ELSEVIER

Journal of Computational and Applied Mathematics 118 (2000) 241–259

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Multiple (multiindex) Mittag–Leffler functions and relations to generalized fractional calculus ☆

Virginia S. Kiryakova

Institute of Mathematics & Informatics, Bulgarian Academy of Sciences, Sofia 1090, Bulgaria

Received 9 March 1999

Abstract

The classical Mittag–Leffler (M–L) functions have already proved their efficiency as solutions of fractional-order differential and integral equations and thus have become important elements of the fractional calculus' theory and applications. In this paper we introduce analogues of these functions, depending on two sets of multiple (m-tuple, $m\geqslant 2$ is an integer) indices. The hint for this comes from a paper by Dzrbashjan (Izv. AN Arm. SSR 13 (3) (1960) 21–63) related to the case $m=2$. We study the basic properties and the relations of the multiindex M–L functions with the operators of the generalized fractional calculus. Corresponding generalized operators of integration and differention of the so-called Gelfond–Leontiev-type, as well as Borel–Laplace-type integral transforms, are also introduced and studied.

MSC: 26A33; 33C60; 33E30; 44A15; 44A30

Keywords: Mittag–Leffler functions; Fox's H-functions; Generalized fractional integrals and derivatives; Gelfond–Leontiev operators; Laplace-type integral transforms

1. Introduction

Till recently, the Mittag–Leffler (M–L) functions have been almost totally ignored in the common handbooks on special functions and existing tables of Laplace transforms (even in the 1991 Mathematics Subject Classification they could not be found). However, a description of their basic properties appeared yet in the third volume of the Bateman Project [9], in a chapter devoted to "miscellaneous functions". In the pioneering works in the field of fractional integral and differential equations, in 1931, Hille and Tamarkin provided a solution of the Abel integral equation of 2nd kind

E-mail address: virginia@math.bas.bg, virginia@diogenes.bg (V.S. Kiryakova)

☆ Partially supported by Research Project MM 708/97, NSF-Bulg. Ministry of Educ. & Science.

PII: S0377-0427(00)00292-2

in terms of M–L functions. Special attention and detailed studies of functions of M–L type had found place in Dzrbashian's works [8], for solving problems of theory of functions, integral transforms, etc. Among the "fractional analysts" these functions nowadays are commonly recognized when looking for the solutions of linear fractional differential equations with constant coefficients (see for example, [24, Section 42]), various kinds of integral equations and more general differential equations of fractional order and practical significance [12,18] or equations involving both Erdélyi–Kober fractional integrals and derivatives [16]. Recently, the attention towards the M–L-type functions and the recognition of their importance have increased from both analytical and numerical points of view, as well as for describing fractional-order control systems, fractional viscoelastic models, etc. Their definition and properties can now be found in many recent books and surveys on fractional calculus, integral and differential equations, mechanics, etc., [12,22], [14, Appendix], etc.

The Mittag–Leffler functions E_α (Mittag–Leffler, 1902–1905) and $E_{\alpha,\beta}$ [1], defined by the power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0, \beta > 0 \tag{1}$$

are natural extensions of the exponential function and trigonometric functions, like the cos-function

$$y_1(z) = E_1(z) = \exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)}, \quad y_2(z) = E_2(-z^2) = \cos z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{\Gamma(2k+1)},$$

satisfying the (integer order) differential equations

$$D^1 y_1(\lambda z) = \lambda y_1(\lambda z), \quad D^2 y_2(\lambda z) = -\lambda^2 y_2(\lambda z).$$

However, in the case of functions (1) with a fractional index α, one has fractional order differential equations, like for example

$$D^\alpha y(z) = \lambda y(z) \quad \text{with } y(z) = z^{\alpha-1} E_{\alpha,\alpha}(\lambda z^\alpha), \quad \alpha > 0. \tag{2}$$

The Mittag–Leffler functions (1) are examples of entire functions of given order $\rho = 1/\alpha$ and type $\sigma = 1$, in a sense, the simplest such functions.

Most of the special functions of mathematical physics are special cases of the generalized hypergeometric functions ${}_pF_q$, and thus, of the more general Meijer's G-functions [9, Vol. 1, Chapter 5]. However, the Mittag–Leffler functions serve as an example of special functions that could not be included in the scheme of the Meijer's G-functions, being a more general Fox's H-function. Namely [27, p. 42],

$$E_{\alpha,\beta}(z) = H^{1,2}_{1,2}\left[-z \,\middle|\, \begin{matrix} (0,1) \\ (0,1),(1-\beta,\alpha) \end{matrix}\right] = \frac{1}{2\pi i} \int_{\mathscr{L}} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\mu-\alpha s)} (-z)^{-s} \, ds \tag{3}$$

and only for rational $\alpha = p/q$, (3) reduces to a G-function.

Functions (1) have been studied in details by Dzrbashjan [6,8]: asymptotic formulas in different parts of complex plane, kernel-functions of inverse Borel-type integral transforms, various relations and representations. He used the denotation $E_\rho(z;\mu)$ for $E_{1/\rho,\mu}(z)$, in terms of (1).

The M–L functions have been also used as generating functions of a class of the so-called Gelfond–Leontiev (G–L) operators of generalized differentiation and integration.

Definition 1.1 (Gelfond and Leontiev [10]). Let the function

$$\varphi(\lambda)=\sum_{k=0}^{\infty}\varphi_k\lambda^k \tag{4}$$

be an entire function with a growth $(\rho>0,\sigma\neq 0)$ such that $\lim_{k\to\infty}k^{1/\rho}\sqrt[k]{|\varphi_k|}=(\sigma e\rho)^{1/\rho}$.

For a function $f(z)=\sum_{k=0}^{\infty}a_kz^k$, analytic in a disk $\Delta_R=\{|z|<R\}$, the correspondence

$$f(z)=\sum_{k=0}^{\infty}a_kz^k \overset{D_\varphi}{\mapsto} D_\varphi f(z)=\sum_{k=1}^{\infty}a_k\,\frac{\varphi_{k-1}}{\varphi_k}\,z^{k-1} \tag{5}$$

is said to be a *Gelfond–Leontiev* (*G–L*) *operator of generalized differentiation with respect to the function* $\varphi(\lambda)$ and the corresponding *G–L operator of integration* is

$$L_\varphi f(z)=\sum_{k=0}^{\infty}a_k\,\frac{\varphi_{k+1}}{\varphi_k}\,z^{k+1}. \tag{6}$$

From the theory of entire functions, it is known that the conditions required for $\varphi(\lambda)$ always hold for $\limsup_{k\to\infty}$. However, here it is supposed that there exists $\lim_{k\to\infty}\sqrt[k]{|\varphi_{k-1}/\varphi_k|}=1$ and therefore, by the Cauchy–Hadamard formula, series (5), (6) for $f(z), D_\varphi f(z), L_\varphi f(z)$ have one and the same radius of convergence $R>0$.

It is evident that if $\varphi(\lambda)=\exp\lambda$, i.e., $\varphi_k=1/\Gamma(k+1)$, $k=0,1,\ldots$, operators (5), (6) become the usual differentiation and integration: $Df(z)=(\mathrm{d}/\mathrm{d}z)f(z)=f'(z), Lf(z)=R^1f(z)=\int_0^z f(\zeta)\,\mathrm{d}\zeta$.

Let $\varphi(\lambda)=E_{1/\rho,\mu}(\lambda)$ be M–L function (1) with $\alpha=1/\rho>0, \beta=\mu>0$. Then $\varphi_k=1/\Gamma(\mu+k/\rho)$, and operators (5), (6) turn into the so-called *Dzrbashjan–Gelfond–Leontiev* (*D–G–L*) *operators of differentiation and integration*:

$$D_{\rho,\mu}f(z)=\sum_{k=1}^{\infty}a_k\frac{\Gamma(\mu+k/\rho)}{\Gamma(\mu+(k-1)/\rho)}\,z^{k-1},\quad L_{\rho,\mu}f(z)=\sum_{k=0}^{\infty}a_k\,\frac{\Gamma(\mu+k/\rho)}{\Gamma(\mu+(k+1)/\rho)}\,z^{k+1} \tag{7}$$

studied by Dimovski and Kiryakova [4,5], Kiryakova [14, Chapter 2]. Briefly, the *following results have been obtained there.*

Denote by $\mathscr{H}(\Omega)$ the space of analytic functions in a complex domain Ω, starlike with respect to the origin $z=0$ and consider the spaces

$$\mathscr{H}_\alpha(\Omega)=\{f(z)=z^p\tilde{f}(z);\,p\geqslant\alpha,\tilde{f}(z)\in\mathscr{H}(\Omega)\},\quad \mathscr{H}_0(\Omega):=\mathscr{H}(\Omega). \tag{8}$$

It happens that D–G–L operators (7) can be analytically continued in the spaces $\mathscr{H}(\Omega)\subset\mathscr{H}_{-\mu\rho}(\Omega)$, with $\Omega\supset\Delta_R$ by means of the *Erdélyi–Kober* (*E–K*) *fractional integrals and derivatives* (see [14,25]):

$$\begin{aligned}I_\beta^{\gamma,\delta}y(z)&=[z^{-(\gamma+\delta)}R^\delta z^\gamma\;y(z^{1/\beta})]|_{z\to z^\beta}\\&=\frac{z^{-\beta(\gamma+\delta)}}{\Gamma(\delta)}\int_0^z(z^\beta-t^\beta)^{\delta-1}t^{\beta\gamma}\;y(t)\,\mathrm{d}(t^\beta)=\frac{1}{\Gamma(\delta)}\int_0^1(1-\sigma)^{\delta-1}\sigma^\gamma y(z\sigma^{1/\beta})\,\mathrm{d}\sigma\end{aligned} \tag{9}$$

and

$$D_\beta^{\gamma,\delta}y(z)=[z^{-\gamma}D^\delta z^{\gamma+\delta}y(z^{1/\beta})]|_{z\to z^\beta}, \tag{10}$$

where $\delta>0,\gamma$ and $\beta>0$ are real parameters and $D_\beta^{\gamma,\delta}I_\beta^{\gamma,\delta}=Id$. Namely,

$$L_{\rho,\mu}f(z)=z^1I_\rho^{\mu-1,1/\rho}f(z)=\frac{z}{\Gamma(1/\rho)}\int_0^1(1-\sigma)^{1/\rho-1}\,\sigma^{\mu-1}f(z\sigma^{1/\rho})\,\mathrm{d}\sigma \tag{11}$$

and

$$D_{\rho,\mu}f(z) = z^{-1}D_\rho^{\mu-1-1/\rho,1/\rho} f(z) - \frac{f(0)\Gamma(\mu)}{\Gamma(\mu-1/\rho)} z^{-1}, \quad \text{since } D_\rho^{\mu-1,1/\rho} z^{-1} = z^{-1}D_\rho^{\mu-1-1/\rho,1/\rho}. \tag{12}$$

Further, transmutation operators relating R–L fractional integrals $R^{1/\rho}$ and D–G–L generalized integrations $L_{\rho,1}, L_{\rho,\mu}$ and represented by E–K operators, have been obtained in above-mentioned papers. They allow finding a convolution, and even a family of convolutions of D–G–L integration operator $L_{\rho,\mu}$, see [4,14, Chapter 2]. By means of these convolutions we give explicit representations of the commutant of $L_{\rho,\mu}$, that is, of all the linear operators commuting with it in $\mathscr{H}(\Omega)$. We have found also a Laplace-type integral transform, the so-called *Borel–Dzrbashjan transform*, corresponding to D–G–L operators and have studied its properties and possible applications.

In [14, Chapter 5] we have introduced multiple analogues of D–G–L operators (7) and studied their analytical continuations in starlike domains by means of generalized fractional differintegrals. These generalized fractional integrals and derivatives happen to be generated by functions $\varphi(\lambda)$ that can be considered as multiindex analogues of M–L functions (1). In this paper we call them "multiple multiindex Mittag–Leffler functions" and study some of their basic properties, relations to fractional calculus operators and to corresponding Laplace-type integral transforms.

In a sense, the multiple M–L functions studied here, are extensions of the already-known hyper-Bessel functions of Delerue (Example 2.7) that generate the hyper-Bessel differential and integral operators [2,14, Chapter 3].

2. Definitions and basic properties of the multiple M–L functions

Definition 2.1. Let $m > 1$ be an integer, $\rho_1, \ldots, \rho_m > 0$ and $\mu_1, \ldots, \mu_m$ be arbitrary real numbers. By means of "multiindices" $(\rho_i), (\mu_i)$ we introduce the so-called *multiindex* (*m-tuple*, *multiple*) *Mittag–Leffler functions*

$$E_{(1/\rho_i),(\mu_i)}(z) = \sum_{k=0}^{\infty} \varphi_k z^k = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + k/\rho_1) \ldots \Gamma(\mu_m + k/\rho_m)}. \tag{13}$$

Theorem 2.2. *For arbitrary sets of indices* $\rho_i > 0, -\infty < \mu_i < +\infty, i = 1, \ldots, m$, *the multiindex Mittag–Leffler function* (13) *is an entire function of order*

$$\rho = \left(\sum_{i=1}^{m} \frac{1}{\rho_i} \right)^{-1},$$

i.e.,

$$\frac{1}{\rho} = \frac{1}{\rho_1} + \cdots + \frac{1}{\rho_m} \tag{14}$$

and type

$$\sigma = \left(\frac{\rho_1}{\rho} \right)^{\rho/\rho_1} \cdots \left(\frac{\rho_m}{\rho} \right)^{\rho/\rho_m}. \tag{15}$$

Moreover, for every positive ϵ, *the asymptotic estimate*

$$|E_{(1/\rho_i),(\mu_i)}(z)| < \exp((\sigma+\epsilon)|z|^{\rho}), \quad |z| \geqslant r_0 > 0 \tag{16}$$

holds with ρ, σ *as in* (14), (15) *for* $|z| \geqslant r_0(\epsilon)$, $r_0(\epsilon)$ *sufficiently large.*

Proof. The proof follows some ideas from Dzrbashjan [7] and Gorenflo, Kilbas and Rogozin [11]. The radius of convergence of series (13), by the Cauchy–Hadamard formula, is $R > 0$, where

$$\frac{1}{R} = \limsup_{k\to\infty} \sqrt[k]{|\varphi_k|} = \limsup_{k\to\infty} \left[\prod_{i=1}^{m} \Gamma\left(\mu_i + \frac{k}{\rho_i}\right)\right]^{-1/k}.$$

By the Stirling's asymptotic formula for the Γ-function $\Gamma(r) \sim \sqrt{2\pi}\, r^{r-1/2}\, \mathrm{e}^{-r}$ for large $r > 0$ (see, e.g., [9, Section 1.18]), we have

$$\Gamma\left(\mu_i + \frac{k}{\rho_i}\right) \sim \sqrt{2\pi}\left(\mu_i + \frac{k}{\rho_i}\right)^{\mu_i + k/\rho_i - 1/2} \mathrm{e}^{-\mu_i}\,\mathrm{e}^{-k/\rho_i} \quad \text{for} \quad k \to \infty$$

and

$$\frac{1}{R} = \limsup_{k\to\infty} \left[\prod_{i=1}^{m} \Gamma\left(\mu_i + \frac{k}{\rho_i}\right)\right]^{-1/k} = \lim_{k\to\infty} \prod_{i=1}^{m} \left[\left(\frac{k}{\rho_i}\right)^{-1/\rho_i} \mathrm{e}^{1/\rho_i}\right]$$
$$= \lim_{k\to\infty} k^{-(1/\rho_1+\cdots+1/\rho_m)}\, \rho_1^{1/\rho_1} \ldots \rho_m^{1/\rho_m}\, \mathrm{e}^{1/\rho_1+\cdots+1/\rho_m}, \tag{17}$$

thus $1/R = 0$, or $R = \infty$.

According to a known formula for the order of an entire function $\sum_{k=0}^{\infty} \varphi_k z^k$,

$$\rho = \limsup_{k\to\infty} \frac{k \ln k}{\ln 1/|\varphi_k|} = \limsup_{k\to\infty} \frac{k \ln k}{\ln[\prod_{i=1}^{m} \Gamma(\mu_i + k/\rho_i)]}.$$

The Stirling's formula in the form

$$\ln \Gamma(r) = \left(r - \frac{1}{2}\right)\ln r - r + \frac{1}{2}\ln(2\pi) + \mathrm{O}\left(\frac{1}{r}\right), \quad r \to \infty$$

now gives

$$\ln\left[\prod_{i=1}^{m} \Gamma\left(\mu_i + \frac{k}{\rho_i}\right)\right] = \sum_{i=1}^{m} \left[\left(\mu_i + \frac{k}{\rho_i} - \frac{1}{2}\right)\ln\left(\mu_i + \frac{k}{\rho_i}\right) - \left(\mu_i + \frac{k}{\rho_i}\right) + \frac{1}{2}\ln(2\pi) + \mathrm{O}\left(\frac{1}{k}\right)\right]$$
$$= \sum_{i=1}^{m} \left(\mu_i + \frac{k}{\rho_i} - \frac{1}{2}\right)\ln\left(\mu_i + \frac{k}{\rho_i}\right)$$
$$- \sum_{i=1}^{m} \mu_i - k\sum_{i=1}^{m} \frac{1}{\rho_i} + \frac{m}{2}\ln(2\pi) + \mathrm{O}\left(\frac{1}{k}\right),$$

then finally,

$$\frac{\ln[\prod_{i=1}^{m} \Gamma(\mu_i + k/\rho_i)]}{k \ln k} = \sum_{i=1}^{m} \frac{1}{\rho_i}\left\{1 + \mathrm{O}\left(\frac{1}{\ln k}\right)\right\}$$

and $1/\rho = \sum_{i=1}^{m} 1/\rho_i$, as in (14).

Further, the type σ of the function $\sum_{k=0}^{\infty} \varphi_k z^k$ of order ρ is determined from the equality

$$(\sigma e\rho)^{1/\rho} = \limsup_{k\to\infty} [k^{1/\rho} \sqrt[k]{|\varphi_k|}].$$

From (17),

$$\limsup_{k\to\infty} [k^{1/\rho} \sqrt[k]{|\varphi_k|}] = \limsup_{k\to\infty} \left[k^{1/\rho} \prod_{i=1}^{m} \Gamma\left(\mu_i + \frac{k}{\rho_i}\right)^{-1/k} \right]$$
$$= \lim_{k\to\infty} [k^{1/\rho} k^{-(1/\rho_1+\cdots+1/\rho_m)} \rho_1^{1/\rho_1} \dots \rho_m^{1/\rho_m} \mathrm{e}^{1/\rho_1+\cdots+1/\rho_m}] = \rho_1^{1/\rho_1} \dots \rho_m^{1/\rho_m} \mathrm{e}^{1/\rho},$$

where $1/\rho = 1/\rho_1 + \cdots + 1/\rho_m$, and

$$\sigma^{1/\rho} = \rho^{-1/\rho}(\rho_1^{1/\rho_1} \dots \rho_m^{1/\rho_m}) = \rho^{-(1/\rho_1+\cdots+1/\rho_m)}(\rho_1^{1/\rho_1} \dots \rho_m^{1/\rho_m})$$

from where (15) holds. The asymptotic estimate (16) follows from the definitions of order and type of an entire functions. □

Next, let us emphasize the place that the multiple M–L functions occupy among the known special functions, especially in the scheme of generalized hypergeometric functions known as Meijer's G-functions (see [9, Vol. 1, 14, Appendix]) and Fox's H-functions (see, e.g., [26,27,23,14, Appendix]).

Definition 2.3. By a *Fox's H-function* we mean a generalized hypergeometric function, defined by means of the Mellin–Barnes-type contour integral

$$H^{m,n}_{p,q}\left[\sigma \left| \begin{matrix} (a_k, A_k)_1^p \\ (b_k, B_k)_1^q \end{matrix} \right.\right] = \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{L}'} \frac{\prod_{k=1}^{m} \Gamma(b_k - B_k s) \prod_{j=1}^{n} \Gamma(1 - a_j + sA_j)}{\prod_{k=m+1}^{q} \Gamma(1 - b_k + sB_k) \prod_{j=n+1}^{p} \Gamma(a_j - sA_j)} \sigma^s \,\mathrm{d}s, \tag{18}$$

where $\mathscr{L}'$ is a suitable contour in $\mathbb{C}$, the orders (m, n, p, q) are integers $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$ and the parameters $a_j \in \mathbb{R}, A_j > 0, j = 1, \dots, p$, $b_k \in \mathbb{R}, B_k > 0$, $k = 1, \dots, q$ are such that $A_j(b_k + l) \neq B_k(a_j - l' - 1)$, $l, l' = 0, 1, 2, \dots$. For various type of contours and conditions for existence and analyticity of function (18) in disks $\subset \mathbb{C}$ whose radii are $\rho = \prod_{j=1}^{p} A_j^{-A_j} \prod_{k=1}^{q} B_k^{B_k} > 0$, one can see [23,26,14, Appendix], etc.

For $A_1 = \cdots = A_p = 1$, $B_1 = \cdots = B_q = 1$ (18) turns into the more popular *Meijer's G-function* (see [9, Vol. 1, Chapter 5,23,14]). The G- and H-functions encompass almost all the elementary and special functions and this makes the knowledge on them very useful. Observe that the generalized hypergeometric functions ${}_pF_q$ are special cases of the G-function:

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; \sigma) = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} G^{1,p}_{p,q+1}\left[-\sigma \left| \begin{matrix} 1 - a_1, \dots, 1 - a_p \\ 0, 1 - b_1, \dots, 1 - b_q \end{matrix} \right.\right], \tag{19}$$

while the M–L functions (1) with irrational parameters $\rho > 0$ and the Wright's generalized hypergeometric functions ${}_p\Psi_q$ with irrational A_j, $B_k > 0$, give examples of *H-functions*, *not reducible to*

G-functions: see representation (3) and

$$ {}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{array}\middle|\,\sigma\right]=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+kA_1)\ldots\Gamma(a_p+kA_p)}{\Gamma(b_1+kB_1)\ldots\Gamma(b_q+kB_q)}\frac{\sigma^k}{k!} $$

$$ =H^{1,p}_{p,q+1}\left[-\sigma\,\middle|\,\begin{array}{c}(1-a_1,A_1),\ldots,(1-a_p,A_p)\\(0,1),(1-b_1,B_1),\ldots,(1-b_q,B_q)\end{array}\right]. \tag{20} $$

However, for $A_1=\cdots=A_p=B_1=\cdots=B_q=1$, cf. (19):

$$ {}_p\Psi_q\left[\begin{array}{c}(a_1,1),\ldots,(a_p,1)\\(b_1,1),\ldots,(b_q,1)\end{array}\middle|\,\sigma\right]=\left[\frac{\prod_{i=1}^{q}\Gamma(b_i)}{\prod_{j=1}^{p}\Gamma(a_j)}\right]\ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;\sigma). \tag{21} $$

Lemma 2.4. *The multiple M–L functions* (13) *are Wright's generalized hypergeometric functions as well as Fox's H-functions of the form*

$$ E_{(1/\rho_i),(\mu_i)}(z)={}_1\Psi_m\left[\begin{array}{c}(1,1)\\(\mu_i,\frac{1}{\rho_i})_1^m\end{array}\middle|\,z\right]=H^{1,1}_{1,m+1}\left[-z\,\middle|\,\begin{array}{c}(0,1)\\(0,1),(1-\mu_i,\frac{1}{\rho_i})_1^m\end{array}\right]. \tag{22} $$

They have the following Mellin–Barnes-type contour integral representation, extending integral formula (3):

$$ E_{(1/\rho_i),(\mu_i)}(z)=\frac{1}{2\pi\mathrm{i}}\int_{\mathscr{L}'}\frac{\Gamma(-s')\Gamma(1+s')}{\prod_{i=1}^{m}\Gamma(\mu_i+s'/\rho_i)}(-z)^{s'}\,\mathrm{d}s' $$

$$ =\frac{1}{2\pi\mathrm{i}}\int_{\mathscr{L}}\frac{\Gamma(s)\Gamma(1-s)}{\prod_{i=1}^{m}\Gamma(\mu_i-s/\rho_i)}(-z)^{-s}\,\mathrm{d}s,\quad z\neq 0, \tag{23} $$

where $\mathscr{L}$ is any contour in $\mathbb{C}$ running from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$ in a way that the poles $s=0,-1,-2,\ldots$ of $\Gamma(s)$ lie to the left of $\mathscr{L}$ and the poles $s=1,2,\ldots$ of $\Gamma(1-s)$ to the right of it (*cf.* [26, *p.* 11]).

Proof. Comparing definitions (13) and (20), we obtain (22). On the other hand, the second equality in (22) and definition (18) of the H-functions yield the integral representation (23), for which the details on the contour $\mathscr{L}$ can be seen, e.g., in [26, p. 11]. □

Representation (22) of the multiple M–L functions as Fox's H-functions allow to describe their asymptotic behaviour as $z\to 0$, $z\to\infty$. In the case of M–L function (1) ($m=1$), Dzrbashjan [6,8] established different asymptotic formulas for $|z|\to\infty$, valid in different parts of the complex plain and under different conditions on ρ,μ. For example, if $\rho>\frac{1}{2}$ and inside angle domains, (1) is $\approx\rho z^{\rho(1-\mu)}\exp(z^{\rho})$. An asymptotic estimate in the case $m>1$ is given by (16) and in more detailed situations, the asymptotics of the multiindex M–L functions could be found from their interpretation as ${}_1\Psi_m$-functions with $1<m$ (the so-called "Bessel"-type generalized hypergeometric functions, see [15]).

Let us mention some interesting *special cases of the multiple* (*multiindex*) *Mittag–Leffler functions*.

Example 2.5. A special function, generalizing classical M–L functions (1), (3) with respect to the number of indices, was considered first by Dzrbashjan [7] in the case $m=2$, see also [14, Appendix]. He denoted it by $\Phi_{\rho_1,\rho_2}(\lambda;\mu_1,\mu_2)$:

$$E_{(1/\rho_1,1/\rho_2),(\mu_1,\mu_2)}(z)=\Phi_{\rho_1,\rho_2}(z;\mu_1,\mu_2)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu_1+k/\rho_1)\Gamma(\mu_2+k/\rho_2)} \tag{24}$$

and showed that it is an entire function of order $\rho=\rho_1\rho_2/(\rho_1+\rho_2)$ and type $\sigma=(\rho_1/\rho)^{\rho/\rho_1}(\rho_2/\rho)^{\rho/\rho_2}$ with the following particular cases:

$$E_{1/\rho,\mu}(z)=E_{(1/\rho,0),(\mu,1)}(z)=\Phi_{\rho,\infty}(z;\mu,1),\qquad \frac{1}{1-z}=E_{(0,0),(1,1)}(z)=\Phi_{\infty,\infty}(z;1,1),$$

$$J_\nu(z)=\left(\frac{z}{2}\right)^{\nu}E_{(1,1),(\nu+1,1)}\left(-\frac{z^2}{4}\right)=\left(\frac{z}{2}\right)^{\nu}\Phi_{1,1}\left(-\frac{z^2}{4};1,\nu+1\right).$$

In [7] Dzrbashjan found also the Borel transform (in [5,14] we called it "Borel–Dzrbashjan transform") of functions (24), some integral relations between them (representing fractional integrals with respect to separate indices) and Mellin transforms on a set of axes. The latter results allowed him to develop a theory of integral transforms in the class L_2, involving kernels related to functions (24) and further, to approximate entire functions in L_2 for an arbitrary finite system of axes starting from the origin.

We like to add to the particular cases of (24): the so-called *Bessel–Maitland* or *Wright's* functions (misnamed after the second name Maitland of E.M. Wright):

$$J_\nu^r(z)={}_0\Psi_1\left[\left.\begin{matrix}-\\(\nu+1,r)\end{matrix}\right|-z\right]=H^{1,0}_{0,2}\left[z\left|\begin{matrix}-\\(0,1),(-\nu,r)\end{matrix}\right.\right]$$
$$=\sum_{k=0}^{\infty}\frac{(-z)^k}{\Gamma(\nu+rk+1)\,k!}=E_{(r,1),(\nu+1,1)}(z) \tag{25}$$

(used by Kalla and Galue (see [15, (4.2)]) as kernel functions of the so-called Wright–Erdélyi–Kober operators of fractional integration and by other authors as kernels of generalized Hankel transforms), as well as the *Struve* and *Lommel* functions (see, e.g., [9, Vol.2, 14, (C.8)]):

$$s_{\mu,\nu}(z)=\frac{1}{4}z^{\mu+1}\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)G^{1,1}_{1,3}\left[\frac{z^2}{4}\left|\begin{matrix}0\\0,(\nu-\mu-1)/2,(-\nu-\mu-1)/2\end{matrix}\right.\right]$$
$$=\frac{1}{4}z^{\mu+1}E_{(1,1),((3-\nu+\mu)/2,(3+\nu+\mu)/2)}\left(-\frac{z^2}{4}\right),$$

$$H_\nu(z)=[\pi 2^{\nu-1}(1/2)_\nu]^{-1}\,s_{\nu,\nu}(z)=\frac{1}{4}z^{\nu+1}E_{(1,1),(3/2,(3+2\nu)/2)}\left(-\frac{z^2}{4}\right).$$

Example 2.6. For arbitrary $m>2$: let $\forall\rho_i=\infty$, i.e., $1/\rho_i=0$ and $\forall\mu_i=1,\ i=1,\dots,m$. From definition (13) it is easily seen that

$$E_{(0,0,\dots,0),(1,1,\dots,1)}(z)=\sum_{k=0}^{\infty}z^k=\frac{1}{1-z}.$$

Example 2.7. Consider the case $m \geqslant 2$ with $\forall \rho_i = 1,\ i = 1, \ldots, m$. Then

$$E_{(1,1,\ldots,1),(\mu_i+1)}(z) = {}_1\Psi_m \left[\begin{array}{c} (1,1) \\ (\mu_i, 1)_1^m \end{array} \middle| z \right] = \left[\prod_{i=1}^{m} \Gamma(\mu_i) \right]^{-1} {}_1F_m(1; \mu_1, \mu_2, \ldots, \mu_m; z)$$

reduces to a ${}_1F_m$- and a *Meijer's* $G^{1,1}_{1,m+1}$-function.

Denote $\mu_i = \gamma_i + 1,\ i = 1, \ldots, m$ and let additionally *one of* μ_i *to be* 1, for example: $\mu_m = 1$, i.e., $\gamma_m = 0$. Then the multiple (multiindex) M–L function becomes a *hyper-Bessel function* (*in a sense of Delerue*, 1953; see [14, Chapter 3; Appendix (D.3)]):

$$\begin{aligned} &\left(\frac{z}{m}\right)^{\sum_{i=1}^{m-1} \gamma_i} E_{(1,1,\ldots,1),(\gamma_1+1,\gamma_2+1,\ldots,\gamma_{m-1}+1,1)}\left(-\left(\frac{z}{m}\right)^m\right) \\ &\quad = \left[\prod_{i=1}^{m-1} \Gamma(\gamma_i + 1)\right]^{-1} \left(\frac{z}{m}\right)^{\sum_{i=1}^{m-1} \gamma_i} {}_0F_{m-1}\left(\gamma_1 + 1, \gamma_2 + 1, \ldots, \gamma_{m-1} + 1; -\left(\frac{z}{m}\right)^m\right) \\ &\quad = J^{(m-1)}_{\gamma_i,\ldots,\gamma_{m-1}}(z). \end{aligned} \tag{26}$$

In general, for *rational values of* $\forall \rho_i,\ i = 1, \ldots, m$, functions (13) are reducible to Meijer's G-functions.

In next section, we show that in analogy with classical M–L functions (1), their multiple analogues (13) satisfy some differential and integral relations and equations of fractional order.

3. Generalized fractional calculus and multiindex Mittag–Leffler functions

The *generalized fractional calculus* [14] is related to the Erdélyi–Kober operators of fractional integration and differentiation (9), (10) and their commutative compositions. The effective use and simple properties of the generalized hypergeometric functions, especially – of a Meijer's G-function and Fox's H-function, as kernel-functions allow to consider and deal with such compositions represented in more suitable way, as single integrals or differ-integrals of special functions.

First, we recall briefly some definitions.

Definition 3.1 (Kiryakova [13,14]). Let $m \geqslant 1$ be an integer; $\beta_i > 0,\ \gamma_i \in \mathbb{R},\ \delta_i > 0,\ i = 1, \ldots, m$. Consider $\gamma = (\gamma_1, \ldots, \gamma_m)$ as a multiweight and resp. $\delta = (\delta_1, \ldots, \delta_m)$ as a *multiorder of fractional integration*. The integral operators defined as follows:

$$I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) = \begin{cases} \displaystyle\int_0^1 H^{m,0}_{m,m}\left[\sigma \middle| \begin{array}{c} (\gamma_i + \delta_i + 1 - \frac{1}{\beta_i}, \frac{1}{\beta_i})_1^m \\ (\gamma_i + 1 - \frac{1}{\beta_i}, \frac{1}{\beta_i})_1^m \end{array}\right] f(z\sigma)\, \mathrm{d}\sigma & \text{if } \sum_{i=1}^m \delta_i > 0, \\ f(x) & \text{if } \delta_1 = \delta_2 = \cdots = \delta_m = 0 \end{cases} \tag{27}$$

are said to be *multiple* (*m-tuple*) *Erdélyi–Kober fractional integration operators* and more generally, all the operators of the form

$$If(z) = z^{\delta_0} I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) \quad \text{with } \delta_0 \geqslant 0 \tag{28}$$

are called briefly a *generalized* (*m-tuple*) *fractional integrals*.

For $\gamma_i > -1 - \alpha/\beta_i,\ \delta_i > 0,\ i = 1,\dots,m$ operators (27) map a space $\mathscr{H}_\alpha(\Omega)$ into itself, preserving the power functions up to a constant multiplier. For $m = 1$ they turn into classical Erdélyi–Kober integrals (9). The main feature of the generalized (m-tuple) fractional integrals is that single integrals (27) involving H-functions (or G-functions in the simpler case of equal $\beta_i = \beta > 0,\ i = 1,\dots,m$) can be equivalently represented by means of *commutative compositions of different E–K integrals* (9), namely for $f \in \mathscr{H}_\alpha(\Omega),\ \gamma_i > -1 - \alpha/\beta_i,\ \delta_i > 0,\ i = 1,\dots,m,$

$$\begin{aligned} I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) &= \left[\prod_{i=1}^{m} I^{\gamma_i,\delta_i}_{\beta_i}\right] f(z) \\ &= \int_0^1 \cdots \int_0^1 \left[\prod_{i=1}^{m} \frac{(1-\sigma_i)^{\delta_i - 1}\sigma_i^{\gamma_i}}{\Gamma(\delta_i)}\right] f(z\sigma_1^{1/\beta_1}\dots\sigma_m^{1/\beta_m})\,\mathrm{d}\sigma_1\dots\mathrm{d}\sigma_m, \end{aligned} \tag{29}$$

while if some of the δ_i are zeros: $\delta_1 = \cdots = \delta_s = 0,\ 1 \leqslant s \leqslant m$, the corresponding multipliers $I^{\gamma_i,\delta_i}_{\beta_i} = I$ are identity operators and the multiplicity of (27) reduces from m to $m - s$ (same for the order of the kernel H-functions). Decomposition (29) is the key to numereous applications of (27), (28), arising from the simple but quite effective tools of the G- and H-functions.

The *generalized fractional derivatives* $D^{(\gamma_i),(\delta_i)}_{(\beta_i),m} = \prod_{i=1}^{m} D^{\gamma_i,\delta_i}_{\beta_i}$, corresponding to (27), are defined by means of explicit differintegral expressions. To this end we denote

$$D_\eta = \left[\prod_{i=1}^{m}\prod_{j=1}^{\eta_i}\left(\frac{1}{\beta_j} x \frac{\mathrm{d}}{\mathrm{d}x} + \gamma_i + j\right)\right], \quad \eta_i = \begin{cases} [\delta_i] + 1, & \text{if } \delta_i \text{ noninteger}, \\ & \text{if } \delta_i \text{ integer}, \end{cases} \quad i = 1,\dots,m \tag{30}$$

and define

$$D^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) := D_\eta\, I^{(\gamma_i+\delta_i),(\eta_i-\delta_i)}_{(\beta_i),m} f(z). \tag{31}$$

The theory of generalized fractional integrals and derivatives (27), (31) has been developed in the monograph [14], together with various applications.

By analogy with the M–L functions, their multiindex analogues (13) can be proved to satisfy several relations involving operators of fractional calculus (E–K operators (9), (10)) or operators of generalized fractional calculus (operators (27), (31)).

First, we consider integral relations involving classical E–K fractional integrals (9).

Lemma 3.2. *For any complex $\lambda \neq 0$ and fixed $l, 1 \leqslant l \leqslant m,\ \alpha_l > 0$*

$$\begin{aligned} &I^{\mu_l - 1,\alpha_l}_{\rho_l} E_{(1/\rho_i),(\mu_1,\dots,\mu_l,\dots,\mu_m)}(\lambda z) \\ &= \frac{1}{\Gamma(\alpha_l)} \int_0^1 (1-\sigma)^{\alpha_l - 1}\sigma^{\mu_l - 1} E_{(1/\rho_i),(\mu_i)}(\lambda z \sigma^{1/\rho_l})\,\mathrm{d}\sigma = E_{(1/\rho_i),(\mu_1,\dots,\mu_l+\alpha_l,\dots,\mu_m)}(\lambda z), \end{aligned} \tag{32}$$

i.e., a fractional integration can transform a multiindex M–L function to another one with a corresponding μ_l-parameter increased by the order of integration.

Proof. Term-by-term integration of the series for entire function $E_{(1/\rho_i),(\mu_i)}(\lambda z)$ gives

$$\text{LHS of (32)} = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\sigma)^{\alpha_l - 1}\sigma^{\mu_l - 1} \left\{ \sum_{k=0}^{\infty} \frac{\lambda^k z^k \sigma^{k/\rho_l}}{\Gamma(\mu_l + k/\rho_l)\prod_{i\neq l}\Gamma(\mu_i + k/\rho_i)} \right\} \mathrm{d}\sigma$$
$$= \sum_{k=0}^{\infty} \frac{\lambda^k z^k}{\Gamma(\mu_l + k/\rho_l)\prod_{i\neq l}\Gamma(\mu_i + k/\rho_i)} \int_0^1 \frac{(1-\sigma)^{\alpha_l-1}}{\Gamma(\alpha_l)} \sigma^{\mu_l + k/\rho_l - 1}\,\mathrm{d}\sigma$$
$$= \sum_{k=0}^{\infty} \frac{\lambda^k z^k}{\Gamma(\mu_l + k/\rho_l)\prod_{i\neq l}\Gamma(\mu_i + k/\rho_i)} \frac{\Gamma(\alpha_l)\Gamma(\mu_l + k/\rho_l)}{\Gamma(\alpha_l)\Gamma(\mu_l + \alpha_l + k/\rho_l)} = \text{RHS of (32)}.$$

This is an extension of the known result ([8, p.120], see also [14, E.28]), $z \to z^{1/\rho}$,

$$I_{\rho}^{\mu-1,\alpha} E_{1/\rho,\mu}(\lambda z^{1\rho}) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\sigma)^{\alpha-1}\sigma^{\mu-1} E_{1/\rho}(\lambda z^{1/\rho}\sigma^{1/\rho})\,\mathrm{d}\sigma = E_{(1/\rho),\mu+\alpha}(\lambda z^{1/\rho}). \qquad \square$$

Applying successively the E–K fractional integrals $I_{\rho_i}^{\mu_i-1,\alpha_i}$, $\alpha_i > 0$, $i = 1, \ldots, m$ to a multiindex M–L function, on the base of (29) and (32), we obtain a relation involving a generalized fractional integral (27):

$$I_{(\rho_i),m}^{(\mu_i-1),(\alpha_i)} E_{(1/\rho_i),(\mu_i)}(\lambda z) = \int_0^1 H_{m,m}^{m,0}\left[\sigma \left| \begin{matrix} (\mu_i + \alpha_i - \frac{1}{\rho_i}, \frac{1}{\rho_i}) \\ (\mu_i - \frac{1}{\rho_i}, \frac{1}{\rho_i}) \end{matrix} \right.\right] E_{(1/\rho_i),(\mu_i)}(\lambda z\sigma)\,\mathrm{d}\sigma$$
$$= \int_0^1 \cdots \int_0^1 \left[\prod_{i=1}^{m} \frac{(1-\sigma_i)^{\alpha_i-1}\sigma_i^{\mu_i-1}}{\Gamma(\alpha_i)}\right] E_{(1/\rho_i),(\mu_i)}(\lambda z \sigma_1^{1/\rho_1} \ldots \sigma_m^{1/\rho_m})\,\mathrm{d}\sigma_1 \ldots \mathrm{d}\sigma_m$$
$$= E_{(1/\rho_i),(\mu_i+\alpha_i)}(\lambda z).$$

Taking in the above $\alpha_i = 1/\rho_i$, $i = 1, \ldots, m$ and multiplying by z, we obtain the following generalized fractional integral relation:

$$zI_{(\rho_i),m}^{(\mu_i-1),(1/\rho_i)} E_{(1/\rho_i),(\mu_i)}(\lambda z) = zE_{(1/\rho_i),(\mu_i+1/\rho_i)}(\lambda z)$$
$$= \frac{1}{\lambda}\sum_{k=0}^{\infty} \frac{\lambda^{k+1}z^{k+1}}{\prod_i \Gamma(\mu_i + (k+1)/\rho_i)} = \frac{1}{\lambda}\sum_{k=1}^{\infty} \frac{\lambda^k z^k}{\prod_i \Gamma(\mu_i + k/\rho_i)}$$
$$= \frac{1}{\lambda} E_{(1/\rho_i),(\mu_i)}(\lambda z) - \frac{1}{\lambda \prod_i \Gamma(\mu_i)}. \tag{33}$$

Then, by the rules of the generalized fractional calculus [14]

$$zI_{(\rho_i),m}^{(\mu_i-1),(1/\rho_i)} = I_{(\rho_i),m}^{(\mu_i-1-1/\rho_i),(1/\rho_i)} z,$$

$$z^{-1} D_{(\rho_i),m}^{(\mu_i-1-1/\rho_i),(1/\rho_i)} I_{(\rho_i),m}^{(\mu_i-1-1/\rho_i),(1/\rho_i)} z f(z) = f(z),$$

$$D_{(\beta_i),m}^{(\gamma_i),(\delta_i)}\{c\} = c\prod_{i=1}^{m} \frac{\Gamma(\gamma_i + \delta_i + 1)}{\Gamma(\gamma_i + 1)}$$

and an application of the operator $\lambda z^{-1} D_{(\rho_i),m}^{(\mu_i-1-1/\rho_i),(1/\rho_i)}$ to (33), it follows that

$$\lambda E_{(1/\rho_i),(\mu_i)}(\lambda z) = z^{-1} D_{(\rho_i),m}^{(\mu_i-1-1/\rho_i),(1/\rho_i)} E_{(1/\rho_i),(\mu_i)}(\lambda z) - z^{-1} D_{(\rho_i),m}^{(\mu_i-1-1/\rho_i),(1/\rho_i)} \left\{ \frac{1}{\prod_i \Gamma(\mu_i)} \right\}$$

from where the following *generalized fractional derivative relation* for the multiple M–L functions follows:

$$D_{(\rho_i),m}^{(\mu_i-1-1/\rho_i),(1/\rho_i)} E_{(1/\rho_i),(\mu_i)}(\lambda z) = \lambda z\, E_{(1/\rho_i),(\mu_i)}(\lambda z) + \left[\prod_{i=1}^{m} \Gamma\left(\mu_i - \frac{1}{\rho_i}\right)\right]^{-1}. \tag{34}$$

The latter can be considered as an extension of the known R–L fractional derivative relation for M–L function (1), $\alpha := 1/\rho$, $\mu = 1$:

$$D^{\alpha} E_{\alpha}(\lambda z) = \lambda E_{\alpha}(\lambda z) + \frac{z^{-\alpha}}{\Gamma(1-\alpha)}.$$

4. Multiple Dzrbashjan–Gelfond–Leontiev operators generated by Mittag–Leffler functions

Relations (33), (34) suggest introducing the following Gelfond–Leontiev operators of generalized integration and differentiation (see Definition 1.1) with respect to functions (13). Paying honour to Dzrbashjan's studies on the classical M–L functions, Gelfond–Leontiev operators generated by them and on the 2-set indices M–L functions (24), we associate these operators also with his name.

Definition 4.1 (Kiryakova [14, Chapter 5]). Let $f(z)$ be an analytic function in a disk $\Delta_R = \{|z| < R\}$ and $\rho_i > 0$, $\mu_i \in \mathbb{R}$, $i = 1, \ldots, m$ be arbitrary parameters. The operators in $\mathscr{H}(\Delta_R)$

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \mapsto \begin{array}{l} D_{(\rho_i),(\mu_i)} f(z) = \displaystyle\sum_{k=1}^{\infty} a_k \frac{\Gamma(\mu_1 + k/\rho_1)\ldots\Gamma(\mu_m + k/\rho_m)}{\Gamma(\mu_1 + (k-1)/\rho_1)\ldots\Gamma(\mu_m + (k-1)/\rho_m)} z^{k-1}, \\ L_{(\rho_i),(\mu_i)} f(z) = \displaystyle\sum_{k=0}^{\infty} a_k \frac{\Gamma(\mu_1 + k/\rho_1)\ldots\Gamma(\mu_m + k/\rho_m)}{\Gamma(\mu_1 + (k+1)/\rho_1)\ldots\Gamma(\mu_m + (k+1)/\rho_m)} z^{k+1} \end{array} \tag{35}$$

are called *multiple Dzrbashjan–Gelfond–Leontiev* (*D–G–L*) *differentiations and integrations*, respectively.

It is seen, from Definition 1.1, that these operators are generated by the multiple Mittag–Leffler functions (13), namely: $\psi_k = [\prod_{i=1}^{m} \Gamma(\mu_i + k/\rho_i)]^{-1}$, and then operators (35) are Gelfond–Leontiev differentiation and integration operators with respect to functions (13).

Let us note that $D_{(\rho_i),(\mu_i)} L_{(\rho_i),(\mu_i)} f(z) = f(z)$ in $\mathscr{H}(\Delta_R)$ and the coincidence of the radii of convergence of $f(z)$ and series (35) follows easily by the Cauchy–Hadamard formula and asymptotic estimation of the Γ-function multipliers, like in [13,14, Theorem 5.5.2]. For $m = 1$ these operators turn into D–G–L operators (7).

The series representations of (35) can be analytically continued for analytic functions in starlike domains $\Omega \supset \Delta_R$ by means of single integral or differintegral expressions, as special cases of the operators (27), (31) of the generalized fractional calculus.

Theorem 4.2. *Let* $\Delta_R \subset \Omega$, $\mu_i \geqslant 0$, $i = 1, \ldots, m$; $\alpha = \max_{1 \leqslant k \leqslant m}\{-\mu_k \rho_k\} \leqslant 0$. *Then, the multiple D–G–L integration operator* (35) *can be analytically continued from* $\mathscr{H}(\Delta_R)$ *into* $\mathscr{H}_{\alpha}(\Omega)$ *by means* *of the single integral operator*

$$L_{(\rho_i),(\mu_i)}f(z) = z\int_0^1 H_{m,m}^{m,0}\left[\sigma \left| \begin{matrix} (\mu_i, \frac{1}{\rho_i})_1^m \\ (\mu_i - \frac{1}{\rho_i}, \frac{1}{\rho_i})_1^m \end{matrix} \right.\right] f(z\sigma)\,\mathrm{d}\sigma = zI_{(\rho_i),m}^{(\mu_i-1),(1/\rho_i)}f(z), \tag{36}$$

that is, by a generalized fractional integral of form (27). *The multiple D–G–L derivative has also a differintegral representation in terms of* (31), *and especially for analytic functions in* $\mathscr{H}(\Omega) \supset \mathscr{H}(\Delta_R)$:

$$D_{(\rho_i),(\mu_i)}f(z) = z^{-1}D_{(\rho_i),m}^{(\mu_i-1-1/\rho_i),(1/\rho_i)}f(z) - \left[\prod_{i=1}^m \frac{\Gamma(\mu_i)}{\Gamma(\mu_i - 1/\rho_i)}\right]\frac{f(0)}{z}. \tag{37}$$

Comparing generalized fractional integrals and derivatives (36), (37) with these involved in relations (33), (34), one easily obtains *integral and differential equations satisfied by* (13), *written in terms of multiple D–G–L operators.*

Lemma 4.3. *The multiple Mittag–Leffler functions* (13) *satisfy the following relations* ($\lambda \neq 0$):

$$L_{(\rho_i),(\mu_i)}E_{(1/\rho_i),(\mu_i)}(\lambda z) = \frac{1}{\lambda}E_{(1/\rho_i),(\mu_i)}(\lambda z) - \frac{1}{\lambda\prod_i \Gamma(\mu_i)}, \tag{38}$$

$$D_{(\rho_i),(\mu_i)}E_{(1/\rho_i),(\mu_i)}(\lambda z) = \lambda E_{(1/\rho_i),(\mu_i)}(\lambda z). \tag{39}$$

Using the tools of the generalized fractional calculus [14] and of the special functions, and extending the results of Dimovski [2] and Luchko and Jakubovich [17], one can find a *family of convolutions* of the multiple D–G–L integrals (35), in the sense of Dimovski [3].

Theorem 4.4. *For* $\lambda \geqslant \max_i(\mu_i\rho_i - 1) \geqslant -1$ *the operations* ($\overset{\lambda}{\star}$), *defined by means of generalized fractional integrals* (27)

$$(f \overset{\lambda}{\star} g)(z) = z^{\lambda+1}I_{(\rho_i),m}^{(2\mu_i-1),((\lambda+1)/\rho_i-\mu_i)}(f \circ g)(z) \tag{40}$$

and of auxiliary operation

$$(f \circ g)(x) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^m [t_i(1-t_i)]^{\mu_i-1} f\left[z\prod_{i=1}^m t_i^{1/\rho_i}\right] g\left[z\prod_{i=1}^m (1-t_i)^{1/\rho_i}\right] \mathrm{d}t_1 \ldots \mathrm{d}t_m \tag{41}$$

are convolutions of the multiple D–G–L integrations (35), (36) *in* $\mathscr{H}_\alpha(\Omega)$ *as well as in the subspace* $\mathscr{H}(\Omega)$.

For a *proof*, see [14, Chapter 5]. This theorem allows to find convolutional representations of the operator $L = L_{(\rho_i),(\mu_i)}$ itself, as well as of all the linear operators M that commute with L in $\mathscr{H}(\Omega)$: $ML = LM$ (the so-called *commutant* of L); as well as to develop other elements of the operational calculus for the multiple D–G–L operators.

Using the simple property of the generalized fractional integrals (27), [14, Chapter 5],

$$I_{(\beta_i),m}^{(\gamma_i),(\delta_i)}\{z^p\} = c_p z^p, \quad c_p = \prod_{i=1}^m \frac{\Gamma(\gamma_i + 1 + p/\beta_i)}{\Gamma(\gamma_i + \delta_i + 1 + p/\beta_i)}, \quad p > -\beta_i(\gamma_i + 1), \tag{42}$$

one can easily find the convolution products of some "basic functions". For simplicity, if $\forall \mu_i \geqslant 0$, $i = 1, \ldots, m$, one can take in (40) $\lambda = -1$ and consider the particular convolution

$$(f \overset{-1}{\star} g)(z) = I^{(2\mu_i - 1),(-\mu_i)}_{(\rho_i),m}(f \circ g)(z) \tag{43}$$

that has the property to "sum up" the exponents of the power functions.

Example 4.5. For $p, q > \alpha$ ($z^p, z^q \in H_\alpha(\Omega)$),

$$z^p \overset{-1}{\star} z^q = z^{p+q} \prod_{i=1}^{m} \frac{\Gamma(\mu_i + p/\rho_i)\Gamma(\mu_i + q/\rho_i)}{\Gamma(\mu_i + (p+q)/\rho_i)}. \tag{44}$$

Example 4.6. For $\alpha, \beta \neq 0$, the convolutional product of two multiindex Mittag–Leffler functions (13) (being entire functions) is

$$E_{(1/\rho_i),(\mu_i)}(\alpha z) \overset{-1}{\star} E_{(1/\rho_i),(\mu_i)}(\beta z) = \frac{\alpha E_{(1/\rho_i),(\mu_i)}(\alpha z) - \beta E_{(1/\rho_i),(\mu_i)}(\beta z)}{(\alpha - \beta)\prod_{i=1}^{m}\Gamma(\mu_i)}. \tag{45}$$

Proof. The proof of (45) follows from relations (38) taken with $\lambda = \alpha, \lambda = \beta$, each of them multiplied convolutionally by $E_{(1/\rho_i),(\mu_i)}(\beta z)$ or $E_{(1/\rho_i),(\mu_i)}(\alpha z)$ resp., and using convolutional property [3]

$$L_{(\rho_i),(\mu_i)} E_{(1/\rho_i),(\mu_i)}(\alpha z) \overset{-1}{\star} E_{(1/\rho_i),(\mu_i)}(\beta z) = E_{(1/\rho_i),(\mu_i)}(\alpha z) \overset{-1}{\star} L_{(\rho_i),(\mu_i)} E_{(1/\rho_i),(\mu_i)}(\beta z)$$

in the same lines as in the proof of Lemma 2.2.9 in [14], for the case $m = 1$. However, (45) can be also directly proved now by using fractional integral relations (32), (33). □

5. Laplace-type integral transform

In [5,14, Chapter 2] we have shown that the role of a Laplace transformation for the D–G–L operators $D_{\rho,\mu}, L_{\rho,\mu}$ ($m = 1$) can be played by the *Borel–Dzrbashjan transform*:

$$\mathscr{B}_{\rho,\mu}\left\{\sum_{k=0}^{\infty} a_k z^k\right\} = \sum_{k=0}^{\infty} \frac{a_k \Gamma(\mu + k/\rho)}{s^{k+1}}; \quad \mathscr{B}_{\rho,\mu}\{f(z); s\} = \rho s^{\mu\rho - 1} \int_0^{\infty} \exp(-s^\rho z^\rho) z^{\mu\rho - 1} f(z)\,\mathrm{d}z \tag{46}$$

usually considered in ρ-convex domains like

$$\Omega = \mathscr{D}_\rho(0; v) = \left\{ s\colon \Re\,(s^\rho) > v, |\arg s| < \frac{\pi}{2\rho} \right\}.$$

There, for $\frac{1}{2} < \mu < 1/\rho$ and $\mathscr{L} := \mathscr{L}_\rho(0; v) = \partial D_\rho(0; v)$ the following *complex inversion formula* holds for (46) (see [8]):

$$f(z) = \frac{1}{2\pi i} \int_{\mathscr{L}} E_{1/\rho,\mu}(sz) \mathscr{B}_{\rho,\mu}\{f; s\}\,\mathrm{d}s, \quad v > v_0. \tag{47}$$

We have proved that (46) has the same convolution as the D–G–L operators, taken with $\lambda = -1$ (operation (43) for $m = 1$), namely,

$$\mathscr{B}_{(\rho),(\mu)}\{f \overset{-1}{\star} g; s\} = \mathscr{B}_{(\rho),(\mu)}\{f(z); s\} \mathscr{B}_{(\rho),(\mu)}\{g(z); s\} \tag{48}$$

and have found some basic operational properties as well as images and convolutions of some particular functions.

Here we introduce a Laplace-type integral transform, corresponding to the multiple D–G–L operators.

Definition 5.1. The Laplace-type integral transform, defined as an H-transform

$$\mathcal{B}(s) = \mathcal{B}_{(\rho_i),(\mu_i)}\{f(z); s\}$$
$$= \int_0^\infty H^{m,0}_{0,m}\left[sz \left| \begin{matrix} - \\ (\mu_i - \frac{1}{\rho_i}, \frac{1}{\rho_i}) \end{matrix} \right.\right] f(z)\,\mathrm{d}z = \frac{1}{s}\int_0^\infty H^{m,0}_{0,m}\left[sz \left| \begin{matrix} - \\ (\mu_i, \frac{1}{\rho_i}) \end{matrix} \right.\right] \frac{f(z)}{z}\,\mathrm{d}z \tag{49}$$

is said to be a *multiple Borel–Dzrbashjan* (*B–D*) *transform*, corresponding to multiple D–G–L operators (35)–(37).

We consider (49) for functions

$$f(z) \in \mathcal{H}_\alpha, \quad \alpha = \max_i(-\mu_i\rho_i);\ f(z) = \mathrm{O}\{\exp(\lambda r^{1/\lambda} z^{1/\lambda})\} \text{ as } |z| \to \infty,$$
$$\text{and for } s \in \left\{\Re\,(s^{1/\lambda}) > v_0;\, |\arg s| < \frac{\pi\lambda}{2}\right\} \tag{50}$$

with some $v_0 \in \mathbb{R}$ and $r := [\prod_{i=1}^m \rho_i^{1/\rho_i}]$, $\lambda := \sum_{i=1}^m 1/\rho_i > 0$.

The restriction for the exponential growth of the originals $f(z)$ as $|z| \to \infty$, arises from the known asymptotic behaviour of the kernel $H^{m,0}_{0,m}$-function, namely, it vanishes exponentially as $|z| \to \infty$, see [20, (1.6.3), 14, (E.17′)].

Example 5.2. If we put $m = 1$ in (49), we obtain the "single" B–D transform (46). Moreover, if $\mu = \rho = 1$, then the *Laplace integral transform* follows as a special case, which justifies the name *Laplace-type integral transform* given to (49).

From definition (49) one can easily evaluate the images of some basic original functions $f(z)$. The aparatus of the H-functions is used.

Example 5.3. Since

$$\mathcal{B}_{(\rho_i),(\mu_i)}\{z^q\} = s^{-(q+1)}\left[\prod_{i=1}^m \Gamma\left(\mu_i + \frac{q}{i}\right)\right], \quad q > \alpha = \max_{1\leqslant i\leqslant m}(-\mu_i\rho_i), \tag{51}$$

then for $\mu_i > 0$, $\rho_i > 0$, $i = 1, \ldots, m$, we obtain the *image of an analytic function*

$$f(z) = \sum_{k=0}^m a_k z^k \in \mathcal{H}(\Delta_R) \overset{\mathcal{B}_{(\rho_i),(\mu_i)}}{\longmapsto} \mathcal{B}(s) = \sum_{k=0}^m \frac{a_k \prod_{i=1}^m \Gamma(\mu_i + k/\rho_i)}{s^{k+1}}. \tag{52}$$

Example 5.4. It is interesting to note that the *image of multiindex M–L function* (13), (22) under (49) is a simple fraction, well-known in the operational calculus:

$$\mathcal{B}_{(\rho_i),(\mu_i)}\{E_{(1/\rho_i),(\mu_i)}(z); s\} = \frac{1}{s-1}. \tag{53}$$

The multiple B–D transform, as defined by (49), (52), plays the same role for the multiple D–G–L derivatives and integrals like the Laplace transform $\mathscr{L}\{f(z);s\}$ for the classical operators of differentiation and integration. First, we have for function (13) the usefull relations (39), (53), analogous to the well-known ones for the exponential function

$$\frac{\mathrm{d}}{\mathrm{d}z}\exp(\lambda z)=\lambda\exp(\lambda z),\quad \mathscr{L}\{\exp z;s\}=\frac{1}{s-1}.$$

Next, it is easy to prove that (49) algebrizes the multiple D–G–L derivatives and integrals, i.e. reduces them to multiplications by fixed rational functions.

Theorem 5.5. *If* $f(z)$ *and* $s\in\mathbb{C}$ *satisfy conditions* (50), *then the multiple D–G–L integration operator* (35), (36) *is algebrized by the multiple B–D transform* (49):

$$\mathscr{B}_{(\rho_i),(\mu_i)}\{L_{(\rho_i),(\mu_i)}f(z);s\}=\frac{1}{s}\mathscr{B}_{(\rho_i),(\mu_i)}\{f(z);s\}\tag{54}$$

and the "*generalized differential law*" *holds*:

$$\mathscr{B}_{(\rho_i),(\mu_i)}\{D_{(\rho_i),(\mu_i)}f(z);s\}=s\mathscr{B}_{(\rho_i),(\mu_i)}\{f(z);s\}-f(0)\left[\prod_{i=1}^{m}\Gamma(\mu_i)\right].\tag{55}$$

The latter relation generalizes the well-known differential law for the Laplace transform. The *proof* follows in an elementary way by replacing (49), (36) into LHS of (54) and the widely used techniques of *evaluation of integrals of products of two H-functions* (see [14, (E.21′), 20, (2.6.8)]). The additional term from (37), depending on $f(0)$, appears now in a modified form in (55).

For the purposes of the operational calculus, to deal with the multiple B–D transform, one should dispose with some inversion formulas. To compare with similar Laplace-type transforms like B–D transform (46) or the more general, so-called *Obrechkoff transform* [2,14, Chapter 3], we can look for various complex inversion formulas, either of the kind of (47), or by means of Mellin transform techniques, shown to be more effective. The following complex inversion formula seems most useful.

Theorem 5.6. *If* $f(z)$, *satisfying* (50), *has a B–D image* (49)

$$\mathscr{B}(s):=\mathscr{B}_{(\rho_i),(\mu_i)}\{f(z);s\},$$

then the inversion formula

$$f(z)=\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{z^{-q}}{\prod_{i=1}^{m}\Gamma(\mu_i-q/\rho_i)}\,\mathrm{d}q\left\{\int_0^\infty s^{-q}\mathscr{B}(s)\,\mathrm{d}s\right\}\tag{56}$$

holds, *provided that the integrals*

$$\int_0^\infty z^{c-1}f(z)\,\mathrm{d}z,\qquad \int_0^\infty s^{-q}\mathscr{B}(s)\,\mathrm{d}s$$

are absolutely convergent for $q=c+\mathrm{i}T$, $-\infty<T<\infty$ *and* c *is a suitably chosen constant*, $c<-\alpha$.

The *proof* is quite similar to that of Theorem 3.9.12, [14] in the case of Obrechkoff transform. Its effectiveness can be easily tested on Example 5.4.

Another complex inversion formula of contour integral type (47), with a kernel-function

$$H^{1,1}_{1,m+1}\left[-sz\,\middle|\,\begin{matrix}(0,1)\\(0,1),(1-\mu_i,\frac{1}{\rho_i})\end{matrix}\right]=E_{(1/\rho_i),(\mu_i)}(sz)$$

can be looked for, too.

Operation (43), being a convolution of D–G–L integration operator $L_{(\rho_i),(\mu_i)}$, is also a convolution of the multiple B–D transform (49), similarly to (48).

The details of all the proofs in this section, as well as another representation of the kernel-function of (49), relationship between (49) and the m-dimensional Laplace transform, etc. we leave for another paper.

6. Relation to fractional hyper-Bessel differential and integral operators and Obrechkoff transform

The wide scope of applications of the multiple D–G–L differintegrals can be seen from the fact that the D–G–L derivatives are not only *multiple Erdélyi–Kober fractional derivatives* $z^{-1}D^{(\mu_i-1-1/\rho_i),(1/\rho_i)}_{(\rho_i),m}$ but also an interesting generalization of the *hyper-Bessel differential operators*, introduced by Dimovski [2] and among the most often encountered operators in problems of mathematical physics:

$$B=z^{\alpha_0}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)z^{\alpha_1}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)z^{\alpha_2}\cdots\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)z^{\alpha_m}. \tag{57}$$

The linear right inverse operators of (57), called *hyper-Bessel integral operators*, have been shown in [14, Chapter 3] to be generalized fractional integrals of form (27) but with all equal $\beta_i=\beta=m-(\alpha_0+\cdots+\alpha_m)>0$, thus their kernel-functions reduced to Meijer's $G^{m,0}_{m,m}$-functions. Denoting $\gamma_i=(\alpha_k+\alpha_{k+1}+\cdots+\alpha_m-m+k)/\beta$, $i=1,\dots,m$, the hyper-Bessel integral operator L ($BLf(z)=f(z)$, $f\in H_\alpha(\Omega)$), can be represented as

$$Lf(z)=(z^\beta/\beta^m)I^{(\gamma_i),(1,1,\dots,1)}_{(\beta,\beta,\dots,\beta),m}f(z). \tag{58}$$

It is seen that (58) follows as a special case of multiple D–G–L integrals (36), by taking for example $\beta=1$ in (58) and $\mu_i=\gamma_i+1$, $\rho_i=1$, $i=1,\dots,m$ in (36); as in Example 2.7, where the hyper-Bessel functions (26) follow from the multiindex M–L functions. Functions (26) are closely related to hyper-Bessel operators (57), (58) and the solutions of hyper-Bessel differential equations $By(x)=\lambda y(x)$ are represented in their terms. Thus, the multiple D–G–L operators of differentiation $D_{(\rho_i),(\mu_i)}$ are natural extensions, a kind of "fractional" analogues (of multi-order of differentiation $(1/\rho_1,\dots,1/\rho_m)$ instead of $(1,\dots,1)$) of the hyper-Bessel differentiations (57).

The Laplace-type integral transform (49), corresponding to (57), (58), i.e., to the case $\rho_1=\cdots=\rho_m=1$, reduces to a G-function transform (see, e.g., [19]). This is the so-called *Obrechkoff integral transform*

$$\mathcal{O}\{f(z);s\}=\beta s^{-\beta(\gamma_m+1)+1}\int_0^\infty G^{m,0}_{0,m}\left[(sz)^\beta\,\middle|\,\begin{matrix}-\\(\gamma_i+1-\frac{1}{\beta})_1^m\end{matrix}\right]f(z)\,\mathrm{d}z, \tag{59}$$

where for simplicity one can put $\beta=1$.

This transformation was introduced by Obrechkoff [21] in 1958 and studied by Dimovski [2] in the modified form

$$\mathcal{O}\{f(z);s\} = \beta \int_0^\infty z^{\beta(\gamma_m+1)-1} K[(sz)^\beta] f(z)\,\mathrm{d}z, \tag{60}$$

where

$$K(z) = \int_0^\infty \cdots \int_0^\infty \exp\left(-u_1 - \cdots - u_{m-1} - \frac{z}{u_1 \ldots u_{m-1}}\right) \prod_{i=1}^m u_i^{\gamma_i - \gamma_i - 1}\,\mathrm{d}u_1 \ldots \mathrm{d}u_{m-1}.$$

It was shown to be a suitable Laplace-type integral transform for building an operational calculus for hyper-Bessel operators (57), (58). Convolutions, differential properties and inversion formulas were found. Later, its new representation (59) by means of the Meijer's G-function, allowed a more detailed study in [14, Chapter 3].

References

[1] R.P. Agarwal, A propos d'une note de M. Pierre Humbert, C.R. Acad. Sci. Paris 236 (1953) 2031–2032.

[2] I. Dimovski, Operational calculus for a differential operator, C.R. Acad. Bulg. Sci. 27 (1) (1966) 513–516.

[3] I. Dimovski, Convolutional Calculus, East European Series, Vol. 43, Kluwer Acad. Publ., Dordrecht, 1990.

[4] I. Dimovski, V. Kiryakova, Convolution and commutant of Gelfond–Leontiev operator of integration, Proceedings of the Constructive Function Theory, Varna'1981, Publ. House BAS, Sofia, 1983, pp. 288–294.

[5] I. Dimovski, V. Kiryakova, Convolution and differential property of the Borel–Dzrbashjan transform, in: Proceedings of the Complex Analysis and Applications, Varna'1981, Publ. House BAS, Sofia, 1984, pp. 148–156.

[6] M.M. Dzrbashjan, On the integral representation and uniqueness of some classes of entire functions (in Russian), Dokl. AN SSSR 85 (1) (1952) 29–32.

[7] M.M. Dzrbashjan, On the integral transformations generated by the generalized Mittag–Leffler function (in Russian), Izv. AN Arm. SSR 13 (3) (1960) 21–63.

[8] M.M. Dzrbashjan, Integral Transforms and Representations of Functions in the Complex Domain (in Russian) Nauka, Moscow, 1966.

[9] A. Erdélyi et al., (Eds.), Higher Transcendental Functions, McGraw-Hill, New York, 1953.

[10] A.O. Gelfond, A.F. Leontiev, On a generalization of the Fourier series (in Russian), Mat. Sbornik 29 (71) (1951) 477–500.

[11] R. Gorenflo, A.A. Kilbas, S. Rogozin, On the generalized Mittag–Leffler type functions, Integral Transforms Special Functions 7 (3–4) (1998) 215–224.

[12] R. Gorenflo, F. Mainardi, Fractional calculus: integral and differential equations of fractional order, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer, Wien and New York, 1997, pp. 223–276.

[13] V. Kiryakova, Generalized $H_{m,m}^{m,0}$-function fractional integration operators in some classes of analytic functions, Bull. Math. (Beograd) 40 (3–4) (1988) 259–266.

[14] V. Kiryakova, Generalized Fractional Calculus and Applications, Research Notes in Math. Series, Vol. 301, Pitman Longman, Harlow & Wiley, New York, 1994.

[15] V. Kiryakova, All the special functions are fractional differintegrals of elementary functions, J. Phys. A 30 (1997) 5083–5103.

[16] V. Kiryakova, B.N. Al-Saqabi, Transmutation method for solving Erdélyi–Kober fractional differintegral equations, J. Math. Anal. Appl. 211 (1997) 347–364.

[17] Yu. Luchko, S. Yakubovich, Operational calculi for the generalized fractional differential operator and applications, Math. Balkanica 4 (2) (1990) 119–130.

[18] F. Mainardi, M. Tomirotti, On a special function arising in the time fractional diffusion wave equation, in: Transform Methods & Special Functions'94, Proceedings of the Internationl Workshop, Sofia 1994, SCTP, Singapore, 1995, pp. 171–183.

[19] O.I. Marichev, V.K. Tuan, The factorization of G-transform in two spaces of functions, in: Proceedings of the Complex Analysis and Applications'85, Publ. House BAS, Sofia, 1986, pp. 418–433.
[20] A.M. Mathai, R.K. Saxena, The H-function with Applications in Statistics and Other Disciplines, Wiley Eastern Ltd., New Delhi, 1978.
[21] N. Obrechkoff, On some integral representations of real functions on the real half-line (in Bulg.), Izv. Mat. Inst. (Sofia) 1 (2) (1958) 3–33.
[22] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.
[23] A.A. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series. More Special Functions, Gordon & Breach Sci. Publ., New York, etc., 1990.
[24] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon & Breach Sci. Publ., New York, etc., 1993.
[25] I.N. Sneddon, The use in mathematical analysis of Erdélyi–Kober operators and some of their application, in: Fractional Calculus and Applications, Lecture Notes in Mathematics, Vol. 457, Springer, New York, 1975, pp. 37–79.
[26] H.M. Srivastava, K.C. Gupta, S.P. Goyal, The H-Functions of One and Two Variables with Applications, South Asian Publ., New Delhi, 1982.
[27] H.M. Srivastava, B.R.K. Kashyap, Special Functions in Queuing Theory And Related Stochastic Processes, Academic Press, New York, 1982.