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Cohomology of split group extensions and characteristic classes

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ABSTRACT

There are characteristic classes that are the obstructions to the vanishing of the differentials in the Lyndon–Hochschild–Serre spectral sequence of a split extension of an integral lattice L by a group G. These characteristic classes exist in the rth page of the spectral sequence provided that the differentials $d_i = 0$ for all $i < r$. When L decomposes into a sum of G-sublattices, we show that there are defining relations between the characteristic classes of L and the characteristic classes of its summands.

1. Introduction

Suppose G is a group and L is a finite rank integral $\mathbb{Z}G$-lattice. Let $\Gamma = L \rtimes G$ be the semidirect product group induced by the action of G on L. Our objective is to analyze the split group extension $0 \to L \to \Gamma \to G \to 1$ and its associated Lyndon–Hochschild–Serre spectral sequence $\{E_*, d_*\}$. In [1] we showed that when G is a cyclic group of prime order, the spectral sequence with integral coefficients collapses at E_2 without extension problems. In fact, our proof stated in [1] applies not only to integral coefficients, but to all coefficient modules in a certain category which we denote by $\mathcal{M}_F(L, G)$. This category is essential in the definition of characteristic classes (see Theorem 2.2). It consists of all finite rank $F\Gamma$-lattices M on which L acts trivially, where F is a given principal ideal domain. In particular, F as a trivial $F\Gamma$-module is an object of $\mathcal{M}_F(L, G)$.

Let us now suppose that L decomposes into a sum of $\mathbb{Z}G$-sublattices; $L = L' \oplus L''$. Let $\{E'_*, d'_*\}$ and $\{E''_*, d''_*\}$ be the Lyndon–Hochschild–Serre spectral sequences associated to the respective semidirect product groups $L' \rtimes G$ and $L'' \rtimes G$. The goal of the present paper is to find necessary and sufficient conditions on the latter defined spectral sequences such that $\{E_*, d_*\}$ has no nonzero differentials up to a given page.

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In Section 2, we define characteristic classes $v_r^t(L)$, introduced by Charlap, Vasquez and Sah (see [2] and [3]), that are the obstructions to the vanishing of the differentials in $\{E_*, d_*\}$. These classes exist and lie in the images of the differentials $d_r^{0,t} : E_r^{0,t}(H_t(L,F)) \to E_r^{r,t-r+1}(H_t(L,F))$ provided that the differentials $d_i^{*,t}$ vanish for all $i < r$ and for all coefficient modules in $\mathcal{M}_F(L,G)$. Our main result is the following decomposition theorem.

Theorem 1.1. *Let G be any group. Assume L' and L'' are $\mathbb{Z}G$-lattices of finite rank and $L = L' \oplus L''$. Let $r \geqslant 2$ and $t \geqslant 0$.*

(a) *Suppose $v_k^i(L') = v_k^j(L'') = 0$ for all $i, j \leqslant t$ and for all $k < r$. Then $d_2^{s,m} = \cdots = d_{r-1}^{s,m} = 0$ for all $s \geqslant 0$, all $m \leqslant t$, and all coefficient modules in $\mathcal{M}_F(L,G)$. Additionally,*

$$v_r^t(L) = \sum_{i+j=t} \big(\big(C_j'^r\big)_* \big(v_r^i(L')\big) + (-1)^i \big(C_i''^r\big)_* \big(v_r^j(L'')\big)\big).$$

(b) *Suppose $v_k^t(L) = 0$ for all $k < r$. Then $d_k'^{s,t} = d_k''^{s,t} = 0$ for all $s \geqslant 0$, all $k < r$, and all coefficient modules in $\mathcal{M}_F(L,G)$. Additionally,*

$$v_r^t(L') = D_*'^r\big(v_r^t(L)\big) \quad \text{and} \quad v_r^t(L'') = D_*''^r\big(v_r^t(L)\big).$$

In Section 3, we define the FG-equivariant homomorphisms $C_j'^r$, $C_i''^r$, D'^r, and D''^r that induce the maps $(C_j'^r)_* : E_2'^{s,i-r+1} \to E_2^{s,t-r+1}$, $(C_i''^r)_* : E_2''^{s,j-r+1} \to E_2^{s,t-r+1}$, $D_*'^r : E_2^{s,t-r+1} \to E_2'^{s,t-r+1}$, and $D_*''^r : E_2^{s,t-r+1} \to E_2''^{s,t-r+1}$, respectively. The homomorphisms $C_j'^2$ and $C_i''^2$ were first considered by Charlap and Vasquez in [2]. They showed that the sum formula holds when $r = 2$. We use a different approach to generalize this result to all pages of the spectral sequence and also to prove a converse.

The problem of establishing the collapse of $\{E_*, d_*\}$ in general can be a difficult one. The spectral sequence can have nonzero differentials even when G is abelian. For instance, Totaro proved in [4] that for any prime number p and $G = C_p^2$, there is a semidirect product group $\Gamma = L \rtimes C_p^2$ such that in the associated Lyndon–Hochschild–Serre spectral sequence with $\mathbb{Z}_p$ coefficients there always exist nonzero differentials at E_p or later.

A question, first posed by Adem (see [1]), that is still open is whether the spectral sequence collapses integrally at E_2 without extension problems when G is an arbitrary finite cyclic group. In view of our results, we can make the following:

Conjecture 1.2. *Let C_n be a cyclic group of order n and let L be a finite rank integral lattice. The Lyndon–Hochschild–Serre spectral sequence $\{E_*, d_*\}$ of any split group extension $0 \to L \to \Gamma \to C_n \to 1$ collapses at E_2 for all coefficient modules in $\mathcal{M}_F(L, C_n)$.*

Note that part (a) of Theorem 1.1 reduces this conjecture to the case where L is an indecomposable $\mathbb{Z}C_n$-lattice (see Corollary 4.2).

2. Preliminary results

Henceforth, let G be any group and let L be any finite rank integral $\mathbb{Z}G$-lattice. Denote by Γ the associated semidirect product $L \rtimes G$. Suppose F is a principal ideal domain. For each $F\Gamma$-module M we have the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{p,q}(M) = H^p\big(G, H^q(L,M)\big) \implies H^{p+q}(\Gamma, M).$$

In the proof of our main result, we make use of the multiplicative structure of this spectral sequence. Namely, any Γ-pairing of $F\Gamma$-modules A, B, and C,

$$\cdot : A \otimes_F B \to C,$$

determines an F-pairing

$$E_r^{p,q}(A) \otimes_F E_r^{s,t}(B) \to E_r^{p+s,q+t}(C).$$

In addition, for the differential d_r and for each $a \in E_r^{p,q}(A)$ and $b \in E_r^{s,t}(B)$ we have the product formula

$$d_r^{p+s,q+t}(a \cdot b) = d_r^{p,q}(a) \cdot b + (-1)^{p+q} a \cdot d_r^{s,t}(b).$$

We denote by $\mathcal{M}_F(L,G)$ the category of all $F\Gamma$-modules M, such that M is a finitely generated free F-module and L acts trivially on M. In particular, F with a trivial Γ-action is a module in this category. $\mathcal{M}_F(L,G)$ has the property that if $M \in \mathcal{M}_F(L,G)$, then $H_i(L,M)$ and $H^i(L,M) \cong \mathrm{Hom}_F(H_i(L,F),M)$ are also in $\mathcal{M}_F(L,G)$. Observe that the projection of Γ onto G induces an equivalence between this category and the category of all finite rank FG-lattices.

Let $t \geqslant 0$ and $r \geqslant 2$. Set $M = H_t(L,F) \cong \bigwedge^t(L) \otimes F$. Assume $d_k^{0,t} = 0$ for all $2 \leqslant k \leqslant r-1$. By applying the Universal Coefficient Theorem it follows

$$\begin{aligned} E_r^{0,t}\big(H_t(L,F)\big) &= E_2^{0,t}\big(H_t(L,F)\big) \\ &= H^0\big(G, H^t\big(L, H_t(L,F)\big)\big) \\ &\cong H^0\big(G, \mathrm{Hom}_F\big(H_t(L,F), H_t(L,F)\big)\big) \quad \text{(by UCT)} \\ &= \mathrm{Hom}_{FG}\big(H_t(L,F), H_t(L,F)\big) \\ &\cong \mathrm{Hom}_{FG}\Big(\bigwedge^t(L) \otimes F, \bigwedge^t(L) \otimes F\Big). \end{aligned}$$

Definition 2.1. With the preceding assumptions, let $\mathrm{id}^t : \bigwedge^t(L) \otimes F \to \bigwedge^t(L) \otimes F$ be the identity homomorphism. Identifying along the above isomorphisms, denote by $[f]$ the class in $E_r^{0,t}(H_t(L,F))$ corresponding to a map $f \in \mathrm{Hom}_{FG}(\bigwedge^t(L) \otimes F, \bigwedge^t(L) \otimes F)$. Then $v_r^t(L) := d_r^{0,t}([\mathrm{id}^t]) \in E_r^{r,t-r+1}(H_t(L,F))$ is said to be a *characteristic class* of the spectral sequence $\{E_r, d_r\}$.

Characteristic classes were first considered by Charlap and Vasquez in [2], but only in the case when $r = 2$. Sah in [3] extended their definition to all $r \geqslant 2$ by proving the following key theorem.

Theorem 2.2. *(See Sah, 1972, [3].) Let $\{E_*, d_*\}$ be the Lyndon–Hochschild–Serre spectral sequence of the extension $0 \to L \to \Gamma \to G \to 1$. Suppose there exist integers $r \geqslant 2$ and $t \geqslant 0$ such that $d_2^{s,t} = \cdots = d_{r-1}^{s,t} = 0$ for all $s \geqslant 0$ and for all coefficient modules in $\mathcal{M}_F(L,G)$.*

(a) *There is a canonical epimorphism*

$$\theta : E_r^{s,0}\big(H^t(L,M)\big) \to E_r^{s,t}(M) \quad \textit{for all } s \geqslant 0 \textit{ and for all } M \in \mathcal{M}_F(L,G).$$

(b) *We have*

$$d_r^{s,t}(x) = (-1)^s y \cdot v_r^t(L),$$

for all $x \in E_r^{s,t}(M)$, all $M \in \mathcal{M}_F(L,G)$, and all $y \in E_r^{s,0}(H^t(L,M))$ with $\theta(y) = x$.

(c) *Let $\sigma : H \to G$ be a group homomorphism which converts L into a $\mathbb{Z}H$-module. Assume $d_2^{s,t} = \cdots = d_{r-1}^{s,t} = 0$ holds for all $s \geqslant 0$ and for all objects in $\mathcal{M}_F(L,H)$. The characteristic class $w_r^t(L)$ for the category $\mathcal{M}_F(L,H)$ is then the image of $v_r^t(L)$ under the map induced on the spectral sequences.*

Note that, with the assumptions of the theorem, characteristic classes $v_r^t(L)$ are obstructions to the vanishing of the differentials $d_r^{s,t}$ for all integers $s \geqslant 0$. The following corollary is an immediate consequence of Sah's theorem.

Corollary 2.3. *Let $t \geqslant 0$. The Lyndon–Hochschild–Serre spectral sequence $\{E_*, d_*\}$ has $d_r^{s,t} = 0$ for all $s \geqslant 0$, all $r < n$, and all $M \in \mathcal{M}_F(L,G)$ if and only if the edge differentials $d_r^{0,t} : E_r^{0,t}(H_t(L,F)) \to E_r^{r,t-r+1}(H_t(L,F))$ are zero for all $r < n$.*

Corollary 2.4. *Suppose $\varphi : L \to \Gamma$ is the natural inclusion. Given an integer $t \geqslant 0$, the following statements are equivalent.*

(a) *In $\{E_*, d_*\}$, the differentials $d_r^{s,t}$ vanish for all $r, s \geqslant 0$ and all $M \in \mathcal{M}_F(L,G)$.*
(b) *$\varphi^* : H^t(\Gamma, M) \to H^t(L,M)$ maps onto $H^t(L,M)^G$ for all $M \in \mathcal{M}_F(L,G)$.*
(c) *$\varphi^* : H^t(\Gamma, H_t(L,F)) \to H^t(L, H_t(L,F))$ maps onto the G-invariants $H^t(L, H_t(L,F))^G$.*

Proof. Clearly (b) implies (c). Note that the map φ^* is given by the composition

$$H^t(\Gamma, M) \twoheadrightarrow E_\infty^{0,t}(M) = E_{t+2}^{0,t}(M) \subset \cdots \subset E_2^{0,t}(M) = H^t(L,M)^G \subset H^t(L,M).$$

If $d_r^{s,t} = 0$ for all $r, s \geqslant 0$ and all $M \in \mathcal{M}_F(L,G)$, then $E_\infty^{0,t}(M) = E_{t+2}^{0,t}(M) = \cdots = E_2^{0,t}(M)$ for all $M \in \mathcal{M}_F(L,G)$. Hence, (a) implies (b). To prove that (a) follows from (c), assume $\varphi^* : H^t(\Gamma, H_t(L,F)) \to H^t(L, H_t(L,F))^G$ is onto. Then by the above composition we have $E_\infty^{0,t}(H_t(L,F)) = E_{t+2}^{0,t}(H_t(L,F)) = \cdots = E_2^{0,t}(H_t(L,F))$. This shows that $d_r^{0,t} : E_r^{0,t}(H_t(L,F)) \to E_r^{r,t-r+1}(H_t(L,F))$ is zero for all $r \geqslant 2$. By the previous corollary $d_r^{s,t} = 0$ for all $s \geqslant 0$, $r \geqslant 2$, and $M \in \mathcal{M}_F(L,G)$. $\square$

3. Decomposition theorem

Suppose L is a direct sum of $\mathbb{Z}G$-sublattices L' and L''. In this section we derive relations between the characteristic classes of L and the characteristic classes of L' and L''.

For convenience, we use $\bigwedge^*(L)$ to denote $\bigwedge^*(L) \otimes F$. Recall that there is a standard FG-module decomposition

$$\bigwedge^n(L) \cong \bigoplus_{i+j=n} \bigwedge^i(L') \otimes \bigwedge^j(L'').$$

Definition 3.1. Given integers $i, j \geqslant 0$ and $r \geqslant 1$, define FG-equivariant homomorphisms

$$C_j'^r : \mathrm{Hom}_F\Big(\bigwedge^i(L'), \bigwedge^{i+r-1}(L')\Big) \to \mathrm{Hom}_F\Big(\bigwedge^{i+j}(L), \bigwedge^{i+j+r-1}(L)\Big)$$

by

$$C_j'^r(f)(x \otimes y) = \begin{cases} f(x) \otimes y & \text{if } x \in \bigwedge^i(L') \text{ and } y \in \bigwedge^j(L''), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$C_i''^r : \mathrm{Hom}_F\Big(\bigwedge^j(L''), \bigwedge^{j+r-1}(L'')\Big) \to \mathrm{Hom}_F\Big(\bigwedge^{i+j}(L), \bigwedge^{i+j+r-1}(L)\Big)$$

by

$$C_i''^r(f)(x\otimes y) = \begin{cases} x\otimes f(y) & \text{if } x\in \bigwedge^i(L') \text{ and } y\in\bigwedge^j(L''), \\ 0 & \text{otherwise.}\end{cases}$$

Let $\Gamma' = L' \rtimes G$ and $\Gamma'' = L'' \rtimes G$. There are associated split exact sequences of groups

$$0\to L'\to \Gamma'\to G\to 1 \quad \text{and} \quad 0\to L''\to\Gamma''\to G\to 1.$$

Suppose $v_*^*(L')$ and $v_*^*(L'')$ are the respective characteristic classes of the Lyndon–Hochschild–Serre spectral sequences $\{E'_*, d'_*\}$ and $\{E''_*, d''_*\}$ corresponding to these extensions. Let $\iota' : L' \to L$ be the natural inclusion and let $p' : L \to L'$ be the natural projection. Similarly, define $\iota'' : L'' \to L$ and $p'' : L \to L''$. As a straightforward application of the definitions, we obtain the following lemma.

Lemma 3.2. *The map* $C_0'^r : \mathrm{Hom}_F(\bigwedge^i(L'), \bigwedge^{i+r-1}(L')) \to \mathrm{Hom}_F(\bigwedge^i(L), \bigwedge^{i+r-1}(L))$ *is given by the composition* $p'^* \circ \iota'_*$, *where*

$$\mathrm{Hom}_F\Big(\bigwedge^i(L'), \bigwedge^{i+r-1}(L')\Big) \xrightarrow{\iota'_*} \mathrm{Hom}_F\Big(\bigwedge^i(L'), \bigwedge^{i+r-1}(L)\Big) \xrightarrow{p'^*} \mathrm{Hom}_F\Big(\bigwedge^i(L), \bigwedge^{i+r-1}(L)\Big),$$

and $C_0''^r$ *is the composition* $p''^* \circ \iota''_*$, *where*

$$\mathrm{Hom}_F\Big(\bigwedge^j(L''), \bigwedge^{j+r-1}(L'')\Big) \xrightarrow{\iota''_*} \mathrm{Hom}_F\Big(\bigwedge^j(L''), \bigwedge^{j+r-1}(L)\Big) \xrightarrow{p''^*} \mathrm{Hom}_F\Big(\bigwedge^j(L), \bigwedge^{j+r-1}(L)\Big).$$

Proof. Suppose $f \in \mathrm{Hom}_F(\bigwedge^i(L'), \bigwedge^{i+r-1}(L'))$. For any $x\in\bigwedge^i(L')$ and $y\in\bigwedge^0(L'')$, $(p'^*\circ\iota'_*)(f)(x\otimes y) = y(\iota'_*(f)(x)) = y(f(x)\otimes 1) = f(x)\otimes y = C_0'^r(f)(x\otimes y)$. The second assertion of the lemma follows analogously. □

Definition 3.3. Given integers $i, j \geqslant 0$ and $r\geqslant 1$, let $\bigwedge^{i+r-1} p' : \bigwedge^{i+r-1}(L) \to \bigwedge^{i+r-1}(L')$ and $\bigwedge^{j+r-1} p'' : \bigwedge^{j+r-1}(L) \to \bigwedge^{j+r-1}(L'')$ be the maps induced by the projections p' and p'', respectively. Define FG-equivariant homomorphisms

$$D'^r : \mathrm{Hom}_F\Big(\bigwedge^i(L), \bigwedge^{i+r-1}(L)\Big) \to \mathrm{Hom}_F\Big(\bigwedge^i(L'), \bigwedge^{i+r-1}(L')\Big)$$

by

$$D'^r(f)(x) = \bigwedge^{i+r-1} p'\big(f(x\otimes 1)\big) \quad \text{for all } x\in\bigwedge^i(L'),$$

and

$$D''^r : \mathrm{Hom}_F\Big(\bigwedge^j(L), \bigwedge^{j+r-1}(L)\Big) \to \mathrm{Hom}_F\Big(\bigwedge^j(L''), \bigwedge^{j+r-1}(L'')\Big)$$

by

$$D''^r(f)(y) = \bigwedge^{j+r-1} p''\big(f(1\otimes y)\big) \quad \text{for all } y\in\bigwedge^j(L'').$$

Lemma 3.4. *The map D'^r is given by the composition $p'_* \circ \iota'^*$, where*

$$\mathrm{Hom}\Big(\bigwedge^i(L), \bigwedge^{i+r-1}(L)\Big) \xrightarrow{\iota'^*} \mathrm{Hom}\Big(\bigwedge^i(L'), \bigwedge^{i+r-1}(L)\Big) \xrightarrow{p'_*} \mathrm{Hom}\Big(\bigwedge^i(L'), \bigwedge^{i+r-1}(L')\Big),$$

and D''^r is the composition $p''_* \circ \iota''^*$, where

$$\mathrm{Hom}\Big(\bigwedge^j(L), \bigwedge^{j+r-1}(L)\Big) \xrightarrow{\iota''^*} \mathrm{Hom}\Big(\bigwedge^j(L''), \bigwedge^{j+r-1}(L)\Big) \xrightarrow{p''_*} \mathrm{Hom}\Big(\bigwedge^j(L''), \bigwedge^{j+r-1}(L'')\Big).$$

Proof. This is an immediate consequence of the definitions of D'^r and D''^r. $\square$

Proposition 3.5. *Let $i \geqslant 0$ and $r \geqslant 2$. Suppose $E'^{s,t}_r(M) = E'^{s,t}_2(M)$ and $E^{s,t}_r(H_i(L,F)) = E^{s,t}_2(H_i(L,F))$ when $(s,t) = (0,i)$ and when $(s,t) = (r, i-r+1)$, for $M = H_i(L',F)$ and for $M = H_i(L,F)$. Then $d^{0,i}_r \circ (C'^1_0)_* = (C'^r_0)_* \circ d'^{0,i}_r$ and $d'^{0,i}_r \circ D'^1_* = D'^r_* \circ d^{0,i}_r$.*

Proof. The first claim asserts that the following diagram commutes.

$$\begin{array}{ccc} H^0(G, H^i(L', H_i(L',F))) & \xrightarrow{d'^{0,i}_r} & H^r(G, H^{i-r+1}(L', H_i(L',F))) \\ {\scriptstyle (C'^1_0)_*}\big\downarrow & & {\scriptstyle (C'^r_0)_*}\big\downarrow \\ H^0(G, H^i(L, H_i(L,F))) & \xrightarrow{d^{0,i}_r} & H^r(G, H^{i-r+1}(L, H_i(L,F))) \end{array}$$

By Lemma 3.2, we know that the map $(C'^*_0)_*$ is induced by the inclusion $\iota' : L' \to L$ and the projection $p' : L \to L'$. Hence, this diagram is the outer square of the commutative diagram below.

$$\begin{array}{ccc} H^0(G, H^i(L', H_i(L',F))) & \xrightarrow{d'^{0,i}_r} & H^r(G, H^{i-r+1}(L', H_i(L',F))) \\ {\scriptstyle \iota'_*}\big\downarrow & & {\scriptstyle \iota'_*}\big\downarrow \\ H^0(G, H^i(L', H_i(L,F))) & \xrightarrow{d'^{0,i}_r} & H^r(G, H^{i-r+1}(L', H_i(L,F))) \\ {\scriptstyle p'^*}\big\downarrow & & {\scriptstyle p'^*}\big\downarrow \\ H^0(G, H^i(L, H_i(L,F))) & \xrightarrow{d^{0,i}_r} & H^r(G, H^{i-r+1}(L, H_i(L,F))) \end{array}$$

The second claim follows by a similar argument using Lemma 3.4. $\square$

As previously noted, the category $\mathcal{M}_F(L,G)$ can be identified with the category of all finite rank FG-lattices. In view of this, we will not distinguish between $\mathcal{M}_F(L',G)$, $\mathcal{M}_F(L'',G)$, and $\mathcal{M}_F(L,G)$.

Lemma 3.6. *There are natural morphisms of spectral sequences $\pi'^* : E'_r \to E_r$, $\phi'^* : E_r \to E'_r$ and $\pi''^* : E''_r \to E_r$, $\phi''^* : E_r \to E''_r$, such that for all $r \geqslant 2$, all $s \geqslant 0$, and all coefficient modules in $\mathcal{M}_F(L,G)$,*

$$d'^{s,t}_r = \phi'^* \circ d^{s,t}_r \circ \pi'^* \quad \textit{and} \quad d''^{s,t}_r = \phi''^* \circ d^{s,t}_r \circ \pi''^*.$$

Proof. We observe that the map ι' induces a natural inclusion $\phi' : \Gamma' \to \Gamma$ and the map p' induces a natural projection $\pi' : \Gamma \to \Gamma'$ such that the composition $\pi' \circ \phi'$ is the identity map on Γ'. Let $\varphi : L \to \Gamma$ and $\varphi' : L' \to \Gamma'$ be the canonical inclusions. It follows that $\phi' \circ \varphi' = \varphi \circ \iota'$ and $\pi' \circ \varphi = \varphi' \circ p'$. Hence, the homomorphisms ϕ' and π' give rise to spectral sequence morphisms $\phi'^* : E_* \to E'_*$

and $\pi'^* : E'_* \to E_*$, such that $\phi'^* \circ \pi'^*$ is the identity morphism on $\{E'_*, d'_*\}$. Then, $d'^{s,t}_r = \phi'^* \circ \pi'^* \circ d'^{s,t}_r = \phi'^* \circ d^{s,t}_r \circ \pi'^*$. Analogously, ι'' and p'' induce an inclusion $\phi'' : \Gamma'' \to \Gamma$ and a projection $\pi'' : \Gamma \to \Gamma''$, respectively, such that the composition $\pi'' \circ \phi''$ is the identity map on Γ'' and $d''^{s,t}_r = \phi''^* \circ d^{s,t}_r \circ \pi''^*$. □

We are now ready to prove our main result.

Theorem 3.7. *Recall that G is an arbitrary group, and that L is a $\mathbb{Z}G$-lattice of finite rank that decomposes into a direct sum of the $\mathbb{Z}G$-lattices L' and L''. Let $r \geqslant 2$ and $t \geqslant 0$.*

(a) *Suppose $v^i_k(L') = v^j_k(L'') = 0$ for all $i, j \leqslant t$ and for all $k < r$. Then $d^{s,m}_2 = \cdots = d^{s,m}_{r-1} = 0$ for all $s \geqslant 0$, all $m \leqslant t$, and all coefficient modules in $\mathcal{M}_F(L, G)$. Additionally,*

$$v^t_r(L) = \sum_{i+j=t} \left(\left(C'^r_j\right)_* \left(v^i_r(L')\right) + (-1)^i \left(C''^r_i\right)_* \left(v^j_r(L'')\right)\right).$$

(b) *Suppose $v^t_k(L) = 0$ for all $k < r$. Then $d'^{s,t}_k = d''^{s,t}_k = 0$ for all $s \geqslant 0$, all $k < r$, and all coefficient modules in $\mathcal{M}_F(L, G)$. Additionally,*

$$v^t_r(L') = D'^r_*\left(v^t_r(L)\right) \quad \textit{and} \quad v^t_r(L'') = D''^r_*\left(v^t_r(L)\right).$$

Proof. To show part (a), we will use induction on $k \geqslant 2$ to prove that $v^m_k(L)$ satisfies the sum formula for all $m \leqslant t$ and all $k \leqslant r$. For each $k < r$, since $v^i_k(L') = v^j_k(L'') = 0$ for all $i, j \leqslant t$, this will imply $v^m_k(L) = 0$ and hence, by Theorem 2.2(b), $d^{s,m}_k = 0$ for all $s \geqslant 0$, all $m \leqslant t$, and all $M \in \mathcal{M}_F(L, G)$.

Suppose $d^{s,m}_2 = \cdots = d^{s,m}_{k-1} = 0$ for all $s \geqslant 0$, all $m \leqslant t$, and all $M \in \mathcal{M}_F(L, G)$. Recall that $v^m_k(L) = d^{0,m}_k([\mathrm{id}^m])$, where $\mathrm{id}^m \in \mathrm{Hom}_{FG}(\bigwedge^m(L), \bigwedge^m(L))$ is the identity map. Similarly, $v^i_k(L') = d'^{0,i}_k([\mathrm{id}'^i])$ and $v^j_k(L'') = d''^{0,j}_k([\mathrm{id}''^j])$. Consider the decomposition $\mathrm{id}^m = \sum_{i+j=m} \mathrm{id}_{ij}$, where $\mathrm{id}_{ij} : \bigwedge^{i+j}(L) \to \bigwedge^{i+j}(L)$ is the F-linear map given by

$$\mathrm{id}_{ij}(x \otimes y) = \begin{cases} x \otimes y & \text{if } x \in \bigwedge^i(L') \text{ and } y \in \bigwedge^j(L''), \\ 0 & \text{otherwise.} \end{cases}$$

Given $i, j \geqslant 0$, let $x \in \bigwedge^i(L')$ and $y \in \bigwedge^j(L'')$. Then, $\mathrm{id}_{ij}(x \otimes y) = x \otimes y = (x \otimes 1) \wedge (1 \otimes y) = C'^1_0(\mathrm{id}'^i)(x \otimes 1) \wedge C''^1_0(\mathrm{id}''^j)(1 \otimes y)$. This implies $[\mathrm{id}_{ij}] = (C'^1_0)_*([\mathrm{id}'^i]) \cdot (C''^1_0)_*([\mathrm{id}''^j]) \in H^0(G, H^{i+j}(L, H_{i+j}(L, F)))$, and thus

$$\left[\mathrm{id}^m\right] = \sum_{i+j=m} \left(C'^1_0\right)_* \left(\left[\mathrm{id}'^i\right]\right) \cdot \left(C''^1_0\right)_* \left(\left[\mathrm{id}''^j\right]\right).$$

By applying the product formula for the differentials, we compute

$$\begin{aligned} v^m_k(L) &= d^{0,m}_k\left(\sum_{i+j=m} \left(C'^1_0\right)_* \left(\left[\mathrm{id}'^i\right]\right) \cdot \left(C''^1_0\right)_* \left(\left[\mathrm{id}''^j\right]\right)\right) \\ &= \sum_{i+j=m} d^{0,m}_k\left(\left(C'^1_0\right)_* \left(\left[\mathrm{id}'^i\right]\right) \cdot \left(C''^1_0\right)_* \left(\left[\mathrm{id}''^j\right]\right)\right) \\ &= \sum_{i+j=m} \left(d^{0,i}_k\left(\left(C'^1_0\right)_* \left(\left[\mathrm{id}'^i\right]\right)\right) \cdot \left(C''^1_0\right)_* \left(\left[\mathrm{id}''^j\right]\right) + (-1)^i \left(C'^1_0\right)_* \left(\left[\mathrm{id}'^i\right]\right) \cdot d^{0,j}_k\left(\left(C''^1_0\right)_* \left(\left[\mathrm{id}''^j\right]\right)\right)\right) \end{aligned}$$

$$\overset{3.5}{=} \sum_{i+j=m} \left(\left(C_0'^{k}\right)_* \left(d_k'^{0,i}\left(\left[\mathrm{id}'^{i}\right]\right)\right) \cdot \left(C_0''^{1}\right)_* \left(\left[\mathrm{id}''^{j}\right]\right) + (-1)^i \left(C_0'^{1}\right)_* \left(\left[\mathrm{id}'^{i}\right]\right) \cdot \left(C_0''^{k}\right)_* \left(d_k''^{0,j}\left(\left[\mathrm{id}''^{j}\right]\right)\right)\right)$$

$$= \sum_{i+j=m} \left(\left(C_0'^{k}\right)_* \left(v_k^i(L')\right) \cdot \left(C_0''^{1}\right)_* \left(\left[\mathrm{id}''^{j}\right]\right) + (-1)^i \left(C_0'^{1}\right)_* \left(\left[\mathrm{id}'^{i}\right]\right) \cdot \left(C_0''^{k}\right)_* \left(v_k^j(L'')\right)\right)$$

$$= \sum_{i+j=m} \left(\left(C_j'^{k}\right)_* \left(v_k^i(L')\right) + (-1)^i \left(C_i''^{k}\right)_* \left(v_k^j(L'')\right)\right).$$

The last equality follows from the definitions of $C_*'^*$ and $C_*''^*$ and the fact that id'^i and id''^j are identity maps.

To prove (b), we will use induction on $k \geqslant 2$ to show that $v_k^t(L') = D_*'^r(v_k^t(L))$ and $v_k^t(L'') = D_*''^r(v_k^t(L))$ for all $k \leqslant r$. By Theorem 2.2, for each $k < r$ and for all $s \geqslant 0$ this will imply that $d_k'^{s,t} = d_k''^{s,t} = 0$ for all coefficient modules in $\mathcal{M}_F(L,G)$. Note that this assertion also follows from Lemma 3.6.

For the identity map $\mathrm{id}'^t : \bigwedge^t(L') \to \bigwedge^t(L')$, we have $\mathrm{id}'^t(x) = x = \bigwedge^t p'(x \otimes 1) = \bigwedge^t p'(\mathrm{id}^t(x \otimes 1)) = D'^1(\mathrm{id}^t)(x)$ for all $x \in \bigwedge^t(L')$. This implies $[\mathrm{id}'^t] = D_*'^1([\mathrm{id}^t]) \in E_k'^{0,t}(H_t(L',F))$. Thus, it follows

$$v_k^t(L') = d_k'^{0,t}\left(\left[\mathrm{id}'^t\right]\right) = d_k'^{0,t} \circ D_*'^1\left(\left[\mathrm{id}^t\right]\right) \overset{3.5}{=} D_*'^k \circ d_k^{0,t}\left(\left[\mathrm{id}^t\right]\right) = D_*'^k\left(v_k^t(L)\right).$$

By an analogous argument, $v_k^t(L'') = D_*''^k(v_k^t(L))$. □

Remark 3.8. Using the same assumptions as in Theorem 3.7(a), the sum formula can be simplified. Since $E_r'^{r,i-r+1} = E_r''^{r,j-r+1} = 0$, $v_r^i(L') = v_r^j(L'') = 0$ hold a priori for all $i, j < r-1$. It is an easy exercise to check that $d_r'^{0,r-1} = 0$ and $d_r''^{0,r-1} = 0$ for all $M \in \mathcal{M}_F(L,G)$, since they are differentials with target in the 0th row (see Proposition 1, [3]). Therefore, $v_r^{r-1}(L') = v_r^{r-1}(L'') = 0$ and we have

$$v_r^t(L) = \sum_{i+j=t} \left(\left(C_j'^{r}\right)_* \left(v_r^i(L')\right) + (-1)^i \left(C_i''^{r}\right)_* \left(v_r^j(L'')\right)\right)$$

$$= \sum_{i+j=t-r} \left(C_j'^{r}\right)_* \left(v_r^{i+r}(L')\right) + \sum_{i+j=t-r} (-1)^i \left(C_i''^{r}\right)_* \left(v_r^{j+r}(L'')\right)$$

$$= \sum_{i+j=t-r} \left(\left(C_j'^{r}\right)_* \left(v_r^{i+r}(L')\right) + (-1)^i \left(C_i''^{r}\right)_* \left(v_r^{j+r}(L'')\right)\right).$$

4. Some corollaries

Theorem 3.7 has particularly interesting applications if characteristic classes of the $\mathbb{Z}G$-lattices L', L'', and L are viewed as obstructions to the vanishing of the differentials in the associated Lyndon–Hochschild–Serre spectral sequences. We use the same notation as before.

Corollary 4.1. *Let $r \geqslant 2$. If $v_k^i(L') = v_k^j(L'') = 0$ for all $i, j \in [k,r]$ and for all $k < r$, then $d_2^{s,m} = \cdots = d_{r-1}^{s,m} = 0$ for all $s \geqslant 0$, all $m \leqslant r$, and all $M \in \mathcal{M}_F(L,G)$. Moreover,*

$$v_r^r(L) = \left(C_0'^{r}\right)_* \left(v_r^r(L')\right) + \left(C_0''^{r}\right)_* \left(v_r^r(L'')\right).$$

Proof. This is a direct consequence of Theorem 3.7 and the preceding remark. □

A consequence of Lemma 3.6 is the fact that when the differentials in the Lyndon–Hochschild–Serre spectral sequence $\{E_*, d_*\}$ are all zero, the same is true for the spectral sequences corresponding to the $\mathbb{Z}G$-sublattices L' and L''. The next corollary gives us a converse.

Corollary 4.2. *Suppose $v_p^t(L') = v_q^t(L'') = 0$ for all $p \leqslant \dim(L')$, all $q \leqslant \dim(L'')$, and all $t \geqslant 0$. Then $d_k^{s,t} = 0$ for all $M \in \mathcal{M}_F(L, G)$, all $s, t \geqslant 0$, and all $k \geqslant 2$. Moreover, if in addition $\{E_*, d_*\}$ has no extension problems, then for every $n \geqslant 0$ and for all $M \in \mathcal{M}_F(L, G)$ we have*

$$H^n(\Gamma, M) = \bigoplus_{i+j=n} H^i\big(G, H^j(L, M)\big).$$

Proof. Since $v_p^t(L') = d_p'^{0,t}([\mathrm{id}'^t]) \in E_p^{p,t-p+1}(H_t(L', F))$, this class lies in the image of the map

$$d_p'^{0,t} : H^0\big(G, H^t\big(L', H_t(L', F)\big)\big) \to H^p\big(G, H^{t-p+1}\big(L', H_t(L', F)\big)\big).$$

Note that $d_p'^{0,t} = 0$ when $t > \dim(L')$ or $p > t + 1$. If $p = t + 1$, then $d_{t+1}'^{0,t} = 0$, since it is a differential with a target in the 0th row (see Proposition 1, [3]). Therefore, if $v_p^*(L') = 0$ for all $p \leqslant \dim(L')$, then all characteristic classes of the spectral sequence $\{E'_*, d'_*\}$ are zero. A similar argument shows that all characteristic classes of $\{E''_*, d''_*\}$ are zero when $v_q^*(L'') = 0$ for all $q \leqslant \dim(L'')$. □

Corollary 4.3. *Let $t \geqslant 0$. Set $\Gamma' = L' \rtimes G$ and $\Gamma'' = L'' \rtimes G$. Let $\varphi' : L' \to \Gamma'$, $\varphi'' : L'' \to \Gamma''$, and $\varphi : L \to \Gamma$ be the natural inclusions.*

(a) *If $\varphi'^* : H^m(\Gamma', H_m(L', F)) \to H^m(L', H_m(L', F))^G$ and $\varphi''^* : H^m(\Gamma'', H_m(L'', F)) \to H^m(L'', H_m(L'', F))^G$ are surjective for all $m \leqslant t$, then $\varphi^* : H^m(\Gamma, M) \to H^m(L, M)^G$ is surjective for all $m \leqslant t$ and for all $M \in \mathcal{M}_F(L, G)$.*
(b) *If $\varphi^* : H^t(\Gamma, M) \to H^t(L, M)^G$ is surjective, then $\varphi'^* : H^t(\Gamma', M) \to H^t(L', M)^G$ and $\varphi''^* : H^t(\Gamma'', M) \to H^t(L'', M)^G$ are surjective for all $M \in \mathcal{M}_F(L, G)$.*

Proof. To prove (a), we observe that Corollary 2.4 implies $v_r^i(L') = v_r^j(L'') = 0$ for all $r \geqslant 0$ and for all $i, j \leqslant t$. Then, by Theorem 3.7, $d_r^{s,m} = 0$ for all $r, s \geqslant 0$, all $m \leqslant t$, and all coefficient modules in $\mathcal{M}_F(L, G)$. Applying again Corollary 2.4 finishes the proof.

For part (b), let $\iota'^* : H^t(L, M)^G \to H^t(L', M)^G$, $\phi'^* : H^t(\Gamma, M) \to H^t(\Gamma', M)$, and $p'^* : H^t(L', M)^G \to H^t(L, M)^G$ be the induced maps of the inclusions $\iota' : L' \to L$, $\phi' : \Gamma' \to \Gamma$, and the projection $p' : L \to L'$, respectively. Then, we have $\varphi'^* \circ \phi'^* = \iota'^* \circ \varphi^*$. Since $\iota'^* \circ p'^*$ is the identity map on $H^t(L', M)^G$, ι'^* is surjective and hence $\iota'^* \circ \varphi^*$ is surjective. The previous equality shows that φ'^* is also surjective. Similarly, it follows that φ''^* is surjective. □

Given an arbitrary finite rank integral $\mathbb{Z}G$-lattice L, Lieberman's result (see Theorem 4, [3]) states that in the associated Lyndon–Hochschild–Serre spectral sequence $\{E_*, d_*\}$, for any $s, t \geqslant 0$ and $r \geqslant 2$, the image of the differential $d_r^{s,t}$ is a torsion group annihilated by the integers $m^{t-r+1}(m^{r-1} - 1)$ for all $m \in \mathbb{Z}$. Using this fact, it was proved in [3] that if F is a field of nonzero characteristic p, then $d_r^{s,t} = 0$ for all $s, t \geqslant 0$ and all $r < p$. The next corollary follows from combining this result with the sum formula of Remark 3.8.

Corollary 4.4. *Let F be a field of nonzero characteristic p. Assume L' and L'' are $\mathbb{Z}G$-lattices of finite rank and $L = L' \oplus L''$.*

(a) *$d_r^{s,t} = 0$ for all $s, t \geqslant 0$, all $r < p$, and all $M \in \mathcal{M}_F(L, G)$.*
(b) *$d_p^{s,t}(x) = (-1)^s y \cdot \sum_{i+j=t-p}((C_j'^p)_*(v_p^{i+p}(L')) + (-1)^i (C_i''^p)_*(v_p^{j+p}(L'')))$ for all $M \in \mathcal{M}_F(L, G)$, for all $x \in E_p^{s,t}(M)$, and for all $y \in E_p^{s,0}(H^t(L, M))$ such that $\theta(y) = x$.*

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