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On the forbidden induced subgraph sandwich problem[☆]

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ABSTRACT

We consider the sandwich problem, a generalization of the recognition problem introduced by Golumbic et al. (1995) [15], with respect to classes of graphs defined by excluding induced subgraphs. We prove that the sandwich problem corresponding to excluding a chordless cycle of fixed length k is NP-complete. We prove that the sandwich problem corresponding to excluding $K_r \setminus e$ for fixed r is polynomial. We prove that the sandwich problem corresponding to $3PC(\cdot,\cdot)$-free graphs is NP-complete. These complexity results are related to the classification of a long-standing open problem: the sandwich problem corresponding to perfect graphs.

1. Introduction

All graphs in this paper are finite, simple and undirected. We say that a graph $G^1 = (V, E^1)$ is a *spanning* subgraph of $G^2 = (V, E^2)$ if $E^1 \subseteq E^2$ and that a graph $G = (V, E)$ is a *sandwich* graph for the pair G^1, G^2 if $E^1 \subseteq E \subseteq E^2$. For notational simplicity in the sequel, we let E^3 be the set of all edges in the complete graph with vertex set V which are not in E^2. Thus every sandwich graph $G = (V, E)$ for the pair G^1, G^2 satisfies $E^1 \subseteq E \subseteq E^2$, or equivalently $E^1 \subseteq E$ and $E \cap E^3 = \emptyset$. We call E^1 the *forced* edge set, $E^2 \setminus E^1$ the *optional* edge set, and E^3 the *forbidden* edge set. The GRAPH SANDWICH PROBLEM FOR PROPERTY Π is defined as follows [15]:

The GRAPH SANDWICH PROBLEM FOR PROPERTY Π

Instance: Vertex set V, and edge sets E^1 and E^2, such that $E^1 \subseteq E^2$.

Question: Is there a graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$ which satisfies property Π?

Both forms (V, E^1, E^2) and (V, E^1, E^3) can be used to refer to an instance of a graph sandwich problem. The *recognition* problem for a class of graphs $\mathcal{C}$ has as instance just the graph $G = (V, E)$ that we want to recognize and the property Π is "belonging to class $\mathcal{C}$". Therefore the recognition problem for $\mathcal{C}$ is equivalent to the particular graph sandwich problem in which the forced edge set $E^1 = E$, and the optional edge set $E^2 \setminus E^1 = \emptyset$ or equivalently the forbidden edge set E^3 consists of all edges not present in E. Graph sandwich problems have attracted much attention lately arising from many applications and as a natural generalization of recognition problems [2,10,12,13,20,21].

Golumbic et al. [15] have considered sandwich problems with respect to several subclasses of perfect graphs, and proved that the GRAPH SANDWICH PROBLEM FOR SPLIT GRAPHS remains in P. On the other hand, they proved that the GRAPH SANDWICH

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PROBLEM FOR CHORDAL GRAPHS turns out to be NP-complete. As remarked in [15], the sandwich problem with respect to $\mathcal{C}$ has the same complexity as the sandwich problem with respect to $\overline{\mathcal{C}}$, the complementary graph class containing for each $G \in \mathcal{C}$ the complement graph $\overline{G}$. Indeed, $G \in \mathcal{C}$ is a sandwich graph for the instance (V, E^1, E^3) if and only if $\overline{G} \in \overline{\mathcal{C}}$ is a sandwich graph for the instance (V, E^3, E^1).

In particular, the classification P versus NP-complete in the seminal paper [15] suggested the investigation of the following properties as regards graph sandwich problems:

1. Let $\mathcal{C}$ be a self-complementary graph class. Is the GRAPH SANDWICH PROBLEM FOR $\mathcal{C}$ in P?
2. Let $\mathcal{C}$ be a graph class defined by a finite family of forbidden induced subgraphs. Is the GRAPH SANDWICH PROBLEM FOR $\mathcal{C}$ in P?

Positive answers for Property 1 have been given for the self-complementary graph property of having a homogeneous set [1], and for several self-complementary graph classes: cographs [15], split graphs [15], P_4-sparse graphs [10], and P_4-reducible graphs [2]. Note that cographs and split graphs are graph classes defined by a finite family of forbidden induced subgraphs and so give for both Properties 1 and 2 positive answers. On the negative side, the NP-completeness for hereditary clique-Helly graphs [12] presents a first example of a graph class $\mathcal{C}$ defined by a finite family of forbidden induced subgraphs for which the GRAPH SANDWICH PROBLEM FOR $\mathcal{C}$ is NP-complete giving for Property 2 a negative answer. Negative evidence for Property 1 has been given for permutation graphs, a self-complementary graph class whose recognition is polynomial whereas the sandwich problem turns out to be NP-complete [15].

We say that a graph G *contains* a graph F if F is isomorphic to an induced subgraph of G. A graph G is F*-free* if it does not contain F. Note that the well-studied class of cographs is precisely the class of P_4-free graphs, where P_4 is the path induced by four vertices. A *hole* is a chordless cycle of length at least 4. A hole is *even* (resp. *odd*) if it contains an even (resp. odd) number of vertices. A hole of length k is also called a k*-hole* and it is denoted by C_k. Let $\mathcal{F}$ be a (possibly infinite) family of graphs. A graph G is $\mathcal{F}$*-free* if it is F-free, for every $F \in \mathcal{F}$. The well-studied class of split graphs is precisely the self-complementary class of $\{C_4, \overline{C_4}, C_5\}$-free graphs [14].

A challenging self-complementary class is the class of Berge graphs, the class of $\{$odd hole, $\overline{\text{odd hole}}\}$-free graphs. The celebrated strong perfect graph theorem characterized the class of perfect graphs as being precisely the class of Berge graphs; the polynomial recognition of Berge graphs was established subsequently [3]. It is surprising that the recognition of even-hole-free graphs can be done in polynomial time [4,8,11] whereas the recognition of odd-hole-free graphs is an outstanding open problem [3].

In the present paper, our result on C_5-free graphs presents a surprising example of a self-complementary graph class $\mathcal{C}$, for which the GRAPH SANDWICH PROBLEM FOR $\mathcal{C}$ is NP-complete. Since the complement of graph C_5 is itself, we have the *first* example of a self-complementary graph class $\mathcal{C}$ defined by just *one* forbidden induced subgraph for which the GRAPH SANDWICH PROBLEM FOR $\mathcal{C}$ is NP-complete.

Remark that the seminal paper on graph sandwich problems [15] left three open problems: the GRAPH SANDWICH PROBLEM FOR STRONGLY CHORDAL GRAPHS recently classified as NP-complete [13], the GRAPH SANDWICH PROBLEM FOR CHORDAL BIPARTITE GRAPHS recently additionally classified as NP-complete [20], and the GRAPH SANDWICH PROBLEM FOR PERFECT GRAPHS, still open.

In the present paper we consider several classes related to perfect graphs contributing towards the classification of this remaining sandwich problem.

By the perfect graph theorem, the class of perfect graphs is defined as a self-complementary class by a parity condition on the size of its holes. Please refer to Fig. 1 where we relate the class of perfect graphs, the parity condition, and graph classes that have been defined and studied with the aim of better understanding the complexity of searching for an even or for an odd hole. A graph is perfect if and only if it contains no odd hole and no complement of an odd hole as an induced subgraph. The class of graphs with no induced C_4 does not have the complement of C_n, $n \geq 6$, as an induced subgraph, either, a fact that relates the structure of C_4-free graphs to the structure of even-hole-free graphs.

A convenient setting for the study of even or odd holes in graphs is the one of signed graphs, since it has been shown in the literature that the essence of even-hole-free (resp. odd-hole-free) graphs is actually captured by their generalization to signed graphs, called the odd-signable (resp. even-signable) graphs. A theorem due to Truemper characterizes the graphs whose edges can be labeled such that all chordless cycles have prescribed parities [9,22]. This theorem has proven to be an essential tool in the study of various objects like balanced matrices, graphs with no even holes and graphs with no odd holes [6,8]. We *sign* a graph by assigning 0, 1 weights to its edges. The weight of a cycle is the sum of the weights of its edges. A graph is *odd-signable* if there exists a signing that makes every triangle odd weight and every hole odd weight. Note that a graph has no even holes if and only if it is odd-signable with all labels equal to 1. A graph is *even-signable* if there exists a signing that makes every triangle odd weight and every hole even weight. Note that a graph has no odd holes if and only if it is even-signable with all labels equal to 1.

Testing whether a graph is even-signable (resp. odd-signable) is open. In fact an algorithm for testing whether a graph is even-signable (resp. odd-signable) implies an algorithm for testing whether a graph is odd-hole-free (resp. even-hole-free) [7]. Odd-signable graphs and even-signable graphs can be characterized in terms of excluded induced subgraphs that we introduce now. Please refer to Fig. 2.

A $3PC$ is a graph induced by three internally disjoint chordless paths $P_1 = x_1, \ldots, y_1$, $P_2 = x_2, \ldots, y_2$ and $P_3 = x_3, \ldots, y_3$ such that any two of them induce a hole. If $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$, we say that $P_1 \cup P_2 \cup P_3$ induces a $3PC(x_1, y_1)$.

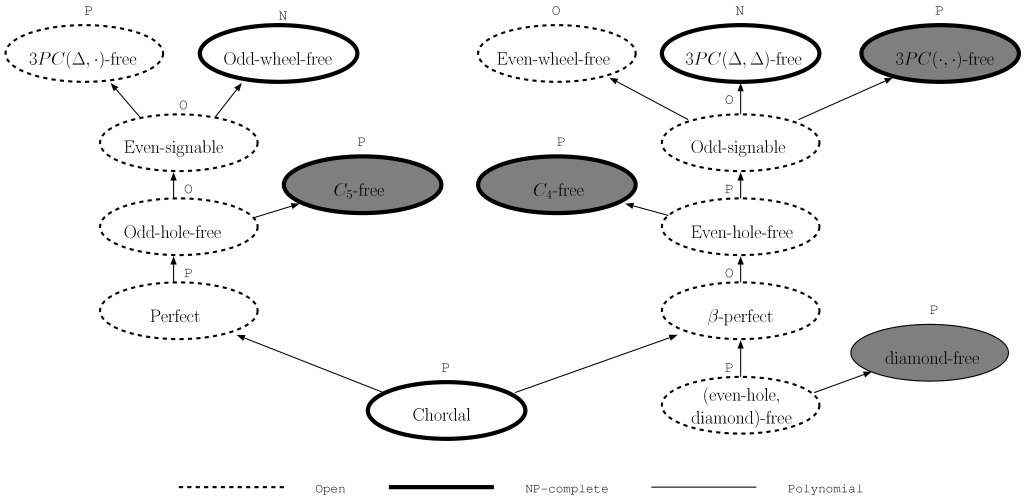

Fig. 1. Graph classes defined by forbidden induced subgraphs (excepting β-perfect graphs, for which such characterization is not known). In the diagram, note that odd-signable graphs are precisely ($3PC(\cdot,\cdot)$, $3PC(\Delta,\Delta)$, even wheel)-free graphs, even-signable graphs are precisely ($3PC(\Delta,\cdot)$, odd wheel)-free graphs and chordal graphs are precisely graphs that are both perfect and β-perfect. Labels O, N and P refer to the complexity of the recognition problem (open, NP-complete and polynomial, respectively). Following the notation of [15], dashed, bold and light classes refer to the complexity of the corresponding sandwich problem (open, NP-complete and polynomial, respectively). The four highlighted classes in the diagram correspond to classes classified in this paper.

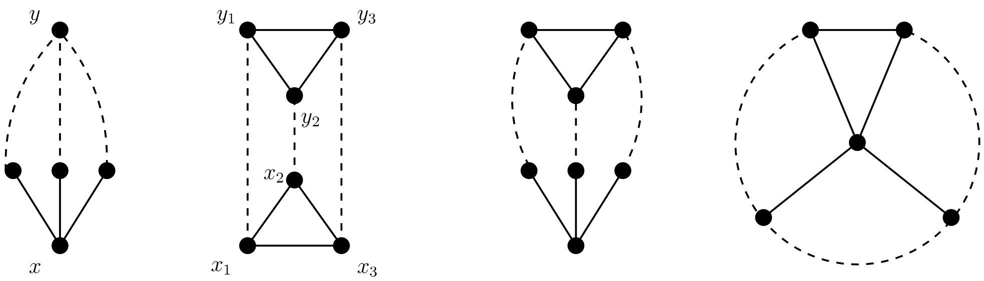

Fig. 2. Path configurations $3PC(\cdot,\cdot)$, $3PC(\triangle,\triangle)$, $3PC(\triangle,\cdot)$ and a wheel. Dashed lines represent paths of length at least 1.

If $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ induce triangles, then we say that $P_1 \cup P_2 \cup P_3$ induces a $3PC(x_1x_2x_3, y_1y_2y_3)$. If $\{x_1, x_2, x_3\}$ induces a triangle and $y_1 = y_2 = y_3$, we say that $P_1 \cup P_2 \cup P_3$ induces a $3PC(x_1x_2x_3, y_1)$. We say that a graph G contains a $3PC(\cdot,\cdot)$ if it contains a $3PC(x, y)$ for some $x, y \in V(G)$. Similarly we say that a graph G contains a $3PC(\Delta,\Delta)$ (resp. $3PC(\Delta,\cdot)$) if it contains a $3PC(x_1x_2x_3, y_1y_2y_3)$ (resp. $3PC(x_1x_2x_3, y)$) for some $x_1, x_2, x_3, y_1, y_2, y_3 \in V(G)$ (resp. $x_1, x_2, x_3, y \in V(G)$). Graphs $3PC(\cdot,\cdot)$, $3PC(\Delta,\Delta)$ and $3PC(\Delta,\cdot)$ are also known as *thetas*, *prisms* and *pyramids*, respectively. Testing for thetas and testing for pyramids can both be done in polynomial time [5] whereas testing for prisms is NP-complete [17].

A *wheel*, denoted by (H, x), is a graph induced by a hole H and a vertex $x \notin V(H)$ having at least three neighbors in H, say $x_1, \ldots, x_n$. If x is adjacent to two consecutive vertices in H, say x_i, x_{i+1}, we say that x_ix_{i+1} is a *short sector* of the wheel. A wheel is *odd* if it has an odd number of short sectors. Wheel (H, x) is *even* if x has an even number of neighbors in H. Detecting an odd wheel is NP-complete [18] whereas detecting an even wheel is open.

It is easy to see that even wheels, $3PC(\cdot,\cdot)$'s and $3PC(\Delta,\Delta)$'s cannot be contained in even-hole-free graphs, while odd wheels and $3PC(\Delta,\cdot)$'s cannot be contained in odd-hole-free graphs. In fact, a graph is odd-signable if and only if it does not contain an even wheel, a $3PC(\cdot,\cdot)$ or a $3PC(\Delta,\Delta)$ and a graph is even-signable if and only if it does not contain an odd wheel or a $3PC(\Delta,\cdot)$ [7,9,22].

Another class introduced in the context of graphs not containing holes of a prescribed parity is the class of *β-perfect* graphs. A graph is β-perfect [19] if for each induced subgraph H of G, $\chi(H) = \beta(H)$, where $\beta(G) = \max\{\delta(G') + 1 : G'$ is an induced subgraph of $G\}$. It is easy to see that β-perfect graphs belong to the class of even-hole-free graphs, and that this containment is proper. It has recently been shown in [16] that (even-hole, diamond)-free graphs are β-perfect, where a *diamond* is a cycle of length 4 that has exactly one chord, or equivalently the graph $K_4 \setminus e$. The recognition of β-perfect graphs is open. Graphs that are both perfect and β-perfect are precisely the well known chordal graphs [19].

In the present paper, we prove that the sandwich problem corresponding to excluding a chordless cycle of fixed length k is NP-complete. We prove that the sandwich problem corresponding to excluding $K_r \setminus e$ for fixed r is polynomial. We prove that the sandwich problem corresponding to $3PC(\cdot,\cdot)$-free graphs is NP-complete. These complexity results are related to the classification of a long-standing open problem: the sandwich problem corresponding to perfect graphs.

2. The C_k-FREE GRAPH SANDWICH PROBLEM

In this section we prove that, for any fixed $k \geq 4$, the C_k-FREE GRAPH SANDWICH PROBLEM is NP-complete. In order to obtain this result, first we prove in Theorem 1 that the C_4-FREE GRAPH SANDWICH PROBLEM (i.e. the particular case $k = 4$) is NP-complete by showing a reduction from the NP-complete problem 3-SATISFIABILITY. Second, in Theorem 2, we show how this reduction can be generalized to prove the NP-completeness of the C_k-FREE GRAPH SANDWICH PROBLEM. These decision problems are defined as follows.

3-SATISFIABILITY (3SAT)
Instance: A pair (X, C), where $X = \{x_1, \ldots, x_n\}$ is the set of variables, and $C = \{c_1, \ldots, c_m\}$ is the collection of clauses over X such that each clause $c \in C$ has $|c| = 3$ literals.
Question: Is there a truth assignment for X such that each clause in C has at least one true literal?

The C_4-FREE GRAPH SANDWICH PROBLEM (C4SP)
Instance: Vertex set V, and edge sets E^1 and E^2, such that $E^1 \subseteq E^2$.
Question: Is there a graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$, and G is C_4-free?

The C_k-FREE GRAPH SANDWICH PROBLEM (CkSP)
Instance: Vertex set V, and edge sets E^1 and E^2, such that $E^1 \subseteq E^2$.
Question: Is there a graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$, and G is C_k-free, for a fixed $k \geq 4$?

Now we prove the following theorem.

Theorem 1. C4SP *is NP-complete.*

Proof. In order to reduce 3SAT to C4SP we need to construct a particular instance (V, E^1, E^2) of C4SP from a generic instance (X, C) of 3SAT, such that C is satisfiable if and only if (V, E^1, E^2) admits a sandwich graph $G = (V, E)$ which is C_4-free. First, we describe the construction of a particular instance (V, E^1, E^2) of C4SP; second, we prove in Lemma 1 that every C_4-free sandwich graph $G = (V, E)$ for particular instance (V, E^1, E^2) defines a satisfying truth assignment for (X, C); third, we prove in Lemma 2 that every satisfying truth assignment for (X, C) defines a C_4-free sandwich graph $G = (V, E)$ for particular instance (V, E^1, E^2). These steps are explained below. $\square$

Construction of a particular instance of C4SP

The vertex set V contains: for each variable $x_i \in X$, a set $X_i = \{x^i_1, x^i_2, \bar{x}^i_1, \bar{x}^i_2\}$; for each clause $c_j = (l^j_1 \vee l^j_2 \vee l^j_3)$ in C, the corresponding set $C_j = A_j \cup B_j$, where A_j, B_j are defined in the following way: $A_j = \{p^j_1, \ldots, p^j_4\}$ and $B_j = \bigcup_{q=1,2,3}\{l^j_q, t^j_q, s^j_{q1}, \ldots, s^j_{q4}, r^j_{q1}, \ldots, r^j_{q4}\}$.

Forced edge set E^1 contains: for each variable x_i, the edges $x^i_1\bar{x}^i_1, \bar{x}^i_1x^i_2, x^i_2\bar{x}^i_2, \bar{x}^i_2x^i_1$; for each clause c_j, $1 \leq j \leq m$, the edges $p^j_1l^j_1, l^j_1p^j_2, p^j_2l^j_2, l^j_2p^j_3, p^j_3l^j_3, l^j_3p^j_4, p^j_4p^j_1, p^j_1t^j_1, t^j_1p^j_2, p^j_2t^j_2, t^j_2p^j_3, p^j_3t^j_3, t^j_3p^j_4, p^j_1s^j_{11}, p^j_2s^j_{13}, p^j_2s^j_{21}, p^j_3s^j_{23}, p^j_3s^j_{31}, p^j_4s^j_{33}$ and for $q \in \{1, 2, 3\}$, the set $Y^j_q = A^j_q \cup B^j_q$ where $A^j_q = \{t^j_qr^j_{q3}, l^j_qr^j_{q1}, s^j_{q1}s^j_{q2}, s^j_{q2}s^j_{q3}, s^j_{q3}s^j_{q4}, s^j_{q4}s^j_{q1}, r^j_{q1}r^j_{q2}, r^j_{q2}r^j_{q3}, r^j_{q3}r^j_{q4}, r^j_{q4}r^j_{q1}\}$ and B^j_q is defined as follows. If $l^j_q = x_i$, then $B^j_q = \{x^i_1s^j_{q2}, x^i_2s^j_{q4}, \bar{x}^i_1r^j_{q2}, \bar{x}^i_2r^j_{q4}\}$; if $l^j_q = \bar{x}_i$, then $B^j_q = \{x^i_1r^j_{q2}, x^i_2r^j_{q4}, \bar{x}^i_1s^j_{q2}, \bar{x}^i_2s^j_{q4}\}$.

The optional edge set $E^0 = E^2 \setminus E^1$ contains: for each variable x_i, the edges $x^i_1x^i_2, \bar{x}^i_1\bar{x}^i_2$, for each clause c_j, the edges $p^j_1p^j_2, p^j_2p^j_3, p^j_3p^j_4$, and for $q \in \{1, 2, 3\}$, the edges $l^j_qt^j_q, s^j_{q2}s^j_{q4}, s^j_{q1}s^j_{q3}, r^j_{q2}r^j_{q4}, r^j_{q1}r^j_{q3}$.

For each x_i, $i \in \{1, \ldots, n\}$, we call the *variable gadget corresponding to* x_i the subgraph of $G^2 = (V, E^2)$ induced by X_i. For each $c_j, j \in \{1, \ldots, m\}$, we call the *clause gadget corresponding to* c_j the subgraph of $G^2 = (V, E^2)$ induced by the set C_j. In Figs. 3 and 4, solid edges are forced E^1-edges, dashed edges are optional E^0-edges and omitted edges are forbidden E^3-edges.

In the clause gadget corresponding to clause c_j, let $Z^j_1 = \{p^j_1, t^j_1, p^j_2, l^j_1\}$ and let R^j_q, $q \in \{1, 2, 3\}$, be the set $\{r^j_{q1}, \ldots, r^j_{q4}\}$.

Lemma 1. *If the particular instance (V, E^1, E^2) of* C4SP *constructed above admits a graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$ and G is C_4-free, then there exists a truth assignment that satisfies (X, C).*

Proof. Suppose there exists a C_4-free sandwich graph $G = (V, E)$. We define the truth assignment for (X, C): for $i \in \{1, \ldots, n\}$, variable x_i is true if and only if $x^i_1x^i_2 \in E(G)$. Suppose that for some $j \in \{1, \ldots, m\}$, the clause $c_j = (l^j_1 \vee l^j_2 \vee l^j_3)$ is false.

Claim 1. $r^j_{q1}r^j_{q3} \in E(G)$, $q \in \{1, 2, 3\}$.

Proof of Claim 1. If $l^j_q = x_i$, then since l^j_q is false, $x^i_1x^i_2 \notin E(G)$. Since X_i cannot induce a 4-hole, $\bar{x}^i_1\bar{x}^i_2 \in E(G)$. Since $\{\bar{x}^i_1, \bar{x}^i_2, r^j_{q2}, r^j_{q4}\}$ and R^j_q cannot induce 4-holes, $r^j_{q1}r^j_{q3} \in E(G)$.

If $l^j_q = \bar{x}_i$, then $x^i_1x^i_2 \in E(G)$. Since $\{x^i_1, x^i_2, r^j_{q2}, r^j_{q4}\}$ and R^j_q cannot induce 4-holes, $r^j_{q1}r^j_{q3} \in E(G)$. This completes the proof of Claim 1. $\square$

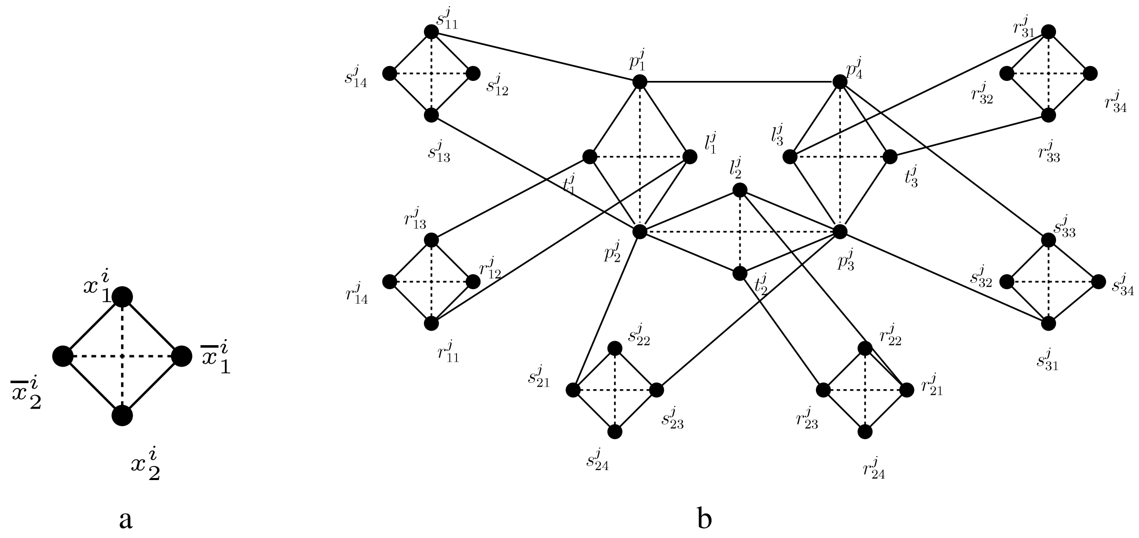

Fig. 3. (a) Variable gadget X_i. (b) Clause gadget C_j. Solid edges are forced E^1-edges, dashed edges are optional $E^0 = E^2 \setminus E^1$ edges and omitted edges are forbidden E^3-edges.

By Claim 1 and since $\{r^j_{11}, r^j_{13}, t^j_1, l^j_1\}$ and Z^j_1 cannot induce 4-holes, $p_1p_2 \in E(G)$. Similarly $p^j_2p^j_3, p^j_3p^j_4 \in E(G)$. But then $\{p^j_1, p^j_2, p^j_3, p^j_4\}$ induces a 4-hole. Therefore all clauses are satisfied. $\square$

The converse of Lemma 1 is given next by Lemma 2.

Lemma 2. *If there exists a truth assignment that satisfies (X, C), then the particular instance (V, E^1, E^2) of* c4sp *constructed above admits a graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$ and G is C_4-free.*

Proof. Suppose that there is a truth assignment that satisfies (X, C). We define a C_4-free sandwich graph G that is the solution for the particular instance (V, E^1, E^2) of c4sp associated with the 3sat instance (X, C).

If variable x_i is true, then include the optional edge $x^i_1x^i_2$, for $i \in \{1, \ldots, n\}$. Otherwise, if variable x_i is false, then include $\overline{x}^i_1\overline{x}^i_2$. For every $c_j = (l^j_1 \vee l^j_2 \vee l^j_3), j \in \{1, \ldots, m\}$, include the following optional edges. If l^j_1 is true, then include the edges $s^j_{11}s^j_{13}, r^j_{12}r^j_{14}$ and $l^j_1t^j_1$. Otherwise, include $s^j_{12}s^j_{14}, r^j_{11}r^j_{13}$ and $p^j_1p^j_2$. If l^j_2 is true, then include the edges $s^j_{21}s^j_{23}, r^j_{22}r^j_{24}$ and $l^j_2t^j_2$. Otherwise, include $s^j_{22}s^j_{24}, r^j_{21}r^j_{23}$ and $p^j_2p^j_3$. If l^j_3 is true, then include the edges $s^j_{31}s^j_{33}, r^j_{32}r^j_{34}$ and $l^j_3t^j_3$. Otherwise, include $s^j_{32}s^j_{34}, r^j_{31}r^j_{33}$ and $p^j_3p^j_4$.

Now we prove that the sandwich graph G constructed above is C_4-free. Clearly each subgraph $G[X_i]$ and each subgraph $G[C_j]$ are C_4-free. Since there are no edges between $G[X_{i_1}]$ and $G[X_{i_2}]$, $i_1 \neq i_2$, and no edges between $G[C_{j_1}]$ and $G[C_{j_2}]$, $j_1 \neq j_2$, it remains to verify that there is no 4-hole that contains an edge with one endnode in a variable gadget and the other endnode in a clause gadget. First consider the next two remarks:

(1) If $l^j_q = x_i$, then $|\{x_1x_2, s^j_{q2}s^j_{q4}\} \cap E| = 1$ and $|\{\overline{x}_1\overline{x}_2, r^j_{q2}r^j_{q4}\} \cap E| = 1$.
(2) If $l^j_q = \overline{x}_i$, then $|\{x_1x_2, r^j_{q2}r^j_{q4}\} \cap E| = 1$ and $|\{\overline{x}_1\overline{x}_2, s^j_{q2}s^j_{q4}\} \cap E| = 1$.

The edges connecting the subgraphs $G[X_i]$ and $G[C_j]$ are: for $q \in \{1, 2, 3\}$, either $\{x^i_1s^j_{q2}, x^i_2s^j_{q4}\} \cup \{\overline{x}^i_1r^j_{q2}, \overline{x}^i_2r^j_{q4}\}$, when $l^j_q = x_i$; or $\{x^i_1r^j_{q2}, x^i_2r^j_{q4}\} \cup \{\overline{x}^i_1s^j_{q2}, \overline{x}^i_2s^j_{q4}\}$, when $l^j_q = \overline{x}_i$. Then, by Remarks (1) and (2), there are no 4-holes containing edges connecting variable gadgets to clause gadgets, and hence G is C_4-free. $\square$

Now we generalize the previous construction to prove the NP-completeness of cksp.

Theorem 2. cksp *is NP-complete.*

Proof. In order to prove that cksp is NP-complete, we show how the variable and the clause gadgets presented in Theorem 1 can be modified to obtain gadgets for every fixed $k \geq 5$. Please refer to Fig. 5, where an example of the construction is given for the particular case where $k = 5$.

In the clause gadget corresponding to clause c_j, $j \in \{1, \ldots, m\}$, replace each edge $p^j_4p^j_1$ in G^1 with the corresponding path $p^j_4, p^j_5, \ldots, p^j_k, p^j_1$ in G^1. For $q = \{1, 2, 3\}$, replace each edge $s^j_{q4}s^j_{q1}$ and each edge $r^j_{q4}r^j_{q1}$ in G^1 with the corresponding paths $s^j_{q4}, s^j_{q5}, \ldots, s^j_{qk}, s^j_{q1}$ and $r^j_{q4}, r^j_{q5}, \ldots, r^j_{qk}, r^j_{q1}$ in G^1. Similarly replace each edge $t^j_qr^j_{q3}, s^j_{13}p^j_2, s^j_{23}p^j_3, s^j_{33}p^j_4, p^j_1t^j_1, p^j_2t^j_2, p^j_3t^j_3$ in G^1 with corresponding paths of length $k-3$ in G^1. If $l^j_q = x_i$, then replace each edge $x^i_1s^j_{q2}$ and each edge $\overline{x}^i_1r^j_{q2}$ in G^1 with

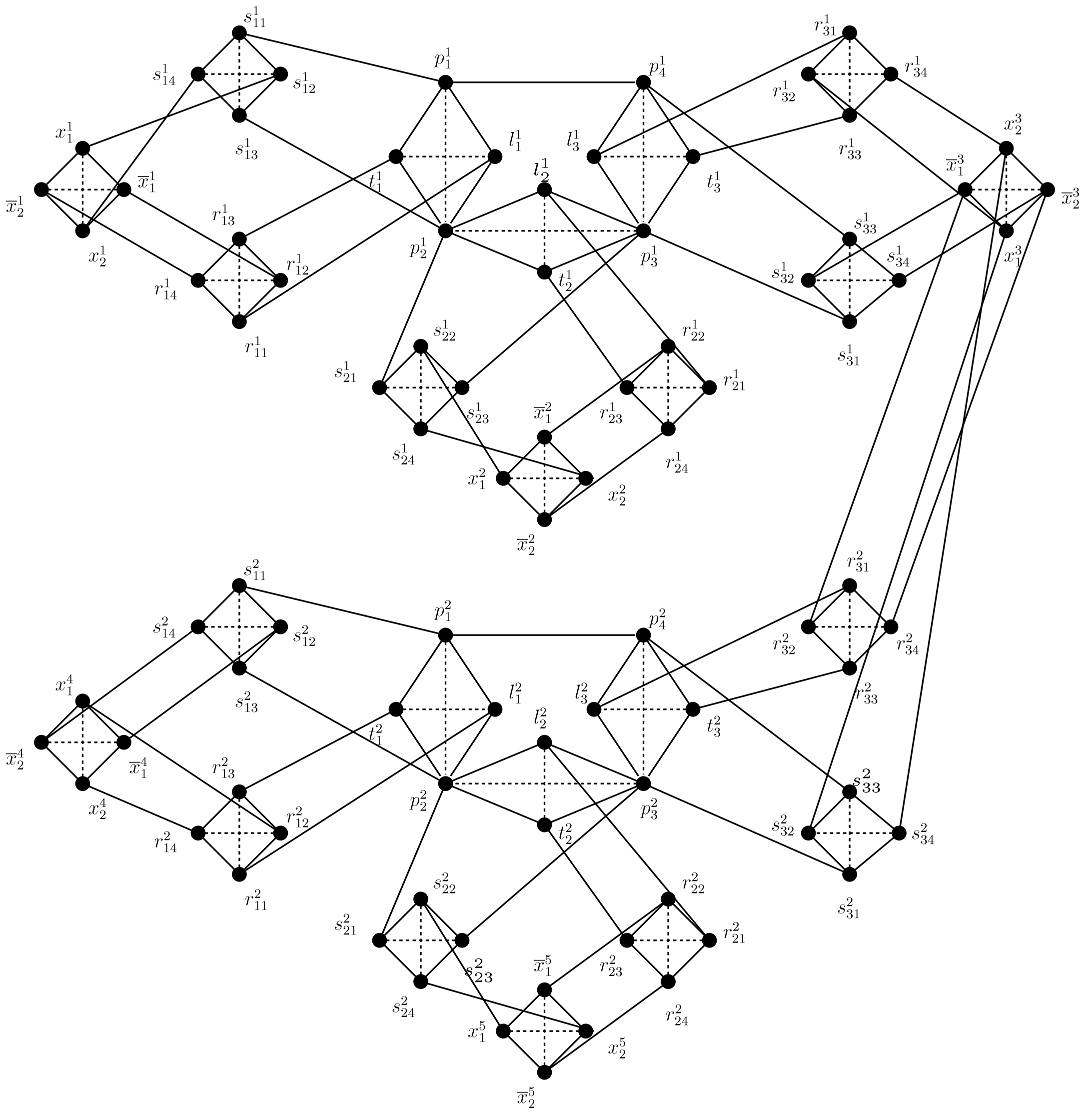

Fig. 4. Example of a particular instance of C4SP for (V, E^1, E^2) obtained from the instance of 3SAT: $I = (X, C) = (\{x_1, x_2, x_3, x_4, x_5\}, \{(x_1 \vee x_2 \vee \bar{x}_3), (\bar{x}_4 \vee x_5 \vee x_3)\})$.

corresponding paths of length $k - 3$ in G^1. If $l_q^j = \bar{x}_i$, then replace the edges $x_1^i r_{q2}^j$ and $\bar{x}_1^i s_{q2}^j$ in G^1 with paths of length $k - 3$ in G^1.

Finally, in the variable gadget corresponding to x_i, $i \in \{1, \ldots, n\}$, replace each edge $\bar{x}_2^i x_1^i$ in G^1 with a corresponding path of length $k - 3$ in G^1. $\square$

3. The $(K_r \setminus e)$-FREE GRAPH SANDWICH PROBLEM

In this section we are interested in the complexity of the $(K_r \setminus e)$-FREE GRAPH SANDWICH PROBLEM. We note that the particular case $r = 4$ is the sandwich problem corresponding to excluding a diamond as an induced subgraph. We present an algorithm that decides in polynomial time whether an instance (V, E^1, E^2) admits a $(K_r \setminus e)$-free sandwich graph. This decision problem and the algorithm are presented below.

The $(K_r \setminus e)$-FREE GRAPH SANDWICH PROBLEM (KrESP)
Instance: Vertex set V, and edge sets E^1 and E^2, such that $E^1 \subseteq E^2$.
Question: Is there a graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$, and G is $(K_r \setminus e)$-free?

The following algorithm runs in time $\mathcal{O}(n^r m)$, where $|V| = n$ and $|E^2| = m$. The algorithm constructs a strictly increasing sequence of edge sets containing E^1 and contained in E^2.

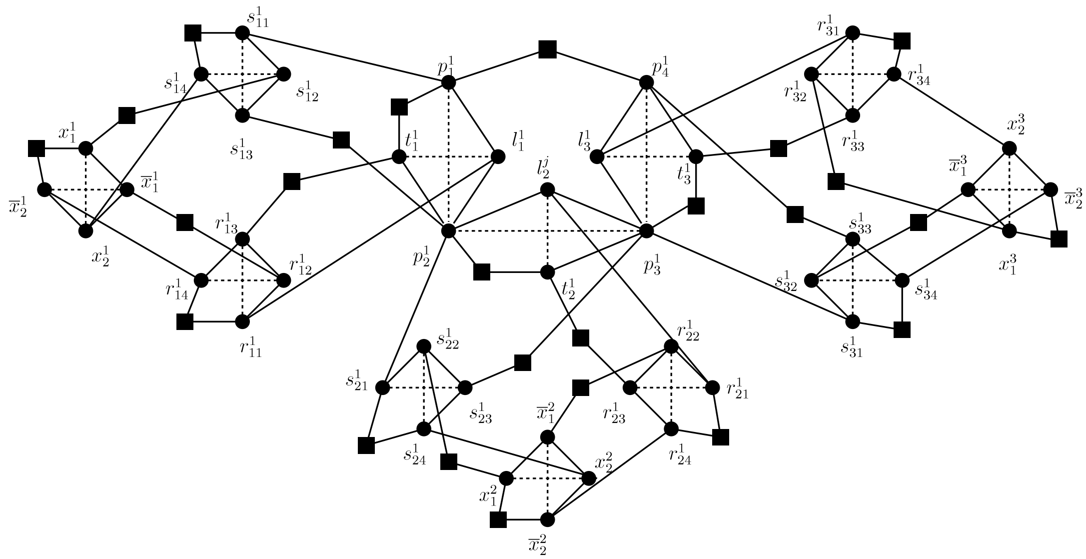

Fig. 5. Example of a particular instance of C5SP for (V, E^1, E^2) obtained from the instance of 3SAT: $I = (X, C) = (\{x_1, x_2, x_3\}, (x_1 \vee x_2 \vee \overline{x}_3))$.

Algorithm 1 KRESP Algorithm

Input: (V, E^1, E^2)
Output: (YES, G) or NO
$E \leftarrow E^1$
while $G = (V, E)$ contains a $K_r \setminus e$ **do**
  $E^* \leftarrow \emptyset$
  for every $K_r \setminus e$ in G **do**
    if e is an optional edge in $E^2 \setminus E^1$ **then**
      $E^* \leftarrow E^* \cup \{e\}$
    else
      return NO
  $E \leftarrow E \cup E^*$
return (YES,G)

Theorem 3. *Algorithm* 1 *correctly decides whether* (V, E^1, E^2) *admits a* $(K_r \setminus e)$*-free sandwich graph G.*

Proof. The proof follows since Algorithm 1 is a greedy algorithm based on the fact that at each step, either there is an edge that must be added (and if it is not among the optional edges, then there is no solution) or the graph is already $K_r \setminus e$-free. See Figs. 7 and 6 for an example of a YES instance and an example of a NO instance, respectively. □

4. The 3*PC*(·, ·)-free graph sandwich problem

In this section we prove that the 3*PC*(·, ·)-FREE GRAPH SANDWICH PROBLEM is NP-complete by showing a reduction from the NP-complete problem CHORDAL GRAPH SANDWICH PROBLEM. These two decision problems are defined as follows.

The CHORDAL GRAPH SANDWICH PROBLEM (CSP)
Instance: Vertex set V, and edge sets E^1 and E^2, such that $E^1 \subseteq E^2$.
Question: Is there a graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$, and G is a chordal graph?
The 3*PC*(·, ·)-FREE GRAPH SANDWICH PROBLEM (3*PC*(·, ·)*SP*)
Instance: Vertex set V, and edge sets E^1 and E^2, such that $E^1 \subseteq E^2$.
Question: Is there a graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$, and G is 3*PC*(·, ·)-free?

Theorem 4. 3*PC*(·, ·)*SP is NP-complete.*

Proof. Let (V^*, E^{1*}, E^{2*}) be an instance of CSP. We construct a particular instance (V, E^1, E^2) of 3*PC*(·, ·)*SP* from (V^*, E^{1*}, E^{2*}) as follows.

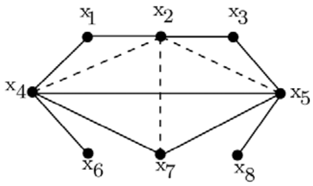

Fig. 6. An example of a NO instance for Algorithm 1.

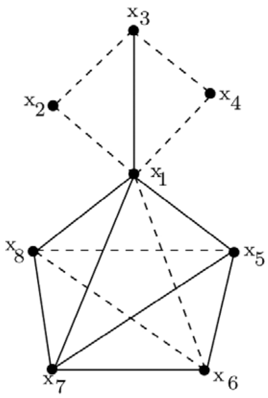

Fig. 7. An example of a YES instance for Algorithm 1.

The vertex set V contains: vertices of V^* and a set S of auxiliary vertices constructed in the following way. For each pair u, v of nonadjacent vertices in G^{1*}, include two vertices x_{uv}, y_{uv} in S. The set of forced edges E^1 contains: all edges in E^{1*}; for each pair $u, v \in V^*$ of nonadjacent vertices in G^{1*}, the edges $\{ux_{uv}, x_{uv}y_{uv}, y_{uv}v\}$; and the edges between every pair of vertices in S. The optional edge set is $E^2 \setminus E^1 = E^{2*} \setminus E^{1*}$. All remaining edges are forbidden.

Suppose that we have a chordal sandwich graph $G^* = (V^*, E^*)$ for (V^*, E^{1*}, E^{2*}). Let $G = (V, E)$ be a sandwich graph for the instance (V, E^1, E^2) such that $V = V^* \cup S$ and E is the union of E^*, all edges between vertices of S and the edges of type ux_{uv} for every $u \in V$ and non-edge uv of G^{1*}. Suppose that G is not $3PC(\cdot, \cdot)$-free and let $T \subseteq V$ be a $3PC(a, b)$ in G. Clearly, T is not entirely contained in V^*; otherwise G^* is not a chordal graph. So at least one vertex of T, say x, is contained in S.

Suppose that $x = a$. Since every vertex of S has at most one neighbor in $G \setminus S$, and since x has three neighbors in T, $N[x] \cap T$ induces a triangle in T, a contradiction. So $x \neq a$, and by symmetry $x \neq b$.

Let H_1, H_2, H_3 be the three holes contained in T. W.l.o.g., suppose that $x \notin H_3$. Since H_3 is not entirely contained in V^* (otherwise G^* is not a chordal graph), H_3 must contain at least one vertex, say y, in $S \setminus \{x\}$. By the same argument as in the previous paragraph, we may assume that $y \neq a, b$. But then, since $xy \in E$, H cannot induce a $3PC(\cdot, \cdot)$, a contradiction. Hence G is $3PC(\cdot, \cdot)$-free.

Now suppose that we have a $3PC(\cdot, \cdot)$-free sandwich graph $G = (V, E)$ for (V, E^1, E^2). Let G^* be a sandwich graph for (V^*, E^{1*}, E^{2*}) with vertex set $V \setminus S$ and edge set $E^* = E \cap (V^* \times V^*)$. Suppose that G^* is not a chordal graph. Let H be a hole in G^*, and let u, v be two nonadjacent vertices in H. But then there is chordless path $P = u, x_{uv}y_{uv}, v$ in G, and hence $H \cup P$ induces a $3PC(u, v)$ in G, a contradiction. Therefore G^* is a chordal graph. □

5. Final remarks

The complexities of sandwich problems corresponding to P_4-free graphs (polynomial [15]), C_4-free graphs (NP-complete) and diamond-free graphs (polynomial) present an interesting non-monotonicity with respect to the number of edges removed from a K_4 graph: three edges (polynomial), two edges (NP-complete) and one edge (polynomial).

The present paper presents complexity results that are related to the classification of the sandwich problem corresponding to perfect graphs, the only remaining open problem suggested in the seminal paper [15]. The investigation of the sandwich problem paradigm proposes interesting and challenging questions itself. For instance, as classified in the paper [15], the sandwich problem for $(C_4, 2K_2, C_5)$-free graphs (the self-complementary class of split graphs) is polynomial, whereas we have established that the sandwich problems for C_4-free graphs and for $2K_2$-free graphs (a consequence of the result for C_4-free graphs) and C_5-free graphs are all NP-complete.

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