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Partition relations for Hurewicz-type selection hypotheses

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ABSTRACT

We give a general method to reduce Hurewicz-type selection hypotheses into standard ones. The method covers the known results of this kind and gives some new ones.

Building on that, we show how to derive Ramsey theoretic characterizations for these selection hypotheses.

1. Introduction

In [6], Menger introduced a hypothesis which generalizes σ-compactness of topological spaces. Hurewicz [3] proved that Menger's property is equivalent to a property of the following type.

$\mathsf{S}_{\mathrm{fin}}(\mathscr{A},\mathscr{B})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathscr{A}$, there exist finite subsets $\mathcal{F}_n\subseteq\mathcal{U}_n$, $n\in\mathbb{N}$, such that $\bigcup_n\mathcal{F}_n\in\mathscr{B}$.

Indeed, Hurewicz observed that X has Menger's property if, and only if, X satisfies $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$, where $\mathcal{O}$ is the collection of all open covers of X. Motivated by a conjecture of Menger, Hurewicz [3] introduced a hypothesis of the following type.

$\mathsf{U}_{\mathrm{fin}}(\mathscr{A},\mathscr{B})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of elements of $\mathscr{A}$ which do not contain a finite subcover, there exist finite (possibly empty) subsets $\mathcal{F}_n\subseteq\mathcal{U}_n$, $n\in\mathbb{N}$, such that $\{\bigcup\mathcal{F}_n\colon n\in\mathbb{N}\}\in\mathscr{B}$.

Hurewicz was interested in $\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$, where Γ is the collection of all open γ-covers of X. ($\mathcal{U}$ is a *γ-cover* of X if it is infinite, and each $x\in X$ is an element in all but finitely many members of $\mathcal{U}$.)

While the Hurewicz-type selection hypotheses $\mathsf{U}_{\mathrm{fin}}(\mathscr{A},\mathscr{B})$ are standard notions in the field of selection principles, they are less standard in the more general field of infinitary combinatorics. The reason for that is that the finite subsets are

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"glued" before considering the resulting object. Moreover, the definition of $\mathsf{U}_{\mathrm{fin}}(\mathscr{A},\mathscr{B})$ is somewhat less elegant than that of $\mathsf{S}_{\mathrm{fin}}(\mathscr{A},\mathscr{B})$, and consequently is less convenient to work with.

$\mathcal{U}$ is an *ω-cover* of X if $X \notin \mathcal{U}$, but for each finite $F \subseteq X$, there is $U \in \mathcal{U}$ such that $F \subseteq U$. One of the main results of [5] is the result that $\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$ is equivalent to $\mathsf{S}_{\mathrm{fin}}(\Omega,\mathscr{B})$ for an appropriate modification $\mathscr{B}$ of Γ. A similar result was established in [1] for $\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Omega)$. In these papers, these reductions were used to obtain Ramsey theoretic characterizations of $\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$ and of $\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Omega)$, respectively.

We generalize these results. As applications, we reproduce the main results of [5], strengthen the main results of [1], and obtain standard and Ramsey theoretic equivalents for a property introduced in [10].

As each cover of a type considered in our Ramsey theoretic results is infinite, each of our Ramsey theoretic results implies Ramsey's classical theorem and can therefore be viewed as a structural extension of Ramsey's theorem.

2. Reduction of selection hypotheses

Convention. To simplify the presentation, by *cover* of X we always mean a *countable* collection $\mathcal{U}$ of open subsets of X, such that $\bigcup\mathcal{U} = X$ and $X \notin \mathcal{U}$. Also, $\mathscr{A}$ and $\mathscr{B}$ always denote families of (such) covers of the underlying space X.

Definition 1. $\mathscr{A}$ is *Ramseyan* if for each $\mathcal{U} \in \mathscr{A}$ and each partition of $\mathcal{U}$ into finitely many (equivalently, two) pieces, one of these pieces belongs to $\mathscr{A}$.

Lemma 2. *(See [7].) Assume that $\mathscr{A}$ is Ramseyan. Then $\mathscr{A} \subseteq \Omega$.*

Proof. Assume that $\mathscr{A}$ is Ramseyan, and $\mathscr{A} \nsubseteq \Omega$. Fix $\mathcal{U} \in \mathscr{A} \setminus \Omega$, and a finite subset $F \subseteq X$ such that F is not contained in any $U \in \mathcal{U}$. Since $\mathcal{U}$ is a cover of X, $|F| \geqslant 2$. For each $C \subsetneq F$, let $\mathcal{U}_C = \{U \in \mathcal{U}\colon\ U \cap F = C\}$. Then $\mathcal{U} = \bigcup_{C \in P(F)\setminus\{F\}} \mathcal{U}_C$ is a partition of $\mathcal{U}$ into finitely many pieces.

As $\mathscr{A}$ is Ramseyan, there is $C \subsetneq F$ such that $\mathcal{U}_C \in \mathscr{A}$. But then the elements of $F \setminus C$ are not covered by any member of $\mathcal{U}_C$. A contradiction. □

Examples of Ramseyan collections of covers are Ω and Γ, defined in the introduction. We will give one more example in Section 5.

A cover $\mathcal{U}$ of X is *multifinite* [11] if there exists a partition of $\mathcal{U}$ into infinitely many finite covers of X.

Definition 3 *(The* Gimel operator *on families of covers).* Let $\mathscr{A}$ be a family of covers of X. $\gimel(\mathscr{A})$ is the family of all covers $\mathcal{U}$ of X such that: Either $\mathcal{U}$ is multifinite, or there exists a partition $\mathcal{P}$ of $\mathcal{U}$ into finite sets such that $\{\bigcup\mathcal{F}\colon\ \mathcal{F} \in \mathcal{P}\} \setminus \{X\} \in \mathscr{A}$.

Remark 4. For each $\mathscr{A}$, $\mathscr{A} \subseteq \gimel(\mathscr{A})$.

An element of $\gimel(\mathscr{A})$ will be called *$\mathscr{A}$-glueable*. This explains our choice of the Hebrew letter *Gimel* (ℷ).

Definition 5. A cover $\mathcal{V}$ is a *finite-to-one derefinement* of a cover $\mathcal{U}$, if there exists a finite-to-one surjection $f : \mathcal{U} \to \mathcal{V}$ such that for each $U \in \mathcal{U}$, $U \subseteq f(U)$.

$\mathscr{A}$ is *finite-to-one derefinable* if for each $\mathcal{U} \in \mathscr{A}$ and each finite-to-one derefinement $\mathcal{V} \in \mathcal{O}$ of $\mathcal{U}$, $\mathcal{V} \in \mathscr{A}$.

A useful tool in the study of selection principles and their relation to Ramsey theory is the game $\mathsf{G}_{\mathrm{fin}}(\mathscr{A},\mathscr{B})$. This game is played by two players, ONE and TWO, with an inning per each natural number n. At the nth inning ONE chooses a cover $\mathcal{U}_n \in \mathscr{A}$ and TWO chooses a finite subset $\mathcal{F}_n$ of $\mathcal{U}_n$. TWO wins if $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \in \mathscr{B}$. Otherwise, ONE wins.

Our goal in this section is proving the following.

Theorem 6. *Let $\mathscr{B}$ be Ramseyan and finite-to-one derefinable. The following are equivalent*:

1. $\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\mathscr{B})$.
2. $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ *and* $\Lambda = \gimel(\mathscr{B})$.
3. *ONE has no winning strategy in* $\mathsf{G}_{\mathrm{fin}}(\Omega,\gimel(\mathscr{B}))$.
4. $\mathsf{S}_{\mathrm{fin}}(\Omega,\gimel(\mathscr{B}))$.

Moreover, in (3) *and* (4)*, Ω can be replaced by any of Λ or Γ.*

We prove this theorem in a sequence of lemmas. As it may be of independent interest, some of these lemmas use weaker (or no) requirements on $\mathscr{B}$ than those posed in Theorem 6.

$\mathcal{U}$ is a *large cover* of X if each point $x \in X$ belongs to infinitely many $U \in \mathcal{U}$. Let Λ be the collection of all countable large covers of X.

The following proof is similar to that of [5, Lemma 8].

Lemma 7. *Assume that $\mathscr{B}$ is Ramseyan and finite-to-one derefinable. Then $\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\mathscr{B})$ implies $\Lambda = \daleth(\mathscr{B})$.*

Proof. By Lemma 2, $\mathscr{B} \subseteq \Omega$. Thus, each $\mathcal{U} \in \daleth(\mathscr{B})$ is large. Let $\mathcal{U}$ be a large cover of X. If $\mathcal{U}$ is multifinite, then $\mathcal{U} \in \daleth(\mathscr{B})$, and we are done.

We now treat the remaining two cases.

Case 1. $\mathcal{U}$ has no finite subcover. Let $\{U_n\colon n \in \mathbb{N}\}$ bijectively enumerate a large cover $\mathcal{U}$ of X. We may assume that no finite subset of $\mathcal{U}$ covers X: If $\mathcal{U}$ contains infinitely many disjoint finite subcovers then it is multifinite.

For $m, n \in \mathbb{N}$, we use the convenient notation

$$U_{[m,n)} = \bigcup_{m \leqslant i < n} U_i,$$

with the convention that $U_{[m,n)} = \emptyset$ whenever $n \leqslant m$. For each n, define $\mathcal{U}_n = \{U_{[n,m)}\colon m \in \mathbb{N}\}$. Each $\mathcal{U}_n$ is a γ-cover of X, and in particular a cover of X.

Applying $\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\mathscr{B})$, choose for each n a finite $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $\{\bigcup \mathcal{F}_n\colon n \in \mathbb{N}\} \in \mathscr{B}$. For each n, there is $m_n \geqslant n$ such that $\bigcup \mathcal{F}_n = U_{[n,m_n)}$.

Let $k_1 = 1$, and $k_2 = m_1 + 1$. Having defined k_{n-1} and k_n, choose k_{n+1} such that:

(1) $k_{n+1} > m_1, m_2, \ldots, m_{k_n}$;
(2) there is i such that $k_n \leqslant i < k_{n+1}$ and $U_{[i,m_i)} \neq \emptyset$; and
(3) $U_{[k_{n-1},k_{n+1})} \notin \{U_{[k_{i-1},k_{i+1})}\colon i < n\}$.

(3) is possible since $U_{[1,k_n)} \neq X$.

For each n, let $\mathcal{V}_n = \{U_{[i,m_i)}\colon k_n \leqslant i < k_{n+1}\}$. As

$$\bigcup_n \mathcal{V}_{2n-1} \cup \bigcup_n \mathcal{V}_{2n} = \{U_{[i,m_i)}\colon i \in \mathbb{N}\} \in \mathscr{B}$$

and $\mathscr{B}$ is Ramseyan, there is $j \in \{0,1\}$ such that $\bigcup_n \mathcal{V}_{2n-j} \in \mathscr{B}$. We consider the case $j = 0$ (the other case can be treated similarly).

For each n, each element of $\mathcal{V}_{2n}$ has the form $U_{[i,m_i)}$ with $k_{2n} \leqslant i < k_{2n+1}$. By (1), $U_{[i,m_i)} \subseteq U_{[k_{2n},k_{2n+2})}$. Thus, $\{U_{[k_{2n},k_{2n+2})}\colon n \in \mathbb{N}\}$ is a finite-to-one derefinement of $\bigcup_n \mathcal{V}_{2n}$, and is therefore a member of $\mathscr{B}$. As $\mathscr{B}$ is finite-to-one derefinable, $\{U_{[1,k_4)}\} \cup \{U_{[k_{2n},k_{2n+2})}\colon n > 1\} \in \mathscr{B}$, either, and this witnesses that the partition of $\mathcal{U}$ into the pieces $\{U_i\colon 1 \leqslant i < k_4\}$ and $\{U_i\colon k_{2n} \leqslant i < k_{2n+2}\}$, $n > 1$, is as required in the statement $\mathcal{U} \in \daleth(\mathscr{B})$.

Case 2. $\mathcal{U}$ has only finitely many disjoint finite subcovers. Let $\mathcal{F}$ be the family of all elements in these finite subcovers. $\mathcal{U} \setminus \mathcal{F}$ is a large cover of X not containing any finite subcover. By what we have just proved, $\mathcal{U} \setminus \mathcal{F} \in \daleth(\mathscr{B})$. As $\mathcal{U} \setminus \mathcal{F}$ is not multifinite, there is a partition $\mathcal{P}$ of $\mathcal{U} \setminus \mathcal{F}$ into finite pieces, such that $\{\bigcup \mathcal{V}\colon \mathcal{V} \in \mathcal{P}\} \in \mathscr{B}$. Fix $\mathcal{V}_0 \in \mathcal{P}$. $\mathcal{P}' = \{\mathcal{V}_0 \cup \mathcal{F}\} \cup \mathcal{P} \setminus \{\mathcal{V}_0\}$ is a partition of $\mathcal{U}$ into finite pieces. Define $f : \{\bigcup \mathcal{V}\colon \mathcal{V} \in \mathcal{P}\} \to \{\bigcup \mathcal{V}\colon \mathcal{V} \in \mathcal{P}'\}$ by $f(\bigcup \mathcal{V}) = \bigcup(\mathcal{V} \cup \mathcal{F})$ if $\bigcup \mathcal{V} = \bigcup \mathcal{V}_0$, and $f(\bigcup \mathcal{V}) = \bigcup \mathcal{V}$ otherwise. As f is finite-to-one and $\mathscr{B}$ is finite-to-one derefinable, $\{\bigcup \mathcal{V}\colon \mathcal{V} \in \mathcal{P}'\} \in \mathscr{B}$, and thus $\mathcal{U} \in \daleth(\mathscr{B})$. □

Note that for each family of covers $\mathscr{B}$, $\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\mathscr{B})$ implies $\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$. Clearly, $\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}) = \mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$. Thus, Lemma 7 shows that the implication (1) ⇒ (2) of Theorem 6 holds for each Ramseyan and bijectively derefinable family of covers $\mathscr{B}$.

The following will be used often.

Lemma 8. *(See [8].)* $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}) = \mathsf{S}_{\mathrm{fin}}(\Lambda,\Lambda) = \mathsf{S}_{\mathrm{fin}}(\Omega,\Lambda) = \mathsf{S}_{\mathrm{fin}}(\Gamma,\Lambda)$.

Lemma 9. *The following are equivalent*:

(1) $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$.
(2) *ONE has no winning strategy in* $\mathsf{G}_{\mathrm{fin}}(\Lambda,\Lambda)$.
(3) *ONE has no winning strategy in* $\mathsf{G}_{\mathrm{fin}}(\Omega,\Lambda)$.
(4) *ONE has no winning strategy in* $\mathsf{G}_{\mathrm{fin}}(\Gamma,\Lambda)$.

Proof. Recall that $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}) = \mathsf{S}_{\mathrm{fin}}(\Lambda,\Lambda)$. $\mathsf{S}_{\mathrm{fin}}(\Lambda,\Lambda)$ is equivalent to (2) [9, Theorem 5].

As $\Gamma \subseteq \Omega \subseteq \Lambda$, (2) ⇒ (3) ⇒ (4). But (4) implies $\mathsf{S}_{\mathrm{fin}}(\Gamma,\Lambda)$, which is the same as (1) [8]. □

Corollary 10. *The conjunction of* $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ *and* $\Lambda = \daleth(\mathscr{B})$ *implies that ONE has no winning strategy in any of the games* $\mathsf{G}_{\mathrm{fin}}(\Lambda,\daleth(\mathscr{B}))$, $\mathsf{G}_{\mathrm{fin}}(\Omega,\daleth(\mathscr{B}))$, *or* $\mathsf{G}_{\mathrm{fin}}(\Gamma,\daleth(\mathscr{B}))$.

Proof. Lemma 9 and the assumption $\Lambda = \daleth(\mathscr{B})$. $\square$

This gives (2) $\Rightarrow$ (3) of Theorem 6. (3) $\Rightarrow$ (4) in that theorem is clear. It remains to show that (4) $\Rightarrow$ (1). As $\Gamma \subseteq \Omega \subseteq \Lambda$, it suffices to prove the following.

Lemma 11. *Assume that* $\mathscr{B}$ *is finite-to-one derefinable. Then* $\mathsf{S}_{\mathrm{fin}}(\Gamma,\daleth(\mathscr{B}))$ *implies* $\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\mathscr{B})$.

Proof. Assume that X satisfies $\mathsf{S}_{\mathrm{fin}}(\Gamma,\daleth(\mathscr{B}))$. As $\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\mathscr{B}) = \mathsf{U}_{\mathrm{fin}}(\Gamma,\mathscr{B})$ [8], it suffices to prove that X satisfies $\mathsf{U}_{\mathrm{fin}}(\Gamma,\mathscr{B})$.

Let $\mathcal{U}_n$, $n \in \mathbb{N}$, be disjoint open γ-covers of X which do not contain a finite subcover. Enumerate each $\mathcal{U}_n$ bijectively as $\{U^n_k\colon k \in \mathbb{N}\}$. For each n, let

$$\mathcal{V}_n = \{U^1_m \cap U^2_m \cap \cdots \cap U^n_m\colon m \in \mathbb{N}\}.$$

For each n, $\mathcal{V}_n$ is an open γ-cover of X. Apply $\mathsf{S}_{\mathrm{fin}}(\Gamma,\daleth(\mathscr{B}))$ to obtain for each n a finite subset $\mathcal{F}_n \subseteq \mathcal{V}_n$ such that $\bigcup_n \mathcal{F}_n \in \daleth(\mathscr{B})$. Each $U \in \bigcup_n \mathcal{F}_n$ is a subset of some element of $\mathcal{U}_1$, hence for each finite subset $\mathcal{F} \subseteq \bigcup_n \mathcal{F}_n$, $\bigcup \mathcal{F} \neq X$. Therefore $\bigcup_n \mathcal{F}_n$ is not multifinite.

Let $\{\mathcal{X}_m\colon m \in \mathbb{N}\}$ be a partition of $\bigcup_n \mathcal{F}_n$ into finite pieces such that $\{\bigcup \mathcal{X}_m\colon m \in \mathbb{N}\} \in \mathscr{B}$. Let

$$f(m) = \min\{k\colon \mathcal{X}_m \cap \mathcal{F}_k \neq \emptyset\},$$

and put

$$\mathcal{Y}_n = \bigcup_{m \in f^{-1}(n)} \mathcal{X}_m.$$

The sets $\{f^{-1}(n)\colon n \in \mathbb{N}\}$ form a partition of $\mathbb{N}$. Since each $\mathcal{F}_k$ is finite and the $\mathcal{X}_m$'s are disjoint, $f^{-1}(n)$ is finite for all n. It follows that each $\mathcal{Y}_n$ is a finite set. Each member of $\mathcal{Y}_n$ belong to some $\mathcal{F}_k \subseteq \mathcal{V}_k$ for some $k \geqslant n$.

For each n, choose $\psi(n) \in \mathbb{N}$ such that:

(1) If $U^n_k \in \mathcal{U}_n$ appear as term in the sets of $\mathcal{Y}_n$ then $\psi(n) \geqslant k$.
(2) The sets $\bigcup_{k \leqslant \psi(n)} U^n_k$ are distinct for different values of n.

This is possible since $\{\bigcup_{k \leqslant m} U^n_k\colon m \in \mathbb{N}\}$, $n \in \mathbb{N}$, are γ-covers.

Define $\mathcal{Z}_n = \{U^n_k\colon k \leqslant \psi(n)\}$. The sets $\mathcal{Z}_n$ are finite and disjoint. For each $n \in \mathbb{N}$, $\bigcup \mathcal{Y}_n \subseteq \bigcup \mathcal{Z}_n \neq X$. Hence $\{\bigcup \mathcal{Z}_n\colon n \in \mathbb{N}\}$ is a finite derefinement of $\{\bigcup \mathcal{X}_n\colon n \in \mathbb{N}\}$. Therefore $\{\bigcup \mathcal{Z}_n\colon n \in \mathbb{N}\} \in \mathscr{B}$ and the sequence $\{\mathcal{Z}_n\}_{n\in\mathbb{N}}$ witnesses that X has the property $\mathsf{U}_{\mathrm{fin}}(\Gamma,\mathscr{B})$. $\square$

This completes the proof of Theorem 6.

3. Partition relations for glueable covers

The symbol $[A]^n$ denotes the set of n-element subsets of A. For a positive integer k, the *Baumgartner–Taylor partition relation* [2]

$$\mathscr{A} \to \lceil \mathscr{B} \rceil^2_k$$

denotes the following statement: For each A in $\mathscr{A}$ and each $f : [A]^2 \to \{1,\dots,k\}$, there are

(1) $B \subseteq A$ such that $B \in \mathscr{B}$;
(2) a partition of B into finite pieces $B = \bigcup_{n\in\mathbb{N}} B_n$; and
(3) $j \in \{1,\dots,k\}$,

such that $f(\{U, V\}) = j$ for all $U, V \in B$ which do not belong to the same B_n.

The Baumgartner–Taylor partition relation is one of the most important partition relations in the studies of open covers and their combinatorial properties—see [4] for a survey of this field.

Lemma 12. *(See [7].) If each member of* $\mathscr{B}$ *is infinite and* $\mathscr{A} \to \lceil \mathscr{B} \rceil^2_2$ *holds, then* $\mathscr{A} \subseteq \Omega$.

Together with Theorem 6, the following gives a Ramsey theoretic characterization of properties of the form $\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\mathscr{B})$.

Theorem 13. *Assume that $\mathscr{B}$ is Ramseyan and finite-to-one derefinable. The following are equivalent*:

(1) $\mathsf{S}_{\mathrm{fin}}(\Omega, \daleth(\mathscr{B}))$.
(2) *For each k*, $\Omega \to \lceil \daleth(\mathscr{B}) \rceil^2_k$ *holds.*
(3) $\Omega \to \lceil \daleth(\mathscr{B}) \rceil^2_2$.

Proof. (1) ⇒ (2) This follows from Theorem 6 and the following.

Lemma 14. *(See [5].) Assume that $\mathscr{A}$ is Ramseyan. If ONE has no winning strategy in the game $\mathsf{G}_{\mathrm{fin}}(\mathscr{A}, \mathscr{B})$, then for each k, $\mathscr{A} \to \lceil \mathscr{B} \rceil^2_k$ holds.*

(3) ⇒ (1) Assume that X satisfies $\Omega \to \lceil \daleth(\mathscr{B}) \rceil^2_2$. By Theorem 6, it suffices to show that X satisfies $\mathsf{U}_{\mathrm{fin}}(\Gamma, \mathscr{B})$.

Let $\mathcal{U}_n$, $n \in \mathbb{N}$, be open γ-covers of X which do not contain a finite subcover. Enumerate each $\mathcal{U}_n$ bijectively as $\{U^n_k\colon k \in \mathbb{N}\}$. For each n, define

$$\mathcal{V}_n = \{U^1_k \cap U^2_k \cap \cdots \cap U^n_k\colon\ k \in \mathbb{N}\},$$

and let $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$. Then, $\mathcal{V}$ is an ω-cover of X. For each element of $\mathcal{V}$ fix a representation of the form $U^1_k \cap U^2_k \cap \cdots \cap U^n_k$. Define a function $f : [\mathcal{V}]^2 \to \{1, 2\}$ by

$$f(\{V_1, V_2\}) = \begin{cases} 1 & \text{if } V_1 \text{ and } V_2 \text{ are from the same } \mathcal{V}_n, \\ 2 & \text{otherwise.} \end{cases}$$

Choose $\mathcal{W} \subseteq \mathcal{V}$ such that $\mathcal{W} \in \daleth(\mathscr{B})$, a partition $\mathcal{W} = \bigcup_k \mathcal{W}_k$ into finite pieces, and a color $j \in \{1, 2\}$, such that for A and B from distinct $\mathcal{W}_k$'s, $f(\{A, B\}) = j$. Consider the possible values of j.

$j = 1$: Then there is an n such that for all $A \in \mathcal{W}$ we have $A \subseteq U^1_n \neq X$. Hence $\mathcal{W}$ is not a cover. Contradiction.

$j = 2$: Let $\mathcal{F}_n = \mathcal{W} \cap \mathcal{V}_n$. Then each $\mathcal{F}_n$ is finite. From this point, the proof continues as in the proof of Lemma 11. □

4. Selecting one element from each cover

We now consider the following selection principle.

$\mathsf{S}_1(\mathscr{A}, \mathscr{B})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of elements of $\mathscr{A}$, there exist $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{U_n\colon n \in \mathbb{N}\} \in \mathscr{B}$.

The corresponding game $\mathsf{G}_1(\mathscr{A}, \mathscr{B})$, is defined as follows: At the nth inning ONE chooses a cover $\mathcal{U}_n \in \mathscr{A}$ and TWO chooses $U_n \in \mathcal{U}_n$. TWO wins if $\{U_n\colon n \in \mathbb{N}\} \in \mathscr{B}$. Otherwise, ONE wins.

The corresponding partition relation, called the ordinary partition relation, is defined as follows. For positive integers n and k,

$$\mathscr{A} \to (\mathscr{B})^n_k$$

means: For each $A \in \mathscr{A}$ and each $f : [A]^n \to \{1, \ldots, k\}$, there is $B \subseteq A$ such that $B \in \mathscr{B}$, and $f|_{[B]^n}$ is constant.

The following theorem was proved in [5] for $\mathscr{B} = \Gamma$, and in [1] for $\mathscr{B} = \Omega$.

Theorem 15. *Let $\mathscr{B}$ be Ramseyan and finite-to-one derefinable. The following are equivalent*:

(1) $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ *and* $\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \mathscr{B})$.
(2) $\mathsf{S}_1(\Lambda, \daleth(\mathscr{B}))$.
(3) $\mathsf{S}_1(\Omega, \daleth(\mathscr{B}))$.
(4) *ONE has no winning strategy in the game* $\mathsf{G}_1(\Omega, \daleth(\mathscr{B}))$.
(5) $\Omega \to (\daleth(\mathscr{B}))^2_2$.
(6) $\Omega \to (\daleth(\mathscr{B}))^2_k$ *for all k.*

Proof. (1) ⇒ (2) $\mathsf{S}_1(\mathcal{O}, \mathcal{O}) = \mathsf{S}_1(\Lambda, \Lambda)$. By Lemma 7, $\Lambda = \daleth(\mathscr{B})$ for X. Thus, X satisfies $\mathsf{S}_1(\Lambda, \daleth(\mathscr{B}))$.

(2) ⇒ (3) $\Omega \subseteq \Lambda$.

(3) ⇒ (1) As $\mathsf{S}_1(\Omega, \daleth(\mathscr{B}))$ implies $\mathsf{S}_{\mathrm{fin}}(\Omega, \daleth(\mathscr{B}))$, we have by Lemma 11 that $\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \mathscr{B})$ holds, and that $\daleth(\mathscr{B}) = \Lambda$. Thus, X satisfies $\mathsf{S}_1(\Omega, \Lambda)$, which is the same as $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ [8].

(1) ⇒ (4) By [9, Theorem 3], $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ implies that ONE does not have a strategy in $\mathsf{G}_1(\Lambda, \Lambda)$, and in particular in $\mathsf{G}_1(\Omega, \Lambda)$. Again, use Theorem 6 to get that $\Lambda = \daleth(\mathscr{B})$.

(4) ⇒ (6) Follows from [5, Theorem 1].

(6) ⇒ (5) is immediate.

(5) ⇒ (3) As $\mathscr{B}$ is Ramseyan, $\mathscr{B} \subseteq \Omega$, and therefore $\daleth(\mathscr{B}) \subseteq \Lambda$. Thus, (5) implies $\Omega \to (\Lambda)^2_k$. Using the methods of [5], one can prove that $\Omega \to (\Lambda)^2_k$ implies $\mathsf{S}_1(\Omega, \Lambda)$ [7]. Clearly, (5) also implies $\Omega \to \lceil \daleth(\mathscr{B}) \rceil^2_2$, and by Theorem 13, we get $\Lambda = \daleth(\mathscr{B})$. □

5. Applications

5.1. γ-covers

As every infinite subset of a γ-cover is again a γ-cover of the same space, Γ is Ramseyan.

Lemma 16. *Γ is finite-to-one derefinable.*

Proof. Assume that $\mathcal{U} \in \Gamma$ and $f : \mathcal{U} \to \mathcal{V}$ is finite-to-one and surjective. As f is finite-to-one and $\mathcal{U}$ is infinite, $\mathcal{V}$ is infinite. Assume that $x \in X$ and $\mathcal{W} = \{V \in \mathcal{V}: x \notin V\}$ is infinite. For each $V \in \mathcal{W}$ and each $U \in f^{-1}(V)$, $U \subseteq V$ and thus $x \notin U$. As f is surjective, $\bigcup_{V \in \mathcal{W}} f^{-1}(V)$ is infinite. A contradiction. □

Thus, we can directly apply Theorems 6, 13, and 15, and obtain the following.

Theorem 17. *(See [5].) The following are equivalent*:

(1) $\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \Gamma)$.
(2) $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$ *and* $\Lambda = \daleth(\Gamma)$.
(3) *ONE has no winning strategy in* $\mathsf{G}_{\mathrm{fin}}(\Omega, \daleth(\Gamma))$.
(4) $\mathsf{S}_{\mathrm{fin}}(\Omega, \daleth(\Gamma))$.
(5) *For each k,* $\Omega \to \lceil \daleth(\Gamma) \rceil^2_k$ *holds.*
(6) $\Omega \to \lceil \daleth(\Gamma) \rceil^2_2$.

Theorem 18. *(See [5].) The following are equivalent*:

(1) $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ *and* $\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \Gamma)$.
(2) $\mathsf{S}_1(\Lambda, \daleth(\Gamma))$.
(3) $\mathsf{S}_1(\Omega, \daleth(\Gamma))$.
(4) *ONE has no winning strategy in the game* $\mathsf{G}_1(\Omega, \daleth(\Gamma))$.
(5) $\Omega \to (\daleth(\Gamma))^2_2$.
(6) $\Omega \to (\daleth(\Gamma))^2_k$ *for all k.*

5.2. ω-covers

Definition 19. A cover $\mathcal{V}$ is a *derefinement* of a cover $\mathcal{U}$ if $\mathcal{U}$ refines $\mathcal{V}$. $\mathscr{A}$ is *derefinable* if for each $\mathcal{U} \in \mathscr{A}$ and each derefinement $\mathcal{V} \in \mathcal{O}$ of $\mathcal{U}$, $\mathcal{V} \in \mathscr{A}$.

Ω is derefinable, and in particular finite-to-one derefinable.

Lemma 20 *(Folklore). Ω is Ramseyan.*

Proof. Assume that $\mathcal{U} \in \Omega$ and $\mathcal{U} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n$ and no $\mathcal{U}_i \in \Omega$. For each i, choose a finite subset F_i of X witnessing $\mathcal{U}_i \notin \Omega$. Then $F = F_1 \cup \cdots \cup F_n$ is not covered by any element of $\mathcal{U}$. A contradiction. □

In the forthcoming Theorem 21, we reproduce the statements of Theorems 2 and 3 of [1]. One direction in the proof of Theorem 3 in [1] uses Theorem 4 of [5], which in turn requires that $\daleth(\Omega)$ is derefinable. Unfortunately, by Theorem 2 of [1], spaces dealt with in this theorem only have $\Lambda = \daleth(\Omega)$. But Λ is not derefinable: Fix distinct $a, b, x_n \in X$, $n \in \mathbb{N}$. Then the large cover $\{X \setminus \{a, x_n\}, X \setminus \{b, x_n\}: n \in \mathbb{N}\}$ refines $\{X \setminus \{a\}, X \setminus \{b\}\}$. Our results give a corrected proof of this direction.

Theorem 21. *The following are equivalent*:

(1) $\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \Omega)$.
(2) $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$ *and* $\Lambda = \daleth(\Omega)$.
(3) *ONE has no winning strategy in* $\mathsf{G}_{\mathrm{fin}}(\Omega, \daleth(\Omega))$.
(4) $\mathsf{S}_{\mathrm{fin}}(\Omega, \daleth(\Omega))$.

(5) *For each k,* $\Omega \to \lceil \gimel(\Omega) \rceil^2_k$ *holds.*
(6) $\Omega \to \lceil \gimel(\Omega) \rceil^2_2$.

Proof. Ω is derefinable and Ramseyan (Lemma 20). Apply Theorems 6 and 13. □

As in the previous theorem, the following Theorem 22 reproduces Theorem 5 of [1] and fixes a problem similar to the above-mentioned one in the original proof of the implication (5) ⇒ (3) below.

Theorem 22. *The following are equivalent.*

(1) $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ *and* $\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \Omega)$.
(2) $\mathsf{S}_1(\Lambda, \gimel(\Omega))$.
(3) $\mathsf{S}_1(\Omega, \gimel(\Omega))$.
(4) *ONE has no winning strategy in the game* $\mathsf{G}_1(\Omega, \gimel(\Omega))$.
(5) $\Omega \to (\gimel(\Omega))^2_2$.
(6) $\Omega \to (\gimel(\Omega))^2_k$ *for all k.*

Proof. Apply Theorem 15. □

5.3. τ^*-covers

Let $[\mathbb{N}]^{\aleph_0} = \{A \subseteq \mathbb{N}\colon\ |A| = \aleph_0\}$. For $A, B \in [\mathbb{N}]^{\aleph_0}$, $A \subseteq^* B$ means that $A \setminus B$ is finite. A family $Y \subseteq [\mathbb{N}]^{\aleph_0}$ is *linearly refinable* if for each $y \in Y$ there exists an infinite subset $\hat{y} \subseteq y$ such that the family $\hat{Y} = \{\hat{y}\colon\ y \in Y\}$ is linearly ordered by $\subseteq^*$.

A countable cover $\mathcal{U} = \{U_n\colon\ n \in \mathbb{N}\}$ of X is a τ^**-cover* of X if $\{\{n\colon\ x \in U_n\}\colon\ x \in X\}$ is linearly refinable. T^* is the collection of all τ^*-covers. $\Gamma \subseteq \mathrm{T}^* \subseteq \Omega$.

T^* is derefinable [10]. In particular, T^* is finite-to-one derefinable. (This latter assertion is easier to see.)

Proposition 23. T^* *is Ramseyan.*

Proof. Let $\{U_n\colon\ n \in \mathbb{N}\}$ be a bijective enumeration of a τ^*-cover $\mathcal{U}$ of X. For each $x \in X$, let $x_{\mathcal{U}} = \{n\colon\ x \in U_n\}$, and let $\hat{x}_{\mathcal{U}}$ be an infinite subset of $x_{\mathcal{U}}$ such that the sets $\hat{x}_{\mathcal{U}}$ are linearly ordered by $\subseteq^*$.

Consider a partition $\mathcal{U} = \mathcal{V} \cup (\mathcal{U} \setminus \mathcal{V})$. Define $A = \{n\colon\ U_n \in \mathcal{V}\}$. We may assume that both A and its complement are infinite.

For each $x \in X$, define $\hat{x}_{\mathcal{V}} = \hat{x}_{\mathcal{U}} \cap A$ and $\hat{x}_{\mathcal{U}\setminus\mathcal{V}} = \hat{x}_{\mathcal{U}} \cap A^c$. If $\{\hat{x}_{\mathcal{V}}\colon\ x \in X\} \subseteq [\mathbb{N}]^{\aleph_0}$ or $\{\hat{x}_{\mathcal{U}\setminus\mathcal{V}}\colon\ x \in X\} \subseteq [\mathbb{N}]^{\aleph_0}$ then we are done. If this is not the case, then there are some $x, y \in X$ such that $\hat{x}_{\mathcal{V}}$ and $\hat{y}_{\mathcal{U}\setminus\mathcal{V}}$ are finite. Without loss of generality, assume that $\hat{y}_{\mathcal{U}} \subseteq^* \hat{x}_{\mathcal{U}}$. Thus,

$$\hat{y}_{\mathcal{V}} = \hat{y}_{\mathcal{U}} \cap A \subseteq^* \hat{x}_{\mathcal{U}} \cap A = \hat{x}_{\mathcal{V}}$$

but $\hat{y}_{\mathcal{V}}$ is infinite and $\hat{x}_{\mathcal{V}}$ is finite. A contradiction. □

By Theorems 6 and 13, we have the following.

Theorem 24. *The following are equivalent*:

(1) $\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \mathrm{T}^*)$.
(2) $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$ *and* $\Lambda = \gimel(\mathrm{T}^*)$.
(3) *ONE has no winning strategy in* $\mathsf{G}_{\mathrm{fin}}(\Omega, \gimel(\mathrm{T}^*))$.
(4) $\mathsf{S}_{\mathrm{fin}}(\Omega, \gimel(\mathrm{T}^*))$.
(5) *For each k,* $\Omega \to \lceil \gimel(\mathrm{T}^*) \rceil^2_k$ *holds.*
(6) $\Omega \to \lceil \gimel(\mathrm{T}^*) \rceil^2_2$.

By Theorem 15, we have the following.

Theorem 25. *The following are equivalent*:

(1) $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ *and* $\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \mathrm{T}^*)$.
(2) $\mathsf{S}_1(\Lambda, \gimel(\mathrm{T}^*))$.
(3) $\mathsf{S}_1(\Omega, \gimel(\mathrm{T}^*))$.

(4) *ONE has no winning strategy in the game* $\mathsf{G}_1(\Omega, \beth(\mathrm{T}^*))$.
(5) $\Omega \to (\beth(\mathrm{T}^*))^2_2$.
(6) $\Omega \to (\beth(\mathrm{T}^*))^2_k$ *for all k*.

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