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Special functions arising from discrete Painlevé equations: A survey

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Abstract

This article is a survey on recent studies on special solutions of the discrete Painlevé equations, especially on hypergeometric solutions of the q-Painlevé equations. The main part of this survey is based on the joint work [K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta, Y. Yamada, Hypergeometric solutions to the q-Painlevé equations, IMRN 2004 47 (2004) 2497–2521, K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta, Y. Yamada, Construction of hypergeometric solutions to the q-Painlevé equations, IMRN 2005 24 (2005) 1439–1463] with Kajiwara, Masuda, Ohta and Yamada. After recalling some basic facts concerning Painlevé equations for comparison, we give an overview of the present status of studies on difference (discrete) Painlevé equations as a source of special functions.

Keywords: Special functions; Discrete Painlevé equation; Basic hypergeometric series

1. Painlevé equations

Let us consider a second-order nonlinear ordinary differential equation

$$y'' = R(t; y, y') \tag{1}$$

for the dependent variable $y = y(t)$, where $R(t; y, \eta)$ is a rational function in (t, y, η) and $' = \mathrm{d}/\mathrm{d}t$ is the derivation with respect to the independent variable t. Such a differential equation is said to have the *Painlevé property* if any solution of (1) has no movable singular point except for poles. It is known that any rational differential equation of second order with the Painlevé property is reduced to one of the six Painlevé equations $P_{\mathrm{I}}, P_{\mathrm{II}}, \ldots, P_{\mathrm{VI}}$ unless it can be integrated algebraically, or transformed into a simpler equation such as the linear differential equations or the differential equations of the elliptic functions. (See Table 1. In this list, $\alpha, \beta, \ldots$ are complex parameters.) Generic solutions of the Painlevé equations are known to be *very transcendental*. For the general background of Painlevé equations, we refer the reader to Umemura's survey [31]. (See also [2,7,10,21,23].)

These Painlevé equations have various characteristic features that could not be imagined only from the Painlevé property, as clarified through the works of Okamoto [24] in 1980s. We remark first that the six Painlevé equations are

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Table 1
The six Painlevé equations

P_{I}:	$y''=6y^2+t$
P_{II}:	$y''=2y^3+ty+\alpha$
P_{III}:	$y''=\frac{1}{y}(y')^2-\frac{1}{t}y'+\frac{1}{t}(\alpha y^2+\beta)+\gamma y^3+\frac{\delta}{y}$
P_{IV}:	$y''=\frac{1}{2y}(y')^2+\frac{3}{2}y^3+4ty^2+2(t^2-\alpha)y+\frac{\beta}{y}$
P_{V}:	$y''=\left(\frac{1}{2y}+\frac{1}{y-1}\right)(y')^2-\frac{1}{t}y'+\frac{(y-1)^2}{t^2}\left(\alpha y+\frac{\beta}{y}\right)+\frac{\gamma}{t}y+\delta\frac{y(y+1)}{y-1}$
P_{VI}:	$y''=\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)(y')^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)y'+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right)$

linked by the *degeneration diagram* (or the coalescence cascade)

$$\begin{array}{ccccccc} P_{\mathrm{VI}} & \to & P_{\mathrm{V}} & \to & P_{\mathrm{III}} & & \\ & & & \searrow & & \searrow & \\ & & & & P_{\mathrm{IV}} & \to & P_{\mathrm{II}} \;\to\; P_{\mathrm{I}}. \end{array} \tag{2}$$

Each of the Painlevé equations P_{J} ($\mathrm{J}=\mathrm{II},\mathrm{III},\mathrm{IV},\mathrm{V},\mathrm{VI}$) except for P_{I} admits a group of *Bäcklund transformations* (transformations of variables and parameters that leave the equation invariant). This group of symmetry is in fact isomorphic to an (extended) *affine Weyl group*; the type of the corresponding affine root system is given as follows.

$$\begin{array}{ccccccc} D_4^{(1)} & \to & A_3^{(1)} & \to & (2A_1)^{(1)} & & \\ & & & \searrow & & \searrow & \\ & & & & A_2^{(1)} & \to & A_1^{(1)} \;\to\; \cdot \end{array} \tag{3}$$

This means that the parameter space of P_{J} is identified with the Cartan subalgebra of a semisimple Lie algebra, and that the natural action of the affine Weyl group on it can be lifted to the level of dependent variables of the differential equation. The points on the walls (the reflecting hyperplanes) in the parameter space are special in the sense that the corresponding P_{J} admits a one-parameter family of special solutions expressed in terms of hypergeometric functions and their confluences. The degeneration diagram (2) of the Painlevé equations is now mapped to the confluence diagram of their *hypergeometric solutions*

$$\begin{array}{ccccccc} \text{Gauss} & \to & \text{Kummer} & \to & \text{Bessel} & & \\ & & & \searrow & & \searrow & \\ & & & & \text{Hermite-Weber} & \to & \text{Airy} \;\to\; \cdot \end{array} \tag{4}$$

Besides the hypergeometric solutions along the walls, the Painlevé equations happen to have algebraic solutions. They arise typically when the parameters take values corresponding to some fixed point of the extended affine Weyl group. The algebraic solutions of the Painlevé equations up to P_{V} are already classified (see [8,19,17]). As to the algebraic solutions of P_{VI}, however, there are still many interesting works in progress [3,1,16] and the final picture has not been clarified yet.

Let us explain how hypergeometric solutions arise from Painlevé equations, by taking the example of P_{II}:

$$P_{\mathrm{II}}:\quad y''=2y^3+ty+b-\tfrac{1}{2}. \tag{5}$$

Supposing that the variable y is subject to this equation, let us consider the variable $\widetilde{y}=y+b/(y'+y^2+t/2)$. Then one can show that $\widetilde{y}$ satisfies the same Painlevé equation P_{II} with the parameter b replaced by $\widetilde{b}=-b$. This transformation of variables $y \to \widetilde{y}$, $b \to \widetilde{b}$ is a fundamental *Bäcklund transformation* of P_{II}, which we now denote by S:

$$S(y)=y+\frac{b}{y'+y^2+t/2}, \quad S(b)=-b, \quad S(t)=t. \tag{6}$$

(We consider S as an automorphism of the differential field $\mathbb{C}(t, y, y', b)$ defined by P_{II}.) Another fundamental Bäcklund transformation of P_{II} is given by

$$T(y)=-y+\frac{b-1}{y'-y^2-t/2}, \quad T(b)=b-1, \quad T(t)=t. \tag{7}$$

The group $\langle S, T\rangle$ generated by these two Bäcklund transformations is the affine Weyl group of type $A_1^{(1)}$. In this case the parameter space for P_{II} is $\mathbb{C}$ with coordinate b, and its integer points are the walls of the affine Weyl group.

If y is a solution of the first-order equation $y'=-y^2-t/2$, then it satisfies $y''=-2yy'-1/2=2y^3+ty-1/2$. Namely, the *Riccati equation* $y'=-y^2-t/2$ implies the Painlevé equation P_{II} with $b=0$. This Riccati equation is reduced to the Airy equation $u''+tu/2=0$ by the change of variables $y=-u'/u$. Accordingly, from the general solution $u(t)=c_1\varphi(t)+c_2\varphi(t)$ of the Airy equation we obtain the solutions

$$y(t)=-\frac{u'(t)}{u(t)}=-\frac{c_1\varphi_1'(t)+c_2\varphi_2'(t)}{c_1\varphi_1(t)+c_2\varphi_2(t)} \tag{8}$$

of P_{II} with $b=0$ parametrized by the ratio $[c_1 : c_2] \in \mathbb{P}^1$. From this *seed solutions* at $b=0$, one can also construct solutions at $b \in \mathbb{Z}$ by Bäcklund transformations. Such a solution is written as the determinant of a matrix whose entries are expressed in terms of Airy functions and their derivatives ([24], see also [18]).

2. Discrete Painlevé equations

It would be natural to ask: What are the counterparts of Painlevé equations in difference equations? What properties would they share? What sort of special functions would arise as their solutions?

In order to fix the idea, we now consider a system of nonautonomous difference equation for two dependent variables $f=f(x)$ and $g=g(x)$ in the form

$$f(x+\delta)=R(x; f(x), g(x)), \quad g(x+\delta)=S(x; f(x), g(x)), \tag{9}$$

with δ being a nonzero complex constant. Here we suppose that $R(x; f, g)$ and $S(x; f, g)$ are rational functions in f, g with coefficients in a field $\mathscr{K}$ of functions in x. Depending on the class of functions appearing in the coefficients, it is customary to consider the following three type of difference equations:

$\mathcal{K}$	type of difference equations
$\mathbb{C}(x)$	rational/ additive
$\mathbb{C}(e^x)$	trigonometric / multiplicative
$\mathbb{C}(\wp(x), \wp'(x))$	elliptic

. (10)

Difference equations of trigonometric type are often called *q-difference equations*; the shift $x \to x+\delta$ is then regarded as the multiplicative shift $t \to qt$ for the variable $t=\mathrm{e}^x$ by $q=\mathrm{e}^{\delta}$.

Since the pioneering work of Grammaticos, Ramani, Papageorgiou and Hietarinta [6,26] in early 1990s, discrete (or difference) Painlevé equations have been studied from various viewpoints. A large class of discrete Painlevé equations, as well as their generalizations, has been discovered through the studies of singularity confinement property (discrete analogue of the Painlevé property), bilinear equations, affine Weyl group symmetries and spaces of initial conditions [27,28,22,30]. For the history of discrete Painlevé equations, we refer the reader to Grammaticos–Ramani [5].

As for discrete Painlevé equations of second order, it seems to be standard nowadays to refer to Sakai's class [30] defined by means of geometry of rational surfaces. Sakai's framework not only fits nicely with discrete Painlevé equations discovered earlier by different approaches, but also clarifies their connection with geometry of plane curves and Cremona transformations. Each equation in this class is defined by an affine Weyl group of Cremona transformations on a certain family of rational surfaces obtained from $\mathbb{P}^2$ by blowing up.

Sakai's list of discrete Painlevé equations consists of one elliptic (*eP*), nine trigonometric (*qP*) and nine rational (*dP*) difference equations. The (only) elliptic difference Painlevé equations is associated with the affine Weyl group of type $E_8^{(1)}$. The diagrams of affine Weyl groups for the nine q-Painlevé equations and for the nine d-Painlevé equations are given as follows, respectively.

$$\begin{array}{lllllllll} qP: & E_8^{(1)} \to E_7^{(1)} \to E_6^{(1)} \to D_5^{(1)} \to A_4^{(1)} \to (A_2+A_1)^{(1)} \to (A_1+A_1')^{(1)} \to & A_1'^{(1)} \\ & & \searrow A_1^{(1)} \\ dP: & E_8^{(1)} \to E_7^{(1)} \to E_6^{(1)} \to D_4^{(1)} \to A_3^{(1)} \to (2A_1)^{(1)} \to A_1'^{(1)} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \searrow \qquad\quad \searrow \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad A_2^{(1)} \to A_1^{(1)} \end{array} \tag{11}$$

We remark that the d-Painlevé equations of type $D_4^{(1)}$ and below arise as Bäcklund (Schlesinger) transformations of differential Painlevé equations. The q-Painlevé equation of type $D_5^{(1)}$ is the q-Painlevé VI equation of Jimbo–Sakai [11]. It is also known from [11] that qP_{VI} of Jimbo–Sakai has q-hypergeometric solutions expressed in terms of ${}_2\phi_1$ series, or the little q-Jacobi polynomials in terminating cases. Hence it would be natural to expect that the hierarchy of q-Painlevé equations admits such q-hypergeometric solutions that correspond to q-hypergeometric orthogonal polynomials in Askey's scheme in terminating cases.

3. q-Hypergeometric functions

Before going further, we recall some notation concerning q-hypergeometric functions from [4].

Fixing a nonzero complex number q with $|q| < 1$, we use the notation of q-shifted factorials

$$(z;q)_N = \prod_{j=0}^{N-1}(1-q^j z), \quad (z;q)_\infty = \prod_{j=0}^{\infty}(1-q^j z) \tag{12}$$

and $(z_1,\ldots,z_m;q)_N = (z_1;q)_N\cdots(z_m;q)_N$. The q-hypergeometric series ${}_r\phi_s$ are defined by

$${}_r\phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}; q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_k}{(q,b_1,\ldots,b_s;q)_k}((-1)^k q^{\binom{k}{2}})^{1+s-r} z^k. \tag{13}$$

The *very well-poised* q-hypergeometric series ${}_{r+3}W_{r+2}$ are defined by

$$\begin{aligned} &{}_{r+3}W_{r+2}(a_0;a_1,\ldots,a_r;q,z) \\ &= {}_{r+3}\phi_{r+2}\left(\begin{matrix} a_0, qa_0^{1/2}, -qa_0^{1/2}, a_1,\ldots,a_r \\ a_0^{1/2}, -a_0^{1/2}, qa_0/a_1,\ldots,qa_0/a_r \end{matrix}; q, z\right) \\ &= \sum_{k=0}^{\infty} \frac{1-q^{2k}a_0}{1-a_0} \frac{(a_0,a_1,\ldots,a_r;q)_k}{(q,qa_0/a_1,\ldots,qa_0/a_r;q)_k} z^k. \end{aligned} \tag{14}$$

The balanced ${}_8W_7$-series

$$\begin{aligned} \Phi(a;b,c,d,e,f) &= {}_8W_7(a;b,c,d,e,f;q,x) \\ &= \sum_{k=0}^{\infty} \frac{1-q^{2k}a}{1-a} \frac{(a,b,c,d,e,f;q)_k\, x^k}{(q,qa/b,qa/c,qa/d,qa/e,qa/f;q)_k}, \quad x = \frac{q^2a^2}{bcdef} \end{aligned} \tag{15}$$

are often called *Askey–Wilson functions*. They satisfy the following *three-term recurrence relation* [9]:

$$
\begin{aligned}
&A_1(\overline{\Phi}-\Phi)+A_2\Phi+A_3(\underline{\Phi}-\Phi)=0, \quad \overline{\Phi}=\Phi(a;b,qc,d/q,e,f), \quad \underline{\Phi}=\Phi(a;b,c/q,qd,e,f),\\
&A_1=\frac{(1-c)(1-a/c)(1-qa/c)(1-qa/bd)(1-qa/de)(1-qa/df)}{e(1-c/d)(1-qc/d)},\\
&A_2=\frac{qa^2}{bcdef}(1-qa/cd)(1-b)(1-e)(1-f),\\
&A_3=\frac{(1-d)(1-a/d)(1-qa/d)(1-qa/bc)(1-qa/ce)(1-qa/cf)}{c(1-d/c)(1-qd/c)}.
\end{aligned} \tag{16}
$$

4. The case of q-Painlevé equation of type $E_7^{(1)}$

As an example, let us consider the q-Painlevé equation associated with the affine Weyl group of type $E_7^{(1)}$ [29,30]:

$$
\begin{aligned}
&\frac{(f\overline{g}-qt^2)(fg-t^2)}{(f\overline{g}-1)(fg-1)}=\frac{(f-b_1t)(f-b_2t)(f-b_3t)(f-b_4t)}{(f-b_5)(f-b_6)(f-b_7)(f-b_8)},\\
&\frac{(fg-t^2)(\underline{f}g-t^2/q)}{(fg-1)(\underline{f}g-1)}=\frac{(g-t/b_1)(g-t/b_2)(g-t/b_3)(g-t/b_4)}{(g-1/b_5)(g-1/b_6)(g-1/b_7)(g-1/b_8)},
\end{aligned} \tag{17}
$$

where $f=f(t)$ and $g=g(t)$ are the dependent variables. For a function $\varphi=\varphi(t)$ of t, we use the notation $\overline{\varphi}=\varphi(qt)$ and $\underline{\varphi}=\varphi(t/q)$. The symbol $b_1,b_2,\ldots,b_8$ stand for parameters with $b_1b_2b_3b_4=q$ and $b_5b_6b_7b_8=1$. ($\overline{b}_j=b_j$ for $j=1,2,\ldots,8$.) The first equation can be considered as defining $\overline{g}$ by (f,g), and the second as defining f by $(\underline{f},g)$. Hence this system can be rewritten into the form $\overline{f}=F(t;f,g)$, $\overline{g}=G(t;f,g)$ of discrete time evolution, where F, G are rational functions in f, g.

We now explain how one can find q-hypergeometric solutions of this q-Painlevé equation. When $b_1b_3=b_5b_7$ (and $b_2b_4=qb_6b_8$), this system is decoupled consistently into the following four equations [20]:

$$
\begin{aligned}
&\frac{f\overline{g}-qt^2}{f\overline{g}-1}=\frac{(f-b_2t)(f-b_4t)}{(f-b_6)(f-b_8)}, \quad \frac{fg-t^2}{fg-1}=\frac{(f-b_1t)(f-b_3t)}{(f-b_5)(f-b_7)},\\
&\frac{\underline{f}g-t^2/q}{fg-1}=\frac{(g-t/b_2)(g-t/b_4)}{(g-1/b_6)(g-1/b_8)}, \quad \frac{fg-t^2}{fg-1}=\frac{(g-t/b_1)(g-t/b_3)}{(g-1/b_5)(g-1/b_7)}.
\end{aligned} \tag{18}
$$

Hence we obtain the *discrete Riccati equation* for g:

$$
\begin{aligned}
&\overline{g}=\frac{(qt^2-1)f+t\{-q(b_6+b_8)t+(b_2+b_4)\}}{\{-(b_6+b_8)+(b_2+b_4)t\}f+b_6b_8(1-qt^2)},\\
&f=\frac{(t^2-1)b_5b_7g+t\{(b_1+b_3)-(b_5+b_7)t\}}{\{t(b_1+b_3)-(b_5+b_7)\}g+(1-t^2)}.
\end{aligned} \tag{19}
$$

Namely, $\overline{g}$ is determined from g by a fractional linear transformation $\overline{g}=(Pg+Q)/(Rg+S)$.

In general let us consider a discrete Riccati equation of the form

$$
\overline{z}=\frac{Az+B}{Cz+D}, \tag{20}
$$

where $z=z(t)$, $\overline{z}=z(qt)$, and A,B,C,D are functions in t. By introducing two dependent variables F, G such that $z=F/G$, the equation for z is linearized as

$$
\overline{F}=(AF+BG)H, \quad \overline{G}=(CF+DG)H. \tag{21}
$$

Namely, if F and G satisfy these two equations for some H, then $z=F/G$ satisfies the original discrete Riccati equation. Hence we obtain two q-difference equations of second order to be satisfied by F and G:

$$\overline{F}+c_1F+c_2\underline{F}=0,\quad \overline{G}+d_1G+d_2\underline{G}=0.$$

By choosing an appropriate factor H, one may expect that these equations for F and G could be identified with the second-order q-difference equations (three-term recurrence relations) for some hypergeometric functions.

In the case of the discrete Riccati equation (19) for g, we need to apply a fractional linear transformation

$$z=\frac{1-b_3/b_5t}{1-b_3/b_1}\,\frac{g-t/b_1}{g-1/b_5} \tag{22}$$

in order that the corresponding discrete Riccati equation can be solved by known special functions. Then by choosing an appropriate factor H, the second-order linear q-difference equations (21) are identified with those for Askey–Wilson functions (very well-poised, balanced ${}_8\phi_7$ series) (16). In this way we obtain the solution

$$z=\frac{{}_8W_7(a;qb,c,d,e,f;x/q)}{{}_8W_7(a;b,c,d,e,f;x)},\quad x=\frac{q^2a^2}{bcdef},$$

where

$$(a;b,c,d,e,f)=\left(\frac{b_1b_8}{b_3b_5};\frac{b_8}{b_5},\frac{b_2}{b_3},\frac{b_4}{b_3},\frac{b_1t}{b_5},\frac{b_1}{b_5t}\right).$$

5. Hypergeometric solutions of the q-Painlevé equations

Each discrete Painlevé equation in Sakai's list can be regarded as defining a second-order nonlinear difference system with respect to n time variables, where n stands for the rank of the translation lattice of the affine Weyl group. If we choose some direction in the lattice for the discrete time evolution, then the other coordinates for the complementary directions are regarded as parameters. In the previous section, we presented an explicit form of the q-Painlevé equation of type (17) of $E_7^{(1)}$ without referring to its geometric origin and the corresponding affine Weyl group action. In this example the variables $t,b_1,\ldots,b_8$ with the constraints $b_1b_2b_3b_4=q$ and $b_5b_6b_7b_8=1$ are the coordinates of a 7-dimensional algebraic torus, among which t is chosen for the time variable. We also remark that the decoupling of (17) into a discrete Riccati equation was carried out after restricting the parameters to the hypersurface $b_1b_3=b_5b_7$ which is in fact a wall of the affine Weyl group.

We now consider the following seven q-Painlevé equations for which the translation lattice of the affine Weyl group has rank $\geqslant 2$:

$$qP:\quad E_8^{(1)}\to E_7^{(1)}\to E_6^{(1)}\to D_5^{(1)}\to A_4^{(1)}\to (A_2+A_1)^{(1)}\to (A_1+A_1')^{(1)}. \tag{23}$$

In each case, among the coordinates for an algebraic torus we choose one direction of discrete time evolution, and regard the other coordinates as parameters. Then by restricting the parameters to an appropriate wall, we obtain Riccati equations that can be linearized to some q-hypergeometric equations. (This procedure of restriction to a wall makes sense only in the cases of rank $\geqslant 2$.) In [13,14], it is shown explicitly that q-hypergeometric functions as in Table 2 arise as Riccati solutions to the q-Painlevé equations of (23) in this way.

For the results concerning the elliptic difference Painlevé equation (of type $E_8^{(1)}$) and its hypergeometric solutions, we refer the reader to [12,15,25].

These results on hypergeometric solutions are starting points of the study of nonlinear special functions in connection with discrete Painlevé equations. If we take hypergeometric solutions as seed solutions, we can construct more special solutions by applying Bäcklund transformations to them. Such solutions would be expressed as determinants of matrices of particular forms whose entries are hypergeometric functions. Also, in order to understand the whole picture of special solutions, it would be necessary to formulate τ-functions for all discrete Painlevé equations in a unified manner. We expect that such a theory of nonlinear special functions will bring out new insights to the world of hypergeometric functions.

Table 2
Hypergeometric solutions of the q-Painlevé equations

Weyl Group symmetry	Hypergeometric function	Terminating case
$E_8^{(1)}$	Balanced ${}_{10}W_9 + {}_{10}W_9$	Ismail–Masson–Rahman
$E_7^{(1)}$	${}_8W_7\left(a; b, c, d, e, f; q, \frac{q^2a^2}{bcdef}\right)$	Askey–Wilson
$E_6^{(1)}$	${}_3\varphi_2\left(\begin{matrix} a, & b, & c \\ & d, & e \end{matrix}; q, \frac{de}{abc}\right)$	Big q-Jacobi
$D_5^{(1)}$ (qP_{VI})	${}_2\varphi_1\left(\begin{matrix} a & b \\ & c \end{matrix}; q, z\right)$	Little q-Jacobi
$A_4^{(1)}$ (qP_{V})	${}_1\varphi_1\left(\begin{matrix} a \\ b \end{matrix}; q, z\right)$	q-Laguerre
$(A_2 + A_1)^{(1)}$ $(qP_{\mathrm{III,IV}})$	${}_1\varphi_1\left(\begin{matrix} a \\ 0 \end{matrix}; q, z\right)$, ${}_1\varphi_1\left(\begin{matrix} 0 \\ b \end{matrix}; q, z\right)$	Stieltjes–Wigert
$(A_1 + A_1')^{(1)}$ (qP_{II})	${}_1\varphi_1\left(\begin{matrix} 0 \\ -q \end{matrix}; q, z\right)$	

References

[1] P.P. Boalch, The fifty-two icosahedral solutions to Painlevé VI, prepint 2004 (math. AG/0406281).
[2] in: R. Conte (Ed.), The Painlevé Property—One Century Later, CRM Series in Mathematical Physics, Springer, Berlin, 1999.
[3] B. Dubrovin, M. Mazzocco, Monodromy of certain Painlevé VI transcendents and reflection groups, Invent. Math. 141 (2000) 55–147.
[4] G. Gasper, M. Rahman, Basic Hypergeometric Series, second ed., Encyclopedia of Mathematics and its Applications, vol. 96, Cambridge University Press, Cambridge, 2004.
[5] B. Grammaticos, A. Ramani, Discrete Painlevé equations: a review, in: B. Grammaticos, Y. Kosmann-Schwarzbach, T. Tamizhmani (Eds.), Discrete Integrable Systems, Lecture Notes in Physics, vol. 644, Springer, Berlin, 2004, pp. 245–321.
[6] B. Grammaticos, A. Ramani, V. Papageorgiou, Do integrable mappings have the Painlevé property?, Phys. Rev. Lett. 67 (1991) 1825–1828.
[7] V.I. Gromak, I. Laine, S. Shimomura, Painlevé Differential Equations in the Complex Plane, de Gruyter Studies in Mathematics, vol. 28, Walter de Gruyter & Co., Berlin, 2002.
[8] V.I. Gromak, N.A. Lukashevich, Special class of solutions of Painlevé equations, Differential Equations 18 (1982) 317–326.
[9] M. Ismail, M. Rahman, The associated Askey–Wilson polynomials, Trans. Amer. Math. Soc. 328 (1991) 201–237.
[10] K. Iwasaki, H. Kimura, S. Shimomura, M. Yoshida, From Gauss to Painlevé—A Modern Theory of Special Functions, Aspects of Mathematics, vol. E16, Vieweg, 1991.
[11] M. Jimbo, H. Sakai, A q-analog of the sixth Painlevé equation, Lett. Math. Phys. 38 (1996) 145–154.
[12] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta, Y. Yamada, ${}_{10}E_9$ solution to the elliptic Painlevé equation, J. Phys. A 36 (2003) L263–L272.
[13] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta, Y. Yamada, Hypergeometric solutions to the q-Painlevé equations, IMRN 2004 47 (2004) 2497–2521.
[14] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta, Y. Yamada, Construction of hypergeometric solutions to the q-Painlevé equations, IMRN 2005 24 (2005) 1439–1463.
[15] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta, Y. Yamada, Point configurations, Cremona transformations and the elliptic difference Painlevé equation, preprint (nlin.SI/0411003).
[16] A.V. Kitaev, Remarks towards the classification of $RS_4^2(3)$-transformations and algebraic solutions of the sixth Painlevé equation, preprint (math.CA/0503082).
[17] A.V. Kitaev, C.K. Law, J.B. McLeod, Rational solutions of the fifth Painlevé equation, Differential Integral Equations 7 (1994) 967–1000.
[18] T. Masuda, Classical transcendental solutions of the Painlevé equations and their degeneration, Tohoku Math. J. 56 (2004) 467–490.
[19] Y. Murata, Rational solutions of the second and the fourth Painlevé equations, Funkcial. Ekvac. 28 (1985) 1–32.
[20] M. Murata, H. Sakai, J. Yoneda, Riccati solutions of discrete Painlevé equations with Weyl group symmetry of type $E_7^{(1)}$, J. Math. Phys. 44 (2003) 1396–1414.
[21] M. Noumi, Painlevé Equations through Symmetry, Translations of Mathematical Monographs, vol. 223, American Mathematical Society, Providence, RI, 2004.
[22] M. Noumi, Y. Yamada, Affine Weyl groups, discrete dynamical systems and Painlevé equations, Comm. Math. Phys. 199 (1998) 281–295.
[23] M. Noumi, Y. Yamada, Symmetries in Painlevé equations, Sugaku Expositions 17 (2004) 203–218 Originally appeared in Japanese in Sugaku 53 (2001) 62–75 (Translated by T. Masuda).
[24] K. Okamoto, Studies on the Painlevé equations I, Ann. Mat. Pura Appl. 146 (1987) 337–381; II, Japan. J. Math. 13 (1987) 47–76; III, Math. Ann. 275 (1986) 221–255; IV, Funkcial. Ekvac. 30 (1987) 305–332.

[25] E. Rains, Recurrences for elliptic hypergeometric integrals, in: M. Noumi, K. Takasaki (Eds.), Elliptic Integrable Systems, Rokko Lectures in Mathematics, vol. 18, Department Mathematics, Kobe University, 2005, pp. 183–199.
[26] A. Ramani, B. Grammaticos, J. Hietarinta, Discrete version of the Painlevé equations, Phys. Rev. Lett. 67 (1991) 1829–1832.
[27] A. Ramani, B. Grammaticos, V. Papageorgiou, Singularity confinement, Symmetries and Integrability of Difference Equations (Estérel, PQ, 1994), CRM Proceedings of Lecture Notes, vol. 9, American Mathematical Society, Providence, RI, 1996, pp. 303–318.
[28] A. Ramani, B. Grammaticos, J. Satsuma, Bilinear discrete Painlevé equations, J. Phys. A 28 (1995) 4655–4665.
[29] A. Ramani, B. Grammaticos, T. Tamizhmani, K.M. Tamizhmani, Special function solution of the discrete Painlevé equations, Comput. Math. Appl. 42 (2001) 603–614.
[30] H. Sakai, Rational surfaces associated with affine root systems and geometry of the Painlevé equations, Comm. Math. Phys. 220 (2001) 165–229.
[31] H. Umemura, Painlevé equations in the past 100 years, Amer. Math. Soc. Transl. (2) 204 (2001) 81–110 (Originally appeared in Japanese in Sugaku 51 (1999) 395–420).