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Quasistatic Motion of a Capillary Drop
I. The Two-Dimensional Case

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A theory is presented for analyzing the nonlinear stability of a drop of incompressible viscous fluid with negligible inertia. The theory is developed here on the two- [illegible]

of the spherical harmonics. Within this context we show that if the free-boundary initiates close to a circle

$$r = 1 + \varepsilon \lambda^0(\theta), \quad |\varepsilon| \quad \text{small},$$

then there exists a global-in-time solution with free boundary

$$r = 1 + \lambda(\theta, t, \varepsilon) = 1 + \sum_{n \geqslant 1} \lambda_n(\theta, t)\, \varepsilon^n,$$

which approaches a circle exponentially fast as $t \to \infty$. Moreover, we prove that if $\lambda^0(\theta)$ is analytic (resp. C^∞) in θ, then the velocity $\mathbf{u}(x, t, \varepsilon)$, the pressure $p(x, t, \varepsilon)$, and the free boundary λ are all jointly analytic (resp. C^∞) in (x, ε).

Key Words: incompressible viscous fluid; Stokes equation; surface tension; stability.

1. THE PROBLEM

Consider an incompressible viscous fluid mass occupying a domain $\Omega(t)$, at time t $\{0 < t < \infty)$. Denote the velocity of the fluid by $\mathbf{u}$ and its pressure by p, so that the stress tensor is

$$T = \nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}} - pI,$$

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or, in terms of components,

$$T_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \delta_{ij} p.$$

If inertial forces can be neglected (e.g. small Reynolds number) the governing equations become

$$\Delta \mathbf{u} - \nabla p = 0 \quad \text{in} \quad \Omega(t), \tag{1.1}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad \Omega(t). \tag{1.2}$$

The boundary $\partial\Omega(t)$ is a free boundary driven by surface tension

$$T\mathbf{n} = -\sigma\kappa\mathbf{n} \quad \text{on} \quad \partial\Omega(t), \tag{1.3}$$

and satisfying the kinematic compatibility condition

$$V_n = \mathbf{u} \cdot \mathbf{n} \quad \text{on} \quad \partial\Omega(t). \tag{1.4}$$

Here V_n is the velocity of the free boundary in the direction of the outward normal $\mathbf{n}$, κ is the mean curvature of $\partial\Omega(t)$ (κ is positive for a circle), and $1/\sigma$ is the capillary number. We consider the evolution of the system (1.1)–(1.4) given an initial shape $\Omega(0)$: we wish to determine both the domains $\Omega(t)$ and the functions $p(x, t)$, $\mathbf{u}(x, t)$. This system needs to be supplemented with several constraints, such as (i)

$$\int_{\Omega(t)} (\mathbf{u} \times \mathbf{x}) = M\mathbf{e}_3, \tag{1.5}$$

where M is a given positive constant and $\mathbf{e}_3 = (0, 0, 1)$, and (ii) either

$$\int_{\Omega(t)} \mathbf{u}\, dx = 0 \tag{1.6}$$

or

$$\frac{1}{|\Omega(t)|} \int_{\Omega(t)} \mathbf{x}\, dx = \mathbf{m}, \tag{1.7}$$

where $|\Omega(t)| = \text{vol.}(\Omega(t))$ and $\mathbf{m}$ is a given constant vector.

We note that for any solution of (1.1), (1.2), (1.4), $|\Omega(t)|$ is independent of t; indeed

$$\frac{d\,|\Omega(t)|}{dt} = \int_{\partial\Omega(t)} V_n = \int_{\partial\Omega(t)} \mathbf{u} \cdot \mathbf{n} = \int_{\Omega(t)} \operatorname{div} \mathbf{u} = 0.$$

We also note that (1.6) is equivalent to (1.7), since

$$\frac{d}{dt}\int_{\Omega(t)} x_i\,dx = \int_{\partial\Omega(t)} x_i V_n = \int_{\partial\Omega(t)} (x_i\mathbf{u})\cdot\mathbf{n}$$
$$= \int_{\Omega(t)} \operatorname{div}(x_i\mathbf{u}) = \int_{\Omega(t)} u_i\,dx.$$

The above free-boundary problem with the constraints (1.5) and (1.6) (or (1.7)) was recently studied in [4, 6, 7] (see also [5]). It was proved in these papers that there exists a unique solution for some time interval $0 < t < T$. In [4, 7] it was also proved that if

$$\Omega(0) \text{ is close to the unit ball and has center of mass at the origin,} \tag{1.8}$$

then there exists a global solution of (1.1)–(1.5) and (1.7) with $\mathbf{m} = 0$ and

$$\Omega(t) \text{ converges to a ball } B_\rho = \{|x| < \rho\} \text{ as } t \to \infty, \text{ where vol. } B_\rho = \text{vol.}(\Omega(0)). \tag{1.9}$$

In the 2-dimensional case (1.5) becomes

$$\int_{\Omega(t)} (x_1 u_2 - x_2 u_1)\,dx = M \tag{1.10}$$

and (1.7) with $\mathbf{m} = 0$ is the condition

$$\int_{\Omega(t)} x_i\,dx = 0 \quad (i = 1, 2). \tag{1.11}$$

The methods developed in [4] and in [6, 7] are entirely different, although both methods are based on transforming the free boundary problem into a problem in the fixed cylinder $B_1 \times \{0 < t < \infty\}$ with, naturally, more complicated PDEs.

In [6, 7] the transformation is done by introducing Langrangean variables, whereas in [4] the authors use an unspecified diffeomorphism from $U_{t>0}\{\Omega(t) \times t\}$ onto $U_{t>0}\{B_1 \times \{t\}\}$. The solution established by both methods is smooth in the spatial variable; in [4] it is proved that the solution is also analytic in a sense that we shall describe later on.

In this paper we develop yet another approach for solving the same free-boundary problem. This approach is based on ideas developed in our earlier paper [3] dealing with the quasi-static Stefan problem. We rewrite the condition (1.8) (in two dimensions) in the form

$$\Omega(0) = \{r < 1 + \varepsilon\lambda^0(\theta)\},\ |\varepsilon| \text{ small}, \tag{1.12}$$

where the origin is chosen as the center of mass of $\Omega(0)$. We prove that there exists a unique solution with

$$\Omega(t) = \{r < 1 + \lambda(\theta, t, \varepsilon)\},$$

satisfying the conditions (1.10), (1.11), such that $\mathbf{u}(x, t, \varepsilon)$, $p(x, t, \varepsilon)$ and $\lambda(\theta, t, \varepsilon)$ are analytic in ε; furthermore,

$$\begin{cases} \text{if } \lambda^0(\theta) \text{ is analytic in } \theta \text{ then} \\ \lambda(\theta, t, \varepsilon) \text{ is analytic jointly in } (\theta, \varepsilon), \text{ and} \\ \mathbf{u}(x, t, \varepsilon), p(x, t, \varepsilon) \text{ are analytic jointly in } (x, \varepsilon). \end{cases} \tag{1.13}$$

The solution is also smooth (x, t, ε), and (as in [4][7]) $\Omega(t)$ converges to B_ρ as $t \to \infty$.

The analyticity result in [4] mentioned above implies, in particular, the analyticity of the solution $\lambda, \mathbf{u}, p$ in the variable ε; however it does not imply the analyticity of λ in (θ, ε) or of $\mathbf{u}, p$ in (x, ε), as asserted in (1.13).

Our method resembles that of [4] in that we first use a diffeomorphism to map $\Omega(t)$ onto the unit disc B_1. However, instead of using just any mapping, we shall take, as in [3], a specific mapping which enables us to get very precise estimates on derivatives of the solution, with respect to both ε and the spatial variable.

In this paper, we deal with the 2-dimensional case. In this case, the problem has also been studied using complex analytic techniques [1], and analyticity of the free boundary has been established (allowing for variable surface tension; see also [2]). These methods, however, do not extend to higher dimensions. In contrast, the procedure we introduce here does not use inherently two-dimensional ideas and can therefore be generalized; in follow up work we will consider the fully three-dimensional case. To exemplify our scheme, however, we have chosen a two-dimensional setting as it presents most of the difficulties of the three dimensional problem while allowing for simpler manipulation of spherical harmonics.

We proceed to outline the structure of the paper. In Section 2 we express the velocity $\mathbf{u}$ in terms of its radial and tangential components,

$$\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta,$$

where $\mathbf{e}_r = (\cos\theta, \sin\theta)$, $\mathbf{e}_\theta = (-\sin\theta, \cos\theta)$ and write the free-boundary problem for the functions $(p, u_r, u_\theta, \lambda)$, in polar coordinates. In Section 3 we reduce the free-boundary problem in $\Omega(t)$ to an equivalent problem in $\Omega_\delta(t) = \Omega(t) \cap \{r > \delta\}$; the reason for doing so is to avoid singularities at the origin that arise as we successively differentiate the system in r and in θ after mapping it onto a fixed domain. In order to be able to extend the solution from $\Omega_\delta(t)$ into $\{r < \delta\}$ we devise special "transparent" boundary conditions at $r = \delta + 0$.

In Section 4 we introduce the mapping

$$r' = \frac{(r-\delta) + \delta(1+\lambda-r)}{1+\lambda-\delta} \quad (\lambda = \lambda(\theta, t, \varepsilon)), \tag{1.14}$$

which transforms the free boundary problem in $\{\Omega_\delta(t), 0 < t < \infty\}$ into a problem in the cylindrical shell

$$\{\delta < r' < 1, 0 < t < \infty\}.$$

In Section 5 we formally expand the solution as power series in ε:

$$u_r = \sum u_n \varepsilon^n, \; u_\theta = \sum v_n \varepsilon^n, \; p = \sum p_n \varepsilon^n, \; \lambda = \sum \lambda_n \varepsilon^n. \tag{1.15}$$

Substituting these into the system of PDEs, boundary conditions and constraints, we obtain, after equating the coefficients of ε^n, a system of equations for u_n, v_n, p_n, λ_n, which we symbolically write in the form

$$\prod (u_n, v_n, p_n, \lambda_n) = F_n; \tag{1.16}$$

the F_n depend only on u_m, v_m, p_m, λ_m for $m < n$.

In Sections 6 and 7 we study the system

$$\prod (u, v, p, \lambda) = F$$

for general F, and derive estimates on u, v, p and λ in appropriate norms. These estimates are used in Sections 8 and 9 in order to prove, by induction, the desired estimates on u_n, r_n, p_n, λ_n that establish convergence of the series in (1.15) and the analyticity assertions of (1.13).

It will be convenient for our work to replace (1.2) by

$$\Delta p = 0 \quad \text{in} \quad \Omega(t), \tag{1.17}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{on} \quad \partial\Omega(t). \tag{1.18}$$

Equation (1.17) is obtained by taking the divergence in (1.1) and using (1.2). Conversely, taking the divergence in (1.1) and using (1.17) we find that div $\mathbf{u}$ is harmonic in $\Omega(t)$ and, then, (1.18) implies div $\mathbf{u} \equiv 0$ in $\Omega(t)$.

2. REFORMULATION IN POLAR COORDINATES

We introduce the orthonormal vectors

$$\mathbf{e}_r = (\cos\theta, \sin\theta), \; \mathbf{e}_\theta = (-\sin\theta, \cos\theta)$$

and write $\mathbf{u} = (u_1, u_2)$ in the form

$$\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta .$$

Then the system (1.1), (1.17) can be written in the form [8; p. 155]

$$\frac{\partial^2}{\partial r^2} u_r + \frac{1}{r}\frac{\partial}{\partial r} u_r + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} u_r - \frac{1}{r^2} u_r - \frac{2}{r^2}\frac{\partial}{\partial \theta} u_\theta - \frac{\partial p}{\partial r} = 0, \tag{2.1}$$

$$\frac{\partial^2}{\partial r^2} u_\theta + \frac{1}{r}\frac{\partial}{\partial r} u_\theta + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} u_\theta - \frac{1}{r^2} u_\theta + \frac{2}{r^2}\frac{\partial}{\partial \theta} u_r - \frac{1}{r}\frac{\partial p}{\partial \theta} = 0, \tag{2.2}$$

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} = 0 \tag{2.3}$$

in $\Omega(t)$. The boundary condition (1.18) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta} u_\theta = 0 \qquad \text{on} \quad \partial\Omega(t). \tag{2.4}$$

In order to write the remaining boundary conditions at the free boundary, we use the relation

$$\partial\Omega(t) = \{r = 1 + \lambda(\theta, t, \varepsilon)\}$$

and write $\partial\Omega(t)$ as

$$(x_1, x_2) = (1+\lambda)\, \mathbf{e}_r \equiv \Lambda \mathbf{e}_r \quad (\Lambda = 1 + \lambda).$$

Then

$$\mathbf{n} = \frac{\mathbf{N}}{|\mathbf{N}|} \qquad \text{where} \quad \mathbf{N} = \Lambda \mathbf{e}_r - \Lambda_\theta \mathbf{e}_\theta \quad \left(\Lambda_\theta = \frac{\partial \Lambda}{\partial \theta}\right).$$

Hence, setting

$$\mathbf{n} = n_r \mathbf{e}_r + n_\theta \mathbf{e}_\theta$$

we have

$$n_r = \mathbf{n} \cdot \mathbf{e}_r = \frac{\Lambda}{(\Lambda^2 + \Lambda_\theta^2)^{1/2}}, \qquad n_\theta = -\frac{\Lambda_\theta}{(\Lambda^2 + \Lambda_\theta^2)^{1/2}},$$

$$V_n = \Lambda_t \mathbf{e}_r \cdot \mathbf{n} = \frac{\Lambda \Lambda_t}{(\Lambda^2 + \Lambda_\theta^2)^{1/2}}.$$

We also have (see, for instance, [3])

$$\kappa = \frac{2\,\Lambda_\theta^2 - \Lambda \Lambda_{\theta\theta} + \Lambda^2}{(\Lambda^2 + \Lambda_\theta^2)^{3/2}}.$$

The equation

$$\begin{pmatrix} 2\dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} \\ \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} & 2\dfrac{\partial u_2}{\partial x_2} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

on the other hand, becomes (cf. [8, p.146])

$$\begin{pmatrix} 2\dfrac{\partial u_r}{\partial r} & \dfrac{\partial u_\theta}{\partial r} + \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} - \dfrac{1}{r} u_\theta \\ \dfrac{\partial u_\theta}{\partial r} + \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} - \dfrac{1}{r} u_\theta & -2\dfrac{\partial u_r}{\partial r} \end{pmatrix} \begin{pmatrix} n_r \\ n_\theta \end{pmatrix} = \begin{pmatrix} F_r \\ F_\theta \end{pmatrix}$$

where

$$(F_1, F_2) = F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta.$$

Since

$$pI \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = p n_r \mathbf{e}_r + p n_\theta \mathbf{e}_\theta,$$

the boundary condition (1.3) then takes the form

$$\begin{pmatrix} 2\dfrac{\partial u_r}{\partial r} - p & \dfrac{\partial u_\theta}{\partial r} + \dfrac{1}{1+\lambda}\dfrac{\partial u_r}{\partial \theta} - \dfrac{1}{1+\lambda} u_\theta \\ \dfrac{\partial u_\theta}{\partial r} + \dfrac{1}{1+\lambda}\dfrac{\partial u_r}{\partial \theta} - \dfrac{1}{1+\lambda} u_\theta & -2\dfrac{\partial u_r}{\partial r} - p \end{pmatrix} \begin{pmatrix} n_r \\ n_\theta \end{pmatrix} = -\sigma\kappa \begin{pmatrix} n_r \\ n_\theta \end{pmatrix}. \tag{2.5}$$

Finally, (1.4) becomes

$$\lambda_t = u_r - \frac{\lambda_\theta}{1+\lambda} u_\theta \qquad \text{on} \quad \partial\Omega(t) \quad \left(\lambda_\theta = \frac{\partial\lambda}{\partial\theta}\right), \tag{2.6}$$

and λ satisfies the initial condition (see (1.12))

$$\lambda\,|_{t=0} = \varepsilon\lambda^0(\theta). \tag{2.7}$$

It is easy to check that the constraints (1.10) and (1.11) become

$$\int_{\Omega(t)} r u_\theta \, dx = M, \tag{2.8}$$

$$\int_0^{2\pi} (1+\lambda)^3 \, e^{\pm i\theta} \, d\theta = 0. \tag{2.9}$$

DEFINITION 2.1. The problem for $(\lambda, p, u_r, u_\theta)$ consisting of (2.1)–(2.9) will be called Problem (A).

In the special case when $\Omega(0)$ is the unit disc, the solution to Problem (A) is

$$p = \sigma, \mathbf{u} = \mu_0(-x_2, x_1) = \mu_0 r \mathbf{e}_\theta,$$

where

$$\mu_0 = \frac{2M}{\pi}. \tag{2.10}$$

We expect the solution for the general case to be of the form

$$\lambda = \sum_{n \geqslant 1} \lambda_n(\theta, t) \, \varepsilon^n, \tag{2.11}$$

$$p = \sigma + \sum_{n \geqslant 1} p_n(r, \theta, t) \, \varepsilon^n, \tag{2.12}$$

$$u_r = \sum_{n \geqslant 1} u_n(r, \theta, t) \, \varepsilon^n, \tag{2.13}$$

$$u_\theta = \mu_0 r + \sum_{n \geqslant 1} v_n(r, \theta, t) \, \varepsilon^n. \tag{2.14}$$

3. REDUCTION TO A TRUNCATED DOMAIN

As mentioned in Section 1, in order to identify and estimate the series in (2.11)–(2.14) we shall transform the free-boundary problem into one defined on a fixed domain. The most natural transformation to this effect is

$$r \mapsto r' = \frac{r}{1+\lambda},$$

which is however singular at the origin. To avoid this difficulty while retaining the explicit nature of the transformation, here we show that Problem (A) is in fact equivalent to a problem posed in

$$\Omega_\delta(t) = \{\delta < r < 1+\lambda(\theta, t, \varepsilon)\} \quad (0 < \delta < \tfrac{1}{2}). \tag{3.1}$$

As we show in Section 4 these domains, in turn, can be regularly and explicitly transformed onto the fixed domain $\{\delta < r < 1\}$.

To reduce Problem (A) to one in Ω_δ we need only find appropriate "transparent" boundary conditions for p, u_r and u_θ at $r = \delta + 0$. In order to determine such conditions suppose p, u_r, u_θ form a solution to

$$\begin{aligned} \Delta p &= 0 \qquad \text{in} \quad r < \delta, \\ p\,|_{r=\delta-0} &= p\,|_{r=\delta+0}; \end{aligned} \tag{3.2}$$

$$\begin{aligned} \Delta \mathbf{u} + \nabla p &= 0 \qquad \text{in} \quad r < \delta, \\ \mathbf{u}\,|_{r=\delta-0} &= \mathbf{u}\,|_{r=\delta+0}. \end{aligned} \tag{3.3}$$

We compute the first derivatives of p, u_r, u_θ at $r = \delta - 0$ and, for simplicity, drop the dependence on t and ε.

We clearly have expansions

$$p = \sum_m a_m r^{|m|} e^{im\theta}, \tag{3.4}$$

$$u_r = \sum_m u_{rm}(r)\, e^{im\theta}, \tag{3.5}$$

$$u_\theta = \sum_m u_{\theta m}(r)\, e^{im\theta}, \tag{3.6}$$

where m varies over all the integers. Introducing the Hilbert transform

$$H\left(\sum_m b_m e^{im\theta}\right) = -i \sum_m b_m (\operatorname{sgn} m)\, e^{im\theta},$$

where sgn $0 = 0$, we have

$$\frac{\partial p}{\partial r} = \frac{1}{\delta} H\left(\frac{\partial p}{\partial \theta}\right) \qquad \text{at} \quad r = \delta - 0. \tag{3.7}$$

To derive analogous formulas for u_r and u_θ, consider first the case

$$p = a_m r^{|m|} e^{im\theta}, \quad 0 \leqslant r < \delta, \tag{3.8}$$

with

$$u_r = u_{rm}(\delta)\, e^{im\theta}, \qquad u_\theta = u_{\theta m}(\delta)\, e^{im\theta} \qquad \text{at} \quad r = \delta - 0.$$

Since

$$u_r = \tfrac{1}{2} a_m r^{|m|+1} e^{im\theta}, \qquad u_\theta = 0$$

is a special solution of (2.1) and (2.2), the general solution can be written in the form

$$u_r = \tfrac{1}{2} a_m r^{|m|+1} e^{im\theta} + \sum_k U_k(r)\, e^{ik\theta},$$

$$u_\theta = \sum_k V_k(r)\, e^{ik\theta}.$$

In view of the boundary conditions in (3.8), U_k and V_k must vanish for all $k \neq m$. The pair $U = U_m(r)$, $V = V_m(r)$ is a solution of the system

$$\begin{aligned} U'' + \frac{1}{r} U' - \frac{m^2+1}{r^2} U - \frac{2im}{r^2} V = 0, \\ V'' + \frac{1}{r} V' - \frac{m^2+1}{r^2} V + \frac{2im}{r^2} U = 0. \end{aligned} \tag{3.9}$$

The most general solution of (3.9), which is regular near $r = 0$, is

$$U = A_1 r^{|m|+1} + A_2 r^{|m|-1}$$
$$V = i(\operatorname{sgn} m)[-A_1 r^{|m|+1} + A_2 r^{|m|-1}]$$

if $m \neq 0$ and

$$U = Ar, \qquad V = Br \qquad \text{if} \quad m = 0,$$

where A_1, A_2 and A, B are arbitrary constants. We thus conclude that, in the special case (3.8), the solution (u_r, u_θ) is given by

$$\begin{aligned} u_r &= \{\tfrac{1}{2} a_m r^{|m|+1} + A_1 r^{|m|+1} + A_2 r^{|m|-1}\} e^{im\theta}, \\ u_\theta &= i(\operatorname{sgn} m)\{-A_1 r^{|m|+1} + A_2 r^{|m|-1}\} e^{im\theta}, \end{aligned} \tag{3.10}$$

if $m \neq 0$, and

$$u_r = Ar, \qquad u_\theta = Br \tag{3.11}$$

if $m = 0$.

We now compute, for $m \neq 0$, the first derivatives of u_r, at $r = \delta$ and determine coefficients P_1, P_2, μ and Q_1, Q_2, τ such that

$$\frac{\partial u_r}{\partial r} = P_1 \frac{\partial u_r}{\partial \theta} + P_2 u_\theta + \mu p, \tag{3.12}$$

$$\frac{\partial u_\theta}{\partial r} = Q_1 \frac{\partial u_\theta}{\partial \theta} + Q_2 u_r + \tau p \tag{3.13}$$

for every choice of A_1, A_2, and a_m. We find that P_1, P_2, μ must satisfy

$$\begin{aligned} (|m|+1)\, \delta^{|m|} &= P_1 m i \delta^{|m|+1} - P_2 i(\operatorname{sgn} m)\, \delta^{|m|+1}, \\ (|m|-1)\, \delta^{|m|-2} &= P_1 m i \delta^{|m|-1} + P_2 i(\operatorname{sgn} m)\, \delta^{|m|-1}, \end{aligned}$$

$$(|m|+1)\, \tfrac{1}{2}\, \delta^{|m|} = P_1\, \tfrac{1}{2}\, i m \delta^{|m|+1} + \mu \delta^{|m|}.$$

The solution to this system is

$$P_1 = -\frac{i}{\delta} (\operatorname{sgn} m), \qquad P_2 = \frac{i}{\delta} (\operatorname{sgn} m), \qquad \mu = \frac{1}{2}.$$

Hence

$$\frac{\partial u_r}{\partial r} = \frac{1}{\delta} H\left(\frac{\partial u_r}{\partial \theta}\right) - \frac{1}{\delta} H(u_\theta) + \frac{1}{2} p \quad \text{at} \quad r = \delta - 0. \tag{3.14}$$

Similarly the coefficients in (3.13) turn out to be

$$Q_1 = -\frac{i}{\delta}(\operatorname{sgn} m), \qquad Q_2 = -\frac{i}{\delta}(\operatorname{sgn} m), \qquad \tau = \frac{i}{2}(\operatorname{sgn} m)$$

so that

$$\frac{\partial u_\theta}{\partial r} = \frac{1}{\delta} H\left(\frac{\partial u_\theta}{\partial \theta}\right) + \frac{1}{\delta} H(u_r) - \frac{1}{2} H(p) \quad \text{at} \quad r = \delta - 0. \tag{3.15}$$

If $m = 0$ then from (3.11) we get

$$\frac{\partial u_r}{\partial r} = \frac{1}{\delta} u_r, \qquad \frac{\partial u_\theta}{\partial r} = \frac{1}{\delta} u_\theta \quad \text{at} \quad r = \delta - 0. \tag{3.16}$$

So far we considered only the special case (3.8). For the general case of (3.4)–(3.6) we obtain by linearity that (3.14) and (3.15) hold for $u_r - u_r^0$, $u_\theta - u_\theta^0$, $p - p^0$, where $(u_r^0, u_\theta^0, p^0) \equiv (u_{r0}, u_{\theta 0}, a_0)$ is the zero mode of (u_r, u_θ, p). Recalling also (3.16) we deduce the following transparency conditions at $r = \delta - 0$:

$$\frac{\partial u_r}{\partial r} = \frac{1}{\delta} H\left(\frac{\partial u_r}{\partial \theta}\right) - \frac{1}{\delta} H(u_\theta - u_\theta^0) + \frac{1}{\delta} u_r^0 + \frac{1}{2}(p - p^0), \tag{3.17}$$

$$\frac{\partial u_\theta}{\partial r} = \frac{1}{\delta} H\left(\frac{\partial u_\theta}{\partial \theta}\right) + \frac{1}{\delta} H(u_r - u_r^0) + \frac{1}{\delta} u_\theta^0 - \frac{1}{2} H(p - p^0). \tag{3.18}$$

The following result shows that, when coupled to Eqs. (2.1)–(2.5), some of the conditions in (3.18) are actually redundant.

THEOREM 3.1. *Suppose* $(\lambda, \mathbf{u}, p)$ *is a solution of* (2.1)–(2.3) *in* $\Omega_\delta(t)$, *satisfying the boundary conditions* (2.4) *and* (2.5). *Then*

(i) (3.18) *is satisfied for mode 0;*

(ii) *If* (3.17) *holds for modes* ± 1 *then* (3.18) *also holds for modes* ± 1.

Proof. We shall invoke the identity (cf. [4])

$$\frac{1}{2}\int_{\Omega_\delta(t)} \sum \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) dx - \int_{\Omega_\delta(t)} p \operatorname{div} \mathbf{v}\, dx$$

$$= \int_{\Omega_\delta(t)} (-\Delta \mathbf{u} + \nabla p) \cdot \mathbf{v}\, dx - \int_{\Omega_\delta(t)} \nabla(\operatorname{div} \mathbf{u}) \cdot \mathbf{v}\, dx + \int_{\partial\Omega_\delta(t)} T(\mathbf{u}, p)\, \mathbf{n} \cdot \mathbf{v}\, ds, \tag{3.19}$$

where $\mathbf{n}$ is the outward normal to $\partial\Omega_\delta(t)$. We first choose $\mathbf{v} = (-x_2, x_1) = r\mathbf{e}_\theta$ and deduce that

$$-\int_{\Omega_\delta(t)} \nabla(\operatorname{div}\mathbf{u})\cdot\mathbf{v}\,dx + \int_{\partial\Omega_\delta(t)} T(\mathbf{u}, p)\,\mathbf{n}\cdot\mathbf{v}\,ds = 0. \tag{3.20}$$

We compute

$$\int_{\Omega_\delta(t)} \nabla(\operatorname{div}\mathbf{u})\cdot\mathbf{v}\,dx = \int_{\partial\Omega_\delta(t)} \mathbf{n}(\operatorname{div}\mathbf{u})\cdot\mathbf{v} = 0 \tag{3.21}$$

since $\operatorname{div}\mathbf{u} = 0$ on $\partial\Omega(t)$ and $\mathbf{n}\cdot\mathbf{v} = 0$ on $r = \delta$. Next

$$\int_{\partial\Omega(t)} T(\mathbf{u}, p)\,\mathbf{n}\cdot\mathbf{v}\,ds = -\sigma\int_{\partial\Omega(t)} \kappa\mathbf{n}\cdot\mathbf{v}\,ds = \sigma\int_{\partial\Omega(t)} \frac{d\boldsymbol{\tau}}{ds}\cdot\mathbf{v}\,ds,$$

where $\boldsymbol{\tau} = r\dot{\theta}\mathbf{e}_\theta + \dot{r}\mathbf{e}_r$ is the tangent vector (s is the arclength parameter and "·" means d/ds), but

$$\frac{d\mathbf{v}}{ds} = \frac{d}{ds}(r(s)\,\mathbf{e}_\theta) = -r\dot{\theta}\mathbf{e}_r + \dot{r}\mathbf{e}_\theta$$

so that $\boldsymbol{\tau}\cdot(d\mathbf{v}/ds) = 0$. Therefore

$$\int_{\partial\Omega(t)} T(\mathbf{u}, p)\,\mathbf{n}\cdot\mathbf{v}\,ds = -\sigma\int_{\partial\Omega(t)} \boldsymbol{\tau}\cdot\frac{d\mathbf{v}}{ds} = 0.$$

Using this and (3.21) in (3.20) and recalling that $T\mathbf{n}$ is given by the left-hand side of (2.5) with $1+\lambda$ replaced by δ, we obtain ($n_r = -1$, $n_\theta = 0$)

$$\begin{aligned} 0 &= \int_{r=\delta} \left[\left(2\frac{\partial u_r}{\partial r} - p\right)\mathbf{e}_r + \left(\frac{\partial u_\theta}{\partial r} + \frac{1}{\delta}\frac{\partial u_r}{\partial\theta} - \frac{1}{\delta}u_\theta\right)\mathbf{e}_\theta\right]\cdot\delta\mathbf{e}_\theta\,ds \\ &= \delta\int_{r=\delta}\left(\frac{\partial u_\theta}{\partial r} - \frac{1}{\delta}u_\theta\right)ds, \end{aligned}$$

i.e., $\partial u_\theta^0/\partial r = (1/\delta)\,u_\theta^0$. This completes the proof of (i).

To prove (ii) we take $v = (1, i)$ in (3.19). Proceeding as before and using the fact that

$$\int_{\partial\Omega(t)} T(\mathbf{u}, p)\,\mathbf{n}\cdot\mathbf{v}\,ds = -\sigma\int_{\partial\Omega(t)} \kappa\mathbf{n}\cdot\mathbf{v}\,ds = \sigma\left(\int_{\partial\Omega(t)} \frac{d\boldsymbol{\tau}}{ds}\,ds\right)\cdot\mathbf{v} = 0,$$

we obtain

$$\int_{r=\delta} T(\mathbf{u}, p)\,\mathbf{n}\cdot\mathbf{v}\,ds - \int_{r=\delta} (\operatorname{div}\mathbf{u})\,\mathbf{n}\cdot\mathbf{v}\,ds = 0. \tag{3.22}$$

We can write

$$\mathbf{v} = e^{i\theta}\mathbf{e}_r + ie^{i\theta}\mathbf{e}_\theta$$

and then

$$\begin{aligned}\int_{r=\delta} T(\mathbf{u}, p)\,\mathbf{n}\cdot\mathbf{v}\,ds &= -\int_{r=\delta}\left[\left(2\frac{\partial u_r}{\partial r} - p\right)\mathbf{e}_r + \left(\frac{\partial u_\theta}{\partial r} + \frac{1}{\delta}\frac{\partial u_r}{\partial\theta} - \frac{1}{\delta}u_\theta\right)\mathbf{e}_\theta\right]\cdot\mathbf{v}\delta\,d\theta\\ &= -\int_{r=\delta}\left[\left(2\frac{\partial u_r}{\partial r} - p\right) + i\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{\delta}\frac{\partial u_r}{\partial\theta} - \frac{1}{\delta}u_\theta\right)\right]e^{i\theta}\delta\,d\theta.\end{aligned} \tag{3.23}$$

On the other hand, since $\mathbf{v}\cdot\mathbf{n} = \mathbf{v}\cdot(-\mathbf{e}_r) = -e^{i\theta}$ at $r = \delta$, we have

$$\int_{r=\delta} (\operatorname{div}\mathbf{u})\,\mathbf{n}\cdot\mathbf{v}\,ds = -\int_{r=\delta}\left(\frac{\partial u_r}{\partial r} + \frac{1}{\delta}u_r + \frac{1}{\delta}\frac{\partial u_\theta}{\partial\theta}\right)e^{i\theta}\delta\,d\theta.$$

Substituting this and (3.23) into (3.22) we deduce, after some cancellations, that

$$\int_{r=\delta}\left(\frac{\partial u_r}{\partial r} + i\frac{\partial u_\theta}{\partial r} - p\right)e^{i\theta}d\theta = 0. \tag{3.24}$$

Similarly by choosing $v = (1, -i)$ in (3.19) we derive the relation

$$\int_{r=\delta}\left(\frac{\partial u_r}{\partial r} - i\frac{\partial u_\theta}{\partial r} - p\right)e^{-i\theta}\,d\theta = 0. \tag{3.25}$$

Denote by u_r^m the mode m of u_r, that is,

$$u_r^m = \frac{1}{2\pi}\int_0^{2\pi} u_r e^{-im\theta}\,d\theta.$$

Similarly we denote by u_θ^m and p^m the m modes of u_θ and p. Then (3.24) and (3.25) yield the relations

$$\frac{\partial u_r^{-1}}{\partial r} + i\frac{\partial u_\theta^{-1}}{\partial r} - p^{-1} = 0, \tag{3.26}$$

$$\frac{\partial u_r^{+1}}{\partial r} - i\frac{\partial u_\theta^{+1}}{\partial r} - p^{+1} = 0. \tag{3.27}$$

The transparent boundary conditions (3.17) and (3.18) for mode -1 are

$$\frac{\partial u_r^{-1}}{\partial r} - \frac{1}{\delta} u_r^{-1} + \frac{i}{\delta} u_\theta^{-1} - \frac{1}{2} p^{-1} = 0,$$

$$\frac{\partial u_\theta^{-1}}{\partial r} - \frac{1}{\delta} u_\theta^{-1} - \frac{i}{\delta} u_r^{-1} + \frac{i}{2} p^{-1} = 0,$$

and, clearly, the second relation follows from the first relation and (3.26). Similarly the transparent boundary condition (3.18) for mode $+1$ follows from the transparent boundary condition (3.17) for mode 1 and (3.27). ∎

Theorem 3.1 suggests that (3.18) should be replaced by

$$\frac{\partial(u_\theta - u_\theta^{(1)})}{\partial r} - \frac{1}{\delta} H\left(\frac{\partial(u_\theta - u_\theta^{(1)})}{\partial \theta}\right) - \frac{1}{\delta} H(u_r - u_r^{(1)}) + \frac{1}{2} H(p - p^{(1)}) = 0 \tag{3.28}$$

where

$$u_\theta^{(1)} = u_\theta^0 + u_\theta^1 e^{i\theta} + u_\theta^{-1} e^{-i\theta}, \qquad u_r^{(1)} = u_r^0 + u_r^1 e^{i\theta} + u_1^{-1} e^{-i\theta},$$

$$p^{(1)} = p^0 + p^1 e^{i\theta} + p^{-1} e^{-i\theta}.$$

DEFINITION 3.1. The problem for $(\lambda, p, u_r, u_\theta)$ consisting of solving (2.1)–(2.3) in $\{\Omega_\delta(t),\ 0 < t < \infty\}$ (where $\Omega_\delta(t)$ is defined by (3.1)) with the boundary conditions (2.4)–(2.6) on $\partial\Omega(t)$, the transparent boundary conditions (3.7), (3.17), (3.28) and the initial condition (2.7) will be called Problem (P_δ).

THEOREM 3.2. *If $(\lambda, u_r, u_\theta, p)$ is a solution to problem (P_δ), then u_r, u_θ, p can be extended uniquely to a solution of* (2.1)–(2.3) *in all of $\Omega(t)$, $0 < t < \infty$.*

Proof. We solve (3.2) and (3.3) in $r \leqslant \delta$. Then u_r, u_θ and p are continuous across $r = \delta$. By Theorem 3.1 their normal derivatives are also continuous across $r = \delta$. It follows that p, $\mathbf{u}$ form a smooth solution of (2.1)–(2.3) in all of $\Omega(t)$. ∎

We finally want to express the constraints (2.8), (2.9) in terms of the solution to problem (P_δ). To do this we note that, by (3.11),

$$\int_{r<\delta} r u_\theta = \int_{r<\delta} r u_\theta^0 = \int_{r<\delta} B r^2 = \frac{\delta}{4} \int_{r=\delta} B r^2 \, ds = \frac{\delta}{4} \int_{r=\delta} r u_\theta^0 \, ds$$

$$= \frac{\delta}{4} \int_{r=\delta} r u_\theta \, ds$$

so that (2.8) coincides with

$$\int_{\Omega_\delta(t)} r u_\theta \, dx + \frac{\delta}{4} \int_{r=\delta} r u_\theta \, ds = M, \tag{3.29}$$

where $ds = \delta \, d\theta$.

The constraint (2.9) is equivalent to

$$\int_0^{2\pi} [(1+\lambda)^3 - 1] \, e^{\pm i\theta} \, d\theta = 0,$$

or

$$\int_0^{2\pi} \lambda e^{\pm i\theta} \, d\theta = -\int_0^{2\pi} \left(\lambda^2 + \frac{\lambda^3}{3}\right) e^{\pm i\theta} \, d\theta. \tag{3.30}$$

DEFINITION 3.2. The problem of finding a solution to problem (P_δ) satisfying the constraints (3.29), (3.30) will be called Problem (A_δ).

We have proved:

THEOREM 3.3. *Any solution to Problem* (A_δ) *can be uniquely extended to a solution of Problem* (A).

In the sequel we focus our efforts on solving Problem (A_δ).

4. THE TRANSFORMED PROBLEM

As in [3] we transform the domains $\Omega_\delta(t)$ into a fixed domain by the change of variables $r \to r'$:

$$r' = \frac{(1-\delta) r + \delta\lambda}{1+\lambda-\delta} \quad \text{or} \quad r = \frac{(1+\lambda-\delta) r' - \delta\lambda}{1-\delta}. \tag{4.1}$$

This transformation maps $\Omega_\delta(t)$ onto $\{\delta < r' < 1\}$. We also introduce the functions

$$\begin{aligned} \tilde{p}(r', \theta, t) &= p(r, \theta, t) - \sigma, \qquad \tilde{u}_{r'}(r', \theta, t) = u_r(r, \theta, t), \\ \tilde{u}_\theta(r', \theta, t) &= u_\theta(r, \theta, t) - \mu_0 r. \end{aligned} \tag{4.2}$$

For any function $q(r, \theta, t)$, if we define

$$q'(r', \theta, t) = q(r, \theta, t),$$

then, using (4.1) and the relation

$$\delta - r' = \frac{1-\delta}{1+\lambda-\delta}(\delta - r),$$

we derive the formulas

$$\frac{\partial q}{\partial r} = \frac{1-\delta}{(1+\lambda-\delta)}\frac{\partial q'}{\partial r'}, \qquad \frac{\partial^2 q}{\partial r^2} = \frac{(1-\delta)^2}{(1+\lambda-\delta)^2}\frac{\partial^2 q'}{\partial r'^2},$$

$$\frac{\partial q}{\partial \theta} = \frac{\partial q'}{\partial \theta} + \frac{(1-\delta)\,\lambda_\theta(\delta-r)}{(1+\lambda-\delta)^2}\frac{\partial q'}{\partial r'} = \frac{\partial q'}{\partial \theta} + \frac{\lambda_\theta(\delta-r')}{(1+\lambda-\delta)}\frac{\partial q'}{\partial r'},$$

$$\begin{aligned}\frac{\partial^2 q}{\partial \theta^2} &= \frac{\partial^2 q'}{\partial \theta^2} + \frac{2(1-\delta)\,\lambda_\theta(\delta-r)}{(1+\lambda-\delta)^2}\frac{\partial^2 q'}{\partial r'\partial \theta} + \frac{\partial}{\partial \theta}\left(\frac{(1-\delta)\,\lambda_\theta(\delta-r)}{(1+\lambda-\delta)^2}\right)\frac{\partial q'}{\partial r'} \\ &\quad + \left(\frac{(1-\delta)\,\lambda_\theta(\delta-r)}{(1+\lambda-\delta)^2}\right)^2\frac{\partial^2 q'}{\partial r'^2} \\ &= \frac{\partial^2 q'}{\partial \theta^2} + \frac{2\lambda_\theta(\delta-r')}{1+\lambda-\delta}\frac{\partial^2 q'}{\partial r'\partial \theta} + \left(\frac{\lambda_\theta(\delta-r')}{1+\lambda-\delta}\right)^2\frac{\partial^2 q'}{\partial r'^2} \\ &\quad + \left[\frac{\partial}{\partial \theta}\left(\frac{\lambda_\theta}{1+\lambda-\delta}\right) - \left(\frac{\lambda_\theta}{1+\lambda-\delta}\right)^2\right](\delta-r')\frac{\partial q'}{\partial r'}\end{aligned}$$

and

$$\frac{1}{r}\frac{\partial q}{\partial r} = \frac{(1-\delta)^2}{[(1+\lambda-\delta)\,r' - \delta\lambda](1+\lambda-\delta)}\frac{\partial q'}{\partial r'}.$$

With the aid of these formulas we transform the problem (A_δ) into a problem for the functions $\tilde{p}(r', \theta, t)$, $\tilde{u}_{r'}(r', \theta, t)$, $\tilde{u}_\theta(r', \theta, t)$, and $\lambda(\theta, t)$. To simplify the notation we shall drop the prime "′" in the independent variable r' and further set

$$\begin{aligned} P(r, \theta, t) &= \tilde{p}(r, \theta, t), && \delta < r < 1, \\ U(r, \theta, t) &= \tilde{u}_r(r, \theta, t), && \delta < r < 1, \\ V(r, \theta, t) &= \tilde{u}_\theta(r, \theta, t), && \delta < r < 1; \end{aligned}$$

in view of (2.11)–(2.14) the power series in ε for P, U, V and λ will not contain zero-order terms.

The system (2.1)–(2.3) becomes

$$\Delta P = F^1, \qquad \delta < r < 1, \tag{4.3}$$

$$\Delta U - \frac{1}{r^2} U - \frac{2}{r^2}\frac{\partial}{\partial\theta} V - \frac{\partial P}{\partial r} = F^2, \qquad \delta < r < 1, \tag{4.4}$$

$$\Delta V - \frac{1}{r^2} V + \frac{2}{r^2}\frac{\partial}{\partial\theta} U - \frac{1}{r}\frac{\partial P}{\partial\theta} = F^3, \qquad \delta < r < 1, \tag{4.5}$$

where

$$\begin{aligned}
F^1 = F^1(P) \equiv {} & \left[1 - \frac{(1-\delta)^2}{(1+\lambda-\delta)^2}\right]\frac{\partial^2 P}{\partial r^2} \\
& + \frac{1}{r}\left[1 - \frac{(1-\delta)^2\, r}{[(1+\lambda-\delta)\, r - \delta\lambda](1+\lambda-\delta)}\right]\frac{\partial P}{\partial r} \\
& + \frac{(1-\delta)^2}{[(1+\lambda-\delta)\, r - \delta\lambda]^2}\left[-2(\delta-r)\frac{\lambda_\theta}{1+\lambda-\delta}\frac{\partial^2 P}{\partial r\partial\theta}\right. \\
& - (\delta-r)\frac{\partial}{\partial\theta}\left(\frac{\lambda_\theta}{1+\lambda-\delta}\right)\frac{\partial P}{\partial r} - (\delta-r)^2\left(\frac{\lambda_\theta}{1+\lambda-\delta}\right)^2\frac{\partial^2 P}{\partial r^2} \\
& \left. + (\delta-r)\left(\frac{\lambda_\theta}{1+\lambda+\delta}\right)^2\frac{\partial P}{\partial r}\right] \\
& + \frac{1}{r^2}\left[1 - \frac{r^2(1-\delta)^2}{[(1+\lambda-\delta)\, r - \delta\lambda]^2}\right]\frac{\partial^2 P}{\partial\theta^2},
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
F^2 = F^1(U) & - \frac{1}{r^2}\left[1 - \frac{(1-\delta)^2\, r^2}{[(1+\lambda-\delta)\, r - \delta\lambda]^2}\right]U \\
& - \frac{2}{r^2}\left[1 - \frac{(1-\delta)^2\, r^2}{[(1+\lambda-\delta)\, r - \delta\lambda]^2}\right]\frac{\partial V}{\partial\theta} \\
& + \frac{2(1-\delta)^2}{[(1+\lambda-\delta)\, r - \delta\lambda]^2}\cdot\frac{\lambda_\theta(\delta-r)}{1+\lambda-\delta}\frac{\partial V}{\partial r} - \frac{\lambda}{1+\lambda-\delta}\frac{\partial P}{\partial r},
\end{aligned} \tag{4.7}$$

$$F^3 = F^1(V) - \frac{1}{r^2}\left[1 - \frac{(1-\delta)^2 r^2}{[(1+\lambda-\delta) r - \delta\lambda]^2}\right] V$$
$$+ \frac{2}{r^2}\left[1 - \frac{(1-\delta)^2 r^2}{[(1+\lambda-\delta) r - \delta\lambda]^2}\right] \frac{\partial U}{\partial \theta}$$
$$- \frac{2(1-\delta)^2}{[(1+\lambda-\delta) r - \delta\lambda]^2} \frac{\lambda_\theta(\delta - r)}{1+\lambda-\delta} \frac{\partial U}{\partial r} - \frac{1}{r}\left[1 - \frac{(1-\delta) r}{(1+\lambda-\delta) r - \delta\lambda}\right] \frac{\partial P}{\partial \theta}$$
$$+ \frac{1-\delta}{[(1+\lambda-\delta) r - \delta\lambda]} \cdot \frac{\lambda_\theta(\delta - r)}{(1+\lambda-\delta)} \frac{\partial P}{\partial r}. \tag{4.8}$$

We next consider the boundary conditions. From (2.4) (at $r = 1$) we obtain

$$\frac{\partial U}{\partial r} + U + \frac{\partial V}{\partial \theta} = G^1 \qquad \text{at} \quad r = 1, \tag{4.9}$$

where

$$G^1 = \frac{\lambda}{1+\lambda-\delta} \frac{\partial U}{\partial r} + \frac{\lambda}{1+\lambda} U + \frac{\lambda}{1+\lambda} \frac{\partial V}{\partial \theta} + \frac{(1-\delta)\lambda_\theta}{(1+\lambda)(1+\lambda-\delta)} \frac{\partial V}{\partial r}. \tag{4.10}$$

Next, from the first component of (2.5) we get

$$\left(2 \frac{\partial U}{\partial r} \frac{1-\delta}{1+\lambda-\delta} - P\right)(1+\lambda)$$
$$- \left[\frac{\partial V}{\partial r} \frac{1-\delta}{1+\lambda-\delta} + \frac{1}{1+\lambda}\left(\frac{\partial U}{\partial \theta} + \frac{\lambda_\theta(\delta - r)}{1+\lambda-\delta} \frac{\partial U}{\partial r}\right) - \frac{1}{1+\lambda} V\right] \lambda_\theta$$
$$= -\sigma\kappa(1+\lambda) + \sigma(1+\lambda) \qquad \text{at} \quad r = 1$$

so that

$$2 \frac{\partial U}{\partial r} - P - \sigma(\lambda_{\theta\theta} + \lambda) = G^2 \qquad \text{at} \quad r = 1, \tag{4.11}$$

where

$$G^2 = \frac{2\lambda}{1+\lambda-\delta} \frac{\partial U}{\partial r} + \frac{\lambda_\theta}{1+\lambda}\left[\frac{1-\delta}{1+\lambda-\delta} \frac{\partial V}{\partial r} + \frac{1}{1+\lambda} \frac{\partial U}{\partial \theta} - \frac{1}{1+\lambda} \cdot \frac{(1-\delta)\lambda_\theta}{1+\lambda-\delta} \frac{\partial U}{\partial r} - \frac{V}{1+\lambda}\right]$$
$$- \sigma\left[\frac{2\lambda_\theta^2 - (1+\lambda)\lambda_{\theta\theta} + (1+\lambda)^2}{[(1+\lambda)^2 + \lambda_\theta^2]^{3/2}} - [1 - (\lambda_{\theta\theta} + \lambda)]\right]. \tag{4.12}$$

Similarly, from the second component of (2.5) we get

$$\frac{\partial V}{\partial r}+\frac{\partial U}{\partial \theta}-V=G^3 \qquad \text{at} \quad r=1, \tag{4.13}$$

where

$$\begin{aligned} G^3 = & \frac{\delta\lambda}{1+\lambda-\delta}\frac{\partial V}{\partial r}-\frac{(1-\delta)\,\lambda_\theta}{1+\lambda-\delta}\frac{\partial U}{\partial r}-\lambda_\theta P \\ & +\sigma\lambda_\theta\left[\frac{2\lambda_\theta^2-(1+\lambda)\,\lambda_{\theta\theta}+(1+\lambda)^2}{[(1+\lambda)^2+\lambda_\theta^2]^{3/2}}-1\right]. \end{aligned} \tag{4.14}$$

The boundary condition (2.6) becomes

$$\lambda_t = U-\frac{\lambda_\theta}{1+\lambda}\left(V+\mu_0\,\frac{(1+\lambda-\delta)-\delta\lambda}{1-\delta}\right),$$

or

$$\lambda_t-U+\mu_0\lambda_\theta=G^4 \qquad \text{at} \quad r=1, \tag{4.15}$$

where

$$G^4=-\frac{\lambda_\theta}{1+\lambda}V. \tag{4.16}$$

Consider next the boundary conditions at $r=\delta$. From (3.7) we get

$$\frac{\partial P}{\partial r}-\frac{1}{\delta}H\left(\frac{\partial P}{\partial \theta}\right)=G^5 \qquad \text{at} \quad r=\delta, \tag{4.17}$$

where

$$G^5=\frac{\lambda}{1+\lambda-\delta}\frac{\partial P}{\partial r}. \tag{4.18}$$

Next, (3.17) yields

$$\frac{\partial U}{\partial r}-\frac{1}{\delta}H\left(\frac{\partial U}{\partial \theta}\right)+\frac{1}{\delta}H(V-V^0)-\frac{1}{\delta}U^0-\frac{1}{2}(P-P^0)=G^6 \qquad \text{at} \quad r=\delta \tag{4.19}$$

where U^0, V^0, P^0 are the zero modes of U, V and P, and

$$G^6 = \frac{\lambda}{1+\lambda-\delta}\frac{\partial U}{\partial r}. \tag{4.20}$$

Finally, (3.28) becomes

$$\frac{\partial(V-V^{(1)})}{\partial r} - \frac{1}{\delta}H\left(\frac{\partial(V-V^{(1)})}{\partial\theta}\right) - \frac{1}{\delta}H(U-U^{(1)})$$
$$+\frac{1}{2}H(P-P^1) = G^7 \qquad \text{at} \quad r=\delta, \tag{4.21}$$

where

$$U^{(1)} = U^0 + U^1 e^{i\theta} + U^{-1}e^{-i\theta}, \qquad V^{(1)} = V^0 + V^1 e^{i\theta} + V^{-1}e^{-i\theta},$$
$$P^{(1)} = P^0 + P^1 e^{i\theta} + P^{-1}e^{-i\theta}$$

are the sums of the zero and ± 1 modes of U, V, and P, and

$$G^7 = \frac{\lambda}{1+\lambda-\delta}\frac{\partial(V-V^{(1)})}{\partial r}. \tag{4.22}$$

Consider, finally, the constraints (3.29), (3.30). Since

$$\int_0^{2\pi} d\theta \int_0^{1+\lambda} r(\mu_0 r)\, r\, dr$$
$$= M + \int_0^{2\pi} d\theta \int_1^{1+\lambda} \mu_0 r^3\, dr = M + \mu_0 \int_0^{2\pi} \frac{(1+\lambda)^4 - 1}{4}\, d\theta$$
$$= M + \mu_0 \int_0^{2\pi} \lambda\, d\theta + \frac{\mu_0}{4}\int_0^{2\pi} \lambda[(2+\lambda)(2+2\lambda+\lambda^2) - 4]\, d\theta,$$

we easily get from (3.29)

$$\int_0^{2\pi} d\theta \int_\delta^1 r^2 V(r,\theta,t)\, dr + \frac{\delta^3}{4}\int_0^{2\pi} V(\delta,\theta,t)\, d\theta + \mu_0 \int_0^{2\pi} \lambda\, d\theta = 2\pi H^1 \tag{4.23}$$

where

$$2\pi H^1 = \int_0^{2\pi} d\theta \int_\delta^1 \left[r^2 - \frac{[(1+\lambda-\delta)\, r - \delta\lambda]^2\, (1+\lambda-\delta)}{(1-\delta)^3} \right] V(r, \theta, t)\, dr$$
$$- \frac{1}{4}\mu_0 \int_0^{2\pi} \lambda[(2+\lambda)(2+2\lambda+\lambda^2)-4]\, d\theta. \tag{4.24}$$

Finally, we write the constraint (3.30) in the form

$$\int_0^{2\pi} \lambda e^{\pm i\theta}\, d\theta = 2\pi H^{2,\pm}, \tag{4.25}$$

where

$$2\pi H^{2,\pm} = -\int_0^{2\pi} \left(\lambda^2 + \frac{\lambda^3}{3} \right) e^{\pm i\theta}\, d\theta. \tag{4.26}$$

DEFINITION 4.1. We shall refer to the transformed problem (4.3)–(4.26) as Problem (T).

5. THE INDUCTIVE SYSTEM

For the transformed problem derived in Section 4, the expansions (2.11)–(2.14) can be written in the form

$$\lambda = \sum_{n \geqslant 1} \lambda_n(\theta, t)\, \varepsilon^n, \tag{5.1}$$

$$P = \sum_{n \geqslant 1} P_n(r, \theta, t)\, \varepsilon^n, \tag{5.2}$$

$$U = \sum_{n \geqslant 1} U_n(r, \theta, t)\, \varepsilon^n, \tag{5.3}$$

$$V = \sum_{n \geqslant 1} V_n(r, \theta, t)\, \varepsilon^n. \tag{5.4}$$

We substitute these series into the differential equations (4.3)–(4.5), the boundary conditions (4.9), (4.11), (4.13), (4.15), (4.17) and (4.19), (4.21) and the constraints (4.23) and (4.25). Upon equating the coefficients of ε^n, we obtain the following system:

$$\Delta P_n = F_n^1, \quad \delta < r < 1, \tag{5.5}$$

$$\Delta U_n - \frac{1}{r^2} U_n - \frac{2}{r^2} \frac{\partial}{\partial \theta} V_n - \frac{\partial P_n}{\partial r} = F_n^2, \quad \delta < r < 1, \tag{5.6}$$

$$\Delta V_n - \frac{1}{r^2} V_n + \frac{2}{r^2} \frac{\partial}{\partial \theta} U_n - \frac{1}{r} \frac{\partial P_n}{\partial \theta} = F_n^3, \quad \delta < r < 1 \tag{5.7}$$

$$\frac{\partial U_n}{\partial r} + \frac{\partial V_n}{\partial \theta} + U_n = G_n^1, \quad r = 1 \tag{5.8}$$

$$2 \frac{\partial U_n}{\partial r} - P_n - \sigma \left(\frac{\partial^2 \lambda_n}{\partial \theta^2} + \lambda_n \right) = G_n^2, \quad r = 1, \tag{5.9}$$

$$\frac{\partial V_n}{\partial r} + \frac{\partial U_n}{\partial \theta} - V_n = G_n^3, \quad r = 1, \tag{5.10}$$

$$\frac{\partial \lambda_n}{\partial t} - U_n + \mu_0 \frac{\partial \lambda_n}{\partial \theta} = G_n^4, \quad r = 1, \tag{5.11}$$

$$\frac{\partial P_n}{\partial r} - \frac{1}{\delta} H \left(\frac{\partial P_n}{\partial \theta} \right) = G_n^5, \quad r = \delta, \tag{5.12}$$

$$\frac{\partial U_n}{\partial r} - \frac{1}{\delta} H \left(\frac{\partial U_n}{\partial \theta} \right) + \frac{1}{\delta} H(V_n - V_n^0) - \frac{1}{\delta} U_n^0 - \frac{1}{2} (P_n - P_n^0) = G_n^6, r = \delta, \tag{5.13}$$

$$\begin{aligned} &\frac{\partial (V_n - V_n^{(1)})}{\partial r} - \frac{1}{\delta} H \left(\frac{\partial (V_n - V_n^{(1)})}{\partial \theta} \right) - \frac{1}{\delta} H(U_n - U_n^{(1)}) \\ &+ \frac{1}{2} H(P_n - P_n^{(1)}) = G_n^7, \quad r = \delta, \end{aligned} \tag{5.14}$$

$$\int_0^{2\pi} d\theta \int_\delta^1 r^2 V_n(r, \theta, t)\, dr + \frac{\delta^3}{4} \int_0^{2\pi} V(\delta, \theta, t)\, d\theta + \mu_0 \int_0^{2\pi} \lambda_n d\theta = 2\pi H_n^1, \tag{5.15}$$

$$\int_0^{2\pi} \lambda_n e^{\pm i\theta} d\theta = 2\pi H_n^{2, \pm}, \tag{5.16}$$

$$\lambda_1 |_{t=0} = \lambda^0(\theta), \qquad \lambda_n |_{t=0} = 0 \qquad \text{if} \quad n \geqslant 2. \tag{5.17}$$

The F_n^j, G_n^j, H_n^1, $H_n^{2,\pm}$ depend only on the λ_m, P_m, U_m, V_m for $m<n$. In the next two sections we shall prove a general lemma for systems of the form (5.5)–(5.17) which establishes for given right-hand sides, existence and uniqueness of a solution as well as estimates on its derivatives. This lemma will be used in Section 8 to establish, by induction, the existence and uniqueness of the λ_n, P_n, U_n, V_n and estimates on their derivatives. These estimates will enable us to complete the proof of (1.9) and (1.13).

6. A FUNDAMENTAL LEMMA

Consider a system of differential equations

$$\Delta P = F_1, \quad \delta < r < 1, \tag{6.1}$$

$$\Delta U - \frac{1}{r^2} U - \frac{2}{r^2}\frac{\partial V}{\partial \theta} - \frac{\partial P}{\partial r} = F_2, \quad \delta < r < 1, \tag{6.2}$$

$$\Delta V - \frac{1}{r^2} V + \frac{2}{r^2}\frac{\partial U}{\partial \theta} - \frac{1}{r}\frac{\partial P}{\partial \theta} = F_3, \quad \delta < r < 1 \tag{6.3}$$

and boundary conditions

$$\frac{\partial U}{\partial r} + \frac{\partial V}{\partial \theta} + U = G_1, \quad r = 1, \tag{6.4}$$

$$2\frac{\partial U}{\partial r} - P - \sigma\left(\frac{\partial^2 \Lambda}{\partial \theta^2} + \Lambda\right) = G_2, \quad r = 1, \tag{6.5}$$

$$\frac{\partial V}{\partial r} + \frac{\partial U}{\partial \theta} - V = G_3, \quad r = 1 \tag{6.6}$$

$$\frac{\partial \Lambda}{\partial t} - U + \mu_0 \frac{\partial \Lambda}{\partial \theta} = G_4, \quad r = 1 \tag{6.7}$$

$$\frac{\partial P}{\partial r} - \frac{1}{\delta} H\left(\frac{\partial P}{\partial \theta}\right) = G_5, \quad r = \delta \tag{6.8}$$

$$\frac{\partial U}{\partial r}-\frac{1}{\delta}H\left(\frac{\partial U}{\partial\theta}\right)+\frac{1}{\delta}H(V-V_0)-\frac{1}{\delta}U_0-\frac{1}{2}(P-P_0)=G_6,\quad r=\delta, \tag{6.9}$$

$$\frac{\partial(V-V_{(1)})}{\partial r}-\frac{1}{\delta}H\left(\frac{\partial(V-V_{(1)})}{\partial\theta}\right)-\frac{1}{\delta}H(U-U_{(1)})+\frac{1}{2}H(P-P_{(1)})=G_7,\quad r=\delta, \tag{6.10}$$

an initial condition

$$\Lambda|_{t=0}=\Lambda_0(\theta), \tag{6.11}$$

and constraints

$$\int_0^{2\pi}d\theta\int_\delta^1 r^2V(r,\theta,t)\,dr+\frac{\delta^3}{4}\int_0^{2\pi}V(\delta,\theta,t)\,d\theta+\mu_0\int_0^{2\pi}\Lambda(\theta,t)\,d\theta=2\pi H_1(t), \tag{6.12}$$

$$\int_0^{2\pi}\Lambda e^{\pm i\theta}d\theta=2\pi H_2^{\pm}(t). \tag{6.13}$$

Here U_0 is the zero mode of U, $U_{(1)}$ is the sum of the 0 and ± 1 modes of U, and V_0, P_0, $V_{(1)}$, $P_{(1)}$ are similarly defined.

We want to prove that, given F_j, G_k, H_1, $H_2^{\pm}$, there exists a unique real-valued solution $\Lambda(\theta,t)$, $P(r,\theta,t)$, $U(r,\theta,t)$, $V(r,\theta,t)$ of (6.1)–(6.13) in a suitable space. To define the appropriate space we introduce the following norms: let

$$B=\{(r,\theta);\,\delta<r<1,\,0\leqslant\theta\leqslant 2\pi\},\qquad \partial B=\{0\leqslant\theta\leqslant 2\pi\},$$

then

$$\|F\|_{s,B}=\left\{\int_0^\infty e^{2\alpha t}\left[\|F(\cdot,t)\|^2_{H^s(B)}+\|F_t(\cdot,t)\|^2_{H^{s-1}(B)}+\|F_{tt}(\cdot,t)\|^2_{H^{s-2}(B)}\right]\right\}^{1/2}, \tag{6.14}$$

$$\|f\|_{s,\partial B}=\left\{\int_0^\infty e^{2\alpha t}\left[\|f(\cdot,t)\|^2_{H^s(\partial B)}+\|f_t(\cdot,t)\|^2_{H^{s-1}(\partial B)}+\|f_{tt}(\cdot,t)\|^2_{H^{s-2}(\partial B)}\right]\right\}^{1/2}, \tag{6.15}$$

$$\|\varphi\|_j=\left\{\int_0^\infty e^{2\alpha t}\sum_{k=0}^{j}|D_t^k\varphi(t)|^2\,dt\right\}^{1/2}, \tag{6.16}$$

where $F = F(x, t)$, $f = f(\theta, t)$. We restrict α by

$$0 < \alpha < \frac{\sigma}{2}. \tag{6.17}$$

We shall be working with norms $\| \ \|_{s, B}$, where s is an integer $\geqslant 2$, and with norms $\| \ \|_{s, \partial B}$, where s is a real number $\geqslant 3$, in which case if

$$f(\theta) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta},$$

then

$$\|f\|_{H^s(\partial B)} = \left[\sum |f_n|^2 (1 + |n|^2)^s \right]^{1/2}.$$

In the fundamental lemma, Lemma 6.1 below, we assume that the norms

$$\|F\| \equiv \|F_1\|_{2, B} + \|F_2\|_{3, B} + \|F_3 - F_3^\infty(r)\|_{3, B},$$
$$\|G\| \equiv \|G_1\|_{3\frac{1}{2}, \partial B} + \|G_2 - G_2^\infty\|_{3\frac{1}{2}, \partial B} + \|G_3 - G_3^\infty\|_{3\frac{1}{2}, \partial B}$$
$$+ \|G_4\|_{4\frac{1}{2}, \partial B} + \|G_5\|_{3\frac{1}{2}, \partial B} + \|G_6\|_{3\frac{1}{2}, \partial B} + \|G_7\|_{3\frac{1}{2}, \partial B},$$

and

$$\|H\| = \|H_1 - H_1^\infty\|_2 + \|H_2^+\|_3 + \|H_2^-\|_3$$

are finite, where $F_3^\infty(r)$ is a function of r that belongs to $H^3(\delta, 1)$ and G_2^∞, G_3^∞, H_1^∞ are constants, and that

$$\|\Lambda_0\|_{H^5(\partial B)} < \infty.$$

LEMMA 6.1. *If $\|F\|$, $\|G\|$, $\|H\|$ are finite and $\Lambda_0 \in H^5(\partial B)$, then there exists a unique solution P, U, V, Λ of* (6.1)–(6.13) *satisfying the estimate*

$$\|\Lambda - \Lambda^\infty\|_{5\frac{1}{2}, \partial B} + \|\Lambda_t\|_{4\frac{1}{2}, \partial B} + \|P - P^\infty\|_{4, B} + \|U\|_{5, B} + \|V - V^\infty(r)\|_{5, B}$$
$$\leqslant C[\|F\| + \|G\| + \|H\| + \|\Lambda_0\|_{H^5(\partial B)}] \tag{6.18}$$

where $V^\infty(r)$ is a function of r, Λ^∞ and P^∞ are constants, and

$$|\Lambda^\infty| + |P^\infty| + \|V^\infty\|_{H^5(\delta, 1)} \leqslant C[\|F\| + \|G\| + \|H\| + \|\Lambda_0\|_{H^5(\partial B)}]; \tag{6.19}$$

the constant C is independent of the data.

To prove Lemma 6.1, we expand the F_j, G_k in Fourier series

$$F_j = \sum_m F_j^m(r, t)\, e^{im\theta} \quad (j = 1, 2, 3),$$

$$G_k = \sum_m G_k^m(t)\, e^{im\theta} \quad (k = 1, 2, ..., 7)$$

and try to find a solution of the form

$$\begin{aligned} \Lambda &= \sum_m \Lambda^m(t)\, e^{im\theta}, \\ P &= \sum_m P^m(r, t)\, e^{im\theta}, \\ U &= \sum_m U^m(r, t)\, e^{im\theta}, \\ V &= \sum_m V^m(r, t)\, e^{im\theta}. \end{aligned} \tag{6.20}$$

Then, with t as a parameter, which we shall often not indicate explicitly,

$$P_{rr}^m + \frac{1}{r} P_r^m - \frac{m^2}{r^2} P^m = F_1^m, \tag{6.21}$$

$$U_{rr}^m + \frac{1}{r} U_r^m - \frac{m^2+1}{r^2} U^m - \frac{2im}{r^2} V^m = f_2^m \equiv F_2^m + \frac{\partial P^m}{\partial r}, \tag{6.22}$$

$$V_{rr}^m + \frac{1}{r} V_r^m - \frac{m^2+1}{r^2} V^m + \frac{2im}{r^2} U^m = f_3^m \equiv F_3^m + \frac{im}{r} P^m. \tag{6.23}$$

Consider first the case $|m| > 1$. Then, one can easily verify that the general solution P_m of (6.21) is

$$P^m = c_1 r^{-|m|} + c_2 r^{|m|} - \frac{r^{|m|}}{2\,|m|} \int_r^1 \rho^{-|m|+1} F_1^m(\rho)\, d\rho - \frac{r^{-|m|}}{2\,|m|} \int_\delta^r \rho^{|m|+1} F_1^m(\rho)\, d\rho. \tag{6.24}$$

The general homogeneous solution of (6.22) and (6.23) is

$$u = A_1 r^{|m|+1} + A_2 r^{|m|-1} + A_3 r^{-|m|+1} + A_4 r^{-|m|-1},$$
$$v = i(\operatorname{sgn} m)[-A_1 r^{|m|+1} + A_2 r^{|m|-1} + A_3 r^{-|m|+1} - A_4 r^{-|m|-1}].$$

A special solution of the inhomogeneous system (6.22), (6.23) is given by

$$u = I_m(f_2^m) + i(\operatorname{sgn} m)\, J_m(f_3^m),$$
$$v = I_m(f_3^m) - i(\operatorname{sgn} m)\, J_m(f_2^m),$$

where

$$I_m(g) = \frac{1}{4}\left\{\int_\delta^r \rho g(\rho)\left[-\frac{(\rho/r)^{|m|+1}}{|m|+1} - \frac{(\rho/r)^{|m|-1}}{|m|-1}\right] + \int_r^1 \rho g(\rho)\left[-\frac{(r/\rho)^{|m|+1}}{|m|+1} - \frac{(r/\rho)^{|m|-1}}{|m|-1}\right]\right\},$$

$$J_m(g) = \frac{1}{4}\left\{\int_\delta^r \rho g(\rho)\left[-\frac{(\rho/r)^{|m|+1}}{|m|+1} + \frac{(\rho/r)^{|m|-1}}{|m|-1}\right] + \int_r^1 \rho g(\rho)\left[-\frac{(r/\rho)^{|m|+1}}{|m|+1} + \frac{(r/\rho)^{|m|-1}}{|m|-1}\right]\right\}.$$

Substituting f_2^m, f_3^m from (6.22), (6.23) and using also (6.24), we find, after some calculations, the general solution of (6.22), (6.23):

$$U^m = A_1 r^{|m|+1} + A_2 r^{|m|-1} + A_3 r^{-|m|+1} + A_4 r^{-|m|-1} + c_1 h_1(r) + c_2 h_2(r) + \Phi_1^m, \tag{6.25}$$

$$V^m = i(\operatorname{sgn} m)[-A_1 r^{|m|+1} + A_2 r^{|m|-1} + A_3 r^{-|m|+1} - A_4 r^{-|m|-1} - c_1 h_1(r) + c_2 h_2(r)] + \Phi_2^m, \tag{6.26}$$

where

$$h_1(r) = \frac{1}{4}\frac{|m|}{(|m|+1)}\left(\frac{|m|+1}{|m|} r^{-|m|+1} - r^{-|m|-1}\delta^2 - \frac{1}{|m|} r^{|m|+1}\right), \tag{6.27}$$

$$h_2(r) = \frac{1}{4}\frac{|m|}{(|m|-1)}\left(\frac{|m|-1}{|m|} r^{|m|+1} + \frac{1}{|m|} r^{-|m|+1}\delta^{2|m|} - r^{|m|-1}\right), \tag{6.28}$$

$$\Phi_1^m = I_m(\hat{F}_2^m) + i(\operatorname{sgn} m)\, J_m(\hat{F}_3^m),$$

$$\Phi_2^m = I_m(\hat{F}_3^m) - i(\operatorname{sgn} m)\, J_m(\hat{F}_2^m),$$

$$\hat{F}_2^m(\rho) = F_2^m(\rho) - \frac{1}{2}\rho^{|m|-1}\int_1^\rho s^{-|m|+1} F_1^m(s)\, ds + \frac{1}{2}\rho^{-|m|-1}\int_\delta^\rho s^{|m|+1} F_1^m(s)\, ds,$$

$$\hat{F}_3^m(\rho) = F_3^m(\rho) - \frac{i}{2}(\operatorname{sgn} m)\left\{\rho^{|m|-1}\int_1^\rho s^{-|m|+1} F_1^m(s)\, ds + \rho^{-|m|-1}\int_\delta^\rho s^{|m|+1} F_1^m(s)\, ds\right\}.$$

We next substitute (6.24)–(6.26) into the boundary conditions (6.4)–(6.10) noting that $\partial/\partial\theta$ corresponds to multiplication by im. We get the following system of seven equations for the seven coefficients A_1, A_2, A_3, A_4, c_1, c_2 and Λ^m:

$$2(|m|+1)\, A_1 - 2(|m|-1)\, A_3 + [h_1'(1) + (|m|+1)\, h_1(1)]\, c_1 + [h_2'(1) - (|m|-1)\, h_2(1)]\, c_2 = \hat{G}_1^m. \tag{6.29}$$

$$2(|m|+1)\, A_1 + 2(|m|-1)\, A_2 - 2(|m|-1)\, A_3 - 2(|m|+1)\, A_4 + (2h_1'(1)-1)\, c_1 + (2h_2'(1)-1)\, c_2 - \sigma(1-m^2)\, \Lambda^m = \hat{G}_2^m, \tag{6.30}$$

$$2(|m|-1)\, A_2 + 2(|m|+1)\, A_4 + [-h_1'(1) + (|m|+1)\, h_1(1)]\, c_1 + [h_2'(1) + (|m|-1)\, h_2(1)]\, c_2 = \hat{G}_3, \tag{6.31}$$

$$\Lambda_t^m + im\mu_0 \Lambda^m - [A_1 + A_2 + A_3 + A_4 + c_1 h_1(1) + c_2 h_2(1)] = \hat{G}_4^m, \tag{6.32}$$

$$c_1 = -\frac{1}{2\,|m|}\,\delta^{|m|+1}\hat{G}_5^m, \tag{6.33}$$

$$-2(|m|-1)\,\delta^{-|m|} A_3 - 2(|m|+1)\,\delta^{-|m|-2} A_4 + \left[h_1'(\delta) - \frac{|m|+1}{\delta}\, h_1(\delta) - \frac{1}{2}\,\delta^{-|m|}\right] c_1 + \left[h_2'(\delta) - \frac{|m|-1}{\delta}\, h_2(\delta) - \frac{1}{2}\,\delta^{|m|}\right] c_2 = \hat{G}_6^m, \tag{6.34}$$

$$-2(|m|-1)\,\delta^{-|m|} A_3 + 2(|m|+1)\,\delta^{-|m|-2} A_4 + \left[-h_1'(\delta) + \frac{|m|+1}{\delta}\, h_1(\delta) - \frac{1}{2}\,\delta^{-|m|}\right] c_1 + \left[h_2'(\delta) - \frac{|m|-1}{\delta}\, h_2(\delta) - \frac{1}{2}\,\delta^{|m|}\right] c_2 = \hat{G}_7^m, \tag{6.35}$$

where

$$\hat{G}_1^m = G_1^m - \left[\frac{\partial \Phi_1^m}{\partial r} + im\Phi_2^m + \Phi_1^m \right] \bigg|_{r=1},$$

$$\hat{G}_2^m = G_2^m - \left[2 \frac{\partial \Phi_1^m}{\partial r} \bigg|_{r=1} + \frac{1}{2\,|m|} \int_\delta^1 \rho^{|m|+1} F_1^m(\rho)\, d\rho \right],$$

$$\hat{G}_3^m = -i(\operatorname{sgn} m)\, G_3^m + i(\operatorname{sgn} m) \left[\frac{\partial \Phi_2^m}{\partial r} - \Phi_2^m \right] \bigg|_{r=1} - |m|\, \Phi_1^m \,|_{r=1},$$

$$\hat{G}_4^m = G_4^m + \Phi_1^m \,|_{r=1}, \qquad \hat{G}_5^m = G_5^m,$$

$$\hat{G}_6^m = G_6^m - \left[\frac{\partial \Phi_1^m}{\partial r} \bigg|_{r=\delta} - \frac{|m|}{\delta}\, \Phi_1^m \,|_{r=\delta} - \frac{i(\operatorname{sgn} m)}{\delta}\, \Phi_2^m \,|_{r=\delta} \right.$$

$$\left. + \frac{1}{2} \frac{\delta^{|m|}}{2\,|m|} \int_\delta^1 \rho^{-|m|+1} F_1^m(\rho)\, d\rho \right],$$

$$\hat{G}_7^m = -i(\operatorname{sgn} m)\, G_7^m + i(\operatorname{sgn} m) \left[\frac{\partial \Phi_2^m}{\partial r} - \frac{|m|}{\delta}\, \Phi_2^m \right] \bigg|_{r=\delta} - \frac{1}{\delta}\, \Phi_1^m \,|_{r=\delta}$$

$$- \frac{1}{2} \frac{\delta^{|m|}}{2\,|m|} \int_\delta^1 \rho^{-|m|+1} F_1^m(\rho)\, d\rho.$$

Using the relations

$$h_1'(\delta) - \frac{|m|+1}{\delta}\, h_1(\delta) = h_2'(\delta) - \frac{|m|-1}{\delta}\, h_2(\delta) = 0,$$

we get from (6.34), (6.35),

$$A_4 = \frac{\delta^{|m|+2}}{4(|m|+1)} (\hat{G}_7^m - \hat{G}_6^m) \tag{6.36}$$

and

$$A_3 = -\frac{\delta^{2\,|m|}}{4(|m|-1)}\, c_2 + \hat{G}_8^m, \tag{6.37}$$

where

$$\hat{G}_8^m = -\frac{\delta^{|m|}}{4(|m|-1)} \left[\hat{G}_6^m + \hat{G}_7^m - \frac{\delta}{2\,|m|}\, \hat{G}_5^m \right].$$

By (6.27), (6.28), the equations (6.29)–(6.31) reduce to

$$2(|m|+1)\,A_1-2(|m|-1)\,A_3+\frac{1}{2}(1-\delta^{2|m|})\,c_2=\hat{G}_1^m, \tag{6.38}$$

$$\begin{gathered}2(|m|+1)\,A_1+2(|m|-1)\,A_2-2(|m|-1)\,A_3-2(|m|+1)\,A_4\\+\left(-1-\frac{|m|}{2}+\frac{|m|}{2}\delta^2\right)c_1-\frac{1}{2}(1+\delta^{2|m|})\,c_2-\sigma\Lambda^m(1-m^2)=\hat{G}_2^m,\end{gathered} \tag{6.39}$$

$$2(|m|-1)\,A_2+2(|m|+1)\,A_4+\frac{1}{2}|m|\,(1-\delta^2)\,c_1=\hat{G}_3^m.$$

From the last equation and (6.33), (6.36),

$$A_2=\hat{G}_9^m\equiv\frac{1}{2(|m|-1)}\left[\hat{G}_3^m-\frac{1}{2}\delta^{|m|+2}(\hat{G}_7^m-\hat{G}_6^m)+\frac{1}{4}\delta^{|m|+1}(1-\delta^2)\,\hat{G}_5^m\right]. \tag{6.40}$$

If we substitute A_3 from (6.37) into (6.38), we get

$$A_1=-\frac{1}{4(|m|+1)}c_2+\frac{1}{2(|m|+1)}(\hat{G}_1^m+2(|m|-1)\,\hat{G}_8^m). \tag{6.41}$$

We next express c_2 in terms of Λ^m by subtracting (6.38) from (6.39) and using (6.36), (6.40),

$$c_2=(m^2-1)\,\sigma\Lambda^m+\hat{G}_{10}^m, \tag{6.42}$$

where

$$\begin{gathered}\hat{G}_{10}^m=(\hat{G}_1^m-\hat{G}_2^m)+2(|m|-1)\,\hat{G}_9^m-\frac{1}{2}\delta^{|m|+2}(\hat{G}_7^m-\hat{G}_6^m)\\-\left(-1-\frac{|m|}{2}-\frac{|m|\,\delta^2}{2}\right)\frac{1}{2\,|m|}\delta^{|m|+1}\hat{G}_5^m.\end{gathered}$$

Substituting (6.42) into (6.37) and (6.41), and then using all these expresions as well as those for A_2, A_4, c_1 in (6.32), we get a differential equation for Λ^m:

$$\Lambda_t^m+\frac{\sigma}{2}|m|\,\Lambda^m+i\mu_0\,|m|\,\Lambda^m=\hat{G}_{11}^m, \tag{6.43}$$

where

$$\hat{G}_{11}^m = \hat{G}_4^m + \frac{1}{2(|m|+1)} (\hat{G}_1^m + 2(|m|-1)\, \hat{G}_8^m) + \hat{G}_9^m + \hat{G}_8^m + \frac{\delta^{|m|+2}}{4(|m|+1)} (\hat{G}_7^m - \hat{G}_6^m)$$
$$- \frac{1-\delta^2}{8(|m|+1)} \delta^{|m|+1} \hat{G}_5^m - \frac{|m|}{2(m^2-1)} \hat{G}_{10}^m.$$

We next consider the case $|m| = 1$. Then

$$P^m = c_1 r^{-1} + c_2 r - \frac{r}{2} \int_r^1 F_1^m(\rho)\, d\rho - \frac{r^{-1}}{2} \int_\delta^r \rho^2 F_1^m(\rho)\, d\rho, \tag{6.44}$$

and the general solution of (6.22), (6.23) is

$$U^m = A_1 r^2 + A_2 + A_3 \log r + A_4 r^{-2} + c_1 h_1(r) + c_2 h_2(r) + \Phi_1^m, \tag{6.45}$$

$$V^m = i(\operatorname{sgn} m)[-A_1 r^2 + A_2 + A_3 \log r - A_4 r^{-2} - c_1 h_1(r) + c_2 h_2(r)] + \Phi_2^m, \tag{6.46}$$

where

$$\Phi_1^m = \tilde{I}_m(\hat{F}_2^m) + i(\operatorname{sgn} m)\, \tilde{J}_m(\hat{F}_3^m),$$
$$\Phi_2^m = \tilde{I}_m(\hat{F}_3^m) - i(\operatorname{sgn} m)\, \tilde{J}_m(\hat{F}_2^m),$$
$\hat{F}_2^m$, $\hat{F}_3^m$ are defined as before,

and

$$\tilde{I}_m(g) = \frac{1}{4} \left\{ \int_\delta^r \rho g(\rho) \left[-\frac{(\rho/r)^2}{2} - \log \frac{\rho}{r} \right] + \int_r^1 \rho g(\rho) \left[-\frac{(r/\rho)^2}{2} - \log \frac{r}{\rho} \right] \right\},$$
$$\tilde{J}_m(g) = \frac{1}{4} \left\{ \int_\delta^r \rho g(\rho) \left[-\frac{(\rho/r)^2}{2} + \log \frac{\rho}{r} \right] + \int_r^1 \rho g(\rho) \left[-\frac{(r/\rho)^2}{2} + \log \frac{r}{\rho} \right] \right\};$$

here

$$h_1(r) = \frac{1}{4} - \frac{\delta^2}{8r^2} - \frac{r^2}{8},$$

$$h_2(r) = -\frac{1}{2} \left[\int_\delta^r \rho \log \frac{\rho}{r} + \int_r^1 \rho \log \frac{r}{\rho} \right].$$

Since $|m|=1$, the boundary condition (6.10) is vacuous, but instead we need to satisfy the constraint (6.13),

$$\Lambda^m = H_2^m, \tag{6.47}$$

where $H_2^1 = H_2^-$ and $H_2^{-1} = H_2^+$.

Next, the boundary conditions (6.4)–(6.9) yield

$$4A_1 + A_3 + \frac{1}{4}(1-\delta^2)\, c_2 = \hat{G}_1^m, \tag{6.48}$$

$$4A_1 + 2A_3 - 4A_4 + \left(\frac{\delta^2}{2} - \frac{3}{2}\right) c_1 - \frac{1}{2}(1+\delta^2)\, c_2 = \hat{G}_2^m, \tag{6.49}$$

$$A_3 + 4A_4 + \frac{1}{2}(1-\delta^2)\, c_1 + \frac{1}{4}(1-\delta^2)\, c_2 = \hat{G}_3^m, \tag{6.50}$$

$$\Lambda_t^m + i(\operatorname{sgn} m)\, \mu_0 \Lambda^m - \left[A_1 + A_2 + A_4 + \frac{1}{8}(1-\delta^2)\, c_1 + h_2(1)\, c_2 \right] = \hat{G}_4^m, \tag{6.51}$$

$$c_1 = -\frac{1}{2}\delta^2 \hat{G}_5^m, \tag{6.52}$$

$$A_3 - \frac{4}{\delta^2} A_4 - \frac{1}{2} c_1 - \frac{1}{4}(1+\delta^2)\, c_2 = \delta \hat{G}_6^m, \tag{6.53}$$

where the $\hat{G}_k^m$ are obtained from the G_j^m in the same way as in the case $|m|>1$, but with the operators I_m, J_m replaced by $\tilde{I}_m$, $\tilde{J}_m$.

The system (6.48)–(6.50), (6.53) for A_1, A_3, A_4 and c_2 has a nonsingular coefficient matrix and can therefore be uniquely solved,

$$A_1 = \tilde{G}_1^m,\ A_3 = \tilde{G}_3^m,\ A_4 = \tilde{G}_4^m,\ c_2 = \tilde{G}_5^m, \tag{6.54}$$

where the $\tilde{G}_j^m$ are linear combinations of the $\hat{G}_k^m$. From (6.47) the function Λ^m and its t-derivatives are determined. Substituting this and (6.52), (6.54) into (6.51), one finds that

$$A_2 = \tilde{G}_2^m, \tag{6.55}$$

where

$$\tilde{G}_2^m = -\hat{G}_4^m - \left[\tilde{G}_1^m + \tilde{G}_4^m - \frac{1}{8}(1-\delta^2)\frac{1}{2}\delta^2\hat{G}_5^m + h_2(1)\,\tilde{G}_5^m \right]$$
$$+\frac{\partial}{\partial t} H_2^m + i(\operatorname{sgn} m)\,\mu_0 H_2^m. \tag{6.56}$$

We finally consider the case of mode 0: in this case

$$P^0 = c_1 + c_2 \log r + \log r \int_\delta^r \rho F_1^0(\rho)\,d\rho - \int_\delta^r \rho \log \rho F_1^0(\rho)\,d\rho \tag{6.57}$$

and

$$U^0 = A_1 r + A_2 r^{-1} + \left[-\frac{1}{4r}(r^2-\delta^2) + \frac{r}{2}\log r \right] c_2 + \Phi_1^0, \tag{6.58}$$

$$V^0 = B_1 r + B_2 r^{-1} + \Phi_2^0, \tag{6.59}$$

where

$$\Phi_1^0 = -\frac{1}{2}\int_\delta^r \frac{\rho^2}{r} F_2^0(\rho)\,d\rho - \frac{1}{2}\int_r^1 r F_2^0(\rho)\,d\rho$$
$$-\frac{1}{4}\int_\delta^r (r^2-\rho^2)\frac{\rho}{r} F_1^0(\rho)\,d\rho + \frac{1}{2}\int_\delta^r \rho F_1^0(\rho)\, r\log r\,d\rho.$$
$$+\frac{1}{2}\int_r^1 \rho F_1^0(\rho)\, r\log\rho\,d\rho,$$
$$\Phi_2^0 = -\frac{1}{2}\int_\delta^r \frac{\rho^2}{r} F_3^0(\rho)\,d\rho - \frac{1}{2}\int_r^1 r F_3^0(\rho)\,d\rho.$$

The boundary conditions (6.4)–(6.9) yield

$$2A_1 = G_1^0, \tag{6.60}$$

$$2A_1 - 2A_2 - \sigma\Lambda^0 - c_1 + \tfrac{1}{2}(1-\delta^2)\,c_2 = \hat{G}_2^0, \tag{6.61}$$

$$-2B_2 = G_3^0 - \int_\delta^1 \rho^2 F_3^0(\rho)\,d\rho, \tag{6.62}$$

$$A_t^0 - A_1 - A_2 + \tfrac{1}{4}(1-\delta^2)\, c_2 = \hat{G}_4^0, \tag{6.63}$$

$$c_2 = \delta G_5^0, \tag{6.64}$$

$$-2A_2\delta^{-2} = G_6^0, \tag{6.65}$$

where

$$\hat{G}_2^0 = G_2^0 - \int_\delta^1 \rho \log \rho F_1^0(\rho)\, d\rho - \int_\delta^1 \rho^2 F_2^0(\rho) - \tfrac{1}{2}\int_\delta^1 (1-\rho^2)\, \rho F_1^0(\rho)\, d\rho,$$

$$\hat{G}_4^0 = G_4^0 - \tfrac{1}{2}\int_\delta^1 \rho^2 F_2^0(\rho)\, d\rho - \tfrac{1}{4}\int_\delta^1 (1-\rho^2)\, \rho F_1^0(\rho)\, d\rho.$$

Substituting A_1, A_2, c_2 from (6.60), (6.65), (6.64) into (6.63), one gets

$$A_t^0 = \tilde{G}_4^0, \tag{6.66}$$

where

$$\tilde{G}_4^0 = \hat{G}_4^0 + \frac{1}{2} G_1^0 - \frac{\delta^2}{2} G_6^0 - \frac{1}{4}(1-\delta^2)\, \delta G_5^0. \tag{6.67}$$

We determine c_1 by integrating (6.66) and substituting the result into (6.61). Finally B_1 is determined by the constraint (6.12), which, as easily seen, yields

$$\frac{1}{4} B_1 = -\frac{2-\delta^2}{4} B_2 - \int_\delta^1 \rho^2 \Phi_2^0\, d\rho - \mu_0 A^0 + H_1 - \frac{\delta^3}{4} \Phi_2^0(\delta). \tag{6.68}$$

In summary, we have proved that the system (6.1)–(6.13) has a unique formal solution of the form (6.20), and have determined the coefficients A^m, P^m, U^m, V^m. In the next section we shall use the differential equation (6.43) to estimate the A^m, A_t^m for $|m| > 1$, (6.51), (6.52), and (6.54), (6.55), to estimate A^m, A_t^m for $m = \pm 1$ and (6.60), (6.63)–(6.65) to estimate A^0, A_t^0. Combining these estimates we shall establish the bounds (6.18), (6.19), and thus complete the proof of Lemma 6.1. (Uniqueness follows from the fact that any solution for which the left-hand sides of (6.18), (6.19) are finite must have a series expansion (6.20).)

7. A FUNDAMENTAL LEMMA (CONTINUED)

LEMMA 7.1. *Consider the initial value problem*

$$\dot{B}(t)+(K+iM)\,B(t)=F(t),\quad t>0,$$
$$B(0)=B_0,$$

where K, M are real numbers and $K>0$. Then, for any $0<\alpha<K$, there hold

$$\int_0^\infty e^{2\alpha s}\,|B(s)|^2\,ds\leqslant\frac{2}{(K-\alpha)^2}\int_0^\infty e^{2\alpha s}\,|F(s)|^2\,ds+\frac{|B_0|^2}{K-\alpha},$$

$$\int_0^\infty e^{2\alpha s}\,|\dot{B}(s)|^2\,ds\leqslant 2\left(\frac{2K^2}{(K-\alpha)^2}+1\right)\int_0^\infty e^{2\alpha s}\,|F(s)|^2\,ds+\frac{2K^2}{K-\alpha}\,|B_0|^2.$$

Proof. The case $M=0$ is proved in [3]; the proof extends with minor changes to $M\neq 0$. ∎

We introduce the notation

$$\langle f\rangle=\int_0^t |f(s)|^2\,e^{2\alpha s}\,ds.$$

Then from the estimates derived in Section 6 we easily get, for $|m|>1$,

$$\langle \hat{G}_{11}^m\rangle\leqslant C\left\{\langle G_4^m\rangle+\frac{C}{m^2}\sum_{j=1}^{7}\langle G_j^m\rangle+N^m\right\},\tag{7.1}$$

where

$$\begin{aligned}N^m&=\sum_{j=1}^{2}\left[\frac{1}{m^2}\left\langle\frac{\partial\Phi_j^m}{\partial r}\right\rangle+\langle\Phi_j^m\rangle\right]_{r=1,\,\delta}\\&\quad+\frac{1}{m^4}\left[\left\langle\int_\delta^1 \rho^{|m|+1}F_1^m(\rho,\cdot)\,d\rho\right\rangle+\left\langle\int_\delta^1 \delta^{|m|}\rho^{-|m|+1}F_1^m(\rho,\cdot)\,d\rho\right\rangle\right]\\&\equiv L_1+\frac{1}{m^4}\,L_2.\end{aligned}\tag{7.2}$$

By the Cauchy–Schwarz inequality,

$$\frac{1}{m^4}\,|L_2|\leqslant\frac{C}{|m|^5}\int_0^t\left[\int_\delta^1 |F_1^m(\rho,s)|^2\,\rho\,d\rho\right]e^{2\alpha s}\,ds.\tag{7.3}$$

Consider next Φ_1^m at $r=1$, and set

$$g_1(\rho,t)=F_2^m(\rho,t), \qquad g_2(\rho,t)=F_3^m(\rho,t),$$

$$g_3(\rho,t)=\rho^{-|m|-1}\int_\delta^\rho \tau^{|m|+1}F_1^m(\tau,t)\,d\tau,$$

$$g_4(\rho,t)=\rho^{|m|-1}\int_\rho^1 \tau^{-|m|+1}F_1^m(\tau,t)\,d\tau.$$

We need to estimate $I_m(g_j)$ and $J_m(g_k)$ for $j=1,3,4$ and $k=2,3,4$. $I_m(g)$ can be written as a sum of four integrals, and we shall first consider the first integral for g_j $(1\leqslant j\leqslant 4)$,

$$D_j=\int_\delta^r \rho g_j(\rho)\frac{(\rho/r)^{|m|+1}}{|m|+1}\,d\rho\,|_{r=1}.$$

By the Cauchy–Schwarz inequality,

$$\langle D_1\rangle \leqslant \frac{c}{|m|^3}\int_0^t\left[\int_\delta^1 |F_2^m(\rho,s)|^2\,\rho\,d\rho\right]e^{2\alpha s}\,ds.$$

D_2 can be estimated in the same way. To estimate D_3 we first use the Cauchy–Schwarz inequality to get

$$\left|\int_\delta^1\left[\rho^{-|m|}\int_\delta^\rho \tau^{|m|+1}F_1^m(\tau,s)\,d\tau\right]\frac{\rho^{|m|+1}}{|m|+1}\,d\rho\right|$$
$$\leqslant \frac{c}{|m|^{3/2}}\left\{\int_\delta^1\left[\rho^{-|m|-2}\int_\delta^\rho \tau^{|m|+1}F_1^m(\tau,s)\,d\tau\right]^2 d\rho\right\}^{1/2}\equiv \frac{C}{|m|^{3/2}}L.$$

Next we substitute $\tau=\rho\lambda$ in L:

$$L=\left\{\int_\delta^1\left|\int_1^{\delta/\rho}\lambda^{|m|+1}F_1^m(\rho\lambda,s)\,d\lambda\right|^2 d\rho\right\}^{1/2}$$
$$\leqslant \int_\delta^1 \lambda^{|m|+1}\,d\lambda\left(\int_{\delta/\lambda}^1 |F_1^m(\rho\lambda,s)|^2\,d\rho\right)^{1/2} \quad \text{by Minkowski's inequality.}$$

Substituting back $\rho=\tau/\lambda$ in the inner integral, we find that

$$L\leqslant \int_\delta^1 \lambda^{|m|}\,d\lambda\left(\int_\delta^1 |F_1^m(\tau,s)|^2\,d\tau\right)^{1/2}$$
$$\leqslant \frac{C}{|m|}\left(\int_\delta^1 |F_1^m(\rho,s)|^2\,\rho\,d\rho\right)^{1/2}.$$

Hence

$$\langle D_3 \rangle \leqslant \frac{C}{|m|^5} \int_0^t \left[\int_\delta^1 |F_1^m(\rho, s)|^2 \, \rho \, d\rho \right] e^{2\alpha s} \, ds.$$

The same bound can similarly be derived for D_4. The other three integrals in $J_m(g_j)$ can be treated in the same way. Hence

$$\langle \Phi_1^m \rangle |_{r=1} \leqslant \frac{C}{|m|^3} \sum_{j=2}^{3} \int_0^t \left[\int_\delta^1 |F_j^m(\rho, s)|^2 \, \rho \, d\rho \right] e^{2\alpha s} \, ds + \frac{C}{|m|^5} \int_0^t \left[\int_\delta^1 |F_1^m(\rho, s)|^2 \, \rho \, d\rho \right] e^{2\alpha s} \, ds.$$

The estimates of $\langle \Phi_2^m \rangle |_{r=1}$ and of $\langle \Phi_j^m \rangle |_{r=\delta}$ are derived in the same way, and so are the estimates of $\frac{1}{|m|^2} \langle \partial \Phi_j^m / \partial r \rangle$. Using the result in (7.2), we then obtain from (7.1) the inequality

$$\langle \hat{G}_{11} \rangle \leqslant C \left\{ \langle G_4^m \rangle + \frac{1}{m^2} \sum_{j=1}^{7} \langle G_j^m \rangle + \frac{1}{|m|^5} \int_0^t \left[\int_\delta^1 |F_1^m(\rho, s)|^2 \, \rho \, d\rho \right] e^{2\alpha s} \, ds + \frac{1}{|m|^3} \sum_{j=2}^{3} \int_0^t \left[\int_0^\delta |F_j^m(\rho, s)|^2 \, \rho \, d\rho \right] e^{2\alpha s} \, ds \right\}. \tag{7.4}$$

We now use Lemma 7.1 to estimate Λ^m and Λ_t^m from the differential equation (6.43) and the bound (7.4). The estimate on Λ_t^m allows us to estimate the derivative $\partial \hat{G}_{11}^m / \partial t$ in the same way that we have estimated $\hat{G}_{11}^m$. Another application of Lemma 7.1 to the equation

$$(\Lambda_t^m)_t + \frac{\sigma}{2} |m| \, (\Lambda_t^m) + i \mu_0 \, |m| \, (\Lambda_t^m) = \frac{\partial \hat{G}_{11}^m}{\partial t}$$

yields an estimate on Λ_{tt}^m. Repeating this process once more we obtain an estimate also on Λ_{ttt}^m. Multiplying the estimates on

$$\frac{\partial^j}{\partial t^j} \Lambda^m \qquad \text{by} \quad |m|^{2(5\frac{1}{2} - j)}$$

and summing on m, $|m| > 1$, and adding also the corresponding estimates for $|m| = 1$, we get

$$\|\Lambda-\Lambda^0\|_{5\frac{1}{2},\partial B}+\|(\Lambda-\Lambda^0)_t\|_{4\frac{1}{2},\partial B}$$
$$\leqslant C\left\{\|F_1-F_1^0\|_{2,B}+\sum_{j=2}^{3}\|F_j-F_j^0\|_{3,B}+\|G_4-G_4^0\|_{4\frac{1}{2},\partial B}\right.$$
$$\left.+\sum_{j=1}^{7}\|G_j-G_j^0\|_{3\frac{1}{2},\partial B}+\|\Lambda_0\|_{H^5(\partial B)}+\|H_2^+\|_3+\|H_2^-\|_3\right\}\equiv R. \tag{7.5}$$

We next recall the formulas for A_j, B_j, c_j for $|m|>1$ and use them together with the estimates on Λ_m established in (7.5) in order to estimate P^m, U^m, V^m (defined in (6.24)–(6.26)). We then sum the estimates of

$$(|m|^{4-j}\,|D_t^jP^m|)^2, \qquad (|m|^{5-j}\,|D_t^jU^m|)^2, \qquad (|m|^{5-j}\,|D_t^jV^m|)^2$$

over m, $|m|>1$, and combine it with the estimates (which are easily obtained) for $|m|=1$. This allows us to conclude that

$$\|P-P^0\|_{4,B}+\|U-U^0\|_{5,B}+\|V-V^0\|_{5,B}\leqslant R, \tag{7.6}$$

where R denotes the right-hand side of (7.5).

Thus, in order to complete the proof of Lemma 6.1, it remains to consider the case of zero mode. From (6.63) we see that

$$\|\Lambda_t^0\|_2^2\leqslant C[\|G_1^0\|_2^2+\|G_4^0\|_2^2+\|G_5^0\|_2^2+\|G_6^0\|_2^2+\|F_1^0\|_2^2+\|F_2^0\|_2^2] \tag{7.7}$$

so that $\Lambda^0\to\Lambda^\infty$ as $t\to\infty$, where Λ^∞ is a constant. We can also estimate the rate of convergence as

$$\left(\int_0^T e^{2\alpha t}\,|\Lambda^0-\Lambda^\infty|^2\,dt\right)^{1/2}$$
$$=\left[\int_0^T e^{2\alpha t}\left(\int_t^\infty \Lambda_t^0(s)\,ds\right)^2 dt\right]^{1/2}$$
$$=\left[\int_0^T \left(\int_t^\infty e^{\alpha(t-s)}e^{\alpha s}\Lambda_t^0(s)\,ds\right)^2 dt\right]^{1/2}$$
$$=\left[\int_0^T \left(\int_0^\infty e^{-\alpha u}e^{\alpha(t+u)}\Lambda_t^0(t+u)\,du\right)^2 dt\right]^{1/2} \quad (s=t+u)$$
$$\leqslant\int_0^\infty e^{-\alpha u}\,du\left[\int_0^T (e^{\alpha(t+u)}\Lambda_t^0(t+u))^2\,dt\right]^{1/2} \quad \text{(by Minkowski's inequality)}$$
$$=\int_0^\infty e^{-\alpha u}\,du\left[\int_u^{T+u} e^{2\alpha t}\,|\Lambda_t^0(t)|^2\,dt\right] \quad (t+u\to t).$$

Combining this with (7.7) we conclude that $\|\Lambda^0-\Lambda^\infty\|_3$ is bounded by the right-hand side of (7.7).

A similar analysis shows that $P^0 \to P^\infty$, $U^0 \to 0$, $V \to V^\infty(r)$ as $t \to \infty$, and

$$\|P^0 - P^\infty\|_{4,\,B}, \qquad \|U^0\|_{5,\,B}, \qquad \|V^0 - V^\infty(r)\|_{5,\,B}$$

can be estimated by the right-hand side of (6.18); note that

$$\frac{\partial}{\partial t}(P^0 - P^\infty) = \frac{\partial P^0}{\partial t}, \qquad \frac{\partial}{\partial t}(V^0 - V^\infty) = \frac{\partial}{\partial t} V^0.$$

Combining these estimates with (7.5), (7.6), the proof of (6.18) is complete.

Finally, the estimates (6.19) follow from the explicit formulas for mode 0, as $t \to \infty$, and this completes the proof of Lemma 6.1.

8. CONVERGENCE

We shall need the following lemma.

LEMMA 8.1. *The following inequality holds for any* $s \geqslant 2$,

$$\|FG\|_{s,\,B} \leqslant M_0\,\|F\|_{s,\,B}\,\|G\|_{s,\,B},$$
$$\|fg\|_{s,\,\partial B} \leqslant M_0\,\|f\|_{s,\,\partial B}\,\|g\|_{s,\,\partial B},$$

where M_0 *is a constant depending only on s.*

The proof is similar to that given in [3] for somewhat different norms.

LEMMA 8.2. *If* $\lambda^0 \in H^5(\partial B)$ *then the system* (5.5)–(5.17) *has a unique solution for every* $n \geqslant 1$, *and the following bounds hold,*

$$\|\lambda_n - \lambda_n^\infty\|_{5\frac{1}{2},\,\partial B} + \|\lambda_{n,\,t}\|_{4\frac{1}{2},\,\partial B} + \|P_n - P_n^\infty\|_{4,\,B} + \|U_n\|_{5,\,B} + \|V_n - V_n^\infty(r)\|_{5,\,B} \leqslant C_0\,\frac{H^{n-1}}{n^2}, \tag{8.1}$$

$$|\lambda_n^\infty| + |P_n^\infty| + \|V_n^\infty\|_{H^5(\delta,\,1)} \leqslant C_0\,\frac{H^{n-1}}{n^2}, \tag{8.2}$$

where λ_n^∞, P_n^∞ *are constants, and* C_0, H *are positive constants.*

Proof. For $n=1$ (8.1) and (8.2) can be verified directly, provided we choose C_0 large enough. Proceeding by induction, we assume that the lemma is true for all indices smaller than n ($n \geqslant 2$) and proceed to prove it for n, by applying Lemma 6.1. Thus we need to express the inhomogeneous terms on the right-hand sides of (5.5)–(5.17) in terms of the λ_m, P_m, U_m, V_m for $m<n$ and then estimate the corresponding norms

$$\|F\|+\|G\|+\|H\|,$$

which occur in (6.18).

We start with F_n^3. From (4.8) and (5.1)–(5.4) we have

$$\begin{aligned} F_n^3(r,\theta,t) = \sum_{k=1}^{n-1} \Bigg[& \psi_k^1(r,\theta,t)\frac{\partial^2 V_{n-k}}{\partial r^2} + \psi_k^2(r,\theta,t)\frac{\partial^2 V_{n-k}}{\partial r\,\partial\theta} + \psi_k^3(r,\theta,t)\frac{\partial^2 V_{n-k}}{\partial\theta^2} \\ & + \psi_k^4(r,\theta,t)\frac{\partial V_{n-k}}{\partial r} + \psi_k^5(r,\theta,t)\,V_{n-k} + \psi_k^6(r,\theta,t)\frac{\partial U_{n-k}}{\partial\theta} \\ & + \psi_k^7(r,\theta,t)\frac{\partial U_{n-k}}{\partial r} + \psi_k^8(r,\theta,t)\frac{\partial P_{n-k}}{\partial\theta} + \psi_k^9(r,\theta,t)\frac{\partial P_{n-k}}{\partial r}\Bigg], \end{aligned}$$

where

$$\begin{aligned} \psi^1(r,\theta,t,\varepsilon) &= \left[1-\frac{(1-\delta)^2}{(1+\lambda-\delta)^2}\right] - \frac{(1-\delta)^2\,(\delta-r)^2}{((1+\lambda-\delta)\,r-\delta\lambda)^2}\left(\frac{\lambda_\theta}{1+\lambda-\delta}\right)^2 \\ &\equiv \sum_{k\geqslant 1} \psi_k^1(r,\theta,t)\,\varepsilon^k, \\ \psi^2(r,\theta,t,\varepsilon) &= -\frac{2(\delta-r)(1-\delta)^2}{((1+\lambda-\delta)\,r-\delta\lambda)^2}\,\frac{\lambda_\theta}{1+\lambda-\delta} \equiv \sum_{k\geqslant 1}\psi_k^2\varepsilon^k, \\ \psi^3(r,\theta,t,\varepsilon) &= \frac{1}{r^2}\left[1-\frac{r^2(1-\delta)^2}{((1+\lambda-\delta)\,r-\delta\lambda)^2}\right] \equiv \sum_{k\geqslant 1}\psi_k^3\varepsilon^k, \\ \psi^4(r,\theta,t,\varepsilon) &= \frac{1}{r}\left[1-\frac{(1-\delta)^2\,r}{((1+\lambda-\delta)\,r-\delta\lambda)(1+\lambda-\delta)}\right] \\ &\quad -\frac{(\delta-r)(1-\delta)^2}{((1+\lambda-\delta)\,r-\delta\lambda)^2}\left[\frac{\partial}{\partial\theta}\left(\frac{\lambda_\theta}{1+\lambda-\delta}\right)-\left(\frac{\lambda_\theta}{1+\lambda-\delta}\right)^2\right] \\ &\equiv \sum_{k\geqslant 1}\psi_k^4\varepsilon^k, \end{aligned}$$

$$\psi^5(r,\theta,t,\varepsilon)=-\frac{1}{r^2}\left[1-\frac{(1-\delta)^2\,r^2}{((1+\lambda-\delta)\,r-\delta\lambda)^2}\right]=-\psi^3(r,\theta,t),$$

$$\psi^6(r,\theta,t,\varepsilon)=2\psi^3(r,\theta,t),$$

$$\psi^7(r,\theta,t,\varepsilon)=-\frac{2(1-\delta)^2}{((1+\lambda-\delta)\,r-\delta\lambda)^2}\frac{\lambda_\theta(\delta-r)}{1+\lambda-\delta}\equiv\sum_{k\geqslant 1}\psi^7_k\varepsilon^k,$$

$$\psi^8(r,\theta,t,\varepsilon)=-\frac{1}{r}\left[1-\frac{(1-\delta)\,r}{(1+\lambda-\delta)\,r-\delta\lambda}\right]\equiv\sum_{k\geqslant 1}\psi^8_k\varepsilon^k,$$

$$\psi^9(r,\theta,t,\varepsilon)=\frac{1-\delta}{((1+\lambda-\delta)\,r-\delta\lambda)}\frac{\lambda_\theta(\delta-r)}{(1+\lambda-\delta)}\equiv\sum_{k\geqslant 1}\psi^9_k\varepsilon^k.$$

It follows from Lemma 10.3 (in the Appendix) and the inductive assumption that, for $k<n$,

$$\|\psi^1_k-\psi^{1,\infty}_k\|_{4,B}\leqslant CC_0\frac{H^{k-1}}{k^2},\qquad |\psi^{1,\infty}_k|\leqslant CC_0\frac{H^{k-1}}{k^2},\tag{8.3}$$

where $\psi^{1,\infty}_k$ is a constant,

$$\|\psi^2_k\|_{4,B}\leqslant CC_0\frac{H^{k-1}}{k^2},$$

$$\|\psi^3_k-\psi^{3,\infty}_k\|_{5,B}\leqslant CC_0\frac{H^{k-1}}{k^2},\qquad \|\psi^{3,\infty}_k(r)\|_{H^5(\delta,1)}\leqslant CC_0\frac{H^{k-1}}{k^2},$$

$$\|\psi^4_k-\psi^{4,\infty}_k\|_{3,B}\leqslant CC_0\frac{H^{k-1}}{k^2},\qquad \|\psi^{4,\infty}_k(r)\|_{H^3(\delta,1)}\leqslant CC_0\frac{H^{k-1}}{k^2},$$

$$\|\psi^7_k\|_{4,B}\leqslant CC_0\frac{H^{k-1}}{k^2},$$

$$\|\psi^8-\psi^{8,\infty}_k\|_{5,B}\leqslant CC_0\frac{H^{k-1}}{k^2},\qquad \|\psi^{8,\infty}_k(r)\|_{H^5(\delta,1)}\leqslant CC_0\frac{H^{k-1}}{k^2},$$

$$\|\psi^9_k\|_{4,B}\leqslant CC_0\frac{H^{k-1}}{k^2}$$

where C is a universal constant, provided H is sufficiently large, independently of k, n.

To see this consider, for instance, the function ψ^1 and write

$$\psi^{1,1}(r,\theta,t)=1-\frac{(1-\delta)^2}{(1+\lambda-\delta)^2},\qquad \psi^{1,2}(r,\theta,t)=\frac{(1-\delta)^2\,(\delta-r)^2}{((1+\lambda-\delta)\,r-\delta\lambda)^2}$$

so that

$$\psi^1 = \psi^{1,1} + \psi^{1,2} \left(\frac{\lambda_\theta}{1+\lambda-\delta} \right)^2. \tag{8.4}$$

Then

$$\begin{aligned}
\psi^{1,1} &= \frac{2\lambda(1-\delta)+\lambda^2}{(1+\lambda-\delta)^2} = \frac{2\lambda(1-\delta)+\lambda^2}{(1-\delta)^2(1+\lambda/(1-\delta))^2} \\
&= \left[\frac{2\lambda}{1-\delta} + \left(\frac{\lambda}{1-\delta} \right)^2 \right] \sum_{p=0}^{\infty} (p+1)(-1)^p \left(\frac{\lambda}{1-\delta} \right)^p \\
&= \sum_{p=1}^{\infty} (-1)^{p-1} \frac{(p+1)}{(1-\delta)^p} \lambda^p
\end{aligned}$$

and

$$\psi^{1,2} = \frac{(\delta-r)^2}{r^2 \left(1 - \frac{(\delta-r)\,\lambda}{(1-\delta)\,r} \right)^2} = \frac{(\delta-r)^2}{r^2} \sum_{p=0}^{\infty} (p+1) \left(\frac{\delta-r}{(1-\delta)\,r} \right)^p \lambda^p.$$

It follows from Lemma 10.3 in the Appendix that

$$\psi^{1,j} = \sum \psi_k^{1,j} \varepsilon^k,$$

where

$$\| \psi_k^{1,j} - \psi_k^{1,j,\infty}(r) \|_{5,B} \leqslant C C_0 \frac{H^{k-1}}{k^2}, \tag{8.5}$$

$$\| \psi_k^{1,j,\infty}(r) \|_{H^5(\delta,1)} \leqslant C C_0 \frac{H^{k-1}}{k^2}, \tag{8.6}$$

provided H is sufficiently large.

Next,

$$\frac{\lambda_\theta}{1+\lambda-\delta} = \frac{\partial}{\partial\theta} \log(1+\lambda-\delta) \equiv \sum_{k=1}^{\infty} \tilde{D}_k(\theta, t)\, \varepsilon^k$$

and, by Lemma 10.3,

$$\log(1+\lambda-\delta) = \log(1-\delta) + \sum_{k\geqslant 1} L_k(\theta, t)\,\varepsilon^k$$

where

$$\|L_k - L_k^\infty\|_{5,\,\partial B} \leqslant CC_0\,\frac{H^{k-1}}{k^2}, \qquad |L_k^\infty| \leqslant CC_0\,\frac{H^{k-1}}{k^2}.$$

Since $\tilde{D}_k = 2\frac{\partial L_k}{\partial \theta}$, we get

$$\|\tilde{D}_k\|_{4,\,\partial B} \leqslant CC_0\,\frac{H^{k-1}}{k^2}. \tag{8.7}$$

Writing

$$\left(\frac{\lambda_\theta}{1+\lambda-\delta}\right)^2 = \sum_{k=1}^{\infty} D_k(\theta, t)\,\varepsilon^k,$$

we have

$$D_k = \sum_{\ell=1}^{k-1} \tilde{D}_{k-\ell}\tilde{D}_\ell$$

so that, by (8.7) and Lemma 10.1,

$$\|D_k\|_{4,\,\partial B} \leqslant CC_0\,\frac{H^{k-1}}{k^2} \tag{8.8}$$

if H is sufficiently large. We write

$$\psi^{1,2}(r,\theta,t)\left(\frac{\lambda_\theta}{1+\lambda-\delta}\right)^2 = \sum_{k\geqslant 1}\left[\left(\sum_{p=1}^{k-1}\psi_p^{1,2}D_{k-p}\right)\varepsilon^k + \frac{(\delta-r)^2}{r^2}\,D_k\varepsilon^k\right] \tag{8.9}$$

By (8.5), (8.6) for $j=2$ and (8.8),

$$\begin{aligned}\left\|\sum_{p=1}^{k-1}\psi_p^{1,2}D_{k-p}\right\|_{4,\,B} &\leqslant \sum_{p=1}^{k-1}\|(\psi_p^{1,2}-\psi_p^{1,2,\infty})\,D_{k-p}\|_{4,\,B} + \sum_{p=1}^{k-1}\|\psi_p^{1,2,\infty}D_{k-p}\|_{4,\,B}\\ &\leqslant 2M_0\sum_{p=1}^{k-1}C^2C_0\,\frac{H^{p-1}}{p^2}\,\frac{H^{k-p-1}}{(k-p-1)^2} \leqslant 2C^2C_0^2A_0M_0\,\frac{H^{k-2}}{k^2}.\end{aligned}$$

Also

$$\left\| \frac{(\delta-r)^2}{r^2} D_k \right\|_{4,B} \leqslant C\, \|D_k\|_{4,\partial B} \leqslant C^2 C_0 \frac{H^{k-1}}{k^2}.$$

Using these estimates in (8.9) and combining the result with (8.5),(8.6) for $j=1$, we obtain, upon recalling (8.4), the assertion (8.3).

Similarly one can prove the asserted bounds on the ψ_k^j for $2 \leqslant j \leqslant 9$. Using these estimates we can now show that

$$\|F_n^3 - F_n^{3,\infty}(r)\|_{3,B} \leqslant CC_0 \frac{H^{n-2}}{n^2}, \qquad \|F_n^{3,\infty}(r)\|_{H^3(\delta,1)} \leqslant CC_0 \frac{H^{n-2}}{n^2}. \tag{8.10}$$

Indeed, consider first the terms with ψ_k^1. Then

$$\begin{aligned}
&\left\| \sum_{k=1}^{n-1} \psi_k^1 \frac{\partial^2 V_{n-k}}{\partial r^2} - \sum_{k=1} \psi_k^{1,\infty}(r) \frac{\partial^2 V_{n-k}^\infty}{\partial r^2} \right\|_{3,B} \\
&\quad \leqslant \left\| \sum_{k=1}^{n-1} (\psi_k^1 - \psi_k^{1,\infty}) \left(\frac{\partial^2 V_{n-k}}{\partial r^2} - \frac{\partial V_{n-k}^\infty}{\partial r^2} \right) \right\|_{3,B} \\
&\quad\quad + \left\| \sum_{k=1}^{n-1} \psi_k^{1,\infty} \left(\frac{\partial^2 V_{n-k}}{\partial r^2} - \frac{\partial^2 V_{n-k}^\infty}{\partial r^2} \right) \right\|_{3,B} + \left\| \sum_{k=1}^{n-1} \frac{\partial^2 V_{n-k}^\infty}{\partial r^2} (\psi_k^1 - \psi_k^{1,\infty}) \right\|_{3,B} \\
&\quad \equiv J_1 + J_2 + J_3.
\end{aligned}$$

By (8.3) and the inductive assumption on the V_k,

$$\begin{aligned}
J_1 &\leqslant M_0 \sum_{k=1}^{n-1} \|\psi_k^1 - \psi_k^{1,\infty}\|_{3,B} \left\| \frac{\partial^2 V_{n-k}}{\partial r^2} - \frac{\partial^2 V_{n-k}^\infty}{\partial r^2} \right\|_{3,B} \\
&\leqslant M_0 \sum_{k=1}^{n-1} (CC_0)^2 \frac{H^{k-1}}{k^2} \frac{H^{n-k-1}}{(n-k-1)^2} \leqslant M_0 A_0 (CC_0)^2 \frac{H^{n-2}}{n^2}.
\end{aligned}$$

The terms J_2, J_3 are estimated in the same way. Hence

$$\left\| \sum_{k=1}^{n-1} \psi_k^1 \frac{\partial^2 V_{n-k}}{\partial r^2} - \sum_{k=1}^{n-1} \psi_k^{1,\infty}(r) \frac{\partial^2 V_{n-k}^\infty}{\partial r^2} \right\|_{3,B} \leqslant 3 M_0 A_0 (CC_0)^2 \frac{H^{n-2}}{n^2}.$$

We can also easily establish the bound

$$\left\| \sum_{k=1}^{n-1} \psi_k^{1,\infty}(r) \frac{\partial^2 V_{n-k}^\infty}{\partial r^2} \right\|_{H^3(1,\delta)} \leqslant C_1 (CC_0)^2 \frac{H^{n-2}}{n^2}$$

for some universal constant C_1.

We can next proceed in the same way to estimate the terms in F_n^3, which involve the ψ_k^j for $j = 2, 3, \ldots, 9$, and thus conclude that the estimates (8.10) are valid.

The functions F_n^1 and F_n^2 can be treated in the same way as F_n^3, and each of the G_n^j, as well as H_n^1 and $H_n^{2,\pm}$, can similarly be estimated in "appropriate" norms; the "appropriate" norms are precisely those needed in the assumptions of Lemma 6.1. (In estimating the G_n^j we use Lemma 10.4.) We can now apply Lemma 6.1 to the system (5.5)–(5.17); the right-hand sides in (6.18), (6.19) are bounded by $C_2 C_0 H^{n-2}/n^2$, where C_2 is a constant independent of n and H. We conclude that if H is sufficiently large (so that $CC_2/H < 1$) then the estimates (8.1), (8.2) hold. This completes the proof of Lemma 8.2. ▮

Lemma 8.2 establishes the existence of a unique global solution:

THEOREM 8.3. *If $\lambda^0 \in H^5(\partial B)$ and $|\varepsilon| \leqslant \varepsilon_0$ for some positive and sufficiently small ε_0, then there exists a unique solution (λ, P, U, V) to problem* (T) *having the form* (5.1)–(5.4) *with*

$$\|\lambda - \lambda_\varepsilon^\infty\|_{5\frac{1}{2}, \partial B} + \|\lambda_t\|_{4\frac{1}{2}, \partial B} + \|P - P_\varepsilon^\infty\|_{4, B} + \|U\|_{5, B} + \|V - V_\varepsilon^\infty(r)\|_{5, B} < \infty, \tag{8.11}$$

where $\lambda_\varepsilon^\infty$, P_ε^∞ are constants

$$\lambda_\varepsilon^\infty = \sum_{n=1}^{\infty} \lambda_n^\infty \varepsilon^n, \qquad P_\varepsilon^\infty = \sum_{n=1}^{\infty} P_n^\infty \varepsilon^n, \tag{8.12}$$

$$V_\varepsilon^\infty(r) = \sum_{n=1}^{\infty} V_n^\infty(r)\, \varepsilon^n, \tag{8.13}$$

and the series (5.1)–(5.4) *and* (8.12), (8.13) *are uniformly convergent for $\delta \leqslant r \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$, $|\varepsilon| \leqslant \varepsilon_0$ and $0 \leqslant t \leqslant T$, for any $T > 0$.*

9. THE MAIN RESULTS

Reversing the transformation (4.1) we obtain the following result:

THEOREM 9.1. *If $\lambda^0 \in H^5(\partial B)$ and $|\varepsilon| \leqslant \varepsilon_0$ then problem* (A) *has a unique global solution with free boundary*

$$r = 1 + \lambda(\theta, t, \varepsilon),$$

where $\lambda(\theta, t, \varepsilon)$ *has the form* (5.1) *where the series is uniformly convergent for* $|\varepsilon| \leqslant \varepsilon_0$ *and*

$$\|\lambda-\lambda_\varepsilon^\infty\|_{5\frac{1}{2}, \partial B}+\|\lambda_t\|_{4\frac{1}{2}, \partial B}<\infty; \tag{9.1}$$

$\lambda_\varepsilon^\infty$ *is a convergent power series as in* (8.12); *furthermore,*

$$p(r, \theta, t, \varepsilon) \to \sigma, \qquad u_r(r, \theta, t, \varepsilon) \to 0, \qquad u_\theta(r, \theta, t, \varepsilon) \to \mu_\varepsilon r \tag{9.2}$$

exponentially fast as $t \to \infty$, *where*

$$\frac{\pi}{2}\mu_\varepsilon(1+\lambda_\varepsilon^\infty)^4 = M. \tag{9.3}$$

Proof. The only assertion that needs to be proved is that the limits of p and u_θ, as $t \to \infty$, are as asserted in (9.2), (9.3), but this follows by noting that the limit of the solution, as $t \to \infty$, is the special solution with free boundary $r = 1+\lambda_\varepsilon^\infty$,

$$p=\sigma,\ u_r=0,\ u_\theta = Cr,$$

where C is a constant determined by the constraint

$$\int_{\{r<1+\lambda_\varepsilon^\infty\}} r u_\theta \, dx = M. \quad \blacksquare$$

Remark 9.1. If $\lambda^0 \in H^{5+m}(\partial B)$ then we can differentiate the system (5.5)–(5.16) up to m times with respect to θ, and each time, apply inductive estimates as before. In particular, if $\lambda^0 \in C^\infty(\partial B)$ then the solution to problem (T) is also in C^∞ in (θ, ε). Using the differential equations for P, U, V we deduce that the solution is C^∞ in (r, θ, ε).

Remark 9.2. In the above analysis we have used norms with two t-derivatives. If $\lambda^0 \in C^\infty$ we can extend the analysis with norms

$$\left(\int_0^\infty e^{2\alpha t} \sum_{j=0}^{s} \|D_t^j F\|_{H^{s-j}}^2\right)^{1/2}$$

for any s, and thus conclude that the solution is in C^∞ in $(r, \theta, t, \varepsilon)$.

Finally, if $\lambda^0(\theta)$ is analytic in θ, then we can establish, inductively on n, estimates of the form

$$\frac{1}{k!}\left[\|D_\theta^k(\lambda_n-\lambda_n^\infty)\|_{5\frac{1}{2}, \partial B}+\|D_\theta^k\lambda_{n,t}\|_{4\frac{1}{2}, \partial B}+\|D_\theta^k(P_n-P_n^\infty)\|_{4, B}\right.$$
$$\left.+\|D_\theta^k U_n\|_{5, B}+\|D_\theta^k(V_n-V_n^\infty(r))\|_{5, B}\right] \leqslant C_0\,\frac{A^{k-1}H^{n-1}}{(k+n)^2}$$

for all k, for some positive constants C_0, A, H with $A/H \ll 1$. (cf. Remark 7.1 in [3]). This implies joint analyticity in (x, ε) both for the solution of problem (T) and of problem (A) (cf. [3]).

By combining Remarks 9.1 and 9.2 we obtain the following result.

THEOREM 9.2. (i) *If* $\lambda^0 \in C^\infty$ *then the solution of problem* (A) *is in* C^∞ *in* (x, t, ε) *for* $x \in \overline{\Omega(t)}$, $0 \leqslant t < \infty$, $|\varepsilon| \leqslant \varepsilon_0$; (ii) *If* $\lambda^0(\theta)$ *is analytic then the solution is also analytic in* (x, ε) *for* $x \in \overline{\Omega(t)}$, $|\varepsilon| \leqslant \varepsilon_0$ *and any* $t \geqslant 0$.

10. APPENDIX

LEMMA 10.1 [3]. *Let* D *be a domain in* $\mathbf{R}^d$ *and let* $\| \ \|$ *denote a norm for functions defined in* D *such that*

$$\|fg\| \leqslant M_0 \|f\| \, \|g\|$$

for some constant $M_0 \geqslant 1$. *Let*

$$\mu(x, \varepsilon) = \sum_{n \geqslant 1} \mu_n(x) \, \varepsilon^n, \qquad x \in D,$$

where

$$\|\mu_n(x) - \mu_n^\infty\| \leqslant C_0 \frac{H^{n-1}}{n^2}, \qquad |\mu_n^\infty| \leqslant C_0 \frac{H^{n-1}}{n^2}$$

for some constants μ_n^∞, C_0, H. *For* $G(s) = s^N$, *where* N *is any positive integer, set*

$$G(\mu(x, \varepsilon)) = \Big(\sum_{n \geqslant 1} \mu_n(x) \, \varepsilon^n \Big)^N = \sum_{n \geqslant N} \Phi_n^N(x) \, \varepsilon^n$$

and

$$\Big(\sum_{n \geqslant 1} \mu_n^\infty \varepsilon^n \Big)^N = \sum_{n \geqslant N} \Phi_n^{N, \infty} \varepsilon^n.$$

Then there exists a universal constant A_0 *such that*

$$|\Phi_n^{N, \infty}| \leqslant C_0 (C_0 A_0)^{N-1} \frac{H^{n-N}}{n^2}, \qquad n \geqslant N,$$

$$\|\Phi_n^N - \Phi_n^{N, \infty}\| \leqslant C_0 (C_0 A_0 M_0)^{N-1} \frac{H^{n-N}}{n^2}, \qquad n \geqslant N.$$

LEMMA 10.2. *Let* $\| \|_{s,B}$, $\| \|_{s,\partial B}$ *be defined as in* (6.14), (6.15) *with* $s \geqslant 2$. *Then there exists a constant* $C(s)$ *depending only on* s *such that for any function*

$$F(r, \theta, t) = \varphi(\theta, t)\, \psi(r)$$

there holds

$$\|F\|_{s,B} \leqslant C(s)\, \|\varphi\|_{s,\partial B}\, \|\psi\|_{H^s(\delta,1)}.$$

The proof follows by the Sobolev imbedding, since the $L^\infty(\Omega)$-norm is bounded by the $H^2(\Omega)$ norm if Ω is a bounded domain in $\mathbf{R}^3$.

LEMMA 10.3. *Let*

$$F(r, \mu) = \sum_{n \geqslant 1} F_n(r)\, \mu^n,$$

where

$$\|F_n\|_{H^s(\delta,1)} \leqslant B^n \qquad \textit{for some constant} \quad B > 0$$

and $s \geqslant 2$, *and let*

$$\lambda(\theta, t, \varepsilon) = \sum_{n \geqslant 1} \lambda_n(\theta, t)\, \varepsilon^n$$

satisfy

$$|\lambda_n^\infty|,\ \|\lambda_n - \lambda_n^\infty\|_{s,\partial B} \leqslant C_0 \frac{H^{n-1}}{n^2}, \qquad n \geqslant 1.$$

Then

$$F(r, \lambda(\theta, t, \varepsilon)) = \sum_{k \geqslant 1} \varphi_k(r, \theta, t)\, \varepsilon^k,$$

where

$$\|\varphi_k(r, \theta, t) - \varphi_k^\infty(r)\|_{s,B} \leqslant 2BC_0 C(s) \frac{H^{k-1}}{k^2},$$

$$|\varphi_k^\infty(r)|_{H^s(\delta,1)} \leqslant 2BC_0 \frac{H^{k-1}}{k^2}$$

provided $H > 2C_0 A_0 B M_0$.

Proof. Using Lemma 10.1 we can write

$$\lambda^N = \sum_{n \geqslant N} \Phi_n^N(\theta, t)\, \varepsilon^n$$

and

$$\left(\sum_{n \geqslant 1} \lambda_n^\infty \varepsilon^n\right)^N = \sum_{n \geqslant N} \Phi_n^{N,\infty}(\theta, t)\, \varepsilon^n,$$

where

$$\|\Phi_n^N - \Phi_n^{N,\infty}\|_{s,\partial B} \leqslant C_0(C_0 A_0 M_0)^{N-1} \frac{H^{n-N}}{n^2}$$

and

$$|\Phi_n^{N,\infty}| \leqslant C_0(C_0 A_0)^{N-1} \frac{H^{n-N}}{n^2}.$$

Then we have

$$F(r, \lambda(\theta, t, \varepsilon)) = \sum_{n \geqslant 1} F_n(r)\, \lambda^n = \sum_{n \geqslant 1} F_n(r) \sum_{k \geqslant n} \Phi_k^n(\theta, t)\, \varepsilon^k$$
$$= \sum_{k \geqslant 1} \varepsilon^k \left(\sum_{n=1}^{k} F_n(r)\, \Phi_k^n(\theta, t)\right).$$

Thus

$$\varphi_k(r, \theta, t) = \sum_{n=1}^{k} F_n(r)\, \Phi_k^n(\theta, t),$$
$$\varphi_k^\infty(r) = \sum_{n=1}^{k} \Phi_k^{n,\infty} F_n(r),$$

and

$$\|\varphi_k - \varphi_k^\infty\|_{s,B} \leqslant \sum_{n=1}^{k} \|F_n(r)(\Phi_k^n - \Phi_k^{n,\infty})\|_{s,B}.$$

Using Lemma 10.2, we get

$$\|\varphi_k - \varphi_k^\infty\|_{s,B} \leqslant C(s) \sum_{n=1}^{k} \|F_n\|_{H^s(\delta,1)} \|\Phi_k^n - \Phi_k^{n,\infty}\|_{s,\partial B}$$
$$\leqslant C(s) \sum_{n=1}^{k} B^n C_0 (C_0 A_0 M_0)^{n-1} \frac{H^{k-n}}{k^2}$$
$$\leqslant BC_0 C(s) \frac{H^{k-1}}{k^2} \sum_{n=1}^{k} \left(\frac{C_0 A_0 M_0 B}{H}\right)^{n-1} \leqslant 2BC_0 C(s) \frac{H^{k-1}}{k^2}$$

provided $H > 2C_0 A_0 M_0 B$. Similarly,

$$\|\varphi_k^\infty\|_{H^s(\delta,1)} \leqslant \sum_{n=1}^{\infty} |\Phi_k^{n,\infty}| \, \|F_n\|_{H^s(\delta,1)}$$
$$\leqslant \sum_{n=1}^{k} B^n C_0 (C_0 A_0)^{n-1} \frac{H^{k-n}}{k^2} \leqslant 2BC_0 \frac{H^{k-1}}{k^2}$$

if $H > 2C_0 A_0 B$. ∎

In the special case where the function F is independent of r, Lemma 10.3 takes the form

LEMMA 10.4 [3]. *Let*

$$F(\mu) = \sum_{n \geqslant 1} F_n \mu^n \qquad \textit{where} \quad |F_n| \leqslant B^n$$

for some constant B and let λ *be as in Lemma* 10.3. *Then*

$$F(\lambda(\theta, t, \varepsilon)) = \sum_{k \geqslant 1} \varphi_k(\theta, t)\, \varepsilon^k,$$

where

$$\|\varphi_k - \varphi_k^\infty\|_{s,\partial B} \leqslant 2BC_0 \frac{H^{k-1}}{k^2}$$

and

$$|\varphi_k^\infty| \leqslant 2BC_0 \frac{H^{k-1}}{k^2},$$

provided $H \geqslant 2C_0 B A_0 M_0$.

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