Matrices with Positive Principal Minors*

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ABSTRACT

A new necessary and sufficient condition is given for all principal minors of a square matrix to be positive. A special subclass of such matrices, called quasidominant matrices, is also examined.

1. INTRODUCTION

Fiedler and Pták [1] have given a number of conditions for all principal minors of a square matrix to be positive. This paper presents another such necessary and sufficient condition. By specializing to a special class of matrices with positive principal minors, called quasidominant matrices, it is also shown how some of the results of Tartar [6] and Araki [2] on M-matrices can be extended.

The following notation will be used. If A is a matrix, its elements will be denoted by a_{ij}, and its transpose and inverse by A' and A^{-1} respectively. Similarly, a vector x has typical component x_i. Except where otherwise stated, all scalars and matrix elements will be assumed to take values in the real field. If A is symmetric, the notation $A>0$ will be used to mean that A is positive definite. For a vector, on the other hand, the notation $x>0$ will mean that all elements of x are real and positive. The determinant of A will be denoted by $\det A$, while $|A|$ will mean the matrix obtained from A by replacing every element by its absolute value. Finally, a signature matrix is any diagonal matrix whose diagonal entries are $+1$ or -1.

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2. MAIN RESULT

THEOREM 1. *All principal minors of F are positive iff, for every signature matrix S, there exists a vector $x>0$ such that $SFSx>0$.*

Proof. It is known [3, p. 364] that, if all principal minors of F are positive, there exists an $x>0$ such that $Fx>0$. Since the principal minors of SFS are identical to those of F, this proves half of the theorem.

The converse can be proved by induction on the size of principal minors of F. Suppose then that for each S there exists $x>0$ such that $SFSx>0$, and suppose also that all $m\times m$ principal minors of F are known to be positive. (The case $m=1$ is of course easily handled.) Partition F as

$$F=\begin{array}{cc} \begin{bmatrix} F_{11} & F_{12} & f_{13} \\ F_{21} & F_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} & \begin{array}{l} \\ \updownarrow m \\ \updownarrow 1 \end{array} \\ \begin{array}{ccc} \quad & \underset{m}{\leftrightarrow} & \underset{1}{\leftrightarrow} \end{array} & \end{array}$$

and choose

$$S=\begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

such that $(f_{32}F_{22}^{-1}F_{21}-f_{31})S_1 \geqslant 0$ and $(-f_{32}F_{22}^{-1})S_2 \geqslant 0$. (Here, the notation $\geqslant 0$ is meant to indicate that the row vector in question has all entries nonnegative.) Notice that, by the inductive hypothesis, $\det F_{22}>0$ and F_{22} is invertible. Now by assumption we have some

$$x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}>0$$

such that

$$SFSx=\begin{bmatrix} S_1F_{11}S_1 & S_1F_{12}S_2 & S_1 f_{13} \\ S_2F_{21}S_1 & S_2F_{22}S_2 & S_2 f_{23} \\ f_{31}S_1 & f_{32}S_2 & f_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \stackrel{\Delta}{=} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}>0.$$

Eliminating x_2, we obtain

$$(f_{33}-f_{32}F_{22}^{-1}f_{23})x_3=y_3+(f_{32}F_{22}^{-1}F_{21}-f_{31})S_1x_1-f_{32}F_{22}^{-1}S_2\,y_2,$$

so that $f_{33}-f_{32}F_{22}^{-1}f_{23}>0$. This means that

$$\det\begin{bmatrix} F_{22} & f_{23} \\ f_{32} & f_{33} \end{bmatrix}=(f_{33}-f_{32}F_{22}^{-1}f_{23})\det F_{22}>0.$$

That is, this particular $(m+1)\times(m+1)$ principal minor is positive. The proof may be completed by permuting the rows and columns of F, so that the above argument applies to every $(m+1)\times(m+1)$ principal minor. ■

Theorem 1 may be partially extended[1] to the complex field, as follows. Let $\mathcal{S}$ be the class of diagonal matrices whose diagonal entries are of the form $\exp(i\theta)$, with θ real. Then if for every $S\in\mathcal{S}$ there exists $x>0$ such that $S^{-1}FSx>0$ (where now F may have complex entries, but the notation $x>0$ still means that elements of x are real and positive), then all principal minors of F are real and positive. The proof of this fact precisely parallels the proof of Theorem 1. However, it is readily shown that the converse statement is false.

Notice that Theorem 1 allows a different x to be chosen for each signature matrix S. For comparison, the following section shows how Theorem 1 is modified when x is required to be independent of S.

3. QUASIDOMINANT MATRICES

Consider now the following special class of matrices.

DEFINITION. Let A be a square matrix. Then A is called *quasidominant* iff there exists a vector $d>0$ such that

$$d_ia_{ii}>\sum_{j\neq i}d_j|a_{ij}|$$

for all i.

This is slightly stronger than the usual definition [4, p. 167], in that all diagonal entries of A are required to be positive. Matrices with nonpositive diagonal entries will be of little interest in the sequel.

[1]This extension was suggested by G. M. Engel.

Another class of matrices, which turns out to be a subclass of the class of quasidominant matrices, will be of particular interest.

DEFINITION. Let A be a square matrix whose off-diagonal entries are nonpositive. Then A will be called an *M-matrix* if it satisfies any of the following (equivalent) conditions:

(a) All leading principal minors of A are positive.
(b) There exists a vector $x>0$ such that $Ax>0$.
(c) There exists a vector $y>0$ such that $A'y>0$.
(d) A is nonsingular, and all entries of A^{-1} are nonnegative.

M-matrices were first introduced by Ostrowski. For a proof of the equivalence of (a)–(d), see [3, pp. 87–105]. Of course, the equivalence only holds if all off-diagonal elements of A are nonpositive.

Notice that, if A has the sign pattern required of an M-matrix, then by the property (b) above A is quasidominant iff it is an M-matrix. This leads to a simple test for quasidominance. If F is an arbitrary square matrix, define $\hat{F}$ via

$$\hat{f}_{ii}=f_{ii}$$

$$\hat{f}_{ij}=-|f_{ij}|, \qquad \text{for} \quad j\neq i.$$

Then clearly F is quasidominant iff $\hat{F}$ is an M-matrix. Notice, too, that if F is quasidominant, then so is F'.

Our next result provides an alternative characterization of quasidominant matrices.

THEOREM 2. *A square matrix F is quasidominant iff there exists a vector $x>0$ such that $SFSx>0$ for every signature matrix S.*

Proof. The condition $SFSx>0$ may be written as

$$f_{ii}x_i > -\sum_{j\neq i} s_{ii}s_{jj}f_{ij}x_j,$$

where $s_{ii}s_{jj}=\pm 1$. If the inequality is to hold for every choice of the $s_{ii}s_{jj}$, then it is clearly equivalent to the condition

$$f_{ii}x_i > \sum_{j\neq i} |f_{ij}|x_j,$$

which is the quasidominance condition. ∎

The essential difference between this result and Theorem 1 is that x is required to be independent of S. As in Theorem 1, a partial extension to matrices with complex elements is possible.

An important corollary of Theorems 1 and 2 is the known result, cf. [5], that all principal minors of a quasidominant matrix are positive, and in particular that every symmetric quasidominant matrix is positive definite. To illustrate the utility of the above ideas, it will now be shown how two important results of Araki [2] can be extended.

THEOREM 3. *If F is quasidominant, there exists a diagonal $P>0$ such that $PF+F'P>0$.*

Proof. Let $x>0$ and $y>0$ be such that $SFSx>0$ and $SF'Sy>0$ for any signature matrix S. Let P be a diagonal matrix with diagonal entries $P_{ii}=y_i/x_i$. Then

$$S(PF+F'P)Sx=PSFSx+SF'SPx=PSFSx+SF'Sy>0,$$

which means, by Theorem 2, that $PF+F'P$ is quasidominant. Since it is symmetric, it is positive definite. ∎

The specialization of Theorem 3 to M-matrices was proved by Tartar [6] and Araki [2, Theorem 1]. Actually, Theorem 3 can also be proved from the result in [2], by using the fact that F is quasidominant iff $\hat{F}$ (see above) is an M-matrix.

THEOREM 4. *If $I-|A|$ is an M-matrix, there exists a diagonal $P>0$ such that $P-A'PA>0$.*

Proof. From [2, Theorem 2], it follows that there exists a diagonal $P>0$ such that $P-|A|'P|A|$ is an M-matrix (and therefore quasidominant). A comparison of the entries of $P-|A|'P|A|$ and $P-A'PA$ then shows that this latter matrix is also quasidominant, for the same P. ∎

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