The product of a Fréchet space and a metrizable space

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Abstract

We give a simple example of a Fréchet space and a metrizable space whose product is not sequential. A consequence of our example is that the product of the sequential fan and a first countable space Y is sequential iff Y is locally countably compact.

Keywords: Fréchet space, sequential space, product.


The purpose of this paper is to give a particularly simple description of the following example, first given by Franklin [4], and Dudley [2].

Example 1. There are a Fréchet space and a metrizable space whose product is not sequential.

The join of two topologies $\mathcal{F}$ and $\mathcal{T}$ on a set is denoted $\mathcal{F} \vee \mathcal{T}$. We need the following known fact, which is useful to study both joins and products:

Lemma. If $\mathcal{F}$ and $\mathcal{T}$ are topologies on a set $X$, then $\langle X, \mathcal{F} \rangle \times \langle X, \mathcal{T} \rangle$ has a subspace homeomorphic to $\langle X, \mathcal{F} \vee \mathcal{T} \rangle$; this subspace is closed if the topology $\mathcal{F} \cap \mathcal{T}$ on $X$ is Hausdorff.

Proof. Let $\Delta$ denote the diagonal $\{(x, x) : x \in X\}$ of the set $X$. The function $x \mapsto (x, x)$ $(x \in X)$, is easily seen to be a homeomorphism of $\langle X, \mathcal{F} \rangle$ onto the subspace $\Delta$ of the space $\langle X, \mathcal{F} \rangle \times \langle X, \mathcal{T} \rangle$ [5, 81].

Since if $\mathcal{U}$ is a topology on $X$ then $\Delta$ is closed in $\langle X, \mathcal{U} \rangle \times \langle X, \mathcal{U} \rangle$ if (and only if) $\mathcal{U}$ is Hausdorff, $\Delta$ is closed in $\langle X, \mathcal{F} \rangle \times \langle X, \mathcal{T} \rangle$ if $\mathcal{F} \cap \mathcal{T}$ is Hausdorff.

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(However, \( \Delta \) can be closed in \( (X, \mathcal{F}) \times (X, \mathcal{F}) \) without \( \mathcal{F} \cap \mathcal{F} \) being Hausdorff. Let \( \mathcal{F} \) be any non-Hausdorff \( T_1 \)-topology and let \( \mathcal{F} \) be the discrete topology.)

Since the property of being sequential is closed hereditary, this implies that Example 1 follows from:

**Example 2.** There are a Fréchet topology \( \mathcal{F} \) and a metrizable topology \( \mathcal{T} \) on a set \( X \) such that \( \mathcal{F} \cap \mathcal{T} \) is Hausdorff and \( \mathcal{F} \cup \mathcal{T} \) is not sequential.

**Proof.** Let \( X = \omega \times \omega \cup \{\infty\} \), where \( \infty \notin \omega \times \omega \). In both \( \mathcal{F} \) and \( \mathcal{T} \) all points of \( \omega \times \omega \) are isolated.

In \( \mathcal{F} \) neighborhoods of \( \infty \) contain all but finitely many points of every column \( \{k\} \times \omega \). Of course \( (X, \mathcal{F}) \) is Fréchet. \( (X, \mathcal{F}) \) is well known: it is the sequential fan.

In \( \mathcal{T} \) neighborhoods of \( \infty \) include all but finitely many columns \( \{k\} \times \omega \), i.e., basic neighborhoods of \( \infty \) have the form \( \{\infty\} \cup \{k, \omega\} \times \omega \), where \( k \in \omega \). This space also is well known: A first countable space \( S \) is not locally countably compact at \( p \) iff there is a closed embedding \( e : (X, \mathcal{T}) \to S \) such that \( e(\infty) = p \).

\( \mathcal{F} \cap \mathcal{T} \) is regular, hence Hausdorff, since all points of \( \omega \times \omega \) are clopen in both \( \mathcal{F} \) and \( \mathcal{T} \), hence in \( \mathcal{F} \cap \mathcal{T} \). (Note that \( \mathcal{F} \cap \mathcal{T} \) is the one-point compactification of discrete \( \omega \times \omega \).)

\( (X, \mathcal{F} \cup \mathcal{T}) \) is not sequential since it is Arens's space \[1\]: Points of \( \omega \times \omega \) are isolated and neighborhoods of \( \infty \) contain all but finitely many points of all but finitely many columns, and so \( (X, \mathcal{F} \cup \mathcal{T}) \) is badly non-Fréchet: it is not discrete but it does not have nontrivial convergent sequences. \( \square \)

**Corollary to Proof.** Let \( \mathcal{F} \) denote the sequential fan, i.e., \( \mathcal{F} = (X, \mathcal{F}) \). If \( Y \) is first countable, then \( \mathcal{F} \times \mathcal{Y} \) is Fréchet iff \( X \) is locally countably compact.

**Proof.** If \( Y \) is locally countably compact, then \( \mathcal{F} \times \mathcal{Y} \) is sequential since the product of a sequential space and a locally countably compact sequential space is sequential, \[3, 3.10.1.(b)\].

If \( Y \) is not locally compact, then, as pointed out above, there is a closed embedding of \( (X, \mathcal{F}) \) into \( \mathcal{Y} \), hence then \( \mathcal{F} \times \mathcal{Y} \) is not sequential since it has \( \mathcal{F} \times (X, \mathcal{F}) \) as closed nonsequential subspace. \( \square \)

This has further consequences, e.g. since \( \mathcal{F} \) embeds as a closed subspace in every nonmetrizable closed image of a metrizable space. We leave these to the reader.

**References**