for all $n \geq N_0$ and

$$\bigcup_{n \geq N_0} \tilde{x}_n \subset \bigcup_{n \geq N_0} P_{\sigma}(\tau, t-s_n) x_n = \bigcup_{n \geq N_0} \bigcup_{\sigma' \in T(t-s_n)} U_{\sigma'}(t-s_n, 0) x_n \subset \bigcup_{t \geq t_\varepsilon} \bigcup_{\sigma \in \Sigma} U_{\sigma}(t, 0) \mathcal{V}.$$

Therefore,

$$k\left(\bigcup_{n \geq N_0} \tilde{x}_n\right) \leq \varepsilon.$$
On the other hand, $\bigcup_{n=N_0'}^{N_0} \tilde{x}_n$ contains only finite number of elements, where $N_0'$ is fixed such that $\tau + s_n - t \geq 0$ as $n \geq N_0'$. Using property (3) for the measure of noncompactness in Lemma 2.10, we have

$$k\left( \bigcup_{n \geq N_0'} \tilde{x}_n \right) = k\left( \bigcup_{n \geq N_0} \tilde{x}_n \right) \leq \varepsilon.$$ 

Let $\varepsilon \to 0$. We then derive that

$$k\left( \bigcup_{n \geq N_0'} \tilde{x}_n \right) = 0.$$ 

This means that $\{\tilde{x}_n\}$ is relatively compact and thus there is a subsequence of $\{\tilde{x}_n\}$ such that $\tilde{x}_n \to x$ as $n \to \infty$ and by Lemma 4.4, we see that $x \in \omega_{\tau, \sigma}(V) = A_\sigma(\tau)$.

Finally, by the norm-to-weak upper semicontinuity of the GMVP $\{P_\sigma(t, \tau)\}$ (recall Theorem 3.5), we can conclude that there exists a subsequence $y_{n_k}$ of $y_n \in P_\sigma(t, \tau)\tilde{x}_n$ and a $y' \in P_\sigma(t, \tau)x$ such that

$$y_{n_k} \rightharpoonup y' \quad \text{as} \quad k \to \infty.$$ 

In view of (4.5), therefore $y = y' \in P_\sigma(t, \tau)x \subset P_\sigma(t, \tau)A_\sigma(\tau)$ and $A_\sigma(t) \subset P_\sigma(t, \tau)A_\sigma(\tau)$. □

**Lemma 4.6.** Assume that $\{V_\sigma(t)\}_{t \in \mathbb{R}}$ is negatively invariant, i.e., $V_\sigma(t) \subset P_\sigma(t, \tau)V_\sigma(\tau)$ for all $t \geq \tau$ and all $\tau \in \mathbb{R}$. Let $W_\sigma(t) = \bigcup_{s \in \mathbb{R}^+} P_\sigma(t, t - s)V_\sigma(t - s)$. Then $\{W_\sigma(t)\}_{t \in \mathbb{R}}$ is invariant, i.e.,

$$P_\sigma(t, \tau)W_\sigma(\tau) = W_\sigma(t), \quad \forall t \geq \tau, \quad \tau \in \mathbb{R}.$$ 

**Proof.** For any $t \geq \tau$ and $\tau \in \mathbb{R}$, observing that

$$P_\sigma(t, \tau)W_\sigma(\tau) = P_\sigma(t, \tau)\bigcup_{s \geq 0} P_\sigma(\tau, \tau - s)V_\sigma(\tau - s) = \bigcup_{s \geq 0} P_\sigma(t, \tau - s)V_\sigma(\tau - s)$$

$$= \bigcup_{s' \geq t - \tau} P_\sigma(t, t - s)V_\sigma(t - s') \quad (s' = t + s - \tau)$$

$$\subset \bigcup_{s' \geq 0} P_\sigma(t, t - s)V_\sigma(t - s') = W_\sigma(t).$$

Therefore, $\{W_\sigma(t)\}_{t \in \mathbb{R}}$ is positively invariant. Let us prove the reverse. Assume that $y \in W_\sigma(t)$, then there exist an $s \in \mathbb{R}^+$ and $x \in V_\sigma(t - s)$ such that $y \in P_\sigma(t, t - s)x$. Two cases may occur.

**Case 1.** $t - s \geq \tau$. Note that $\{V_\sigma(t)\}_{t \in \mathbb{R}}$ is negatively invariant, so we can find a $x_0 \in V_\sigma(\tau) \subset W_\sigma(\tau)$ such that $x \in P_\sigma(t - s, \tau)x_0$. Hence

$$y \in P_\sigma(t, t - s)x \subset P_\sigma(t, t - s)P_\sigma(t - s, \tau)x_0 = P_\sigma(t, \tau)x_0 \subset P_\sigma(t, \tau)W_\sigma(\tau).$$
Case 2. $t - s < \tau$. In this case,

$$y \in P_\sigma(t, t - s)x = P_\sigma(t, \tau)P_\sigma(\tau, t - s)x.$$  
Therefore there exists an $x_0 \in P_\sigma(\tau, t - s)x = P_\sigma(\tau, \tau - (s + \tau - t))x \subset W_\sigma(\tau)$, such that

$$y \in P_\sigma(t, \tau)x = P_\sigma(t, \tau)x_0 \subset P_\sigma(t, \tau)W_\sigma(\tau).$$

In conclusion, $W_\sigma(t) \subset P_\sigma(t, \tau)W_\sigma(\tau)$ for all $t \geq \tau$ and all $\tau \in \mathbb{R}$. □

Lemma 4.7. $P_\sigma(t, \tau)A_\sigma(\tau) = A_\sigma(t)$, $\forall t \geq \tau, \tau \in \mathbb{R}$.

Proof. Only the positive invariance needs to be verified. Let $W_\sigma(t) = \bigcup_{s \geq 0} P_\sigma(t, t - s) \times A_\sigma(t - s)$. By Lemmas 4.5 and 4.6, we know that $\{W_\sigma(t)\}_{t \in \mathbb{R}}$ is invariant. Recall that $A_\sigma(t) = \omega_{t, \sigma}(\mathcal{V}) \subset \mathcal{V}$ for all $t \in \mathbb{R}$. We can conclude that for any $t \geq \tau$ and $\tau \in \mathbb{R}$,

$$W_\sigma(t) = P_\sigma(t, \tau)W_\sigma(\tau) = P_\sigma(t, \tau)\bigcup_{s \geq 0} P_\sigma(\tau, \tau - s)A_\sigma(\tau - s)$$

$$= \bigcup_{s \geq 0} P_\sigma(t, \tau - s)A_\sigma(\tau - s)$$

$$\subset \bigcup_{s' \geq t - \tau} P_\sigma(t, t - s')\mathcal{V} \quad (s' = s - \tau + t)$$

which means that

$$\bigcup_{s \geq 0} P_\sigma(t, t - s)A_\sigma(t - s) = W_\sigma(t) \subset \omega_{t, \sigma}(\mathcal{V}) = A_\sigma(t), \quad \text{i.e.,}$$

$$A_\sigma(t) \subset \bigcup_{s \geq 0} P_\sigma(t, t - s)A_\sigma(t - s).$$

Thus writing $\tau = t - s$, we have

$$P_\sigma(t, \tau)A_\sigma(\tau) \subset A_\sigma(t), \quad \forall t \geq \tau, \tau \in \mathbb{R}.$$  

The proof of the lemma is finished. □

Now we are ready to complete the proof of Theorem 4.3. Using Lemma 4.7, let us show that $A_\sigma(t) \subset K_\sigma(t)$ for all $t \in \mathbb{R}$. Indeed, let $u_t$ be an arbitrary element of $A_\sigma(t)$. We shall construct a bounded complete trajectory $u(s)$, $s \in \mathbb{R}$, of the GMVP $\{P_\sigma(s, t)\}$ such that $u(t) = u_t$. We take $u(s) \in P_\sigma(s, t)u_t$, where $s \geq t$. Let us extend $u(s)$ to $s \leq t$. Lemma 4.7 implies that there exists $u_{t - 1} \in A_\sigma(t - 1)$ such that $u_t \in P_\sigma(t, t - 1)u_{t - 1}$. If we now take $u(s) \in P_\sigma(s, t - 1)u_{t - 1}$ for $s \in [t - 1, t]$, then we have $u(s) \in A_\sigma(s) \subset \mathcal{V}$ for all $s \geq t - 1$. Applying the above procedure several times we can construct $u(s) \in A_\sigma(s) \subset \mathcal{V}$ for all $s \geq t - n$, $n \in \mathbb{N}$. Letting $n \to \infty$, we obtain a bounded complete trajectory $u(s)$, $s \in \mathbb{R}$, of the GMVP $\{P_\sigma(s, t)\}$ such that $u(s) \in A_\sigma(s) \subset \mathcal{V}$ for all $s \in \mathbb{R}$ and $u(t) = u_t$. Therefore, $u_t = u(t) \in K_\sigma(t)$ and $A_\sigma(t) \subset K_\sigma(t)$. Taking into account (4.4), thus we have $A_\sigma(t) = K_\sigma(t)$ for all $t \in \mathbb{R}$.
Finally, it suffices to prove that for every bounded set \( B \subset X \) and for each fixed \( t \in \mathbb{R} \),
\[
\lim_{s \to +\infty} H^+_X(P_\sigma(t, t - s)B, K_\sigma(t)) = 0.
\]
Assume, otherwise, then there exist a bounded subset \( B_0 \) of \( X \) and \( \tilde{t}_0 \in \mathbb{R} \), such that
\[
H^+_X(P_\sigma(\tilde{t}_0, \tilde{t}_0 - s)B_0, K_\sigma(\tilde{t}_0)) \not\to 0 \quad (s \to +\infty).
\]
Thus there exist \( \varepsilon' > 0 \) and sequences \( \{x_n\} \subset B_0, \{s_n\} \subset \mathbb{R}^+ \), \( s_n \to +\infty \) \((n \to \infty)\), and \( y_n \in P_\sigma(\tilde{t}_0, \tilde{t}_0 - s_n)x_n \), such that
\[
\text{dist}_X(y_n, K_\sigma(\tilde{t}_0)) \geq \varepsilon' > 0, \quad \forall n \in \mathbb{N}.
\]
(4.6)

We observe that \( \mathcal{V} \) is a uniformly absorbing set, therefore it is easy to check that for \( n \) sufficiently large, \( \bigcup_{\sigma \in \Sigma} U_\sigma(s_n, 0)B_0 \) and \( \bigcup_{\sigma \in \Sigma} U_\sigma(s_n, 0)x_n \) belong to \( \mathcal{V} \). As in the proof above, by the uniformly \( \omega \)-limit compactness, we can verify that \( y_n \) is relatively compact and possesses at least one cluster point \( y_0 \). On the other hand,
\[
y_n \in P_\sigma(\tilde{t}_0, \tilde{t}_0 - s_n)x_n = P_\sigma(\tilde{t}_0, \tilde{t}_0 - s_n + t_0)P_\sigma(\tilde{t}_0 - s_n + t_0, \tilde{t}_0 - s_n)x_n
\[
\subset P_\sigma(\tilde{t}_0, \tilde{t}_0 - s_n + t_0) \bigcup_{\sigma' \in T(\tilde{t}_0 - s_n, t_0)} U_{\sigma'}(t_0, 0)x_n
\[
\subset P_\sigma(\tilde{t}_0, \tilde{t}_0 - s_n + t_0) \mathcal{V} = \bigcup_{\sigma' \in T(\tilde{t}_0 - s_n + t_0, 0)} U_{\sigma'}(s_n - t_0, 0)\mathcal{V},
\]
where \( t_0 \) is a positive constant such that \( \bigcup_{\sigma \in \Sigma} U_\sigma(t_0, 0)B_0 \subset \mathcal{V} \). Hence, \( y_0 \) belongs to \( A_\sigma(\tilde{t}_0) = K_\sigma(\tilde{t}_0) = \omega_{t_0, \sigma}(\mathcal{V}) \) and this contradicts (4.6). Thus the proof of Theorem 4.3 is completed. \( \square \)

**Remark 4.8.** From Theorem 4.3, it seems that the kernel sections
\[
K_\sigma(t) = \bigcap_{T \geq 0, t} \bigcup_{s \geq T} P_\sigma(t, t - s)\mathcal{V}^{\text{WS}}
\]
is larger than \( \bigcap_{T \geq 0, t} \bigcup_{s \geq T} P_\sigma(t, t - s)\mathcal{V} \), which is obtained by the usual method (see [9,10, 12,15,17,29,32–34], etc.). However, under the assumption that the family of MVSPs \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \), is uniformly \( \omega \)-limit compact, we can show that
\[
K_\sigma(t) = \bigcap_{T \geq 0, t} \bigcup_{s \geq T} P_\sigma(t, t - s)\mathcal{V}^{\text{WS}} = \bigcap_{T \geq 0, t} \bigcup_{s \geq T} P_\sigma(t, t - s)\mathcal{V}.
\]
Now we present the relation between the uniform flattening and the uniform \( \omega \)-limit compactness of the family of MVSPs \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \). First, we need the following definition:
Definition 4.9. A family of multi-valued semiprocesses \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \), on a Banach space \( X \) is said to be uniform flattening if for any fixed \( \tau \in \mathbb{R}^+ \), any bounded set \( B \subset X \) and \( \varepsilon > 0 \), there exist \( \tau_0 = \tau_0(B, \varepsilon) > 0 \) independent of \( \sigma \in \Sigma \) and a finite-dimensional subspace \( X_1 \) of \( X \) such that

\[
\begin{align*}
(1) & \quad \mathcal{P}\left( \bigcup_{t \geq \tau_0} \bigcup_{\sigma \in \Sigma} U_\sigma(t + \tau, \tau)B \right) \text{ is bounded;} \\
(2) & \quad \left\| (I - \mathcal{P})\left( \bigcup_{t \geq \tau_0} \bigcup_{\sigma \in \Sigma} U_\sigma(t + \tau, \tau)B \right) \right\|_X < \varepsilon,
\end{align*}
\]

where \( \mathcal{P} : X \to X_1 \) is the canonical projector.

Theorem 4.10. Let \( X \) be a Banach space and \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \), is a family of multi-valued semiprocesses on \( X \).

1. If the family of MVSPs \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \) is uniform flattening, then \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \), is also uniformly \( \omega \)-limit compact.

2. Let \( X \) be a uniformly convex Banach space, in particular, \( X \) be a Hilbert space. Then \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \), is uniformly \( \omega \)-limit compact if and only if \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \), is uniform flattening.

Proof. (1) It follows from property (8) in Lemma 2.10, we can deduce that

\[
k\left( \bigcup_{t \geq \tau_0} \bigcup_{\sigma \in \Sigma} U_\sigma(t + \tau, \tau)B \right) \leq 2\varepsilon.
\]

Therefore, \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \), is uniformly \( \omega \)-limit compact.

(2) Let \( X \) be a uniformly convex Banach space and \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \), is uniformly \( \omega \)-limit compact. Then for each fixed \( \tau \in \mathbb{R}^+ \), any bounded set \( B \subset X \) and \( \varepsilon > 0 \), there exist \( \tilde{T}_0 = \tilde{T}_0(B, \varepsilon) > 0 \) independent of \( \sigma \in \Sigma \) and finite number of subsets \( A_1, A_2, \ldots, A_n \) with diameter less than \( \varepsilon/2 \), such that

\[
\bigcup_{t \geq \tilde{T}_0} \bigcup_{\sigma \in \Sigma} U_\sigma(t + \tau, \tau)B \subset \bigcup_{i=1}^n A_i.
\]

Let \( x_i \in A_i \). Then

\[
\bigcup_{t \geq \tilde{T}_0} \bigcup_{\sigma \in \Sigma} U_\sigma(t + \tau, \tau)B \subset \bigcup_{i=1}^n \mathcal{N}\left( x_i, \frac{\varepsilon}{2} \right).
\]

Let \( X_1 = \text{span}[x_1, \ldots, x_n] \). Since \( X \) is uniformly convex, there exists a projection \( \mathcal{P} : X \to X_1 \), such that for any \( x \in X \), \( \| x - \mathcal{P}x \| = \text{dist}_X(x, X_1) \). Hence

\[
\left\| (I - \mathcal{P})\left( \bigcup_{t \geq \tilde{T}_0} \bigcup_{\sigma \in \Sigma} U_\sigma(t + \tau, \tau)B \right) \right\|_X \leq \frac{\varepsilon}{2} < \varepsilon.
\]
Namely, \( \{ U_\sigma(t, \tau) \}, \sigma \in \Sigma \), is uniform flattening.

The proof of the theorem is complete. \( \square \)

The following theorem follows directly from Theorems 4.3 and 4.10.

**Theorem 4.11.** Let \( X \) be a uniformly convex Banach space, in particular, \( X \) be a Hilbert space. Let \( \{ U_\sigma(t, \tau) \mid t \geq \tau, \tau \in \mathbb{R}^+ \} \), \( \sigma \in \Sigma \), be a family of multi-valued semiprocesses on \( X \) possessing jointly norm-to-weak upper semicontinuity on \( X \times \Sigma \). Also let \( T(t) \) be a nonempty compact subset of \( \mathcal{E} \) for each \( t < 0 \) and \( \sigma \in \Sigma \) and let \( \{ T(t) \} \) be a continuous invariant \( (T(t)\mathcal{E} = \mathcal{E} \text{ for all } t \in \mathbb{R}^+) \) semigroup on \( \mathcal{E} \) satisfying the translation identity (1.1). Suppose that \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \), is uniformly dissipative and uniform flattening.

Then the kernel \( K_\sigma \) of the general multi-valued process \( \{ P_\sigma(t, \tau) \mid t \geq \tau, \tau \in \mathbb{R}^+ \} \) with \( \sigma \in \Sigma \) is nonempty, the kernel sections \( K_\sigma(t) \) are all compact, invariant and pullback attract every bounded set \( B \subset X \).

In particular, let \( X = L^p(D) \) with \( D \) is a bounded smooth domain in \( \mathbb{R}^n \). We will give a efficient method to prove the existence of kernel sections in \( L^p(D) \) (\( p > 0 \)).

**Lemma 4.12.** [35] For any \( \varepsilon > 0 \), the bounded subset \( B \) of \( L^p(D) \) (\( p > 0 \)) has a finite \( \varepsilon \)-net in \( L^p(D) \) if there exists a positive constant \( M_1 = M_1(\varepsilon) \) which depends on \( \varepsilon \) such that

1. \( B \) has a finite \( (3M_1)^{(q-p)/q} (\xi)^{q/p} \)-net in \( L^q(D) \) for some \( q, 0 < q \leq p \);
2. \( \int_{D(|u| \geq M_1)} |u|^p \frac{1}{|u(x)|} < 2^{-2p^2/p} \varepsilon \) for any \( u \in B \), where \( D(|u| \geq M_1) = \{ x \in D \mid |u(x)| \geq M_1 \} \).

From the definition of uniform \( \omega \)-limit compactness of the family of MVSPs and the lemma above, we have the following result.

**Corollary 4.13.** Let \( \{ U_\sigma(t, \tau) \}_{\sigma \in \Sigma} \) be a family of MVSPs on \( L^p(D) \) and \( L^q(D) \), respectively, where \( p \geq q > 0 \) and \( D \subset \mathbb{R}^n \) is bounded, and \( \{ U_\sigma(t, \tau) \}_{\sigma \in \Sigma} \) satisfy the following two assumptions:

1. \( \{ U_\sigma(t, \tau) \}, \sigma \in \Sigma \), is uniformly \( \omega \)-limit compact in \( L^q(D) \);
2. for any \( \varepsilon > 0 \) and any bounded subset \( B \subset L^p(D) \), there exist positive constants \( M_1 = M_1(B, \varepsilon) \) and \( T_2 = T_2(B, \varepsilon) \) which are all independent of \( \sigma \in \Sigma \), such that

\[
\int_{D(|u| \geq M_1)} |u|^p < \varepsilon, \quad \forall u_0 \in B, \ t \geq T_2, \ \sigma \in \Sigma, \ u \in U_\sigma(t, \tau)u_0.
\]

Then \( \{ U_\sigma(t, \tau) \}_{\sigma \in \Sigma} \) is uniformly \( \omega \)-limit compact in \( L^p(D) \).

Combining Theorem 4.3 and Corollary 4.13, we have the following result which can be used easily to prove the existence of kernel sections for some GMVP in \( L^p(D) \).
Theorem 4.14. Assume that $p \geq q > 0$ and $D \subset \mathbb{R}^n$ is bounded. Let $\{U_\sigma(t, \tau)\}_{\sigma \in \Sigma}$ be a family of multi-valued semiprocesses on $L^p(D)$ and $L^q(D)$, respectively, possessing jointly norm-to-weak upper semicontinuity on $X \times \Sigma$. Also let $T(t)$ be a nonempty compact subset of $\Xi$ for each $t < 0$ and $\sigma \in \Xi$ and let $(T(t))$ be a continuous invariant ($T(t)\Xi = \Xi$ for all $t \in \mathbb{R}^+ \big)$ semigroup on $\Xi$ satisfying the translation identity (1.1). Assume that $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$, is

1. uniformly dissipative in $L^p(D)$;
2. uniformly $\omega$-limit compact in $L^q(D)$;
3. for any $\varepsilon > 0$ and any bounded subset $B \subset L^p(D)$, there exist positive constants $M_1 = M_1(B, \varepsilon)$ and $T_2 = T_2(B, \varepsilon)$ which are all independent of $\sigma \in \Sigma$, such that

$$\int_{D(|u| \geq M_1)} |u|^p < \varepsilon, \quad \forall u_0 \in B, \quad t \geq T_2, \quad \sigma \in \Sigma, \quad u \in U_\sigma(t, \tau)u_0.$$

Then the kernel $K_\sigma$ of the general multi-valued process $\{P_\sigma(t, \tau)\} | t \geq \tau, \quad \tau \in \mathbb{R}$ with $\sigma \in \Xi$ is nonempty, the kernel sections $K_\sigma(t)$ are all compact, invariant ($P_\sigma(t, \tau)K_\sigma(\tau) = K_\sigma(t)$ for all $t \geq \tau$ and all $\tau \in \mathbb{R}$) and pullback attract every bounded set $B \subset L^p(D)$.

Similar to the proof of [32, Theorem 3.7], we obtain

Theorem 4.15. If the family of MVSPs $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$, is uniformly dissipative, then for each $\sigma \in \Xi$,

1. for any uniformly absorbing sets $V_1, V_2 \subset X$,

$$\omega_{t,\sigma}(V_1) = \omega_{t,\sigma}(V_2), \quad \forall t \in \mathbb{R};$$

2. for any bounded set $B \subset X$ and any uniformly absorbing set $V_1 \subset X$,

$$\omega_{t,\sigma}(B) \subset \omega_{t,\sigma}(V_1), \quad \forall t \in \mathbb{R},$$

where $\omega_{t,\sigma}(V_i), \ i = 1, 2$, is the pullback $\omega$-limit set of the general multi-valued process $\{P_\sigma(t, \tau)\}$ with $\sigma \in \Xi$ defined in (1.2).

Remark 4.16. From Theorem 4.15, we easily see that the kernel sections $K_\sigma(t) = \omega_{t,\sigma}(V)$ given as in Theorem 4.3 are independent of the choice of uniformly absorbing set $V$, therefore they are unique.

A family of MVSPs $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$, is said to be uniformly asymptotically upper-compact in $X$ if for any fixed $\tau \in \mathbb{R}^+$ and any bounded set $B \subset X$, any sequence $y_n \in U_\sigma(T_n + \tau, \tau)B$ with $\sigma_n \in \Sigma$ and $T_n \to +\infty \ (n \to \infty)$ is precompact in $X$. The following theorem shows that uniformly $\omega$-limit compactness equals to uniformly asymptotically upper compactness.

Analogous to the arguments in the proof of [32, Theorem 3.9], we have:
Theorem 4.17. Let \( \{U_\sigma(t, \tau)\} \), \( \sigma \in \Sigma \), be a family of multi-valued semiprocesses on \( X \). Then \( \{U_\sigma(t, \tau)\} \), \( \sigma \in \Sigma \), is uniformly asymptotically upper-semicompact if and only if \( \{U_\sigma(t, \tau)\} \), \( \sigma \in \Sigma \), is uniformly \( \omega \)-limit compact.

Remark 4.18. Let \( X \) be a uniformly convex Banach space, in particular, \( X \) be a Hilbert space, and \( \{U_\sigma(t, \tau)\} \), \( \sigma \in \Sigma \), be a family of MVSPs on \( X \). We can deduce from Theorems 4.10 and 4.17 that \( \{U_\sigma(t, \tau)\} \), \( \sigma \in \Sigma \), is uniformly asymptotically upper-semicompact if and only if \( \{U_\sigma(t, \tau)\} \), \( \sigma \in \Sigma \), is uniform flattening.

5. Uniform attractors and uniform forward attraction of kernel sections

In this section, we consider the structure of uniform attractors of a family of multi-valued semiprocesses and uniform forward attraction of the inflated kernel sections defined in (1.3) of a family of general multi-valued processes generated by the family of MVSPs. For this purpose, we first consider a (autonomous) multi-valued semidynamical system \( \{F(t)\} \) on \( X \) and present the necessary and sufficient conditions of the existence of global attractors for the MVSS \( \{F(t)\} \).

Let \( \mathcal{K} \) be the kernel of the MVSS \( \{F(t)\} \). The kernel \( \mathcal{K} \) consists of all bounded complete trajectories of the MVSS \( \{F(t)\} \), i.e.,

\[
\mathcal{K} = \left\{ u(\cdot) \mid \sup_{t \in (-\infty, +\infty)} \| u(t) \|_X \leq C_u, \ u(t + \tau) \in F(\tau)u(t), \ \forall t \in \mathbb{R}, \ \tau \in \mathbb{R}^+ \right\}.
\]

As usual, \( \mathcal{K}(s) \) denotes the kernel section at a time moment \( s \in \mathbb{R} \):

\[
\mathcal{K}(s) = \left\{ u(s) \mid u(\cdot) \in \mathcal{K} \right\}, \ \mathcal{K}(s) \subset X.
\]

Obviously,

\[
\mathcal{K}(t + \tau) \subseteq F(\tau)\mathcal{K}(t), \ \forall t \in \mathbb{R}, \ \tau \in \mathbb{R}^+.
\]

Definition 5.1. Let \( \{F(t)\} \) be a (autonomous) multi-valued semidynamical system on \( X \). For any subset \( B \) of \( X \), the \( \omega \)-limit set \( \omega(B) \) defined by

\[
\omega(B) = \bigcap_{s \in \mathbb{R}^+} \bigcup_{t \geq s} F(t)B^{\text{WS}}.
\]

Theorem 5.2. Let \( \{F(t)\} \) be a (autonomous) multi-valued semidynamical system on \( X \). Then \( \{F(t)\} \) has a unique global attractor \( \mathcal{A} = \omega(U) \); moreover, \( \mathcal{A} \) coincides with the kernel section at time \( \tau \), i.e., \( \mathcal{A} = \mathcal{K}(\tau) \) for any \( \tau \in \mathbb{R} \) if and only if \( \{F(t)\} \) is

(1) dissipative, i.e., there exists a bounded subset \( U \) of \( X \) so that for any bounded set \( B \subset X \), there exists a \( T_0 = T_0(B) \in \mathbb{R}^+ \), such that

\[
F(t)B \subset U, \ \forall t \geq T_0;
\]

(2) \( \omega \)-limit compact.

The proof of Theorem 5.2 is similar to the one of Theorem 4.3, and the arguments here are easier than those in the proof of Theorem 4.3, so they are omitted.
Corollary 5.3. Let \{F(t)\} be a (autonomous) multi-valued semidynamical system on X. Then the following statements are equivalent:

1. There is a compact attracting set \(\Theta\) for \{F(t)\}, i.e., there is a compact set \(\Theta\) which attracts any bounded set \(B \subset X\) under \{F(t)\}.
2. \{F(t)\} is dissipative and \(\omega\)-limit compact.
3. \{F(t)\} has a unique global attractor \(A\).

Theorem 5.4. Let \{F(t)\} be a (autonomous) multi-valued semidynamical system on X. Moreover, \(F(t)x\) is upper-semicontinuous in \(x\) for fixed \(t \in \mathbb{R}^+\), i.e., if \(x_n \to x\) in \(X\), then

\[
H_X^*(F(t)x_n, F(t)x) \to 0 \quad \text{as} \quad n \to \infty.
\]

Then the following statements are equivalent:

1. There is a compact attracting set \(\Theta\) for \{F(t)\}.
2. \{F(t)\} is dissipative and \(\omega\)-limit compact.
3. \{F(t)\} has a unique global attractor \(A\) and

\[
A = \omega^s(\Theta) = \bigcap_{s \in \mathbb{R}^+} \bigcup_{t \geq s} F(t)\Theta.
\]

Proof. Thanks to Corollary 5.3, we only need to prove \((1) \Rightarrow (3)\). Analogous to the proof of Theorem 4.3, we can check that \(A = \omega^s(\Theta)\) is compact and invariant. Let us show that \(A\) attracts all bounded subset \(B\) of \(X\). Clearly, \(\omega^s(\Theta)\) attracts \(\Theta\), i.e., for any \(\varepsilon > 0\), there exists a \(\tau_1' > 0\) such that

\[
H_X^*(F(t)\Theta, \omega^s(\Theta)) < \frac{\varepsilon}{2}, \quad \forall t \geq \tau_1'.
\]  
(5.1)

Since \(F(\tau_1')x\) is upper-semicontinuous in \(x\), there exists a \(\delta > 0\) such that for any \(y \in N(x, \delta)\),

\[
H_X^*(F(\tau_1')y, F(\tau_1')x) < \frac{\varepsilon}{2}.
\]  
(5.2)

Noticing that compact set \(\Theta\) attracts any bounded set \(B \subset X\), hence there exists a \(\tilde{T}_1 > 0\) such that

\[
H_X^*(F(t)B, \Theta) < \delta, \quad \forall t \geq \tilde{T}_1.
\]  
(5.3)

Combining (5.2) and (5.3) together, we can deduce that

\[
H_X^*(F(t+\tau_1')B, F(\tau_1')\Theta) < \frac{\varepsilon}{2}, \quad \forall t \geq \tilde{T}_1.
\]  
(5.4)

It follows from (5.1) and (5.4) that

\[
H_X^*(F(t+\tau_1')B, \omega^s(\Theta)) \leq H_X^*(F(t+\tau_1')B, F(\tau_1')\Theta) + H_X^*(F(\tau_1')\Theta, \omega^s(\Theta)) \lesssim \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall t \geq \tilde{T}_1.
\]

We have completed the proof of Theorem 5.4. \(\square\)
**Definition 5.5.** Let \( \{U_\sigma(t, \tau) \mid t \geq \tau, \tau \in \mathbb{R}^+\}, \sigma \in \Sigma \), be a family of multi-valued semiprocesses on \( X \). For every nonempty subset \( B \) of \( X \) and any \( \tau \in \mathbb{R}^+ \), the uniform \( \omega \)-limit set \( \omega_{\tau, \Sigma}(B) \) defined by

\[
\omega_{\tau, \Sigma}(B) = \bigcap_{T \geq 0} \bigcup_{i \geq T} \bigcup_{\sigma \in \Sigma} U_\sigma(t + \tau, \tau)B^{WS}.
\]

**Remark 5.6.** \( y \in \omega_{\tau, \Sigma}(B) \iff \) there exist sequences \( x_n \in B, t_n \in \mathbb{R}^+, t_n \to +\infty \) \((n \to \infty)\), \( \sigma_n \in \Sigma \), \( y_n \in U_{\sigma_n}(t_n + \tau, \tau)x_n \), such that \( y_n \rightharpoonup y \) as \( n \to \infty \).

Now we recapitulate the following result which will be used in the proof of Theorem 5.10; see [5] for more details.

**Theorem 5.7.** Let \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \), be a family of multi-valued semiprocesses on \( X \) satisfying the translation identity (1.1). Then the following statements are equivalent:

1. There is a compact uniformly attracting set for the family of MVSPs \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \).
2. \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \), is uniformly dissipative, i.e., for any fixed \( \tau \in \mathbb{R}^+ \), there exists a bounded subset \( V \) of \( X \) so that for any bounded set \( B \subset X \), there exists \( \bar{\tau} = \bar{\tau}(B) \in \mathbb{R}^+ \) independent of \( \sigma \in \Sigma \), such that

\[
U_\sigma(t + \tau, \tau)B \subset V, \quad \forall \sigma \in \Sigma, \ t \geq \bar{\tau},
\]

and \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \), is uniformly \( \omega \)-limit compact.
3. \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \), has a unique uniform attractor \( \mu = \omega_{0, \Sigma}(V) = \omega_{\tau, \Sigma}(V) \) for any \( \tau \in \mathbb{R}^+ \).

**Remark 5.8.** It is worth noticing that

1. in [5], authors present the sufficient conditions of the existence of uniform attractors for the family of MVSPs \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \), here we clarify that it is also necessary;
2. it follows from the conclusion (3) in Theorem 5.7 that

\[
\mu = \omega_{0, \Sigma}(V) = \omega_{\tau, \Sigma}(V) = \bigcap_{T \geq 0} \bigcup_{i \geq T} \bigcup_{\sigma \in \Sigma} U_\sigma(t + \tau, \tau)\bigcap_{T \geq 0} \bigcup_{i \geq T} \bigcup_{\sigma \in \Sigma} U_\sigma(t + \tau, \tau)V^{WS}.
\]

it seems to be larger than \( \bigcap_{T \geq 0} \bigcup_{i \geq T} \bigcup_{\sigma \in \Sigma} U_\sigma(t + \tau, \tau)V \) as in [5], however, under the assumption that the family of MVSPs \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \), is uniformly \( \omega \)-limit compact, we can prove that

\[
\mu = \omega_{0, \Sigma}(V) = \omega_{\tau, \Sigma}(V) = \bigcap_{T \geq 0} \bigcup_{i \geq T} \bigcup_{\sigma \in \Sigma} U_\sigma(t + \tau, \tau)\bigcap_{T \geq 0} \bigcup_{i \geq T} \bigcup_{\sigma \in \Sigma} U_\sigma(t + \tau, \tau)V.
\]

As a direct result of Theorem 5.7, we have:

**Corollary 5.9.** Let \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \), be a family of multi-valued semiprocesses on \( X \) satisfying the translation identity (1.1), and let \( \{T(h)\} \) be a continuous invariant semigroup on a subset \( \Xi \).
of $\Sigma$. Then we can construct a family of general multi-valued processes $\{P_\sigma(t, \tau)\}$, $\sigma \in \Xi$, and the following statements are equivalent:

1. There is a compact uniformly attracting set for the family of GMVPs $\{P_\sigma(t, \tau)\}$, $\sigma \in \Xi$.
2. $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, is uniformly dissipative, i.e., for any fixed $\tau \in \mathbb{R}^+$, there exists a bounded subset $\mathcal{V}$ of $X$ so that for any bounded set $B \subset X$, there exists $\bar{\tau} = \bar{\tau}(B) \in \mathbb{R}^+$ independent of $\sigma \in \Sigma$, such that

$$U_\sigma(t + \tau, \tau)B \subset \mathcal{V}, \quad \forall \sigma \in \Sigma, \quad t \geq \bar{\tau},$$

and $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, is uniformly $\omega$-limit compact.
3. $\{P_\sigma(t, \tau)\}$, $\sigma \in \Xi$, has a unique uniform attractor $\mu = \omega_{0, \Xi}(\mathcal{V}) = \omega_{\tau, \Xi}(\mathcal{V})$ for any $\tau \in \mathbb{R}$, where the uniform $\omega$-limit set $\omega_{\tau, \Xi}(\mathcal{V})$ defined by

$$\omega_{\tau, \Xi}(\mathcal{V}) = \bigcap_{T \geq 0} \bigcup_{t \geq T} \bigcup_{\sigma \in \Xi} P_\sigma(t + \tau, \tau)V^{WS}.$$
(4) the uniform attractor satisfies

\[ \Pi_1 A = A_{\Sigma} = \bigcup_{\sigma \in \omega(\Sigma)} K_\sigma(0); \]  

(5) for any fixed \( \epsilon_0 > 0 \), the family of inflated kernel sections \( \{ K_{\sigma}^{[\epsilon_0]}(0) \} \), \( \sigma \in \omega(\Sigma) \), defined in (1.3) uniformly (w.r.t. \( \sigma \in \omega(\Sigma) \)) pullback (respectively forward) attracts each bounded subset \( B \) of \( X \), i.e., for any \( \epsilon > 0 \), there is a \( T_1 = T_1(B, \epsilon) > 0 \) independent of \( \sigma \in \omega(\Sigma) \) such that

\[ H^\epsilon_X \left( P_{\sigma}(0, -t) B, K_{\sigma}^{[\epsilon_0]}(0) \right) < \epsilon, \quad \forall \sigma \in \omega(\Sigma), \ t \geq T_1 \]  

(respectively \( H^\epsilon_X \left( P_{\sigma}(t, 0) B, K_{\sigma}^{[\epsilon_0]}(T(t) \sigma) \right) < \epsilon, \quad \forall \sigma \in \omega(\Sigma), \ t \geq T_1 \)).

Here \( K_\sigma(0) \) is the section at \( t = 0 \) of the kernel \( K_\sigma \) of the GMVP \( \{ P_{\sigma}(t, \tau) \} \) with \( \sigma \in \omega(\Sigma) \).

**Proof.**  Write \( Y = X \times \Sigma \) and endow \( Y \) with the norm \( \| \cdot \|_Y \) defined by

\[ \| (x, \sigma_1) - (y, \sigma_2) \|_Y = \| x - y \|_X + \| \sigma_1 - \sigma_2 \|_\Sigma, \quad \forall (x, \sigma_1), (y, \sigma_2) \in Y. \]  

(5.8)

Clearly, \((Y, \| \cdot \|_Y)\) is a Banach space.

Now we consider the multi-valued semidynamical system \( \{ F(t) \} \) on \( Y \) over \( \mathbb{R}^+ \) defined by

\[ F(t)(x, \sigma) = \left( U_{\sigma}(t, 0)x, T(t) \sigma \right), \quad \forall (x, \sigma) \in Y. \]  

(5.9)

Then \( F(t) \) is well defined on \( Y \) and due to the jointly norm-to-weak upper semicontinuity of \( U_{\sigma}(t, \tau)x \) in \( (x, \sigma) \) and the continuity of \( T(t) \sigma \) in \( \sigma \), we see that \( F(t)(x, \sigma) \) is norm-to-weak upper-semicontinuous in \( (x, \sigma) \) for any fixed \( t \in \mathbb{R}^+ \). Since the family of MVSPs \( \{ U_{\sigma}(t, \tau) \}, \sigma \in \Sigma \), is uniformly dissipative, there exists a bounded subset \( \mathcal{V} \) of \( X \) so that for any bounded set \( B \subset X \), one can find a \( \tilde{\tau} > 0 \) independent of \( \sigma \in \Sigma \), such that

\[ U_{\sigma}(t, 0)B \subset \mathcal{V}, \quad \forall \sigma \in \Sigma, \ t \geq \tilde{\tau}. \]

It follows that \( \mathcal{U} := \mathcal{V} \times \Sigma \) is a bounded absorbing set of \( F(t) \), i.e., \( F(t) K \subset \mathcal{U} \) for \( t \) sufficiently large for each bounded subset \( K \) of \( \mathcal{Y} \). The uniform \( \omega \)-limit compactness of the family of MVSPs \( \{ U_{\sigma}(t, \tau) \}, \sigma \in \Sigma \), implies that \( \{ F(t) \} \) is \( \omega \)-limit compact (recall that \( T(t) \) has a attractor \( \omega(\Sigma) \)). Thus, according to Theorem 5.2, the multi-valued semidynamical system \( \{ F(t) \} \) has a unique global attractor \( \mathcal{A} = \omega(\mathcal{V} \times \Sigma) \) which is compact and invariant. Obviously,

\[ \Pi_2 \mathcal{A} = \Pi_2 \omega(\mathcal{V} \times \Sigma) = \Pi_2 \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} F(t)(\mathcal{V} \times \Sigma)^{WS} = \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} T(t)\Sigma^{WS} = \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} T(t)\Sigma \]

\[ = \omega(\Sigma). \]

Hence (2) is proved.

Consider the nonempty compact sets

\[ \tilde{A}_{\sigma}(0) = \{ x \in X \mid (x, \sigma) \in \mathcal{A} \}. \]
We have:

**Lemma 5.11.** \( \{ \tilde{A}_\sigma (0) \}_{\sigma \in \omega (\Sigma)} \) are positively and accumulatively invariant.

**Proof.** By the invariance of the attractor \( A \), i.e., \( F(t)A = A \) for all \( t \geq 0 \), we have

\[
F(t)(\tilde{A}_\sigma (0), \sigma) = (U_\sigma (t, 0)\tilde{A}_\sigma (0), T(t)\sigma) \subset A,
\]

which implies that

\[
U_\sigma (t, 0)\tilde{A}_\sigma (0) \subset \tilde{A}_{T(t)\sigma} (0),
\]

i.e., \( \{ \tilde{A}_\sigma (0) \}_{\sigma \in \omega (\Sigma)} \) is positively invariant. In consequence, we can deduce that \( U_{\sigma'} (t, 0)\tilde{A}_{\sigma'} (0) \subset \tilde{A}_\sigma (0) \) for all \( \sigma' \in T(-t)\sigma \), i.e.,

\[
\bigcup_{\sigma' \in T(-t)\sigma} U_{\sigma'} (t, 0)\tilde{A}_{\sigma'} (0) \subset \tilde{A}_\sigma (0).
\]

To obtain the opposite inclusion, let \( x \in \tilde{A}_\sigma (0) \), i.e., \( (x, \sigma) \in A \). Fix \( t \geq 0 \). It follows from the invariance \( F(t)A = A \) for all \( t \geq 0 \) that there exists \( (y, \sigma') \in A \) with \( y \in \tilde{A}_{\sigma'} (0) \) such that \( (x, \sigma) \in F(t)(y, \sigma') \). This means that \( T(t)\sigma' = \sigma \) and \( x \in U_{\sigma'} (t, 0)y \). Thus \( \sigma' \in T(-t)\sigma \). Taking the union over all \( x \in \tilde{A}_\sigma (0) \), we have

\[
\tilde{A}_\sigma (0) \subset \bigcup_{\sigma' \in T(-t)\sigma} U_{\sigma'} (t, 0)\tilde{A}_{\sigma'} (0).
\]

So,

\[
\tilde{A}_\sigma (0) = \bigcup_{\sigma' \in T(-t)\sigma} U_{\sigma'} (t, 0)\tilde{A}_{\sigma'} (0),
\]

i.e., accumulatively invariance holds true. \( \square \)

**Lemma 5.12.** Let \( A_\sigma (0) = \omega_{0,\sigma} (\mathcal{V}) \). Then \( \tilde{A}_\sigma (0) = A_\sigma (0) \) for all \( \sigma \in \omega (\Sigma) \).

**Proof.** First, let us prove that \( \tilde{A}_\sigma (0) \subset A_\sigma (0) \). Noticing that

\[
\tilde{A}_\sigma (0) = \bigcup_{\sigma' \in T(-t)\sigma} U_{\sigma'} (t, 0)\tilde{A}_{\sigma'} (0), \quad \forall \sigma \in \omega (\Sigma), \quad t \geq 0,
\]

and

\[
\bigcup_{\sigma \in \omega (\Sigma)} \tilde{A}_\sigma (0) = \text{Proj}_{\mathcal{X}} A \subset \mathcal{V}.
\]

Therefore for all \( t \in \mathbb{R}^+ \),

\[
\tilde{A}_\sigma (0) \subset \bigcup_{\sigma' \in T(-t)\sigma} U_{\sigma'} (t, 0)\mathcal{V},
\]
which implies that
\[
\widetilde{A}_\sigma (0) \subset \bigcap_{s \geq 0} \bigcup_{t \geq s} \bigcup_{\sigma' \in T (-t)\sigma} U_{\sigma'} (t, 0) \mathcal{V}_{WS} = \omega_{0,\sigma} (\mathcal{V}) = A_\sigma (0).
\]

As for the converse, observing that
\[
A_\sigma (0) \times \{\sigma\} = \bigcap_{s \geq 0} \bigcup_{t \geq s} \bigcup_{\sigma' \in T (-t)\sigma} U_{\sigma'} (t, 0) \mathcal{V}_{WS} \times \{\sigma\} = \bigcap_{s \geq 0} \bigcup_{t \geq s} \bigcup_{\sigma' \in T (-t)\sigma} U_{\sigma'} (t, 0) \mathcal{V} \times \{\sigma\} = \bigcap_{s \geq 0} \bigcup_{t \geq s} \bigcup_{\sigma' \in T (-t)\sigma} U_{\sigma'} (t, 0) \mathcal{V} \times \{T (t)\sigma'\} \mathcal{V} \times \{\sigma\}.
\]

(5.10)

On the other hand, we easily verify that
\[
\mathcal{A} = \bigcap_{s \geq 0} \bigcup_{t \geq s} F (t) (\mathcal{V} \times \Sigma) \mathcal{V}_{WS} \supset \bigcap_{s \geq 0} \bigcup_{t \geq s} F (t) (\mathcal{V} \times \omega (\Sigma)) \mathcal{V}_{WS} \supset \bigcup_{s \geq 0} \bigcup_{t \geq s} \bigcup_{\sigma \in \omega (\Sigma)} U_\sigma (t, 0) \mathcal{V} \times \{T (t)\sigma\} \mathcal{V}_{WS}.
\]

(5.11)

Combining (5.10) and (5.11) together, hence we have \(A_\sigma (0) \times \{\sigma\} \subset \mathcal{A}\), which implies that \(A_\sigma (0) \times \{\sigma\} \subset \widetilde{A}_\sigma (0) \times \{\sigma\}\) and therefore \(A_\sigma (0) \subset \widetilde{A}_\sigma (0)\). Thus \(A_\sigma (0) = \widetilde{A}_\sigma (0)\) for all \(\sigma \in \omega (\Sigma)\). \(\Box\)

In conclusion, we have
\[
\mathcal{A} = \bigcup_{\sigma \in \omega (\Sigma)} A_\sigma (0) \times \{\sigma\}.
\]

Invoking Theorem 4.3, we know that \(A_\sigma (0) = \omega_{0,\sigma} (\mathcal{V}) = \mathcal{K}_\sigma (0)\). Hence
\[
\mathcal{A} = \bigcup_{\sigma \in \omega (\Sigma)} \mathcal{K}_\sigma (0) \times \{\sigma\}
\]

and (3) is proved.

Now let us show that \(\Pi_1 \mathcal{A} = \mathcal{A}_\Sigma\) is the uniform attractor of the family of MVSPs \(\{U_\sigma (t, \tau)\}, \sigma \in \Sigma\). We see that \(\Pi_1 \mathcal{A} = \mathcal{A}_\Sigma\) is compact because of the compactness of \(\mathcal{A}\). Let us verify that \(\mathcal{A}_\Sigma\) is a uniformly attracting set. Let \(B\) be a bounded subset of \(X\). Since \(\mathcal{A}\) attracts \(B \times \Sigma\) under \(\{F (t)\}\),

\[
H^*_B (F (t) (B \times \Sigma), \mathcal{A}) \rightarrow 0 \quad \text{as } t \rightarrow +\infty.
\]

(5.12)

By (5.9), we have
\[
\sup_{\sigma \in \Sigma} H^*_B (U_\sigma (t, 0) B, \mathcal{A}_\Sigma) \rightarrow 0 \quad \text{as } t \rightarrow +\infty.
\]

(5.13)
In fact, we can replace 0 in (5.13) by any \( s \in \mathbb{R}^+ \). Indeed, since the family of MVSPs \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \), is uniformly dissipative, the set \( B_1 = \bigcup_{\sigma \in \Sigma} U_\sigma(\tilde{\tau}_1 + s, s)B \) is bounded in \( X \) for some positive constant \( \tilde{\tau}_1 \). Let \( h = \tilde{\tau}_1 + s \). Therefore if \( t \geq \tilde{\tau}_1 \), then

\[
U_\sigma(t+s,s)B = U_\sigma(t+s, \tilde{\tau}_1+s)U_\sigma(\tilde{\tau}_1+s, s)B \subset U_\sigma(t+s, \tilde{\tau}_1+s)B_1
\]

By (5.13), we find that

\[
\sup_{\sigma \in \Sigma} H^*_X(U_\sigma(t+s,s)B, A_\Sigma) \leq \sup_{\sigma \in \Sigma} H^*_X(U_\sigma(t-\tilde{\tau}_1,0)B_1, A_\Sigma) \rightarrow 0 \quad (t \to +\infty).
\]

Let us verify the minimality property. We shall establish the inclusion \( \Pi_1A = A_\Sigma \subset \omega(0, \Sigma(V)) \), where \( \omega(0, \Sigma(V)) \) is a uniform attractor of the family of MVSPs \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \) (recall Theorem 5.7). Indeed, the set \( V \times \Sigma \) is a absorbing set of the MVSS \( \{ F(t) \} \). Then by Theorem 5.2, we have

\[
A = \omega(0, \Sigma(V)) = \bigcap_{t \geq 0} \bigcup_{h \geq t} F(h)(V \times \Sigma)^{WS}.
\]

Note that \( (y, \sigma) \in \omega(0, \Sigma(V)) \iff \text{there exists sequences } x_n \in V, \sigma_n \in \Sigma, t_n \in \mathbb{R}^+, t_n \to +\infty (n \to \infty), y_n \in U_{\sigma_n}(t_n,0)x_n \text{ and } T(t_n)\sigma_n, \text{ such that }\]

\[
y_n \rightharpoonup y \quad \text{as } n \to \infty, \quad T(t_n)\sigma_n \to \sigma \quad \text{as } n \to \infty. \quad (5.14)
\]

It follows from Remark 5.6 and (5.14) that \( y \in \omega(0, \Sigma(V)) \), i.e., \( \Pi_1A \subset \omega(0, \Sigma(V)) \). Hence, \( \Pi_1A = A_\Sigma = \omega(0, \Sigma(V)) \) is the uniform attractor of the family of MVSPs \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \). We have proved (1) and (4).

Let \( \epsilon_0 > 0 \). Now we remain to show that for any bounded subset \( B \) of \( X \) and \( \epsilon > 0 \) (we can assume that \( \epsilon < \epsilon_0 \)), there is a \( T_1 = T_1(B, \epsilon) > 0 \) independent of \( \sigma \in \omega(\Sigma) \) such that

\[
H^*_X(P_\sigma(0,-t)B, K_0^{[\epsilon_0]}(0)) < \epsilon, \quad \forall \sigma \in \omega(\Sigma), \quad t \geq T_1, 
\]

\[
H^*_X(P_\sigma(t,0)B, K_0^{[\epsilon_0]}(0)) < \epsilon, \quad \forall \sigma \in \omega(\Sigma), \quad t \geq T_1.
\]

Let \( \Omega = B \times \omega(\Sigma) \). By (5.12), there is a \( T_1 = T_1(\Omega, \epsilon) > 0 \) such that when \( t \geq T_1 \), we have

\[
\inf_{v \in A} H^*_Y(F(t)u, v) < \frac{\epsilon}{2}, \quad \forall u := (u', \sigma) \in \Omega, \quad \text{i.e.,}
\]

\[
\inf_{v \in A} H^*_Y((U_\sigma(t,0)u', T(t)\sigma), v) < \frac{\epsilon}{2}, \quad \forall u' \in B, \quad \sigma \in \omega(\Sigma), \quad t \geq T_1. \quad (5.15)
\]

Now for each \( T(t)\sigma \), we divide \( A \) into two parts:

\[
A = A_{T(t)\sigma}[\epsilon_0] \cup A_{T(t)\sigma}^C[\epsilon_0].
\]
where
\[ A_{T(t)\sigma}[\epsilon_0] = \bigcup_{\sigma' \in \omega(\Sigma), \|\sigma' - T(t)\sigma\|_{\Sigma} \leq \epsilon_0} (K_{\sigma'}(0) \times \{\sigma'\}) , \]
\[ A_{C_{T(t)\sigma}}[\epsilon_0] = \bigcup_{\sigma' \in \omega(\Sigma), \|\sigma' - T(t)\sigma\|_{\Sigma} > \epsilon_0} (K_{\sigma'}(0) \times \{\sigma'\}) . \]

Let \((u', \sigma) \in B \times \omega(\Sigma), t \geq T_1\) and \(x \in U_{\sigma}(t, 0)u'\). Note that, if \(v := (y, \sigma') \in A_{C_{T(t)\sigma}}[\epsilon_0]\), then by the definition of \(\|\cdot\|_Y\),

\[ \|(x, T(t)\sigma) - (y, \sigma')\|_{Y} = \|x - y\|_{X} + \|T(t)\sigma - \sigma'\|_{\Sigma} \geq \|T(t)\sigma - \sigma'\|_{\Sigma} > \epsilon_0 > \epsilon . \] (5.16)

So, by (5.15) and (5.16), we necessarily have

\[ \inf_{v \in A_{T(t)\sigma}[\epsilon_0]} \|(x, T(t)\sigma) - v\|_{Y} < \frac{\epsilon}{2}. \]

Thus, in particular, there exists a point \(v' := (y', \sigma'') \in A_{T(t)\sigma}[\epsilon_0]\) such that

\[ \|(x, T(t)\sigma) - (y', \sigma'')\|_{Y} \leq \frac{2\epsilon}{3} . \] (5.17)

Since \(\|\sigma'' - T(t)\sigma\|_{\Sigma} \leq \epsilon_0\), we can conclude that \(y' \in K_{\sigma''}(0) \subseteq K_{T(t)\sigma}[\epsilon_0](0)\). From this and (5.17) it follows that

\[ \text{dist}_{X}(x, K_{\sigma''}(0)) \leq \|x - y'\|_{X} \leq \|(x, T(t)\sigma) - (y', \sigma'')\|_{Y} \leq \frac{2\epsilon}{3} \]

and hence that

\[ \text{dist}_{X}(x, K_{T(t)\sigma}[\epsilon_0](0)) \leq \text{dist}_{X}(x, K_{\sigma''}(0)) \leq \frac{2\epsilon}{3} < \epsilon . \]

Observing that \(u' \in B, \sigma \in \omega(\Sigma), t \geq T_1\) and \(x \in U_{\sigma}(t, 0)u'\) are otherwise arbitrary, we obtain

\[ H_X^*(U_{\sigma}(t, 0)B, K_{T(t)\sigma}[\epsilon_0](0)) < \epsilon, \quad \forall t \geq T_1, \sigma \in \omega(\Sigma). \] (5.18)

In view of (1.2), we have

\[ H_X^*(P_{\sigma}(t, 0)B, K_{T(t)\sigma}[\epsilon_0](0)) < \epsilon, \quad \forall t \geq T_1, \sigma \in \omega(\Sigma). \]

It follows from (5.18) that

\[ H_X^*(U_{\sigma''}(t, 0)B, K_{T(t)\sigma''}[\epsilon_0](0)) < \epsilon, \quad \forall t \geq T_1, \sigma'' \in T(-t)\sigma, \sigma \in \omega(\Sigma), \]

which means that

\[ H_X^*\left( \bigcup_{\sigma'' \in T(-t)\sigma} U_{\sigma''}(t, 0)B, K_{\sigma''}[\epsilon_0](0) \right) < \epsilon, \quad \forall t \geq T_1, \sigma \in \omega(\Sigma). \]
By (1.2), we can deduce that \( P_\sigma(0, -t) = \bigcup_{\sigma'' \in T(-t)\sigma} U_{\sigma''}(t, 0) \) and
\[
H^+_X\left( P_\sigma(0, -t) B, K^{[\varepsilon_0]}(0) \right) < \varepsilon, \quad \forall t \geq T_1, \; \sigma \in \omega(\Sigma).
\]

The proof of Theorem 5.10 is complete. \( \square \)

**Theorem 5.13.** In addition to the hypotheses in Theorem 5.10, assume that \( U_\sigma(t, \tau)x \) is upper-semicontinuous in \( \sigma \) and \( x \) for fixed \( t \geq \tau, \; \tau \in \mathbb{R}^+ \), i.e., if \( x_n \rightarrow x \) in \( X \) and \( \sigma_n \rightarrow \sigma \) in \( \Sigma \), then for any \( y_n \in U_{\sigma_n}(t, \tau)x_n \), there exists \( y \in U_\sigma(t, \tau)x \) such that \( y_n \rightarrow y \) as \( n \rightarrow \infty \). Then we can define a family of general multi-valued processes \( \{ P_\sigma(t, \tau), \sigma \in \omega(\Sigma) \} \) and the multi-valued semidynamical system \( \{ F(t) \} \) corresponding to the family of multi-valued semidynamical processes \( \{ U_\sigma(t, \tau), \sigma \in \Sigma \} \), acting on \( X \times \Sigma \) possesses a unique compact attractor \( \tilde{A} \) which is strictly invariant with respect to \( \{ F(t) \} \). For all \( t \geq 0 \).

1. \( \Pi_1 \tilde{A} = A_{\Sigma} = \text{the uniform (w.r.t. } \sigma \in \Sigma\text{)} \text{ attractor of the family of multi-valued semidynamical processes } \{ U_\sigma(t, \tau), \sigma \in \Sigma \}; \)
2. \( \Pi_2 \tilde{A} = \omega(\Sigma) = \text{the attractor of the semigroup } \{ T(t) \}\text{ acting on } \Sigma: T(t)\omega(\Sigma) = \omega(\Sigma) \text{ for all } t \geq 0; \)
3. \( \text{the global attractor satisfies} \)
\[
\mathcal{A} = \bigcup_{\sigma \in \omega(\Sigma)} K_\sigma(0) \times \{ \sigma \};
\]
4. \( \text{the uniform attractor satisfies} \)
\[
\Pi_1 \tilde{A} = A_{\Sigma} = A_{\omega(\Sigma)} = \bigcup_{\sigma \in \omega(\Sigma)} K_\sigma(0);
\]
5. \( \text{for any fixed } \varepsilon_0 > 0, \text{ the family of inflated kernel sections } \{ K^{[\varepsilon_0]}(0) \}, \; \sigma \in \omega(\Sigma), \text{ defined in (1.3) uniformly (w.r.t. } \sigma \in \omega(\Sigma)\text{)} \text{ pullback (respectively forward) attracts each bounded subset } B \text{ of } X, \text{ i.e., for any } \varepsilon > 0, \text{ there is a } T_1 = T_1(B, \varepsilon) > 0 \text{ independent of } \sigma \in \omega(\Sigma) \text{ such that} \)
\[
H^+_X\left( P_\sigma(0, -t) B, K^{[\varepsilon_0]}(0) \right) < \varepsilon, \quad \forall \sigma \in \omega(\Sigma), \; t \geq T_1
\]
\[
(\text{respectively } \quad H^+_X\left( P_\sigma(t, 0) B, K^{[\varepsilon_0]}(0) \right) < \varepsilon, \quad \forall \sigma \in \omega(\Sigma), \; t \geq T_1).
\]

Here \( K_\sigma(0) \text{ is the section at } t = 0 \text{ of the kernel } K_\sigma \text{ of the GMVP } \{ P_\sigma(t, \tau) \}\text{ with } \sigma \in \omega(\Sigma) \text{ and } A_{\omega(\Sigma)} \text{ is the uniform attractor of the family of general multi-valued processes } \{ P_\sigma(t, \tau), \sigma \in \omega(\Sigma) \}.\)

**Proof.** It only remains to show that \( A_{\Sigma} = A_{\omega(\Sigma)} \). Consider the MVSS \( \{ \tilde{F}(t) \} \) corresponding to the family of GMVPs \( \{ P_\sigma(t, \tau), \sigma \in \omega(\Sigma) \}, \text{ acting on } X \times \omega(\Sigma) \) by formula
\[
\tilde{F}(t)(x, \sigma) = \left( P_\sigma(t, 0)x, T(t)\sigma \right), \quad \forall (x, \sigma) \in X \times \omega(\Sigma).
\]
Evidently, the MVSS \( \{ \tilde{F}(t) \} \) and \( \{ F(t) \} \) coincide on \( X \times \omega(\Sigma) \). Analogous to the proof of (1), in view of Theorem 3.8, we can show that the MVSS \( \{ \tilde{F}(t) \} \) possesses an attractor \( \tilde{A} \) and \( \Pi_1 \tilde{A} = A_{\omega(\Sigma)} \), where \( A_{\omega(\Sigma)} \) is the uniform attractor of the family of GMVPs \( \{ P_\sigma(t, \tau), \sigma \in \omega(\Sigma) \}.\)
\[ \sigma \in \omega(\Sigma) \]. Note that \( \Pi_1 \mathcal{A} = \mathcal{A}_\Sigma \) is the uniform attractor of the family of MVSPs \( \{U_\sigma(t, \tau)\} \), \( \sigma \in \Sigma \). We only need to prove that \( \widetilde{\mathcal{A}} = \mathcal{A} \). It is easy to check that \( \mathcal{A}_\Sigma \) is a compact uniformly (w.r.t. \( \sigma \in \omega(\Sigma) \)) attracting set of the family of GMVPs \( \{P_\sigma(t, \tau)\} \), \( \sigma \in \omega(\Sigma) \). Hence, the set \( \mathcal{A}_\Sigma \times \omega(\Sigma) \) is also a compact attracting set of GMVPs \( \{\tilde{F}(t)\} \). This set is a compact attracting set of MVSS \( \{\tilde{F}(t)\} \), because \( \omega(\Sigma) \) is the attractor of \( \{T(\tau)\} \) on \( \Sigma \). Observing that \( \{\tilde{F}(t)\} \) coincides with \( \{F(t)\} \) on \( X \times \omega(\Sigma) \). Therefore the \( \omega \)-limit sets \( \omega^s(\mathcal{A}_\Sigma \times \omega(\Sigma)) \) of \( \mathcal{A}_\Sigma \times \omega(\Sigma) \) coincide as well. By Theorem 5.4, we can conclude that \( \widetilde{\mathcal{A}} = \omega^s(\mathcal{A}_\Sigma \times \omega(\Sigma)) = \mathcal{A} \) and thus the proof is complete.

**Remark 5.14.** Conclusions (1)–(4) in Theorems 5.10 and 5.13 can be seen as a generalization of Theorem VII.4.1 in [12].

As a simple consequence of Theorem 5.10, we have:

**Theorem 5.15.** Under the hypotheses of Theorem 5.10, the set-valued mapping \( \sigma \rightarrow \mathcal{K}_\sigma(0) \) is upper-semicontinuous.

**Proof.** Invoking Theorem 5.10, we see that the graph of the set-valued mapping \( \omega(\Sigma) \ni \sigma \rightarrow \mathcal{K}_\sigma(0) \) is the global attractor \( \mathcal{A} \). Thus the conclusion can be easily obtained by making use of the equivalence between the closedness of the graph and the upper semicontinuity of the mapping \( \sigma \rightarrow \mathcal{K}_\sigma(0) \); see [2]. □

In particular, if \( \mathcal{K}_\sigma(0) \) is lower-semicontinuous in \( \sigma \) in the sense of Hausdorff semidistance, then we can show that the pullback (forward) attraction rate of the family of kernel sections \( \{\mathcal{K}_\sigma(0)\}, \sigma \in \omega(\Sigma) \), is uniform with respect to \( \sigma \in \omega(\Sigma) \).

**Theorem 5.16.** Suppose that the hypotheses in Theorem 5.10 hold and let \( \mathcal{K}_\sigma(0) \) is lower-semicontinuous in \( \sigma \). Then the family of kernel sections \( \{\mathcal{K}_\sigma(0)\}, \sigma \in \omega(\Sigma) \), uniformly (w.r.t. \( \sigma \in \omega(\Sigma) \)) pullback (forward) attracts every bounded subset \( B \) of \( X \), i.e., for any \( \varepsilon > 0 \), there is a \( \tau_2 = \tau_2(B, \varepsilon) > 0 \) independent of \( \sigma \in \omega(\Sigma) \) such that

\[ H_X^p(P_\sigma(0, -t)B, \mathcal{K}_\sigma(0)) < \varepsilon, \quad \forall \sigma \in \omega(\Sigma), \quad t \geq \tau_2 \]

(respectively \( H_X^p(P_\sigma(t, 0)B, K_{\sigma}(t)\sigma(0)) < \varepsilon, \quad \forall \sigma \in \omega(\Sigma), \quad t \geq \tau_2 \)).

**Proof.** \( \mathcal{K}_\sigma(0) \) is uniformly continuous in \( \sigma \in \omega(\Sigma) \) with respect to the Hausdorff distance (note that \( \omega(\Sigma) \) is compact). The conclusion follows immediately from this and conclusion (5) in Theorem 5.10. □

**Remark 5.17.** A simple and interesting case is that \( \mathcal{K}_\sigma(0) \) is a singleton for each \( \sigma \in \omega(\Sigma) \). In such a situation, the upper semicontinuity of \( \mathcal{K}_\sigma(0) \) (see Theorem 5.15) reduces to continuity.

6. Mixed wave systems with supercritical exponent and ordinary differential equations

For illustrating our abstract theory developed in Sections 4 and 5, in this section we investigate mixed wave systems with supercritical exponent and ordinary differential equations.
We consider the system
\[ u_{tt} + h(u_t) = \Delta u - f(u, y(t)) + J(x, y(t)), \quad u|_{\partial D} = 0, \quad t \geq \tau, \quad (6.1) \]
\[ u|_{t=\tau} = u_0, \quad u_t|_{t=\tau} = u_1, \quad \tau \geq 0; \quad (6.2) \]
\[ \frac{dy}{dt} = g(y), \quad y|_{t=0} = y_0, \quad y_0, y \in \mathbb{R}, \quad (6.3) \]

where \( D \) is a smooth bounded domain in \( \mathbb{R}^3 \).

We assume that for some bounded interval \( I_0 \subset \mathbb{R} \), the nonlinear term \( h \) and \( f(v, s) \in \mathcal{C}(\mathbb{R} \times I_0) \) satisfy the following conditions:

\[(H_1) \quad h \in \mathcal{C}^1(\mathbb{R}), \quad h(0) = 0, \quad h \text{ is strictly increasing}; \]
\[(H_2) \quad \lim \inf_{|v| \to \infty} h'(v) > 0; \]
\[(H_3) \quad C_1(|v|^{p-1} - C_2) \leq |h(v)| \leq C_3(1 + |v|^{p-1}), \quad \forall v \in \mathbb{R}; \]
\[(F_1) \quad |f(v, s)| \leq C_4(|v|^{p-1} + 1); \]
\[(F_2) \quad F(v, s) = \int_0^s f(v, w) \, dw, \quad F(v, s) \geq C_5|v|^p - C_6, \quad |F'_s(v, s)| \leq \alpha_0 F(v, s) + C_7, \quad \forall v \in \mathbb{R}, \quad s \in I_0, \quad \alpha_0 \text{ is sufficiently small, } 2 \leq p < 6; \]
\[(F_3) \quad f(v, s) \geq C_8 F(v, s) - C_9, \quad \forall v \in \mathbb{R}, \quad s \in I_0. \]

We assume that \( g \in \mathcal{C}(\mathbb{R}) \) satisfies the local Lipschitz condition

\[(L) \quad |g(v_1) - g(v_2)| \leq L|v_1 - v_2|, \quad \forall v_1, v_2 \in \mathbb{R}, \]

where \( L = L(M) < +\infty \), if \( |v_1|, |v_2| \leq M \), and that \( g \) satisfies dissipative condition

\[(P) \quad |g(v)|v| \leq -m|y|^2 + C, \quad m > 0. \]

It is easy to show that under conditions (L) and (P), problem (6.3) generates a continuous semigroup \( \{T(t)\}, \quad T(t) : \mathbb{R} \to \mathbb{R}, \quad y(t) = T(t)y_0, \) which has a compact set \( \Sigma = [-\sqrt{\frac{C}{m}}, \sqrt{\frac{C}{m}}] \) such that

\[ T(t) \Sigma \subset \Sigma, \quad \forall t \geq 0. \quad (6.4) \]

Thus, by basic results in the theory of dynamical systems (see [12,17,26,29], etc.), the semigroup \( T(t) \) on \( \Sigma \) has a unique attractor \( \omega(\Sigma) \).

We now introduce several notations. Denote by \( k_X(A) \) the Kuratowski measure of noncompactness of \( A \) in the sense of the norm of \( X \). Let \( H = L^2(D) \) and \( V = H^1_0(D) \). The inner products and norms of \( H \) and \( V \) are denoted by \( (\cdot, \cdot), \| \cdot \|_2 \) and \((\cdot, \cdot), \| \cdot \| \), respectively. Let \( E = V \cap L^p(D) \times H \). Denoted by \( | \cdot |_p \) the norm of \( L^p(D) \).

Note that condition (H_3) implies that

\[ |h(v)|^{\frac{1}{p-1}} \leq C_{10}(1 + |v|), \]

therefore, we have

\[ |h(v)|^{\frac{p}{p-1}} = |h(v)|^{\frac{1}{p-1}} \cdot |h(v)| \leq C_{10}(1 + |v|)|h(v)| \leq C_{10}|h(v)| + C_{10}|h(v)| \cdot |v|, \]
using Young’s inequality and (H1), we obtain
\[ |h(v)|^p \leq C_{11} (1 + h(v) \cdot v), \quad \forall v \in \mathbb{R}. \]  
(6.5)

It follows from (H1) and (H3) that \( h(v)v \geq 0 \) for all \( v \in \mathbb{R} \), and
\[ |v|^{p-1} \leq C_{12}(|h(v)| + 1), \]
hence, by Young’s inequality,
\[ |v|^p \leq C_{12}(|h(v)| + 1)|v| \leq C_{12}h(v)v + C_{12}|v| \leq C_{12}h(v)v + \frac{1}{2}|v|^p + C_{13}, \]
which implies that
\[ \frac{C_{14}}{2}|v|^p \leq h(v)v + C_{15}, \quad \forall v \in \mathbb{R}. \]  
(6.6)

Lemma 6.1. Suppose that \( h \) satisfies (H1)–(H3), \( f \in C(\mathbb{R} \times \Sigma) \) satisfies (F1)–(F3), \( g \in C(\mathbb{R}) \) satisfies (L) and (P), \( J \in C(\Sigma; H) \), \((u_0, u_1) \in E \) and \( y_0 \in \Sigma \). Then there exist \( C_{18} > 0 \), \( \alpha > 0 \) and \( \beta > 0 \), such that for any \( t \geq \tau \),
\[ |v(t)|^2 - 3\alpha^2|u(t)|^2 + \|u(t)\|^2 + 2 \int_D F(u(t), y(t)) \]
\[ \leq \left( |v(\tau)|^2 - 3\alpha^2|u(\tau)|^2 + \|u(\tau)\|^2 + 2 \int_D F(u(\tau), y(\tau)) \right) e^{-2\beta(t-\tau)} \]
\[ + \frac{C_{18} + \frac{3}{8\alpha} \|J\|^2_{L^\infty(\mathbb{R}^+; H)}}{2\beta}, \]  
(6.7)

where \( v = u_t + \alpha u \).

Proof. Let \( \alpha \) is a positive number which will be fixed later and let \( v = u_t + \alpha u \). Taking the scalar product in \( H \) of (6.1) and \( v \) we obtain
\[ \frac{1}{2} \frac{d}{dt}|v|^2 + \alpha|v|^2 + (h(u_t), v) \]
\[ = (\Delta u, v) - (f(u, y(t)), v) + 2\alpha(u_t, v) + \alpha^2(u, v) + (J(x, y(t)), v). \]  
(6.8)

Using Young’s inequality and (6.5), we have
\[ \alpha |(h(u_t), u)| \leq \eta \int_D |h(u_t)|^p \leq C_{12} \int_D |u|^p \leq \eta C_{11} \int_D (1 + h(u_t)u_t) + C_\eta \alpha^p \int_D |u|^p \]
\[ \leq \eta C_{11} |D| + \eta C_{11} \int_D h(u_t)u_t + C_\eta \alpha^p \int_D |u|^p. \]  
(6.9)
Noticing that
\[ (\Delta u, v) = (\Delta u, u_t + \alpha u) = -\frac{1}{2} \frac{d}{dt} \|u\|^2 - \alpha \|u\|^2, \tag{6.10} \]
\[ 2\alpha (u_t, v) = 2\alpha (u_t, u_t + \alpha u) = 2\alpha |u_t|^2 + \alpha^2 \frac{d}{dt} |u|^2. \tag{6.11} \]
\[ \alpha^2 (u, v) = \alpha^2 (u, u_t + \alpha u) = \frac{1}{2} \alpha^2 \frac{d}{dt} |u|^2 + \alpha^3 |u|^2. \tag{6.12} \]

By (F2), (F3) and (L),
\[ (f(u, y(t)), v) = (f(u, y(t)), u_t) + \alpha (f(u, y(t)), u) \]
\[ = \frac{d}{dt} \int_D F(u, y(t)) - \int_D F'_y(u, y(t)) y(t) + \alpha \int_D f(u, y(t)) u \]
\[ \geq \frac{d}{dt} \int_D F(u, y(t)) - \int_D |F'_y(u, y(t))| |g(y)| + \alpha C_8 \int_D F(u, y(t)) - \alpha C_9 |D| \]
\[ \geq \frac{d}{dt} \int_D F(u, y(t)) + \left( \alpha C_8 - \alpha_0 L \sqrt{\frac{C}{m}} \right) \int_D F(u, y(t)) - C_7 L \sqrt{\frac{C}{m}} |D| - \alpha C_9 |D|. \tag{6.13} \]

It follows from Young’s inequality that
\[ |(J(x, y(t)), v)| \leq |(J(x, y(t)), u_t)| + \alpha |(J(x, y(t)), u)| \]
\[ \leq \alpha |u_t|^2 + 2\alpha^2 |u|^2 + \frac{3}{8\alpha} |J(x, y(t))|^2. \tag{6.14} \]

Combining (6.8)–(6.14) together, we have
\[ \frac{1}{2} \frac{d}{dt} \left( |v|^2 - 3\alpha^2 |u|^2 + \|u\|^2 + 2 \int_D F(u, y(t)) \right) \]
\[ + \alpha |v|^2 - (3\alpha^3) |u|^2 + \alpha \|u\|^2 + \left( \alpha C_8 - \alpha_0 L \sqrt{\frac{C}{m}} \right) \int_D F(u, y(t)) - 3\alpha |u_t|^2 + \int_D h(u_t) u_t \]
\[ \leq \eta C_{11} \int_D h(u_t) u_t + \eta C_{11} |D| + C_\eta \alpha^p \int_D |u|^p + C_7 L \sqrt{\frac{C}{m}} |D| + \alpha C_9 |D| + \frac{3}{8\alpha} |J(x, y(t))|^2. \tag{6.15} \]

From (6.6), we can deduce that
\[ \frac{C_{14}}{2} |v|^2 \leq h(v) v + C_{16}, \]
which implies that if we take \( \eta \) and \( \alpha \) sufficiently small, then we have

\[
\int_D h(u_t)u_t - \eta C_{11} \int_D h(u_t)u_t \geq 3\alpha |u_t|^2 - C_{17}|D|.
\] (6.16)

We observe that if we take \( \alpha \) and \( \alpha_0 \) sufficiently small, then by (F2), we obtain

\[
\alpha C_8 - \alpha_0 L\sqrt{\frac{C}{m}} > 0 \quad \text{and}
\]

\[
\frac{\alpha C_8 - \alpha_0 L\sqrt{\frac{C}{m}}}{2} \int_D F(u, y(t)) \geq \frac{(\alpha C_8 - \alpha_0 L\sqrt{\frac{C}{m}})C_5}{2} \int_D |u|^p - \frac{(\alpha C_8 - \alpha_0 L\sqrt{\frac{C}{m}})C_6|D|}{2}
\]

\[
\geq C_\eta \alpha^p \int_D |u|^p - \frac{(\alpha C_8 - \alpha_0 L\sqrt{\frac{C}{m}})C_6|D|}{2}.
\] (6.17)

Let \( C'_8 = C_8 - \frac{\alpha_0 L}{\alpha} \sqrt{\frac{C}{m}} \). Two cases may occur.

**Case 1.** \( C'_8 \leq 4 \). Note that

\[
\|u\|^2 \geq \lambda_1 |u|^2.
\] (6.18)

Here we choose \( \alpha \) and \( \alpha_0 \) sufficiently small, such that (6.16) and (6.17) hold true, and

\[
\alpha \left(1 - \frac{C'_8}{4}\right) \lambda_1 \geq 3\alpha^3 - \frac{3\alpha^3 C'_8}{4} = 0.
\]

From this and (6.18), we can deduce that

\[
\alpha |v|^2 - (3\alpha^3)|u|^2 + \alpha\|u\|^2 \geq \frac{\alpha C'_8}{4} |v|^2 - \frac{3\alpha^3 C'_8}{4} |u|^2 + \frac{\alpha C'_8}{4} \|u\|^2.
\] (6.19)

By (6.15)–(6.17) and (6.19),

\[
\frac{1}{2} \frac{d}{dt} \left(|v|^2 - 3\alpha^2|u|^2 + \|u\|^2 + 2 \int_D F(u, y(t))\right)
\]

\[
+ \frac{\alpha C'_8}{4} \left(|v|^2 - 3\alpha^2|u|^2 + \|u\|^2 + 2 \int_D F(u, y(t))\right)
\]

\[
\leq \frac{\alpha C'_8 C_6|D|}{2} + C_{17}|D| + \eta C_{11}|D| + C_7 L\sqrt{\frac{C}{m}}|D| + \alpha C_9 |D| + \frac{3}{8\alpha} |J(x, y(t))|^2.
\] (6.20)
Case 2. $C_8' > 4$. In view of (F2), we get
\[
\left(\frac{\alpha C_8'}{2} - 2\alpha\right) \int_D F(u, y(t)) \geq \left(\frac{\alpha C_8'}{2} - 2\alpha\right) C_5 \int_D |u|^p - \left(\frac{\alpha C_8'}{2} - 2\alpha\right) C_6 |D|.
\] (6.21)

Hence, (6.15)–(6.17) and (6.21) mean that
\[
\frac{1}{2} \frac{d}{dt} \left( |v|^2 - 3\alpha^2 |u|^2 + \|u\|^2 + 2 \int_D F(u, y(t)) \right) \\
+ \alpha \left( |v|^2 - 3\alpha^2 |u|^2 + \|u\|^2 + 2 \int_D F(u, y(t)) \right) \\
\leq \left(\frac{\alpha C_8'}{2} - 2\alpha\right) C_6 |D| + C_{17} |D| + \eta C_{11} |D| + C_7 L \sqrt{\frac{C}{m}} |D| + \alpha C_9 |D| \\
+ \frac{\alpha C_8' C_6}{2} |D| + \frac{3}{8\alpha} |J(x, y(t))|^2.
\] (6.22)

Let $\beta = \min\{\frac{\alpha C_8'}{4}, \alpha\}$. We can conclude from (6.20) and (6.22) that there exists a $C_{18} > 0$ such that
\[
\frac{d}{dt} \left( |v|^2 - 3\alpha^2 |u|^2 + \|u\|^2 + 2 \int_D F(u, y(t)) \right) \\
+ 2\beta \left( |v|^2 - 3\alpha^2 |u|^2 + \|u\|^2 + 2 \int_D F(u, y(t)) \right) \leq C_{18} + \frac{3}{8\alpha} |J(x, y(t))|^2.
\]

By $J \in C(\Sigma; H)$, we have $J(x, y(t)) \in L^\infty(\mathbb{R}^+; H)$. Thus by Gronwall’s lemma, we can deduce that for any $t \geq \tau$,
\[
|v(t)|^2 - 3\alpha^2 |u(t)|^2 + \|u(t)\|^2 + 2 \int_D F(u(t), y(t)) \\
\leq \left( |v(\tau)|^2 - 3\alpha^2 |u(\tau)|^2 + \|u(\tau)\|^2 + 2 \int_D F(u(\tau), y(\tau)) \right) e^{-2\beta(t-\tau)} \\
+ \frac{C_{18} + \frac{3}{8\beta} \|J\|_{L^\infty(\mathbb{R}^+; H)}}{2\beta}.
\]

The proof of the lemma is completed. □
Theorem 6.2. [12] Assume that $h$ satisfies $(H_1)$–$(H_3)$, $f \in C(\mathbb{R} \times \Sigma)$ satisfies $(F_1)$–$(F_3)$, $g \in C(\mathbb{R})$ satisfies $(L)$ and $(P)$, $J \in C(\Sigma; H)$, $(u_0, u_1) \in E$ and $y_0 \in \Sigma$. Then system (6.1)–(6.3) has a weak solution $(u(t), y(t))$, and $u(t)$ satisfies

$$u \in L^\infty([\tau, +\infty); L^p(D) \times V) \quad \text{and} \quad u_1 \in L^\infty([\tau, +\infty); H).$$

In addition, by the similar arguments in [12], we can define a family of multi-valued mappings $U_{\gamma_0}(t, \tau) : E \to 2^E$, $t \geq \tau$, $\tau \in \mathbb{R}^+$, with $\gamma_0 \in \Sigma$ by setting $U_{\gamma_0}(t, \tau)(u_0, u_1) = \{(u(t), u_1(t)) \mid u(\cdot) \text{ is a weak solution of system (6.1), (6.2), where } y(t) = T(t)y_0 \text{ is the solution of (6.3)}\}.$

It is easy to verify that properties (1), (2) in Definition 2.6 hold true. Let us check that $U_{\gamma_0}(t, \tau)(u_0, u_1)$ with $y_0 \in \Sigma$ and $(u_0, u_1) \in E$ is jointly norm-to-weak upper-semicontinuous in $y_0$ and $(u_0, u_1)$ for any fixed $t \geq \tau$, $\tau \in \mathbb{R}^+$. Let $(u_{n0}, u_{n1}) \to (u_0, u_1)$ in $E$ and $y_{n0} \to y_0$ in $\Sigma$. It remains to show that for any fixed $t \geq \tau$, $\tau \in \mathbb{R}^+$, and any $(u_n(t), u_n'(t)) \in U_{\gamma_0}(t, \tau)(u_{n0}, u_{n1})$, there exists a $(u(t), u'(t)) \in U_{\gamma_0}(t, \tau)(u_0, u_1)$ such that $(u_n(t), u_n'(t)) \to (u(t), u'(t))$ in $E$. Similarly to the proof of the existence of weak solutions [see [12] for details], in view of Lemma 6.1, the jointly norm-to-weak upper semicontinuity can be obtained and thus the detailed arguments are omitted here. Hence, the family of multi-valued mappings $\{U_{\gamma_0}(t, \tau)\}, y_0 \in \Sigma$, forms a family of multi-valued semiprocesses on $E$ satisfying jointly norm-to-weak upper semicontinuity. What is more, we easily see that the family of MVSPs $\{U_{\gamma_0}(t, \tau)\}, y_0 \in \Sigma$, satisfies the translation identity (1.1).

Lemma 6.3. [28] Let $B$ be a bounded subset in $L^p(D)$ ($p \geq 1$). If $B$ has a finite $\varepsilon$-net in $L^p(D)$, then there exists $M = M(B, \varepsilon)$ such that for any $u \in B$, the following estimate is valid:

$$\int_{D(|u| \geq M)} |u|^p \leq 2^{p+1}\varepsilon^p.$$ 

Lemma 6.4. Suppose that the hypotheses in Theorem 6.2 hold. Then the family of MVSPs $\{U_{\gamma_0}(t, \tau)\}, y_0 \in \Sigma$, is uniformly dissipative and for any $\varepsilon > 0$ and any bounded subset $B \subset E$, there is a $T_4 \geq \tau$ such that

$$k_{L^p(D)}\left( \bigcup_{s \geq T_4} \bigcup_{y_0 \in \Sigma} U_{\gamma_0}^1(s, \tau)B \right) \leq C_{28}\varepsilon,$$

where $C_{28}$ is independent of $\varepsilon$ and $T_4$, and $U_{\gamma_0}^1(s, \tau) = \pi \circ U_{\gamma_0}(s, \tau)$,

$$\pi : H^1_0(D) \cap L^p(D) \times L^2(D) \to H^1_0(D) \cap L^p(D)$$

is the projector.

Proof. We can conclude from (6.7) and (F2) that the family of MVSPs $\{U_{\gamma_0}(t, \tau)\}, y_0 \in \Sigma$, has a uniformly absorbing set $B_1$ in $E$, i.e., for any fixed $\tau \in \mathbb{R}^+$ and any bounded set $B \subset E$, there is a $t_0 > 0$ independent of $y_0 \in \Sigma$ such that

$$U_{\gamma_0}(t + \tau, \tau)B \subset B_1, \quad \forall t \geq t_0, \quad y_0 \in \Sigma.$$

(6.23)

Thus the family of MVSPs $\{U_{\gamma_0}(t, \tau)\}, y_0 \in \Sigma$, is uniformly dissipative in $E$. 

Since \( J \in C(\Sigma; H) \) and
\[
\Sigma = \left[ -\sqrt{\frac{C}{m}}, \sqrt{\frac{C}{m}} \right],
\] for any \( \varepsilon > 0 \), there exists an \( \eta > 0 \), such that
\[
\left| \frac{1}{2} \| J(y) \|_2^2 - \frac{1}{2} \| J(y') \|_2^2 \right| < \frac{\varepsilon}{3}, \quad \forall y, y' \in \Sigma, \ |y - y'| < \eta.
\]

Clearly, \( \Sigma \) has a finite \( \eta \)-net, that is \( \Sigma \subseteq \bigcup_{i=1}^{n} \mathcal{N}(y_i, \eta) \). For each \( y_i \in \Sigma \), there exists a \( \delta_i > 0 \), such that if \( e \subset D \) and \( m(e) \leq \delta_i \), then
\[
\int_e \| J(x, y_i) \|_2^2 < \frac{\varepsilon}{3},
\]
where \( m(e) \) denotes the Lebesgue measure of \( e \subset D \). Let \( \delta = \min\{\delta_i, i = 1, 2, \ldots, n\} \) and \( y \in \Sigma \) be given arbitrarily. Then there is an \( i \), such that
\[
y \in \mathcal{N}(y_i, \eta),
\]
and if \( e \subset D \) and \( m(e) \leq \delta \), then we have
\[
\int_e \| J(x, y) \|_2^2 = \int_e \left( \| J(x, y) \|_2^2 - \| J(x, y_i) \|_2^2 \right) + \int_e \| J(x, y_i) \|_2^2 \\
\leq 2 \left| \frac{1}{2} \| J(y) \|_2^2 - \frac{1}{2} \| J(y_i) \|_2^2 \right| + \int_e \| J(x, y_i) \|_2^2 < \varepsilon,
\]
which implies that if \( e \subset D \) and \( m(e) \leq \delta \), then
\[
\int_e \| J(x, T(t)y_0) \|_2^2 < \varepsilon, \quad \forall t \in \mathbb{R}^+, \ y_0 \in \Sigma.
\] (6.24)

It follows from (6.23) that there exists a \( \rho_1 > 0 \) such that
\[
|u|^p + \| u \|^2 + \| u_r \|^2 \leq \rho_1^2, \quad \forall t \geq t_0, \ y_0 \in \Sigma, \ u \in U_{y_0}(t + \tau, \tau) B.
\] (6.25)

Especially, we can conclude from Lemma 6.3 that there exists a \( \tilde{M}_1 > 1 \), such that
\[
\int_{D(\| u \| \geq \tilde{M}_1)} |u|^2 < \varepsilon, \quad \forall t \geq t_0, \ y_0 \in \Sigma, \ u \in U_{y_0}(t + \tau, \tau) B.
\] (6.26)

By (6.25), we have
\[
\rho_1^2 \geq \int_{D(\| u \| \geq \tilde{M}_1)} |u|^p \geq \int_{D(\| u \| \geq M_1)} |u|^p \geq \int_{D(\| u \| \geq M_1)} M_1^p m(D(\| u \| \geq M_1)),
\]
where \( D(|u| \geq M_1) = \{ x \in D \mid |u(x)| \geq M_1 \} \). Therefore we can choose \( M_1 > \tilde{M}_1 \) large enough such that

\[
m(D(|u| \geq M_1)) \leq \min\{ \varepsilon, \delta \}, \quad \forall t \geq t_0, \; y_0 \in \Sigma, \; u \in U_{y_0}(t + \tau, \tau)B. \tag{6.27}
\]

Let \( y_0 \in \Sigma \). Observing that for any \( s \geq \tau \), \( U_{y_0}(s, \tau)(u_0, u_1) = \{(u(s), u_s(s)) \mid u(\cdot) \) is a weak solution of problem (6.1), (6.2), where \( y(s) = T(s)y_0 \) is the solution of (6.3)\}. We consider \( (u(s), u_s(s)) \in U_{y_0}(s, \tau)(u_0, u_1) \) with \( s \geq t_0 + \tau \) and \( (u(\tau), u_s(\tau)) = (u_0, u_1) \in B. \) We easily see that (6.1) is equivalent to the following equation:

\[
(u_s + \alpha_1 u_s + \alpha_1 u + h(u_s) - 2\alpha_1 u_s - \Delta u + f(u, y(s)) - \alpha_1^2 u = \tilde{J}(x, y(s)). \tag{6.28}
\]

Taking \( (u - M_2)_+ + \alpha_1 (u - M_2)_+ = v \) as the test function, where \( M_2 = 2M_1 \), we have

\[
\begin{align*}
(u_s + \alpha_1 u_s, v) + \alpha_1 (u_s + \alpha_1 u, v) + & \left( h(u_s), v \right) - 2\alpha_1 (u_s, v) - (\Delta u, v) \\
+ & \left( f(u, y(s)), v \right) - \alpha_1^2 (u, v) = (J(x, y(s)), v), \tag{6.29}
\end{align*}
\]

where \( \alpha_1 < \alpha \) is a positive constant which will be fixed later, \((u - M_2)_+\) denotes the positive part of \( u - M_2 \), that is,

\[
(u - M_2)_+ = \begin{cases} u - M_2, & u \geq M_2, \\ 0, & u \leq M_2. \end{cases}
\]

We will deal with each term of (6.29) one by one as follows. First, from (6.25) and (6.26), we have

\[
\begin{aligned}
2\alpha_1 \int_{D(u \geq M_2)} M_1 v & \leq 2\alpha_1 \int_{D(u \geq M_2)} |u - M_1| \cdot |v| \leq 2\alpha_1 \left( \int_{D(u \geq M_2)} |u - M_1|^2 \right)^{1/2} \cdot |v|_2 \leq C_{19} \varepsilon,
\end{aligned}
\]

therefore,

\[
\begin{aligned}
(u_s + \alpha_1 u_s, v) &= \int_{D(u \geq M_2)} |v|^2 + 2\alpha_1 \int_{D(u \geq M_2)} M_1 \cdot v \geq \int_{D(u \geq M_2)} |v|^2 - C_{19} \varepsilon. \tag{6.30}
\end{aligned}
\]

Secondly, by Young’s inequality, (6.5) and (6.27),

\[
\begin{aligned}
\alpha_1 \left| (h(u_s), (u - M_2)_+) \right| & \leq \alpha_1 \int_{D(u \geq M_2)} |h(u_s)| |u| \leq \eta \int_{D(u \geq M_2)} |h(u_s)| \frac{p}{p-1} + C_\eta \alpha_1^p \int_{D(u \geq M_2)} |u|^p \\
& \leq \eta C_{11} \int_{D(u \geq M_2)} (1 + h(u_s)u_s) + C_\eta \alpha_1^p \int_{D(u \geq M_2)} |u|^p \\
& \leq \eta C_{11} \varepsilon + \eta C_{11} \int_{D(u \geq M_2)} h(u_s)u_s + C_\eta \alpha_1^p \int_{D(u \geq M_2)} |u|^p. \tag{6.31}
\end{aligned}
\]
Using Young’s inequality and (6.26), we obtain

\[
2\alpha_1 (u_s, v) = 2\alpha_1 \int_{D(u \geq M_2)} |u_s|^2 + 2\alpha_1^2 \int_{D(u \geq M_2)} u_s (u - M_2) \\
\leq 3\alpha_1 \int_{D(u \geq M_2)} |u_s|^2 + \alpha_1^3 \int_{D(u \geq M_2)} |u - M_2|^2 \\
\leq 3\alpha_1 \int_{D(u \geq M_2)} |u_s|^2 + \alpha_1^3 \int_{D(u \geq M_2)} |u|^2 \leq 3\alpha_1 \int_{D(u \geq M_2)} |u_s|^2 + \alpha_1^3 \varepsilon. \quad (6.32)
\]

In the sequel, we deal with the residual four terms:

\[
(-\Delta u, (u - M_2)_+ + \alpha_1 (u - M_2)_+) = \frac{1}{2} \frac{d}{ds} \int_{D(u \geq M_2)} |\nabla u|^2 + \alpha_1 \int_{D(u \geq M_2)} |\nabla u|^2 \quad (6.33)
\]

Using Young’s inequality, (F1)–(F3), (L) and (6.26), (6.27), we have

\[
-\left( f (u, y(s)), (u - M_2)_+ + \alpha_1 (u - M_2)_+ \right) = -\frac{d}{ds} \int_{D(u \geq M_2)} F(u, y(s)) + \int_{D(u \geq M_2)} F'_y(u, y(s)) y'(s) - \alpha_1 \int_{D(u \geq M_2)} f(u, y(s)) (u - M_2),
\]

\[
\int_{D(u \geq M_2)} F'_y(u, y(s)) y'(s) \leq \int_{D(u \geq M_2)} |F'_y(u, y(s))| |g(y(s))| \\
\leq \alpha_0 L \sqrt{\frac{C}{m}} \int_{D(u \geq M_2)} F(u, y(s)) + L \sqrt{\frac{C}{m}} C_7 \varepsilon,
\]

\[-\alpha_1 \int_{D(u \geq M_2)} f(u, y(s)) u \leq -\alpha_1 C_8 \int_{D(u \geq M_2)} F(u, y(s)) + \alpha_1 C_9 \varepsilon, \quad \text{and}
\]

\[
\alpha_1 M_2 \int_{D(u \geq M_2)} f(u, y(s)) \\
\leq \alpha_1 M_2 C_4 \int_{D(u \geq M_2)} |u|^{p-1} + 2\alpha_1 C_4 \int_{D(u \geq M_2)} M_1 \\
\leq \eta_1 C_4 \int_{D(u \geq M_2)} |u|^p + C_1 \alpha_1^p 2^p \int_{D(u \geq M_2)} |u|^p + C_4 \alpha_1 \int_{D(u \geq M_2)} |u|^2 + C_4 \alpha_1 \varepsilon \\
< \eta_1 C_4 \int_{D(u \geq M_2)} |u|^p + C_1 \alpha_1^p 2^p \int_{D(u \geq M_2)} |u|^p + 2\alpha_1 C_4 \varepsilon.
\]
Hence,
\[
- \left( f(u, y(s)), (u - M_2)_+ \right) + \alpha_1 (u - M_2)_+ \\
\leq - \frac{d}{ds} \int_{D(u \geq M_2)} F(u, y(s)) - \left( \alpha_1 C_8 - \alpha_0 L \sqrt{\frac{C}{m}} \right) \int_{D(u \geq M_2)} F(u, y(s)) \\
+ \left( \eta_1 C_4^{\frac{p}{p+1}} + C_\eta_1 \alpha_1^2 \right) \int_{D(u \geq M_2)} |u|^p + \left( L \sqrt{\frac{C}{m}} C_7 + \alpha_1 C_9 + 2\alpha_1 C_4 \right) \varepsilon.
\] (6.34)

At the same time, from (6.24)–(6.26), we get
\[
\left| \alpha_1^2 (u, v) \right| = \alpha_1^2 \int_{D(u \geq M_2)} u \cdot v |v|_2 \leq \alpha_1^2 |v|_2 \left( \int_{D(u \geq M_2)} |u|^2 \right)^{\frac{1}{2}} < \alpha_1^2 C_{20} \varepsilon \quad \text{and} \quad (6.35)
\]

\[
\left| (J(x, y(s)), v) \right| \leq \int_{D(u \geq M_2)} |J(x, y(s))| |v| \leq |v|_2 \left( \int_{D(u \geq M_2)} |J(x, y(s))|^2 \right)^{\frac{1}{2}} < C_{21} \varepsilon.
\] (6.36)

Combining (6.29)–(6.36) together, we obtain
\[
\frac{1}{2} \frac{d}{ds} \left( \int_{D(u \geq M_2)} |v|^2 + \int_{D(u \geq M_2)} |\nabla u|^2 + 2 \int_{D(u \geq M_2)} F(u, y(s)) \right) \\
+ \alpha_1 \int_{D(u \geq M_2)} |v|^2 + \alpha_1 \int_{D(u \geq M_2)} |\nabla u|^2 + \left( \alpha_1 C_8 - \alpha_0 L \sqrt{\frac{C}{m}} \right) \int_{D(u \geq M_2)} F(u, y(s)) \\
< - \int_{D(u \geq M_2)} h(u_s) u_s + \eta C_{11} \int_{D(u \geq M_2)} h(u_s) u_s + \left( \eta_1 C_4^{\frac{p}{p+1}} + C_\eta_1 \alpha_1^2 \right) \int_{D(u \geq M_2)} |u|^p \\
+ \eta_1 \alpha_1^2 \int_{D(u \geq M_2)} |u|^p + 3\alpha_1 \int_{D(u \geq M_2)} |u_s|^2 \\
+ \left( L \sqrt{\frac{C}{m}} C_7 + \alpha_1 C_9 + 2\alpha_1 C_4 + \alpha_1^2 C_{20} + C_{21} + \alpha_1^3 + \eta C_{11} + \alpha_1 C_{19} \right) \varepsilon.
\] (6.37)

Similar to (6.16) and (6.17), if we take \( \eta > 0, \eta_1 > 0, \alpha_1 > 0 \) and \( \alpha_0 > 0 \) sufficiently small, in view of (F2) and (6.27), then we have
\[
\int_{D(u \geq M_2)} h(u_s) u_s - \eta C_{11} \int_{D(u \geq M_2)} h(u_s) u_s \geq 3\alpha_1 \int_{D(u \geq M_2)} |u_s|^2 - C_{22} \varepsilon,
\]

\[
\alpha_1 C_8 - \alpha_0 L \sqrt{\frac{C}{m}} > 0 \quad \text{and}
\]
\[
\frac{\alpha_1 C_8 - \alpha_0 L \sqrt{C_m}}{2} \int_{D(u \geq M_2)} F(u, y(s)) \\
\geq \frac{(\alpha_1 C_8 - \alpha_0 L \sqrt{C_m}) C_5}{2} \int_{D(u \geq M_2)} |u|^p - \frac{(\alpha_1 C_8 - \alpha_0 L \sqrt{C_m}) C_6 \varepsilon}{2} \\
\geq (\eta_1 C_4^{p-1} + C_\eta \alpha_1^p 2^p) \int_{D(u \geq M_2)} |u|^p + C_\eta \alpha_1^p \int_{D(u \geq M_2)} |u|^p - \frac{(\alpha_1 C_8 - \alpha_0 L \sqrt{C_m}) C_6 \varepsilon}{2}.
\]

Therefore,
\[
\frac{1}{2} \frac{d}{ds} \left( \int_{D(u \geq M_2)} |v|^2 + \int_{D(u \geq M_2)} |\nabla u|^2 + 2 \int_{D(u \geq M_2)} F(u, y(s)) \right) \\
+ \alpha_1 \int_{D(u \geq M_2)} |v|^2 + \alpha_1 \int_{D(u \geq M_2)} |\nabla u|^2 + \frac{\alpha_1 C_8 - \alpha_0 L \sqrt{C_m}}{2} \int_{D(u \geq M_2)} F(u, y(s)) < C_{23} \varepsilon.
\]

(6.38)

Let \( \tilde{C}_8 = C_8 - \frac{\alpha_0 L}{\alpha_1} \sqrt{C_m} \). Two cases may occur.

**Case 1.** \( \tilde{C}_8 \leq 4 \). Observing that
\[
\alpha_1 \int_{D(u \geq M_2)} |v|^2 + \alpha_1 \int_{D(u \geq M_2)} |\nabla u|^2 \geq \frac{\alpha_1 \tilde{C}_8}{4} \left( \int_{D(u \geq M_2)} |v|^2 + \int_{D(u \geq M_2)} |\nabla u|^2 \right).
\]

Thus, we can conclude from (6.38) that
\[
\frac{1}{2} \frac{d}{ds} \left( \int_{D(u \geq M_2)} |v|^2 + \int_{D(u \geq M_2)} |\nabla u|^2 + 2 \int_{D(u \geq M_2)} F(u, y(s)) \right) \\
+ \frac{\alpha_1 \tilde{C}_8}{4} \left( \int_{D(u \geq M_2)} |v|^2 + \int_{D(u \geq M_2)} |\nabla u|^2 + 2 \int_{D(u \geq M_2)} F(u, y(s)) \right) < C_{23} \varepsilon. \quad (6.39)
\]

**Case 2.** \( \tilde{C}_8 > 4 \). It follows from (F_2) and (6.27) that
\[
\left( \frac{\alpha_1 \tilde{C}_8}{2} - 2 \alpha_1 \right) \int_{D(u \geq M_2)} F(u, y(s))
\]
\[
\begin{align*}
\geq & \left( \alpha_1 \tilde{C}_8^2 - 2\alpha_1 \right) C_5 \int_{D(u \geq M_2)} |u|^p - \left( \alpha_1 \tilde{C}_8^2 - 2\alpha_1 \right) C_6 \\
\geq & -\left( \alpha_1 \tilde{C}_8^2 - 2\alpha_1 \right) C_6 \varepsilon. 
\end{align*}
\]

(6.40)

By (6.38) and (6.40), we get

\[
\frac{1}{2} \frac{d}{ds} \left( \int_{D(u \geq M_2)} |v|^2 + \int_{D(u \geq M_2)} |\nabla u|^2 + 2 \int_{D(u \geq M_2)} F(u, y(s)) \right) + \alpha_1 \left( \int_{D(u \geq M_2)} |v|^2 + \int_{D(u \geq M_2)} |\nabla u|^2 + 2 \int_{D(u \geq M_2)} F(u, y(s)) \right) \\
< C_{23} \varepsilon + \left( 2\alpha_1 - \frac{\alpha_1 \tilde{C}_8^2}{2} \right) \int_{D(u \geq M_2)} F(u, y(s)) \\
< \left( C_{23} + \left( \frac{\alpha_1 \tilde{C}_8^2}{2} - 2\alpha_1 \right) C_6 \varepsilon. 
\end{array}
\]

(6.41)

Let \( \beta_1 = \min\{\frac{\alpha_1 \tilde{C}_8^2}{4}, \alpha_1\} \). Then (6.39) and (6.41) mean that

\[
\begin{align*}
\frac{1}{2} \frac{d}{ds} \left( \int_{D(u \geq M_2)} |v|^2 + \int_{D(u \geq M_2)} |\nabla u|^2 + 2 \int_{D(u \geq M_2)} F(u, y(s)) \right) \\
+ \beta_1 \left( \int_{D(u \geq M_2)} |v|^2 + \int_{D(u \geq M_2)} |\nabla u|^2 + 2 \int_{D(u \geq M_2)} F(u, y(s)) \right) < C_{24} \varepsilon.
\end{align*}
\]

Taking account of Gronwall’s lemma, \((F_1)\), \((F_2)\) and (6.25), we can conclude that

\[
\begin{align*}
\int_{D(u \geq M_2)} |v(s)|^2 + \int_{D(u \geq M_2)} |\nabla u(s)|^2 + 2 \int_{D(u \geq M_2)} F(u(s), y(s)) \\
< \left( \int_{D(u \geq M_2)} |v(t_0 + \tau)|^2 + \int_{D(u \geq M_2)} |\nabla u(t_0 + \tau)|^2 + 2 \int_{D(u \geq M_2)} F(u(t_0 + \tau), y(t_0 + \tau)) \right) \\
\times e^{-2\beta_1(s-t_0-\tau)} + \frac{C_{24} \varepsilon}{2\beta_1} \\
< C_{25} e^{-2\beta_1(s-t_0-\tau)} + \frac{C_{24} \varepsilon}{2\beta_1}.
\end{align*}
\]

(6.42)
We can take $T_3 = t_0 + \tau + \frac{1}{2\beta_1} \ln \left( \frac{C_{25}}{\varepsilon} \right)$ such that when $s \geq T_3$,

$$
\int_{D(u \geq M_2)} |v(s)|^2 + 2 \int_{D(u \geq M_2)} |\nabla u(s)|^2 + 2 \int_{D(u \geq M_2)} F(u(s), y(s)) < \left( \frac{C_{24}}{2\beta_1} + 1 \right) \varepsilon, \tag{6.43}
$$

where $v(s) = (u(s) - M_2)_+ + \alpha_1 (u(s) - M_2)_+$.

Similarly, taking $v(s) = (u(s) + M_2)_- + \alpha_1 (u(s) + M_2)_-$ as the test function, we have for $s$ sufficiently large,

$$
\int_{D(u(s) \leq -M_2)} \left| (u(s) + M_2)_- + \alpha_1 (u(s) + M_2)_- \right|^2 + \int_{D(u(s) \leq -M_2)} |\nabla u(s)|^2 + 2 \int_{D(u(s) \leq -M_2)} F(u(s), y(s)) < C_{26} \varepsilon, \tag{6.44}
$$

where $(u + M_2)_-$ denotes the negative part of $u + M_2$.

Using (F$_2$) and (6.27), we can conclude from (6.43) and (6.44) that for $s$ sufficiently large,

$$
\int_{D(|u(s)| \geq M_2)} |u(s)|^p \leq C_{27} \varepsilon.
$$

Note that $H^1_0(D)$ is compactly embedded in $L^2(D)$. Thanks to (6.25) and Lemma 4.12, we know that there is a $T_4 \geq \tau$ such that

$$
k_{L^p(D)} \left( \bigcup_{s \geq T_4} \bigcup_{y_0 \in \Sigma} U^1_{y_0}(s, \tau) B \right) < C_{28} \varepsilon.
$$

Thus the proof of this lemma is finished. \(\Box\)

Similar to what $u$ done in [16,28], let $u = v + w$, we decompose Eq. (6.1) as follows:

$$
\begin{align*}
\begin{cases}
v_{tt} + h(v_t + w_t) - h(w_t) - \Delta v = 0, \\
v|_{\partial D} = 0, \quad v(\tau) = u_0, \quad v_t(\tau) = u_1, \quad \tau \geq 0;
\end{cases}
\end{align*}
$$

$$
\begin{align*}
\begin{cases}
w_{tt} + h(w_t) - \Delta w = -f(u, y(t)) + J(x, y(t)), \\
w|_{\partial D} = 0, \quad w(\tau) = 0, \quad w_t(\tau) = 0, \quad \tau \geq 0.
\end{cases}
\end{align*}
$$

We borrow some techniques from [16,28] to obtain the following lemmas.

**Lemma 6.5.** Under the hypotheses of Theorem 6.2, there exists a function $\gamma = \gamma(t)$ such that $\gamma(t) \to 0$ as $t \to \infty$ and for any $t \geq \tau$,

$$
|v_t(t)|^2 + \|\nabla v(t)\| \leq \gamma(t),
$$

where $v$ is any solution of (6.45) with $(u_0, u_1) \in B$ and $B$ is any bounded subset of $E$.

In particular, $\gamma$ is independent of $w_t$. 
Proof. Multiplying (6.45) by $v_t$ and integrating by parts we get
\[
E_v(t_4) - E_v(t_3) + \int_{t_3}^{t_4} \int_D \left( h(v_t + w_t) - h(w_t) \right) v_t \, dx \, dt = 0,
\]
(6.47)
where $E_v(t) = \frac{1}{2} \{ |v_t(t)|^2 + \|v\|^2 \}$, and thus the estimate
\[
\|v(t)\|, |v_t(t)| \leq C_{29}(B), \quad \forall t \geq \tau.
\]
(6.48)
Thanks to Lemma 6.1, we get
\[
\|u(t)\|, |u(t)|_p, |u_t(t)| \leq C_{30}(B), \quad \forall t \geq \tau.
\]
(6.49)
Consequently, since $w = u - v$, we deduce that
\[
\|w(t)\|, |w_t(t)| \leq C_{31}(B), \quad \forall t \geq \tau.
\]
(6.50)
Taking the scalar product in $H$ of (6.1) and $u_t$ we have
\[
\frac{d}{dt} \left( E_u(t) + \int_D F(u, y(t)) \right) + \int_D h(u_t) u_t = \int_D F_y(u, y(t)) y'(t) + \int_D J(x, y(t)) u_t.
\]
(6.51)
Making use of (F1), (F2), (6.49), (6.50) and $J(x, y(t)) \in L^\infty(\mathbb{R}^+; H)$ and Young’s inequality, we can conclude from (6.51) that
\[
\int_{t_3}^{t_4} \int_D h(u_t) u_t \, dx \, dt \leq C_{32}(B, |t_4 - t_3|).
\]
(6.52)
Similarly, we multiply (6.46) by $w_t$ and integrate by parts, in view of (F1), (6.6), (6.49), (6.50) and $J(x, y(t)) \in L^\infty(\mathbb{R}^+; H)$, we can deduce that
\[
\frac{d}{dt} \left( E_w(t) \right) + \int_D h(w_t) w_t = -\int_D f(u, y(t)) w_t + \int_D J(x, y(t)) w_t \quad \text{and}
\]
\[
\int_{t_3}^{t_4} \int_D h(w_t) w_t \, dx \, dt \leq C_{33}(B, |t_4 - t_3|).
\]
(6.53)
Similarly, we multiply (6.46) by $w_t$ and integrate by parts, in view of (F1), (6.6), (6.49), (6.50) and $J(x, y(t)) \in L^\infty(\mathbb{R}^+; H)$, we can deduce that
\[
\frac{d}{dt} \left( E_w(t) \right) + \int_D h(w_t) w_t = -\int_D f(u, y(t)) w_t + \int_D J(x, y(t)) w_t \quad \text{and}
\]
\[
\int_{t_3}^{t_4} \int_D h(w_t) w_t \, dx \, dt \leq C_{33}(B, |t_4 - t_3|).
\]
(6.54)
In the following, fully analogous to the proof of Lemma 3 in [16], the conclusion can be obtained, thus the detailed arguments are omitted here. \qed

Lemma 6.6. Assume that the hypotheses in Theorem 6.2 hold. Let the couple $(u, u_t)$ belong to the set of solutions of (6.1), (6.2) starting from $(u_0, u_1) \in B$ for any bounded set $B \subset E$, where
\[ y(t) = T(t)y_0 \] be a solution of (6.3). Then for any \( t_5 \geq \tau \), there is a compact set \( \mathcal{K}(t_5) \subset V \times H \) such that

\[ (w(t_5), w_t(t_5)) \in \mathcal{K}(t_5) \quad (6.55) \]

for any couple \((w, w_t)\) solving (6.46).

**Proof.** According to Lemma 6.1 and (F1),

\[ f(u(t), T(t)y_0) \]

is bounded in \( L^{\frac{p}{p-1}}(D \times [\tau, t_5]) \) for any \( y_0 \in \Sigma \).

Recall that

\[ \{ L^\infty(0, T; H^1_0(D)) \cap W^{1,\infty}(0, T; L^2(D)) \} \hookrightarrow L^m(0, T; L^s(D)) \]

compactly for any \( 1 < m < \infty \), \( 1 \leq s < 6 \). Hence,

\[ f(u(t), T(t)y_0) \]

is a compact set in \( L^r(D \times [\tau, t_5]) \) for any fixed \( y_0 \in \Sigma \), \( t_5 \geq \tau \) and \( 1 \leq r < \frac{p}{p-1} \).

On the other hand, thanks to Fubini’s theorem, we know that for any \( f(u(t), T(t)y_0) \in L^\infty(\tau, t_5; L^{\frac{p}{p-1}}(D)) \) and any \( M_3 \geq 0 \), we have

\[
\int_{D_{t_5}} |f(u(t), T(t)y_0)|^{\frac{p}{p-1}} \, dx \, dt \leq C_{34} \epsilon.
\]

Using Lemma 4.12, we can deduce that \( f(u(t), T(t)y_0) \) is a compact set in \( L^{\frac{p}{p-1}}(D \times [\tau, t_5]) \).

Note that

\[ \Sigma = \left[ -\sqrt{\frac{C}{m}}, \sqrt{\frac{C}{m}} \right] \quad \text{and} \quad J \in C(\Sigma; H). \]

Consequently, it suffices to show that

\[ (f, y_0) \rightarrow (w(t_5), w_t(t_5)), \quad w \text{ solves } (6.46), \quad (6.56) \]

is continuous as a mapping from \( L^{\frac{p}{p-1}}(D \times [\tau, t_5]) \times \Sigma \) into \( V \times H \).
To this end, we consider a couple of solutions \( w^1, w^2 \) corresponding to \( f^1, f^2 \) and \( y_0^1 \in \Sigma, y_0^2 \in \Sigma \), respectively. The energy inequality yields

\[
||w^1_t(t_5) - w^2_t(t_5)||^2 + \left\| w^1(t_5) - w^2(t_5) \right\|^2 \\
\leq C_{35} \int_\tau^{t_5} \int_D |f^1 - f^2| |w^1_t - w^2_t| \, dx \, dt \\
+ C_{35} \int_\tau^{t_5} \int_D (J(x, T(t)y_0^1) - J(x, T(t)y_0^2))(w^1_t - w^2_t) \, dx \, dt \\
\leq C_{35} \left\| f^1 - f^2 \right\|_{L^p(D \times [\tau, t_5])} \left\| w^1_t - w^2_t \right\|_{L^p(D \times [\tau, t_5])} \\
+ C_{35} \left\| J(x, T(t)y_0^1) - J(x, T(t)y_0^2) \right\|_{L^2(D \times [\tau, t_5])} \left\| w^1_t - w^2_t \right\|_{L^2(D \times [\tau, t_5])},
\]  

(6.57)

On the other hand, we have

\[
\frac{1}{2} \left( ||w^i_t(t_5)||^2 + \left\| w^i(t_5) \right\|^2 \right) + \int_\tau^{t_5} \int_D h(w^i_t) w^i_t \, dx \, dt \\
\leq \left\| f^i \right\|_{L^{p-1}(D \times [\tau, t_5])} \left\| w^i_t \right\|_{L^p(D \times [\tau, t_5])} + \left\| J(x, T(t)y_0^i) \right\|_{L^2(D \times [\tau, t_5])} \left\| w^i_t \right\|_{L^2(D \times [\tau, t_5])},
\]

\( i = 1, 2, \)  

(6.58)

and hence, in view of (6.6), we can deduce that

\[
\left\| w^i_t \right\|_{L^p(D \times [\tau, t_5])} \text{ is bounded for } i = 1, 2.
\]

(6.59)

Combining (6.57) and (6.59), by \( J \in C(\Sigma; H) \), we obtain the continuity of the operator in (6.56).

**Lemma 6.7.** Under the hypotheses of Theorem 6.2, for any \( \varepsilon > 0 \) and any bounded set \( B \subset E \), there exists \( \tau_4 > 0 \) independent of \( y_0 \in \Sigma \) such that

\[
k_E \left( \bigcup_{y_0 \in \Sigma} U_{y_0}(t + \tau, \tau)B \right) \leq C_{36} \varepsilon, \quad \forall t \geq \tau_4.
\]

where the constant \( C_{36} \) is independent of \( \varepsilon \) and \( \tau_4 \).

**Proof.** From Lemma 6.5, we know that for \( t \) large enough, \( v \) is uniformly small. By Lemma 6.6, \( w \) belongs to compact set \( \mathcal{K}(t_5 + \tau) \) for any \( t_5 \geq 0 \). On the other hand,

\[
\bigcup_{y_0 \in \Sigma} U_{y_0}(t_5 + t_0 + t + \tau, \tau)B = \bigcup_{y_0 \in \Sigma} U_{y_0}(t_5 + t_0 + t + \tau, t_0 + t + \tau)U_{y_0}(t_0 + t + \tau, \tau)B
\]
where $B_1$ is a uniformly absorbing set such that $U_{y_0}(t_0 + t + \tau, t_0 + t + \tau)B_1 \subset U_{y_0}(t, \tau)B_1\subset \bigcup_{y_0 \in \Sigma} U_{y_0}(t, \tau)B_1$, for all $t \geq t_5 + t_0$ and all $y_0 \in \Sigma$. The conclusion follows immediately from Lemma 6.4 and the definition of the noncompactness measure. \(\square\)

**Remark 6.8.** It is worth noticing that we need $p < 6$ only in the proof of Lemma 6.5, however, the conclusions in other lemmas also hold for $p = 6$. It is quite natural to ask whether we can obtain the following theorem under $p \leq 6$, this is rather a problem which deserves to be further clarified, and it is still undergoing investigations. The results in this direction can be expected to be reported in forthcoming papers.

Now let us apply the abstract results in Sections 4 and 5 to obtain:

**Theorem 6.9.** Assume that $h$ satisfies (H1)–(H3), $f \in C(\mathbb{R} \times \Sigma)$ satisfies (F1)–(F3), $g \in C(\mathbb{R})$ satisfies (L) and (P), $J \in C(\Sigma; H)$, $(u_0, u_1) \in E$ and $y_0 \in \Sigma$. Then the semigroup $\{T(t)\}$ generated by problem (6.3) has the attractor $\omega(\Sigma)$ and we can construct a family of general multi-valued processes $\{P_{y_0}(t, \tau)\}$, $y_0 \in \omega(\Sigma)$, which possess nonempty kernel $K_{y_0}$ for each $y_0 \in \omega(\Sigma)$ and the multi-valued semidynamical system $\{F(t)\}$ corresponding to the family of multi-valued semiprocesses $\{U_{y_0}(t, \tau)\}$, $y_0 \in \Sigma$, and acting on $E \times \Sigma$ possesses a unique compact attractor $A$ which is strictly invariant with respect to $\{F(t)\}$: $F(t)A = A$ for all $t \geq 0$.

Furthermore,

1. $\Pi_1A = A_\Sigma$ is the uniform (w.r.t. $y_0 \in \Sigma$) attractor of the family of multi-valued semiprocesses $\{U_{y_0}(t, \tau)\}$, $y_0 \in \Sigma$;
2. $\Pi_2A = \omega(\Sigma)$ is the attractor of the semigroup $\{T(t)\}$ acting on $\Sigma$: $T(t)\omega(\Sigma) = \omega(\Sigma)$ for all $t \geq 0$;
3. the global attractor satisfies $A = \bigcup_{y_0 \in \omega(\Sigma)} K_{y_0}(0) \times \{y_0\}$;
4. the uniform attractor satisfies $\Pi_1A = A_\Sigma = \bigcup_{y_0 \in \omega(\Sigma)} K_{y_0}(0)$;
5. for any fixed $\varepsilon_0 > 0$, the family of inflated kernel sections $\{K_{y_0}^{[\varepsilon_0]}(0)\}$, $y_0 \in \omega(\Sigma)$, uniformly (w.r.t. $y_0 \in \omega(\Sigma)$) pullback (respectively forward) attracts each bounded subset $B$ of $E$.

Here $K_{y_0}(0)$ is the section at $t = 0$ of the kernel $K_{y_0}$ of the GMVP $\{P_{y_0}(t, \tau)\}$ with $y_0 \in \omega(\Sigma)$.
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References


