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ORIGINAL ARTICLE

Steady Poiseuille flow and heat transfer of couple stress fluids between two parallel inclined plates with variable viscosity

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Brinkman number;
Perturbation technique;
Heat transfer

Abstract The purpose of this paper is to study the non-isothermal Poiseuille flow between two heated parallel inclined plates using incompressible couple stress fluids. Reynold's model is used for temperature dependent viscosity. We have developed highly non-linear coupled ordinary differential equations from momentum and energy equations. The Perturbation technique is used to obtain the approximate analytical expressions for velocity and temperature distributions. Expressions for velocity field, temperature distribution, dynamic pressure, volume flow rate, average velocity and shear stress on the plates are obtained. The influence of various emerging parameters on the flow problem is discussed and presented graphically.

1. Introduction

In recent years, scientists have shown their interest in non-Newtonian fluids because of their applications in many natural, industrial and technological problems. Several authors cited a wide range of applications of non-Newtonian fluids that cover the flow of polymer solutions, food stuffs, drilling oil and gas wells, synthetic fibers and the extrusion of molten plastics. Tan and Xu (2002), Tan and Masuoka (2005a,b), Farooq et al. (2011, 2012), Shah et al. (2011), Chen et al. (2004) and Fetecau and Fetecau (2002, 2003, 2005) have discussed some of the interesting fluid flow problems involving non-Newtonian fluids.

In order to explain the behavior of non-Newtonian fluids, different constitutive equations have been suggested. Among these, the couple stress fluid model introduced by Stokes (1966) has distinct characteristics, such as the presence of couple stresses, non-symmetric stress tensor and body couples. The couple stress fluid theory presented by Stokes suggests models for those fluids whose microstructure is mechanically momentous. The effect of microstructure on a liquid can be felt, if the characteristic geometric dimension of the problem considered is of the same order of magnitude as the size of the microstructure (Srinivasacharya and Kaladhar, 2011). To introduce a size dependent effect is one of the main features of couple stresses. The subject of classical continuum mechanics ignores the effect of size of material particles within the continua. This is unswerving with neglecting the rotational interaction among the particles of the fluid, which results in a symmetry of the force–stress tensor. However, this

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Nomenclature

$\mathbf{V}$	velocity vector	B_r	Brinkman number
U	reference velocity	M	viscosity index
H	height of plate	m	constant number
$\mathbf{f}$	body force	n	viscosity parameter
$\mathbf{T}$	Cauchy stress tensor		
$\mathbf{I}$	unit tensor	*Greek symbols*	
$\mathbf{A}_1$	first Rivlin-Ericksen tensor	ϵ	small parameter
$\mathbf{L}$	gradient of $\mathbf{V}$	η	couple stress parameter
c_p	specific heat	Θ	dimensional temperature
p	non-dimensional dynamic pressure	Θ^*	non-dimensional temperature
p^*	dimensional dynamic pressure	Θ_0	lower plate temperature
u	dimensional velocity in the x-direction	Θ_1	upper plate temperature
u^*	non-dimensional velocity in the x-direction	κ	thermal conductivity
x	dimensional x-coordinate	μ	dimensional coefficient of viscosity
x^*	non-dimensional x-coordinate	μ^*	non-dimensional coefficient of viscosity
y	dimensional y-coordinate	μ_0	reference viscosity
y^*	non-dimensional y-coordinate	ρ	constant density of the fluid
B	non-dimensional parameter	$\boldsymbol{\tau}$	extra stress tensor

cannot be true and a size dependent couple-stress theory is needed in some important cases for instance fluid flow with suspended particles. The spin field due to microrotation of these freely suspended particles set up an antisymetric stress, which is known as couple-stress, and thus forming couple-stress fluid. The couple stress fluids are proficient of describing different types of lubricants, suspension fluids, blood etc. These fluids have applications in various processes that take place in the industry such as solidification of liquid crystals, extrusion of polymer fluids, colloidal solutions and cooling of metallic plate in a bath etc. Stokes has also written a review of couple stress fluid dynamics (Stokes, 1984) which contains an extensive study about these fluids. Basic ideas and techniques for both steady and unsteady flow problems of Newtonian and non-Newtonian fluids are given by Ellahi (Ellahi, 2009). The basic equations governing the flow of couple stress fluids are non-linear in nature and even of higher order than the Navier Stokes equations. Thus an exact solution of these equations is not easy to find. Different perturbation techniques are commonly used for obtaining approximate solutions of these equations.

Heat transfer flow has importance in different engineering applications such as the design of thrust bearings and radial diffusers' transpiration cooling, drag reduction and thermal recovery of oil. Heat transfer plays an important role in processing and handling of non-Newtonian mixtures (Tsai et al., 1988). The mechanics of nonlinear fluid flows is a challenge to mathematicians, engineers and scientists since the nonlinearity can manifest itself in different ways as is the case in the analysis of reactive variable viscosity flows in a slit with wall injection or suction. In our case, one of the reasons of the nonlinearity of the coupled ordinary differential equations is the temperature dependent viscosity. Flows with temperature dependent viscosity are studied by various researchers. (Yurusoy and Pakdemirli, 2002; Makinde, 2006, 2009, 2010).

In this paper, we study the heat transfer flow of incompressible couple stress fluids with temperature dependent viscosity between two parallel inclined plates kept at different temperatures. The basic governing equations for couple stress fluids are given in Section 2. In Section 3, the Poiseuille flow is formulated and perturbation solutions are obtained for velocity field and temperature distribution. In Section 4, we compute volume flux, average velocity and shear stress on the plates. Section 5 is devoted to results and discussion and conclusion is provided in Section 6.

2. Basic equations

The basic equations governing the flow of an incompressible couple stress fluid are (Siddiqui et al., 2006, 2008; Islam and Zhou, 2007, 2009; El-Dabe and El-Mohandis, 1995; El-Dabe et al., 2003)

$$div\mathbf{V} = 0, \tag{1}$$

$$\rho\frac{D\mathbf{V}}{Dt} = div\boldsymbol{T} - \eta\nabla^4\mathbf{V} + \rho\mathbf{f}, \tag{2}$$

$$\rho c_p\frac{D\boldsymbol{\Theta}}{Dt} = \kappa\nabla^2\Theta + \boldsymbol{T}\cdot\mathbf{L}, \tag{3}$$

where $\mathbf{V}$ is the velocity vector, ρ is the constant density, f is the body force per unit mass, $\boldsymbol{T}$ is the Cauchy stress tensor, Θ is the temperature, κ is the thermal conductivity, c_p is the specific heat, $\mathbf{L}$ is the gradient of $\mathbf{V}$, η is the couple stress parameter and the operator $\frac{D}{Dt}$ denotes the material derivative which is defined as:

$$\frac{D}{Dt}(*) = \left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)(*).$$

The Cauchy stress tensor $\boldsymbol{T}$ can be defined as:

$$\boldsymbol{T} = -p\ \mathbf{I} + \boldsymbol{\tau}, \quad \boldsymbol{\tau} = \mu\mathbf{A}_1, \tag{4}$$

where p is the dynamic pressure, $\mathbf{I}$ is the unit tensor, μ is the coefficient of viscosity and $\mathbf{A}_1$ is the first Rivlin-Ericksen tensor defined as:

$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T$, $\mathbf{L}^T$ is the transpose of $\mathbf{L}$.

3. Formulation and solution of plane Poiseuille flow

Consider the steady flow of couple fluid between two infinite parallel inclined plates which are placed at $y = -H$ (lower

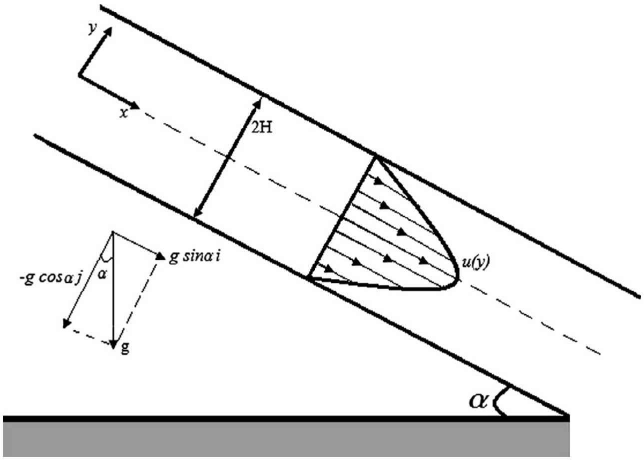

Figure 1 Geometry of the problem.

plate) and $y = H$ (upper plate). Plates are at rest and motion of the fluid is maintained due to both constant pressure gradient and gravity. The temperatures of lower and upper plates are kept at Θ_0 and Θ_1 respectively. The chosen coordinate system is shown in Fig. 1. The angle made by the plates and the horizontal direction is α. Viscosity of the fluid μ, is assumed to be a function of Θ. Velocity and temperature fields are of the form:

$$\mathbf{V} = \mathbf{V}(u, 0, 0), \quad u = u(y), \quad \text{and} \quad \Theta = \Theta(y). \tag{5}$$

Using these assumptions, we observe that the continuity Eq. (1) is identically satisfied and the momentum Eq. (2) reduces to

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}(\tau_{xy}) - \eta\frac{d^4u}{dy^4} + \rho g \sin\alpha, \tag{6}$$

$$0 = -\frac{\partial p}{\partial y} - \rho g \cos\alpha, \tag{7}$$

$$0 = -\frac{\partial p}{\partial z}. \tag{8}$$

The pressure p, is obtained from Eq. (7) as

$$p = \rho g[x \sin\alpha - y\cos\alpha] + C, \tag{9}$$

where C is a constant of integration and can be calculated by applying the appropriate boundary condition. Eq. (8) implies that $p \neq p(z)$. Using Eq. (5) in Eq. (4), the non-zero components of the extra stress tensor $\boldsymbol{\tau}$ are

$$\tau_{xy} = \mu\frac{du}{dy} = \tau_{yx}. \tag{10}$$

Substituting Eq. (10) into Eq. (6) we obtain

$$\eta\frac{d^4u}{dy^4} - \mu\frac{d^2u}{dy^2} - \frac{d\mu}{dy}\frac{du}{dy} + \frac{\partial p}{\partial x} - \rho g \sin\alpha = 0. \tag{11}$$

Eqs. (4), (5) and (10) transforms the energy Eq. (3) to the form

$$\frac{d^2\Theta}{dy^2} + \frac{\mu}{\kappa}\left(\frac{du}{dy}\right)^2 + \frac{\eta}{\kappa}\left(\frac{d^2u}{dy^2}\right)^2 = 0. \tag{12}$$

The associated boundary conditions are

$$u(-H) = u(H) = 0, \tag{13}$$

$$u''(-H) = u''(H) = 0, \tag{14}$$

$$\Theta(-H) = \Theta_0, \quad \Theta(H) = \Theta_1. \tag{15}$$

Eq. (13) is the usual no-slip boundary conditions. Eq. (14) implies that couple stresses are zero at the plates. We introduce the following non-dimensional parameters:

$$u^* = \frac{u}{U}, \; y^* = \frac{y}{H}, \; x^* = \frac{x}{H}, \; \Theta^* = \frac{\Theta - \Theta_0}{\Theta_1 - \Theta_0}, \; \mu^* = \frac{\mu}{\mu_0}, \; p^* = \frac{p}{\mu_0 U/H},$$
$$B_r = \frac{\mu_0 U^2}{\kappa(\Theta_1 - \Theta_0)}, \; B^2 = \frac{\mu_0 H^2}{\eta}, \; G = -\frac{B^2H^5}{\mu_0 U}\frac{\partial p}{\partial x} + \frac{\rho g H^4}{\eta U}\sin\alpha,$$

where U is the reference velocity, μ_0 is the reference viscosity and B_r is the Brinkman number. Using these dimensionless parameters, Eqs. (11) and (12) take the form (dropping asterisks)

$$\frac{d^4u}{dy^4} - B^2\mu\frac{d^2u}{dy^2} - B^2\frac{d\mu}{dy}\frac{du}{dy} - G = 0, \tag{16}$$

$$\frac{d^2\Theta}{dy^2} + B_r\mu\left(\frac{du}{dy}\right)^2 + \frac{B_r}{B^2}\left(\frac{d^2u}{dy^2}\right)^2 = 0, \tag{17}$$

and the corresponding boundary conditions (13)–(15) become

$$u(-1) = 0, \quad u(1) = 0, \tag{18}$$

$$u''(-1) = 0, \quad u''(1) = 0, \tag{19}$$

$$\Theta(-1) = 0, \quad \Theta(1) = 1. \tag{20}$$

Assume that the temperature dependent fluid viscosity μ is given by Reynold's model (Aksoy and Pakdemirli, 2010; Farooq et al., 2011, 2012; Massoudi and Christie, 1995; Chinyoka and Makinde, 2011; Reynolds, 1886; Szeri and Rajagopal, 1985). The dimensionless form of this model is

$$\mu = \exp(-M\Theta), \tag{21}$$

where $M = n(\Theta_1 - \Theta_0)$. Let $M = \epsilon\, m$, where ϵ is a small perturbation parameter. Using the Taylor series expansion, Eq. (21) reduces to

$$\mu \cong 1 - \epsilon\, m\Theta, \quad \frac{d\mu}{dy} \cong -\epsilon\, m\frac{d\Theta}{dy}. \tag{22}$$

Substituting Eq. (22) in the governing Eqs. (16) and (17), the following coupled system is obtained:

$$\frac{d^4u}{dy^4} - B^2(1 - \epsilon\, m\Theta)\frac{d^2u}{dy^2} + B^2\epsilon\, m\frac{d\Theta}{dy}\frac{du}{dy} - G = 0, \tag{23}$$

$$\frac{d^2\Theta}{dy^2} + B_r(1 - \epsilon\, m\Theta)\left(\frac{du}{dy}\right)^2 + \frac{B_r}{B^2}\left(\frac{d^2u}{dy^2}\right)^2 = 0. \tag{24}$$

In order to solve these coupled ordinary differential equations with associated boundary conditions (18)–(20), we use the perturbation technique. Taking the approximate velocity and temperature profiles as

$$u = \sum_{i=0}^{\infty}\epsilon^i u_i \quad \text{and} \quad \Theta = \sum_{i=0}^{\infty}\epsilon^i\Theta_i. \tag{25}$$

Inserting Eq. (25) into Eqs. (23), (24), (18), (19) and (20) and then separating at each order of approximation, we obtain the following systems of equations along with the corresponding boundary conditions.

Zeroth order equations:

$$\frac{d^4u_0}{dy^4} - B^2\frac{d^2u_0}{dy^2} - G = 0, \tag{26}$$

$$\frac{d^2\Theta_0}{dy^2} + B_r\left(\frac{du_0}{dy}\right)^2 + \frac{B_r}{B^2}\left(\frac{d^2u_0}{dy^2}\right)^2 = 0, \tag{27}$$

$$u_0 = 0, \quad \frac{d^2u_0}{dy^2} = 0, \quad \Theta_0 = 0, \quad \text{at} \quad y = -1 \tag{28}$$

$$u_0 = 0, \quad \frac{d^2u_0}{dy^2} = 0, \quad \Theta_0 = 1 \quad \text{at} \quad y = 1. \tag{29}$$

1st order equations:

$$\frac{d^4u_1}{dy^4} - B^2\frac{d^2u_1}{dy^2} + B^2m\Theta_0\frac{d^2u_0}{dy^2} + B^2m\left(\frac{du_0}{dy}\right)\left(\frac{d\Theta_0}{dy}\right) = 0, \tag{30}$$

$$\frac{d^2\Theta_1}{dy^2} + 2B_r\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right) - B_rm\Theta_0\left(\frac{du_0}{dy}\right)^2 + \frac{2B_r}{B^2}\left(\frac{d^2u_0}{dy^2}\right)\left(\frac{d^2u_1}{dy^2}\right) = 0, \tag{31}$$

$$u_1 = 0, \quad \frac{d^2u_1}{dy^2} = 0, \quad \Theta_1 = 0, \quad \text{at} \quad y = -1 \tag{32}$$

$$u_1 = 0, \quad \frac{d^2u_1}{dy^2} = 0, \quad \Theta_1 = 0 \quad \text{at} \quad y = 1. \tag{33}$$

The second order equations have not been considered because of lengthy calculations. Solving Eqs. (26) and (27) with the corresponding boundary conditions (28) and (29) we have

$$u_0(y) = \Gamma_0 - \Gamma_1y^2 + \Gamma_2\cosh[By], \tag{34}$$

$$\Theta_0(y) = \Upsilon_0 - \Upsilon_1y^2 - \Upsilon_2y^4 - \Upsilon_3\cosh[By] - \Upsilon_4\cosh[2By] + \Upsilon_5y\sinh[By]. \tag{35}$$

Substituting Eqs. (34) and (35) into Eqs. (30) and (31) and then solving with respect to the boundary conditions (32) and (33), we obtain

$$\begin{aligned} u_1(y) = {} & (\Gamma_3 - \Gamma_4\cosh[By] + \Gamma_5\cosh[2By] + \Gamma_6\cosh[3By])y^0 \\ & + (\Gamma_7\sinh[By] - \Gamma_8\sinh[2By])y \\ & + (\Gamma_9 - \Gamma_{10}\cosh[By])y^2 + (\Gamma_{11}\sinh[By])y^3 \\ & + (\Gamma_{12} - \Gamma_{13}\cosh[By])y^4 + (\Gamma_{14}\sinh[By])y^5 \\ & + \Gamma_{15}y^6, \end{aligned} \tag{36}$$

$$\begin{aligned} \Theta_1(y) = {} & (\Upsilon_6 + \Upsilon_7\cosh[By] + \Upsilon_8\cosh[2By] - \Upsilon_9\cosh[3By] - \Upsilon_{10}\cosh[4By])y^0 \\ & + (-\Upsilon_{11}\sinh[By] + \Upsilon_{12}\sinh[2By] + \Upsilon_{13}\sinh[3By])y \\ & + (-\Upsilon_{14} + \Upsilon_{15}\cosh[By] - \Upsilon_{16}\cosh[2By])y^2 \\ & + (-\Upsilon_{17}\sinh[By] + \Upsilon_{18}\sinh[2By])y^3 \\ & + (\Upsilon_{19} + \Upsilon_{20}\cosh[By] - \Upsilon_{21}\cosh[2By])y^4 \\ & - (\Upsilon_{22}\sinh[By] + \Upsilon_{23}\sinh[2By])y^5 + (\Upsilon_{24} + \Upsilon_{25}\cosh[By])y^6 \\ & - \Upsilon_{26}y^8. \end{aligned} \tag{37}$$

Inserting Eqs. (34)–(37) into Eq. (25), the perturbation solutions upto order one are:

$$\begin{aligned} u(y) = {} & \Gamma_0 - y^2\Gamma_1 + \cosh[By]\Gamma_2 \\ & + \epsilon\{(\Gamma_3 - \Gamma_4\cosh[By] + \Gamma_5\cosh[2By] + \Gamma_6\cosh[3By])y^0 \\ & + (\Gamma_7\sinh[By] - \Gamma_8\sinh[2By])y + (\Gamma_9 - \Gamma_{10}\cosh[By])y^2 \\ & + (\Gamma_{11}\sinh[By])y^3 + (\Gamma_{12} - \Gamma_{13}\cosh[By])y^4 \\ & + (\Gamma_{14}\sinh[By])y^5 + \Gamma_{15}y^6\}, \end{aligned} \tag{38}$$

$$\begin{aligned} \Theta(y) = {} & \Upsilon_0 - y^2\Upsilon_1 - y^4\Upsilon_2 - \cosh[By]\Upsilon_3 - \cosh[2By]\Upsilon_4 + y\sinh[By]\Upsilon_5 \\ & + \epsilon\{(\Upsilon_6 + \Upsilon_7\cosh[By] + \Upsilon_8\cosh[2By] - \Upsilon_9\cosh[3By] - \Upsilon_{10}\cosh[4By])y^0 \\ & + (-\Upsilon_{11}\sinh[By] + \Upsilon_{12}\sinh[2By] + \Upsilon_{13}\sinh[3By])y \\ & + (-\Upsilon_{14} + \Upsilon_{15}\cosh[By] - \Upsilon_{16}\cosh[2By])y^2 + (-\Upsilon_{17}\sinh[By] + \Upsilon_{18}\sinh[2By])y^3 \\ & + (\Upsilon_{19} + \Upsilon_{20}\cosh[By] - \Upsilon_{21}\cosh[2By])y^4 - (\Upsilon_{22}\sinh[By] + \Upsilon_{23}\sinh[2By])y^5 \\ & + (\Upsilon_{24} + \Upsilon_{25}\cosh[By])y^6 - \Upsilon_{26}y^8\}, \end{aligned} \tag{39}$$

where Γ_i and Υ_i are constants which are given in appendix.

4. Volume flux, average velocity and shear stress on the plates

The volume flux in the non-dimensional form is given by

$$Q = \int_{-1}^{1} u(y)dy. \tag{40}$$

Using Eq. (38) in Eq. (40) we obtain

$$\begin{aligned} Q = {} & \frac{1}{210B^6}\{140B^5\epsilon\sinh[3B]\Gamma_6 - 210B^5\epsilon\cosh[2B]\Gamma_8 + 105B^4\epsilon\sinh[2B](2B\Gamma_5 + \Gamma_8) \\ & + 420B\epsilon\cosh[B](B^4\Gamma_7 + 2B^3\Gamma_{10} + B(6 + B^2)(B\Gamma_{11} + 4\Gamma_{13}) + (120 + 20B^2 + B^4)\Gamma_{14}) \\ & - 420\sinh[B](-B^5\Gamma_2 + B\epsilon(B^4\Gamma_4 + B^3\Gamma_7 + B(2 + B^2)(B\Gamma_{10} + 3\Gamma_{11}) \\ & + (24 + 12B^2 + B^4)\Gamma_{13}) + 5(24 + 12B^2 + B^4)\epsilon\Gamma_{14}) \\ & + 4B^6(7(15\Gamma_0 - 5\Gamma_1 + \epsilon(15\Gamma_3 + 5\Gamma_9 + 3\Gamma_{12})) + 15\epsilon\Gamma_{15})\}. \end{aligned} \tag{41}$$

The average velocity, $\overline{u}$, of the couple stress fluid is:

$$\bar{u} = \frac{Q}{H}, \tag{42}$$

which in the non-dimensional form coincides with flow rate given in Eq. (41). The dimensionless shear stress τ_p, on the surface of the upper plate is given by the formula

$$\tau_p = -\tau_{xy}\big|_{y=1} = -\mu\frac{du}{dy}\bigg|_{y=1}. \tag{43}$$

The minus sign accounts for the upper plate facing the negative y-direction of the coordinate system as shown in Fig. 1 (Papanastasiou et al., 2000). Using Eq. (38) in Eq. (43) we have,

$$\begin{aligned} \tau_p = {} & \mu\{2\Gamma_1 - B\sinh[B]\Gamma_2 + \epsilon(-3B\sinh[3B]\Gamma_6 + 2B\cosh[2B]\Gamma_8 \\ & + \sinh[2B](-2B\Gamma_5 + \Gamma_8) + \sinh[B](-\Gamma_7 - 3\Gamma_{11} + B(\Gamma_4 + \Gamma_{10} + \Gamma_{13}) - 5\Gamma_{14}) \\ & - \cosh[B](-2(\Gamma_{10} + 2\Gamma_{13}) + B(\Gamma_7 + \Gamma_{11} + \Gamma_{14})) - 2(\Gamma_9 + 2\Gamma_{12} + 3\Gamma_{15}))\}. \end{aligned} \tag{44}$$

Similarly, shear stress on the lower plate can be calculated.

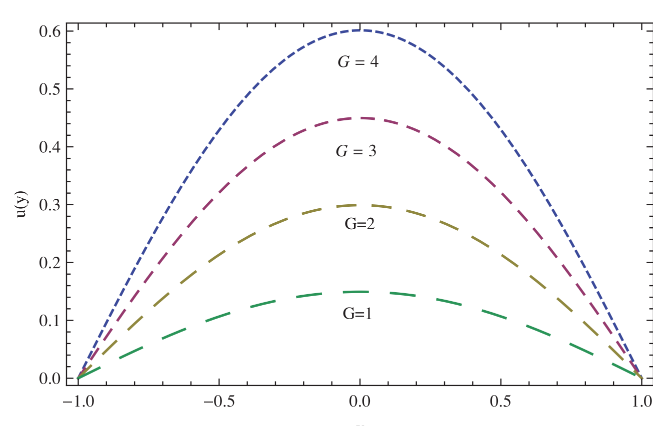

Figure 2 Effect of parameter G on velocity field, $u(y)$, for $B_r = 1$, $m = 3$ and $\epsilon = 0.1$.

5. Results and discussion

In this work, we have studied the heat transfer flow of couple stress fluids between two heated parallel inclined plates. The approximate analytical solutions of velocity field and temperature distribution are obtained by using the perturbation technique. The effect of various non-dimensional parameters on velocity field, temperature distribution, volumetric flow rate and shear stress is investigated graphically as shown below. In Figs. 2 and 3, velocity, u, of the fluid is plotted against

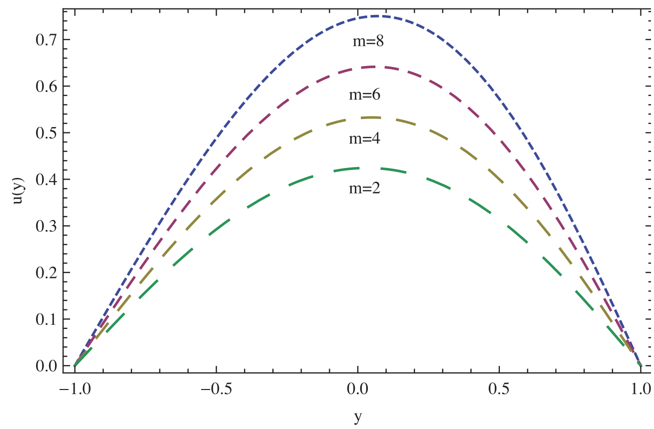

Figure 3 Effect of parameter m on velocity field, $u(y)$, for $B_r = 1$, $G = 4$ and $\epsilon = 0.1$.

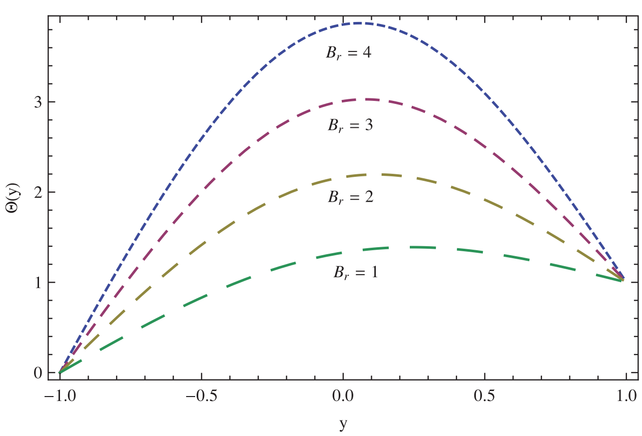

Figure 4 Effect of the Brinkman number, B_r, on temperature distribution, $\Theta(y)$, for $G = 2$, $m = 3$ and $\epsilon = 0.1$.

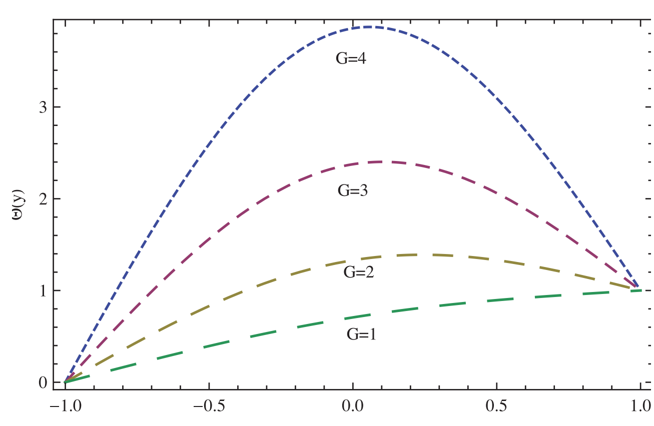

Figure 5 Effect of parameter G on temperature distribution, $\Theta(y)$, for $B_r = 1$, $m = 3$ and $\epsilon = 0.1$.

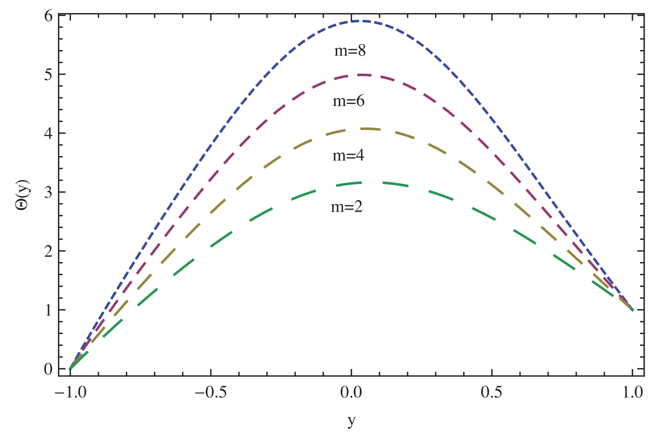

Figure 6 Effect of parameter m on temperature distribution, $\Theta(y)$, for $B_r = 2$, $G = 4$ and $\epsilon = 0.1$.

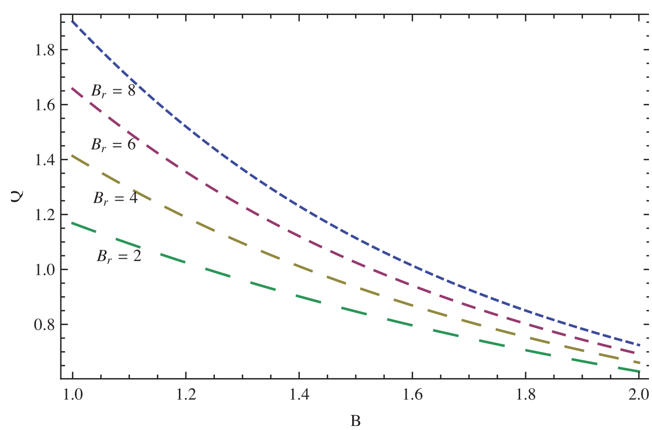

Figure 7 Effect of the Brinkman number, B_r, on flow rate, Q, for $G = 4$, $m = 3$ and $\epsilon = 0.1$.

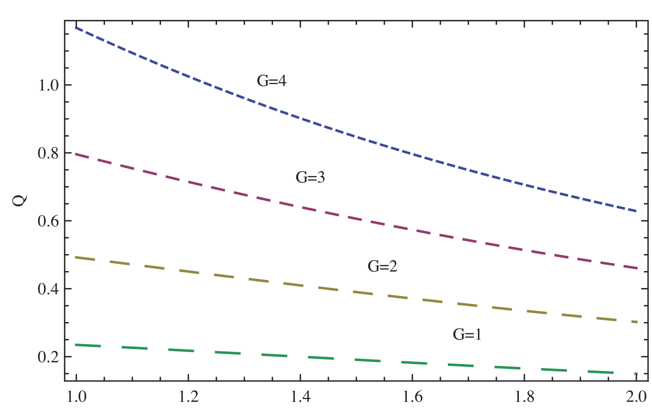

Figure 8 Effect of parameter, G, on flow rate, Q, for $B_r = 2$, $m = 3$ and $\epsilon = 0.1$.

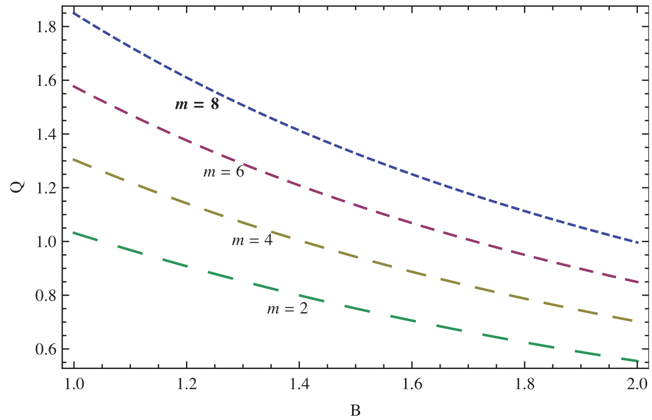

Figure 9 Effect of parameter, m, on flow rate, Q, for $B_r = 2$, $G = 4$ and $\epsilon = 0.1$.

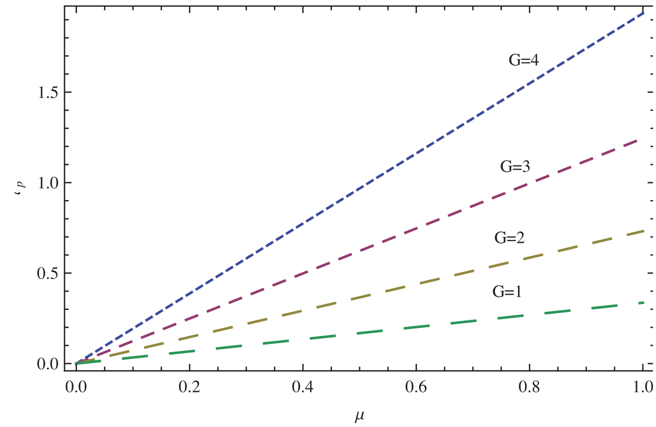

Figure 11 Effect of parameter G on shear stress, τ_p, for $B_r = 2$, $m = 3$ and $\epsilon = 0.1$.

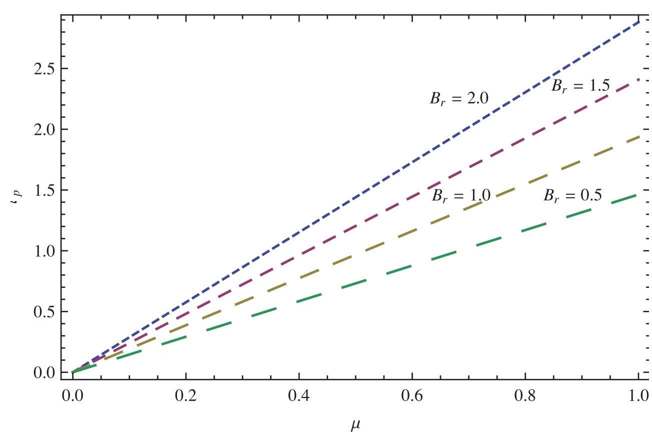

Figure 10 Effect of the Brinkman number, B_r, on shear stress, τ_p, for $G = 4$, $m = 3$ and $\epsilon = 0.1$.

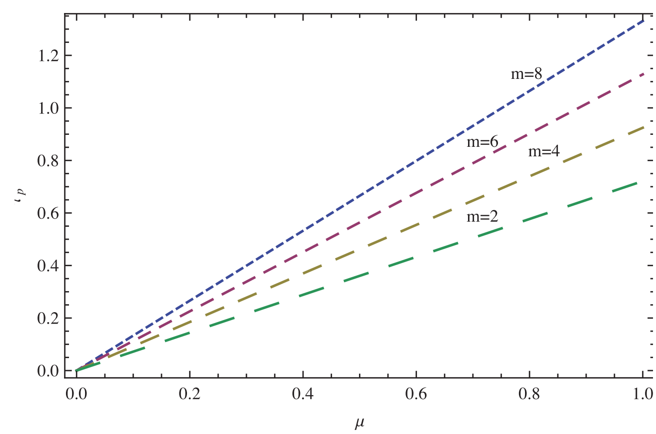

Figure 12 Effect of parameter m on shear stress, τ_p, for $B_r = 1$, $G = 4$ and $\epsilon = 0.1$.

the independent variable y. Both these profiles are parabolic and satisfy the given boundary conditions. These graphs show an increasing behavior of velocity of the fluid by increasing the non-dimensional parameters G and m, respectively.

Figs. 4–6 are plotted to visualize the effect of different parameters on thermal profiles $\Theta(y)$. It can be seen in Fig. 4 that, an increase in B_r increases the temperature of the fluid while Figs. 5 and 6 also depict a direct relation between the temperature distribution $\Theta(y)$ and the dimensionless quantities G and m, respectively. The volume flow rate of the fluid is investigated in Figs. 7–9 and effects of the Brinkman number B_r and parameters G and m can be observed in these figures. In order to observe the behavior of the shear stress τ_p in the Poiseuille flow while changing the values of three parameters B_r, G and m, we have sketched τ_p against the viscosity μ in Figs. 10–12. Again B_r shows a direct relation with τ_p in Fig. 10. Figs. 11 and 12 report that the shear stress is strongly dependent on the physical quantities G and m, respectively. It is clear from the figures that as these parameters increase the shear stress also increases on the plates.

6. Conclusion

In this paper, we have studied the heat transfer flow of couple stress fluids between two heated parallel inclined plates for Reynold's viscosity model. The strongly nonlinear and coupled differential equations are solved with the help of the perturbation technique for fluid velocity and temperature distribution. Analytical expressions for velocity field, temperature distribution, dynamic pressure, volumetric flow rate, average velocity of fluid and shear stress on the plates are obtained. It is shown graphically that velocity, temperature, volume flow rate and shear stress on the plates are strongly dependent on the dimensionless parameters B_r, G and m.

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Appendix A.

$$\Gamma_0 = \frac{G}{2B^4}(-2+B^2), \quad \Gamma_1 = \frac{G}{2B^2}, \quad \Gamma_2 = \frac{G\mathrm{sech}[B]}{B^4},$$

$$\begin{aligned}\Gamma_3 = {} & \frac{1}{24B^6}m(B^2\Gamma_2(-48B\sinh[B](B^2\Upsilon_1 + 2(6+B^2)\Upsilon_2) + 4B^4\cosh[3B]\Upsilon_4 \\ & + 12\cosh[B](48\Upsilon_2 + B^2(4(\Upsilon_1 + 6\Upsilon_2) - B^2(2\Upsilon_0 - 2(\Upsilon_1 + \Upsilon_2) + \Upsilon_4))) \\ & + 6B^3(-2+B^2)\Upsilon_5 - 6B^4\sinh[2B]\Upsilon_5 + 3B^3\cosh[2B](2B\Upsilon_3 + \Upsilon_5)) + 4\Gamma_1(1440\Upsilon_2 \\ & + B^2(72\Upsilon_1 + B(B(6(-2+B^2)\Upsilon_0 + 3(4\cosh[B]\Upsilon_3 + \cosh[2B]\Upsilon_4) - B(B(3\Upsilon_1 + 2\Upsilon_2) \\ & + 12\sinh[B](\Upsilon_3 + \cosh[B]\Upsilon_4))) + 12((2+B^2)\cosh[B] - 2B\sinh[B])\Upsilon_5)))),\end{aligned}$$

$$\Gamma_4 = \frac{m\Gamma_1}{B^4}(-\Upsilon_0 B^4 + 6(B^2\Upsilon_1 + 20\Upsilon_2)) - \frac{1}{4}Bm\Gamma_2\Upsilon_5,$$

$$\Gamma_5 = \frac{m\Gamma_1}{2B^2}(\Upsilon_1 B^2 + 20\Upsilon_2), \quad \Gamma_6 = \frac{1}{3}m\Gamma_1\Upsilon_2,$$

$$\begin{aligned}\Gamma_7 = {} & \frac{m\mathrm{sech}[B]}{720B^6}(B^2\Gamma_2(-360B^4(2\cosh[B] + B\sinh[B])\Upsilon_0 + 135B^4\cosh[3B]\Upsilon_4 \\ & + 12B\sinh[B](5B^2(-3+2B^2)\Upsilon_1 + (-45 + 6B^2(-5+B^2))\Upsilon_2 - 15B^4\Upsilon_4) \\ & + 180\cosh[B](B^2(8+B^2)\Upsilon_1 + (96 + 3B^2 + B^4)\Upsilon_2 - 2B^4\Upsilon_4) - 360B^3\Upsilon_5 \\ & - 240B^4\sinh[2B]\Upsilon_5 + 40B^3\cosh[2B](6B\Upsilon_3 + 5\Upsilon_5)) + 40\Gamma_1(4320\Upsilon_2 + B^2(216(\Upsilon_1 + 10\Upsilon_2) \\ & + B(B(-36(\Upsilon_0 - 3\Upsilon_1 - 5\Upsilon_2) - 9((-4+B^2)\cosh[B] + B\sinh[B])\Upsilon_3 \\ & + 4(5\cosh[2B] - 6B\sinh[2B])\Upsilon_4) + 3(3(8+B^2)\cosh[B] + B(-3+2B^2)\sinh[B])\Upsilon_5)))),\end{aligned}$$

$$\Gamma_8 = \frac{m}{4B^2}(3\Gamma_2(B^2\Upsilon_1 + 15\Upsilon_2) + 2B\Gamma_1(B\Upsilon_3 + 3\Upsilon_5)), \quad \Gamma_9 = \frac{3}{4}m\Gamma_2\Upsilon_2,$$

$$\Gamma_{10} = \frac{m}{4B^3}(\Gamma_2(-2B^4\Upsilon_0 + 7B^2\Upsilon_1 + 93\Upsilon_2 - B^4\Upsilon_4) + 2B\Gamma_1(3B\Upsilon_3 + 7\Upsilon_5)),$$

$$\Gamma_{11} = \frac{m}{6B}(\Gamma_2(B^2\Upsilon_1 + 21\Upsilon_2) + 2B\Gamma_1\Upsilon_5), \quad \Gamma_{12} = \frac{1}{10}Bm\Gamma_2\Upsilon_2,$$

$$\Gamma_{13} = \frac{m}{72B^2}(44\Gamma_1\Upsilon_4 + B\Gamma_2(6B\Upsilon_3 + 11\Upsilon_5)), \quad \Gamma_{14} = \frac{m}{12B}(4\Gamma_1\Upsilon_4 + B\Gamma_2\Upsilon_5), \quad \Gamma_{15} = \frac{1}{48}m\Gamma_2\Upsilon_4$$

$$\Upsilon_0 = \frac{1}{12B^2}(6B^2 + B_r(4(6+B^2)\Gamma_1^2 + 48(\cosh[B] - B\sinh[B])\Gamma_1\Gamma_2 + 3B^2\cosh[2B]\Gamma_2^2)),$$

$$\Upsilon_1 = -\frac{2B_r\Gamma_1^2}{B^2}, \quad \Upsilon_2 = -\frac{1}{3}B_r\Gamma_1^2, \quad \Upsilon_3 = -\frac{4B_r\Gamma_1\Gamma_2}{B^2}, \quad \Upsilon_4 = -\frac{1}{4}B_r\Gamma_2^2, \quad \Upsilon_5 = \frac{1}{2} + \frac{4B_r\Gamma_1\Gamma_2}{B},$$

$$\Upsilon_6 = \frac{B_r}{60480B^7}(945B^7\cosh[4B]\Gamma_2(48\Gamma_6 + m\Gamma_2\Upsilon_4) + 560B^5\cosh[3B](-336\Gamma_1\Gamma_6$$
$$+ 4\Gamma_2(18B^2\Gamma_5 - 3B\Gamma_8 + 4m\Gamma_1\Upsilon_4) + Bm\Gamma_2^2(3B\Upsilon_3 + 2\Upsilon_5)) - 1680B^6\sinh[3B](8\Gamma_1(6\Gamma_6$$
$$+ m\Gamma_2\Upsilon_4) + B\Gamma_2(24\Gamma_8 + m\Gamma_2\Upsilon_5)) + 15120B\sinh[B](8\Gamma_1(2640\Gamma_{14} + 2B(216\Gamma_{13}$$
$$+ B(6(7\Gamma_{11} + 30\Gamma_{14}) + B(10\Gamma_{10} + 28\Gamma_{13} + B(2\Gamma_7 + 4\Gamma_{11} + 6\Gamma_{14} + B(\Gamma_{10} + \Gamma_{13}$$
$$+ m\Gamma_2\Upsilon_0)) - (18 + B^2)m\Gamma_2\Upsilon_1)) - (600 + 60B^2 + B^4)m\Gamma_2\Upsilon_2) + B^5m\Gamma_2\Upsilon_4)$$
$$- 16B^2m\Gamma_1^2(4B\Upsilon_3 + (18 + B^2)\Upsilon_5) + B\Gamma_2(16B^4\Gamma_9 + 32B^2(6 + B^2)\Gamma_{12}$$
$$+ 48(120 + 20B^2 + B^4)\Gamma_{15} + 3B^5m\Gamma_2\Upsilon_5)) + 15120B^3\sinh[2B](\Gamma_2(-15\Gamma_{14}$$
$$- B(6\Gamma_{13} + B(3\Gamma_{11} + 10\Gamma_{14} + B(2\Gamma_{10} + 4\Gamma_{13} - 2B(\Gamma_7 + \Gamma_{11} + \Gamma_{14}) + m\Gamma_2\Upsilon_1)))$$
$$- 2B(3 + B^2)m\Gamma_2\Upsilon_2) - 8Bm\Gamma_1^2\Upsilon_4 - 2B^2\Gamma_1(4B\Gamma_5 - 2\Gamma_8 + m\Gamma_2(B\Upsilon_3 + 2\Upsilon_5)))$$
$$+ 72B^5(-4\Gamma_1(140(6 + B^2)\Gamma_9 + 840(\Gamma_{12} + \Gamma_{15}) + B^2(112\Gamma_{12} + 90\Gamma_{15}$$
$$+ m\Gamma_1(70\Upsilon_0 - 28\Upsilon_1 - 15\Upsilon_2))) + 7B^4m\Gamma_2^2(30\Upsilon_0 - 5\Upsilon_1 - 2\Upsilon_2 + 15\Upsilon_4)$$
$$+ 70B\Gamma_2(48B_r\Gamma_1\Gamma_4\Gamma_8 + B(-12(\Gamma_{10} + \Gamma_{13}) + B(6\Gamma_7 + 3\Gamma_{11} + 2\Gamma_{14} - 2m\Gamma_1\Upsilon_5))))$$
$$- 3780B^2\cosh[2B](8B\Gamma_1(4B^2\Gamma_5 - 4(B + B^3)\Gamma_8 - (3 + 2B^2)m\Gamma_1\Upsilon_4) + Bm\Gamma_2^2(-15\Upsilon_2$$
$$+ B^2(-3(\Upsilon_1 + 6\Upsilon_2) + 2B^2(\Upsilon_0 - \Upsilon_1 - \Upsilon_2 + \Upsilon_4))) + 2\Gamma_2(4B^4\Gamma_4(B + 8B_r\Gamma_1\Gamma_8) - 15\Gamma_{14}$$
$$- B(6\Gamma_{13} + B(3(\Gamma_{11} + 10\Gamma_{14}) + 2B(\Gamma_{10} + 6\Gamma_{13} + B(6B\Gamma_6 + \Gamma_7 + 3\Gamma_{11}$$
$$- 2B(\Gamma_{10} + \Gamma_{13}) + 5\Gamma_{14}) + 2m\Gamma_1\Upsilon_3) + 2(3 + 2B^2)m\Gamma_1\Upsilon_5))))$$
$$- 15120\cosh[B](16\Gamma_1(B^5\Gamma_4 + B^4(4 + B^2)\Gamma_7 + 1440\Gamma_{14} + B(3B^2(4 + B^2)\Gamma_{10}$$
$$+ B(48 + 18B^2 + B^4)\Gamma_{11} + 240\Gamma_{13} + 96B^2\Gamma_{13} + 5B^4\Gamma_{13} + 600B\Gamma_{14} + 40B^3\Gamma_{14}$$
$$+ B^5\Gamma_{14} + 2B^4m\Gamma_2\Upsilon_0 - 24B^2m\Gamma_2\Upsilon_1 - 6B^4m\Gamma_2\Upsilon_1 - 720m\Gamma_2\Upsilon_2 - 240B^2m\Gamma_2\Upsilon_2$$
$$- 10B^4m\Gamma_2\Upsilon_2 + B^4m\Gamma_2\Upsilon_4)) - 16B^2m\Gamma_1^2(B(6 + B^2)\Upsilon_3 + 6(4 + B^2)\Upsilon_5)$$
$$+ B\Gamma_2(5760\Gamma_{15} + B^2(192(\Gamma_{12} + 15\Gamma_{15}) + B^2(16(\Gamma_9 + 6\Gamma_{12} + 15\Gamma_{15})$$
$$+ B(-8B\Gamma_5 - 4\Gamma_8 + m\Gamma_2(B\Upsilon_3 + 6\Upsilon_5))))))),$$

$$\Upsilon_7 = \frac{B_r}{4B^7}(16\Gamma_1(B^5\Gamma_4 + 4B^4\Gamma_7 + 1440\Gamma_{14} + B(12B^2\Gamma_{10} + 48B\Gamma_{11} + 240\Gamma_{13}$$
$$+ m\Gamma_2(2B^4\Upsilon_0 - 24B^2\Upsilon_1 - 720\Upsilon_2 + B^4\Upsilon_4))) - 96B^2m\Gamma_1^2(B\Upsilon_3 + 4\Upsilon_5)$$
$$+ B\Gamma_2(-8B^6\Gamma_5 - 4B^5\Gamma_8 + 16B^4\Gamma_9 + 192B^2\Gamma_{12} + 5760\Gamma_{15} + B^6m\Gamma_2\Upsilon_3 + 6B^5m\Gamma_2\Upsilon_5)),$$

$$\Upsilon_8 = \frac{B_r}{16B^5}(Bm\Gamma_2^2(2B^4\Upsilon_0 - 3B^2\Upsilon_1 - 15\Upsilon_2 + 2B^4\Upsilon_4) + 8B\Gamma_1(4B^2\Gamma_5 - 4B\Gamma_8 - 3m\Gamma_1\Upsilon_4)$$
$$+ 2\Gamma_2(4B^5\Gamma_4 - 15\Gamma_{14} - B(6\Gamma_{13} + B(3\Gamma_{11} + 2B(B(6B\Gamma_6 + \Gamma_7) + \Gamma_{10} + 2m\Gamma_1\Upsilon_3)$$
$$+ 6m\Gamma_1\Upsilon_5)))),$$

$$\Upsilon_9 = \frac{B_r}{108B^2}(-336\Gamma_1\Gamma_6 + 4\Gamma_2(18B^2\Gamma_5 - 3B\Gamma_8 + 4m\Gamma_1\Upsilon_4) + Bm\Gamma_2^2(3B\Upsilon_3 + 2\Upsilon_5)),$$

$$\Upsilon_{10} = \frac{B_r\Gamma_2}{64}(48\Gamma_6 + m\Gamma_2\Upsilon_4),$$

$$\Upsilon_{11} = \frac{B_r}{4B^6}(8\Gamma_1(2640\Gamma_{14} + 2B(B^4\Gamma_4 + 216\Gamma_{13} + B(42\Gamma_{11} + B(2B\Gamma_7 + 10\Gamma_{10}$$
$$+ m\Gamma_2(B^2\Upsilon_0 - 18\Upsilon_1))) - 600m\Gamma_2\Upsilon_2) + B^5m\Gamma_2\Upsilon_4) - 32B^2m\Gamma_1^2(2B\Upsilon_3 + 9\Upsilon_5)$$
$$+ B\Gamma_2(-8B^5\Gamma_8 + 16B^4\Gamma_9 + 192B^2\Gamma_{12} + 5760\Gamma_{15} + 3B^5m\Gamma_2\Upsilon_5)),$$

$$\Upsilon_{12} = \frac{B_r}{4B^4}(\Gamma_2(-2B^4\Gamma_7 + 15\Gamma_{14} + B(6\Gamma_{13} + B(2B\Gamma_{10} + 3\Gamma_{11} + Bm\Gamma_2\Upsilon_1) + 6m\Gamma_2\Upsilon_2))$$
$$+ 8Bm\Gamma_1^2\Upsilon_4 + 2B^2\Gamma_1(4B\Gamma_5 - 2\Gamma_8 + m\Gamma_2(B\Upsilon_3 + 2\Upsilon_5))),$$

$$\Upsilon_{13} = \frac{B_r}{36B}(8\Gamma_1(6\Gamma_6 + m\Gamma_2\Upsilon_4) + B\Gamma_2(24\Gamma_8 + m\Gamma_2\Upsilon_5)),$$

$$\Upsilon_{14} = \frac{B_r}{8B^2}(-32\Gamma_1\Gamma_9 + B^2\Gamma_2(4B\Gamma_7 - 8\Gamma_{10} + B^2m\Gamma_2(2\Upsilon_0 + \Upsilon_4))),$$

$$\Upsilon_{15} = \frac{4B_r}{B^5}(6B\Gamma_2(B^2\Gamma_{12} + 30\Gamma_{15}) + \Gamma_1(600\Gamma_{14} + B(B^3\Gamma_7 + 96\Gamma_{13} + 3B(B\Gamma_{10} + 6\Gamma_{11} - 2Bm\Gamma_2\Upsilon_1) - 240m\Gamma_2\Upsilon_2)) - B^2m\Gamma_1^2(B\Upsilon_3 + 6\Upsilon_5)),$$

$$\Upsilon_{16} = \frac{B_r}{8B^3}(\Gamma_2(-4B^3\Gamma_{10} + 30\Gamma_{14} + B(6B\Gamma_{11} + 12\Gamma_{13} + m\Gamma_2(B^2\Upsilon_1 + 9\Upsilon_2))) + 8Bm\Gamma_1^2\Upsilon_4 + 4B^2\Gamma_1(4\Gamma_8 + m\Gamma_2\Upsilon_5)),$$

$$\Upsilon_{17} = \frac{4B_r}{B^4}(2B\Gamma_2(B^2\Gamma_{12} + 30\Gamma_{15}) + \Gamma_1(B^3\Gamma_{10} + 180\Gamma_{14} + B(4B\Gamma_{11} + 28\Gamma_{13} - m\Gamma_2(B^2\Upsilon_1 + 60\Upsilon_2))) - B^2m\Gamma_1^2\Upsilon_5),$$

$$\Upsilon_{18} = \frac{B_r\Gamma_2}{2B^2}(5\Gamma_{14} + B(-B\Gamma_{11} + 2\Gamma_{13} + m\Gamma_2\Upsilon_2)),$$

$$\Upsilon_{19} = \frac{B_r}{24B^2}(8B^2m\Gamma_1^2\Upsilon_0 + B^2\Gamma_2(-6B\Gamma_{11} + 24\Gamma_{13} + B^2m\Gamma_2\Upsilon_1) + 4\Gamma_1(4B^2\Gamma_9 + 24\Gamma_{12} + B^3m\Gamma_2\Upsilon_5)),$$

$$\Upsilon_{20} = \frac{4B_r}{B^3}(15B\Gamma_2\Gamma_{15} + \Gamma_1(40\Gamma_{14} + B(B\Gamma_{11} + 5(\Gamma_{13} - 2m\Gamma_2\Upsilon_2)))),$$

$$\Upsilon_{21} = \frac{B_r\Gamma_2}{8B}(-4B\Gamma_{13} + 10\Gamma_{14} + Bm\Gamma_2\Upsilon_2),$$

$$\Upsilon_{22} = \frac{4B_r}{B^2}(3B\Gamma_2\Gamma_{15} + \Gamma_1(B\Gamma_{13} + 6\Gamma_{14} - Bm\Gamma_2\Upsilon_2)), \quad \Upsilon_{23} = \frac{1}{2}B_r\Gamma_2\Gamma_{14},$$

$$\Upsilon_{24} = \frac{B_r}{60B^2}(16\Gamma_1(2B^2\Gamma_{12} + 15\Gamma_{15}) - 8B^2m\Gamma_1^2\Upsilon_1 + B^3\Gamma_2(-10\Gamma_{14} + Bm\Gamma_2\Upsilon_2)),$$

$$\Upsilon_{25} = \frac{4B_r\Gamma_1\Gamma_{14}}{B}, \quad \Upsilon_{26} = \frac{1}{14}B_r\Gamma_1(-6\Gamma_{15} + m\Gamma_1\Upsilon_2).$$

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