



On a subclass of close-to-convex functions

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ABSTRACT

We consider a subclass of the class of close-to-convex functions. We show the relationship between our class and the appropriate subordination. Moreover, we give the coefficient estimates and a sufficient condition for functions to belong to the class investigated. Finally, we obtain the distortion and the growth theorems. The results presented are a generalization of the results obtained by Gao and Zhou.

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1. Introduction

Let f be a complex analytic function in $U = \{z : |z| < 1\}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

Let \mathcal{S} , \mathcal{S}^* and \mathcal{K} be the usual classes of functions which members are univalent (analytic and one-to-one), starlike and close-to-convex in U , respectively. We also denote by $S^*(\alpha)$ the class of starlike functions of order α , where $0 \leq \alpha < 1$. Suppose that functions g and F are analytic in U . The function g is said to be subordinate to F , written $g \prec F$ (or $g(z) \prec F(z)$ for $z \in U$), if there exists a function ω analytic in U , with $\omega(0) = 0$, $|\omega(z)| < 1$, and such that $g(z) = F(\omega(z))$, $z \in U$. If F is univalent in U , then $g \prec F$ if and only if $g(0) = F(0)$ and $g(U) \subset F(U)$. In this paper, we introduce a new class $K_s(\gamma)$ of analytic functions related to the starlike functions.

Definition 1. Let f be an analytic function in U defined by (1). We say that $f \in K_s(\gamma)$, $0 \leq \gamma < 1$, if there exists a function $g \in S^*(\frac{1}{2})$ such that

$$\operatorname{Re} \frac{-z^2 f'(z)}{g(z)g(-z)} > \gamma, \quad z \in U. \quad (2)$$

Moreover, we say that the function f is generated by the function g .

We see that $K_s(0) = K_s$, where K_s is the class of functions which was defined by Gao and Zhou in the paper [1]. By simple calculations we see that the inequality (2) is equivalent to

$$\left| \frac{z^2 f'(z)}{g(z)g(-z)} + 1 \right| < \left| \frac{z^2 f'(z)}{g(z)g(-z)} - 1 + 2\gamma \right|, \quad z \in U. \quad (3)$$

Now we present two examples of functions belonging to this class.

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Example 1. A member of the family $K_s(\gamma)$ is the function f_1 of the form

$$f_1(z) = (1 - \gamma) \ln \frac{1+z}{\sqrt{1+z^2}} + \gamma \arctan z, \quad z \in U. \quad (4)$$

Indeed, f_1 is analytic in U and $f_1(0) = 0$. Moreover,

$$f_1'(z) = \frac{1 - (1 - 2\gamma)z}{(1+z)(1+z^2)}, \quad z \in U.$$

Let us put

$$g_1(z) = \frac{z}{\sqrt{1+z^2}}, \quad z \in U. \quad (5)$$

Then $g_1 \in S^*(\frac{1}{2})$ and

$$\operatorname{Re} \frac{-z^2 f_1'(z)}{g(z)g(-z)} = \operatorname{Re} \frac{1 - (1 - 2\gamma)z}{1+z}, \quad z \in U.$$

Then the inequality (2) holds and $f_1 \in K_s(\gamma)$. We may say that f_1 is generated by g_1 .

Example 2. The function

$$f_2(z) = \frac{\gamma}{2} \ln \frac{1+z}{1-z} + (1 - \gamma) \frac{z}{1-z}, \quad z \in U \quad (6)$$

belongs to the class $K_s(\gamma)$, too. We see that f_2 is analytic in U , $f_2(0) = 0$ and

$$f_2'(z) = \frac{1 + (1 - 2\gamma)z}{(1-z)(1-z^2)}, \quad z \in U.$$

If we put

$$g_2(z) = \frac{z}{1-z}, \quad z \in U, \quad (7)$$

then $g_2 \in S^*(\frac{1}{2})$ and

$$\operatorname{Re} \frac{-z^2 f_2'(z)}{g(z)g(-z)} = \operatorname{Re} \frac{1 + (1 - 2\gamma)z}{1-z}, \quad z \in U.$$

Hence, we have the inequality (2). It means that $f_2 \in K_s(\gamma)$ and is generated by g_2 .

In the examples presented we assume that the logarithm of functions and power of functions are main branches (i.e. $\log 1 = 0$, $\sqrt{1} = 1$). For our further investigations we use the following lemma.

Lemma 1 ([1]). Assume that $g \in S^*(\frac{1}{2})$ and given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in U. \quad (8)$$

If we put

$$G(z) = \frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1}, \quad (9)$$

where for $n = 2, 3, \dots$

$$B_{2n-1} = 2b_{2n-1} - 2b_2 b_{2n-2} + \dots + (-1)^n 2b_{n-1} b_{n+1} + (-1)^{n+1} b_n^2, \quad (10)$$

then $G \in S^*$.

Remark 1. Since $g \in S^*(\frac{1}{2})$, then from Lemma 1 we obtain that G given by (9) belongs to \mathcal{S}^* . Then by (2) we see that our class $K_s(\gamma)$ is a subclass of the class \mathcal{K} of close-to-convex functions.

2. Main results

First of all, we show in which way our class is associated with the appropriate subordination.

Theorem 1. An analytic function f belongs to the class $K_s(\gamma)$ ($0 \leq \gamma < 1$) if and only if there exists $g \in \mathcal{S}^*(\frac{1}{2})$ such that

$$\frac{-z^2 f'(z)}{g(z)g(-z)} \prec \frac{1 + (1 - 2\gamma)z}{1 - z}, \quad z \in U. \quad (11)$$

Proof. Let $f \in K_s(\gamma)$. Then, there exists function $g \in \mathcal{S}^*(\frac{1}{2})$ such that

$$\operatorname{Re} \frac{-z^2 f'(z)}{g(z)g(-z)} > \gamma, \quad z \in U.$$

In terms of subordinations this means that

$$\frac{-z^2 f'(z)}{g(z)g(-z)} \prec \frac{1 + (1 - 2\gamma)z}{1 - z}, \quad z \in U,$$

because the function $F(z) = \frac{1 + (1 - 2\gamma)z}{1 - z}$ is univalent in U and $H(0) = F(0) = 1$, where $H(z) = \frac{-z^2 f'(z)}{g(z)g(-z)}$. Conversely, we assume that the subordination (11) holds. Then, there exists an analytic function ω in U such that $\omega(0) = 0$, $|\omega(z)| < 1$ and

$$\frac{-z^2 f'(z)}{g(z)g(-z)} = \frac{1 + (1 - 2\gamma)\omega(z)}{1 - \omega(z)}.$$

Hence, by using condition $|\omega(z)| < 1$ we obtain (3), which is equivalent to (2), so $f \in K_s(\gamma)$. \square

Now, we prove another sufficient condition for functions to belong to the class investigated, $K_s(\gamma)$.

Theorem 2. Let $g \in \mathcal{S}^*(\frac{1}{2})$ be a function given by (8) and $0 \leq \gamma < 1$. If an analytic function f in U defined by (1) satisfies the inequality

$$2 \sum_{n=2}^{\infty} n|a_n| + (|1 - 2\gamma| + 1) \sum_{n=2}^{\infty} |B_{2n-1}| \leq 2(1 - \gamma), \quad (12)$$

where for $n = 2, 3, \dots$ the coefficients B_{2n-1} are given by (10), then $f \in K_s(\gamma)$ and it is generated by g . In particular, if

$$\sum_{n=2}^{\infty} n|a_n| \leq 1 - \gamma,$$

then $f \in K_s(\gamma)$ and it is generated by $g(z) = z$.

Proof. We set for f given by (1) and g defined by (8)

$$\begin{aligned} \Lambda &= \left| z f'(z) - \frac{-g(z)g(-z)}{z} \right| - \left| z f'(z) + \frac{-(1 - 2\gamma)g(z)g(-z)}{z} \right| \\ &= \left| \sum_{n=2}^{\infty} n a_n z^n - \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \right| - \left| (2 - 2\gamma)z + \sum_{n=2}^{\infty} n a_n z^n + (1 - 2\gamma) \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \right|. \end{aligned} \quad (13)$$

Hence, for $z \in U$, using (12) we have the inequalities

$$\begin{aligned} \Lambda &\leq \sum_{n=2}^{\infty} n|a_n||z|^n + \sum_{n=2}^{\infty} |B_{2n-1}||z|^{2n-1} - \left((2 - 2\gamma)|z| - \sum_{n=2}^{\infty} n|a_n||z|^n - |1 - 2\gamma| \sum_{n=2}^{\infty} |B_{2n-1}||z|^{2n-1} \right) \\ &= -(2 - 2\gamma)|z| + \sum_{n=2}^{\infty} 2n|a_n||z|^n + (|1 - 2\gamma| + 1) \sum_{n=2}^{\infty} |B_{2n-1}||z|^{2n-1} \\ &< \left(-(2 - 2\gamma) + \sum_{n=2}^{\infty} 2n|a_n| + (|1 - 2\gamma| + 1) \sum_{n=2}^{\infty} |B_{2n-1}| \right) |z| \\ &\leq 0. \end{aligned}$$

From the above calculation we obtain that $\Lambda < 0$. Thus by (13) we have

$$\left| z f'(z) - \frac{-g(z)g(-z)}{z} \right| < \left| z f'(z) + \frac{-(1 - 2\gamma)g(z)g(-z)}{z} \right|, \quad z \in U,$$

which is equivalent to the inequality (3) and also to the inequality (2). Thus $f \in K_s(\gamma)$ and it completes the proof. \square

The next theorem gives the estimate of the coefficients.

Theorem 3. Let $0 \leq \gamma < 1$. Suppose that an analytic function f given by (1) and $g \in S^*(\frac{1}{2})$ given by (8) are such that the condition (2) holds. Then, for $n = 2, 3, \dots$ we have

$$2n^2|a_{2n}|^2 - 2(1 - \gamma)^2 \leq (1 - \gamma) \sum_{k=2}^n (2(2k - 1)|a_{2k-1}B_{2k-1}| + (|2\gamma - 1| + 1)|B_{2k-1}|^2), \quad (14)$$

where B_{2n-1} is defined by (10). In particular, if $g(z) = z$, then

$$n|a_{2n}| \leq 1 - \gamma.$$

Proof. Since $f \in \mathcal{K}_s(\gamma)$, for some $g \in S^*(\frac{1}{2})$ the inequality (3) holds. From the lemma, which was proved by Owa (see [2]) with $\alpha = \beta = 1$, we have

$$\frac{zf'(z)}{G(z)} = \frac{1 + (2\gamma - 1)z\phi(z)}{1 + z\phi(z)}, \quad z \in U,$$

where ϕ is an analytic function in U , $|\phi(z)| \leq 1$ for $z \in U$ and G is given by (9). Then

$$(zf'(z) - (2\gamma - 1)G(z))z\phi(z) = G(z) - zf'(z).$$

Now, we put

$$z\phi(z) = \sum_{n=1}^{\infty} t_n z^n.$$

We see that $|z\phi(z)| \leq |z|$ for $z \in U$. Thus

$$\left((2 - 2\gamma)z + \sum_{n=2}^{\infty} na_n z^n - (2\gamma - 1) \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \right) \sum_{n=1}^{\infty} t_n z^n = \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} - \sum_{n=2}^{\infty} na_n z^n. \quad (15)$$

We compare coefficients in (15). Hence, we can write for $n \geq 2$

$$\begin{aligned} & \left((2 - 2\gamma)z + \sum_{k=1}^{n-1} 2ka_{2k} z^{2k} + \sum_{k=2}^n ((2k - 1)a_{2k-1} - (2\gamma - 1)B_{2k-1}) z^{2k-1} \right) z\phi(z) \\ &= \sum_{k=2}^n (B_{2k-1} - (2k - 1)a_{2k-1}) z^{2k-1} - \sum_{k=1}^n 2ka_{2k} z^{2k} + \sum_{k=2n+1}^{\infty} c_k z^k. \end{aligned}$$

Then, we square the modulus of the both sides of the above equality and then we integrate along $|z| = r < 1$. After using the fact that $|z\phi(z)| \leq |z| < 1$, we obtain

$$\begin{aligned} & \sum_{k=2}^n |B_{2k-1} - (2k - 1)a_{2k-1}|^2 r^{4k-2} + \sum_{k=1}^n |2ka_{2k}|^2 r^{4k} + \sum_{k=2n+1}^{\infty} |c_k|^2 r^{2k} \\ & < |2 - 2\gamma|^2 r^2 + \sum_{k=1}^{n-1} |2ka_{2k}|^2 r^{4k} + \sum_{k=2}^n |(2k - 1)a_{2k-1} - (2\gamma - 1)B_{2k-1}|^2 r^{4k-2}. \end{aligned}$$

Letting $r \rightarrow 1$ on the left side of this inequality, we have

$$\sum_{k=2}^n |B_{2k-1} - (2k - 1)a_{2k-1}|^2 + \sum_{k=1}^n |2ka_{2k}|^2 \leq (2 - 2\gamma)^2 + \sum_{k=1}^{n-1} |2ka_{2k}|^2 + \sum_{k=2}^n |(2k - 1)a_{2k-1} - (2\gamma - 1)B_{2k-1}|^2.$$

Hence,

$$\begin{aligned} 4n^2|a_{2n}|^2 - 4(1 - \gamma)^2 & \leq \sum_{k=2}^n (|(2k - 1)a_{2k-1} - (2\gamma - 1)B_{2k-1}|^2 - |B_{2k-1} - (2k - 1)a_{2k-1}|^2) \\ & \leq 2(1 - \gamma) \sum_{k=2}^n (2(2k - 1)|a_{2k-1}B_{2k-1}| + (|2\gamma - 1| + 1)|B_{2k-1}|^2). \end{aligned}$$

Thus we have the inequality (14) which finishes the proof. \square

In the last part of this paper, we provide the growth and the distortion theorems for the class of functions considered, $\mathcal{K}_s(\gamma)$.

Theorem 4. If $f \in K_s(\gamma)$, where $0 \leq \gamma < 1$, then we have

$$\frac{1 - (1 - 2\gamma)r}{(1+r)(1+r^2)} \leq |f'(z)| \leq \frac{1 + (1 - 2\gamma)r}{(1-r)(1-r^2)}, \quad (16)$$

$$(1-\gamma) \ln \frac{1+r}{\sqrt{1+r^2}} + \gamma \arctan r \leq |f(z)| \leq \frac{\gamma}{2} \ln \frac{1+r}{1-r} + (1-\gamma) \frac{r}{1-r}, \quad (17)$$

where $|z| = r$, $0 \leq r < 1$. These results are sharp.

Proof. If $f \in K_s(\gamma)$, then there exists a function $g \in S^*(\frac{1}{2})$ such that (2) holds. It follows from Lemma 1 that the function given by (9) is an odd starlike function. Then we have

$$\frac{r}{1+r^2} \leq |G(z)| \leq \frac{r}{1-r^2}, \quad |z| = r, 0 \leq r < 1 \quad (18)$$

(see [3], p. 70). From (2) we obtain that there exists a function p having real part greater than γ such that

$$\frac{zf'(z)}{G(z)} = p(z), \quad z \in U. \quad (19)$$

It is known that

$$\frac{1 - (1 - 2\gamma)r}{1+r} \leq |p(z)| \leq \frac{1 + (1 - 2\gamma)r}{1-r}, \quad |z| = r, 0 \leq r < 1 \quad (20)$$

(see [4], p. 105). Thus, from (18), (19), (20) we get the inequalities (16). From the right-hand side of the estimate (16) we have for $z = qe^{i\theta}$

$$\begin{aligned} |f(z)| &= \left| \int_0^z f'(t) dt \right| \leq \int_0^r |f'(qe^{i\theta})| dq \leq \int_0^r \frac{1 + (1 - 2\gamma)q}{(1-q)^2(1+q)} dq \\ &= \frac{\gamma}{2} \ln \frac{1+r}{1-r} + (1-\gamma) \frac{r}{1-r}. \end{aligned}$$

This gives us the right-hand side of the inequality (17). To prove the left-hand side of the inequality (17) we must show that it holds for the nearest point $f(z_0)$ from zero, where $|z_0| = r$ and $0 < r < 1$. Moreover, we have $|f(z)| \geq |f(z_0)|$ for $|z| = r$. Since $f \in \mathcal{K}$ we know that the function f is univalent in the open unit disc U . We conclude that the original image of the line segment $[0, f(z_0)]$ is a piece of arc Γ in $\{z : |z| \leq r\}$. Using (16) we obtain

$$\begin{aligned} |f(z)| &= \int_{f(\Gamma)} |dw| = \int_{\Gamma} |f'(z)| |dz| \geq \int_0^r \frac{1 - (1 - 2\gamma)q}{(1+q)(1+q^2)} dq \\ &= (1-\gamma) \ln \frac{1+r}{\sqrt{1+r^2}} + \gamma \arctan r. \end{aligned}$$

This finishes the proof of the inequality (17). The estimates presented above are sharp. The extremal functions in this case are f_1 and f_2 given by (4) and (6), respectively. The first function is generated by g_1 and the second function is generated by g_2 ; these are given by (5) and (7), respectively. \square

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