Initial Boundary Value Problem for the System of Compressible Adiabatic Flow Through Porous Media

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We study the initial value problem for the system of compressible adiabatic flow through porous media in the one space dimension with fixed boundary condition. Under the restriction on the oscillations in the initial data, we establish the global existence and large time behavior for the classical solutions via the combination of characteristic analysis and energy estimate methods. The time-asymptotic states for the solution are found and the exponential convergence to the asymptotic states is proved. It is also shown that this problem can be approximated very well time-asymptotically by an initial boundary problem for the related diffusion equations obtained from the original hyperbolic system by Darcy’s law, provided that the initial total mass is the same. The difference between these two solutions tends to zero exponentially fast as time goes to infinity in the sense of $H^1$. The diffusive phenomena caused by damping mechanism with boundary effects are thus observed.

1. INTRODUCTION

The motion of the adiabatic gas flow through porous media can be modeled by the following damped hyperbolic system

\[
\begin{cases}
  v_t - u_x = 0 \\
  u_t + p(v, s)_x = - xu, \quad x > 0, \\
  (e(v, s) + \frac{1}{2}u^2)_t + (pu)_x = - xu^2,
\end{cases}
\]

(1.1)

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where \( \nu \) denotes the specific volume, \( u \) is velocity, \( s \) stands for entropy, \( p \) denotes pressure with \( p_s(\nu, s) < 0 \) for \( \nu > 0 \), and \( e \) is the specific internal energy for which \( e_s \neq 0 \) and \( e_s + p = 0 \) holds due to the second law of thermodynamics. System (1.1) is equivalent to the following system for smooth solutions

\[
\begin{cases}
v_t - u_x = 0 \\ u_t + p(\nu, s)_x = -\nu u, \quad \nu > 0, \\ s_t = 0,
\end{cases}
\]  

which is strictly hyperbolic with eigenvalues \( \lambda_1 = \lambda_3 = \sqrt{-p}, \) and \( \lambda_2 = 0 \).

For the isentropic flow, namely \( s = \text{const.} \), (1.2) takes the form

\[
\begin{cases}
v_t - u_x = 0 \\ u_t + p(\nu)_x = -\nu u, \quad \nu > 0.
\end{cases}
\]  

The diffusive effect created by the damping mechanism has been investigated a lot for the Cauchy problem of (1.3) with the initial data

\[
(v(x, 0), u(x, 0)) = (\nu_*(x), u_*(x))
\]  

such that

\[
\lim_{x \to \pm \infty} (v_*(x), u_*(x)) = (\nu_\pm, u_\pm).
\]

It has been proved in [3] that the smooth solution of (1.3)-(1.4) can be described by the solution of the problem

\[
\begin{cases}
\ddot{v}_t = -\frac{1}{\nu} p(\dot{\nu})_{xx} \\ \ddot{u} = -\frac{1}{\nu} p(\dot{\nu})_x \\ \dot{\nu}(x, 0) = \nu_*(x + d_0),
\end{cases}
\]  

time-asymptotically, where, the system in (1.6) is obtained from (1.3) by approximating the momentum equation in (1.3) with Darcy’s law and \( \nu_* \) is the similarity solution of (1.6) with \( \nu_*(-\infty) = \nu_+ \), \( d_0 \) is a constant determined by the initial data. For other results related to [3], we refer to [2, 4, 5, 7, 8, 15] for smooth solutions, and to [1, 2, 6, 9, 10, 11, 14, 17] for weak solutions.
Recently, the system (1.3) in a quarter plane $R_+ \times R_+$ with different kinds of boundary conditions given on $x = 0$ has been studied by Marcati and Mei [13] and by Nishihara and Yang [16], respectively. The asymptotic behavior of solutions and the relation to the corresponding nonlinear diffusion waves are given.

The global existence of smooth solutions for Cauchy problem of (1.2) with small initial data have been studied in [2, 5, 7, 18]. The diffusive effect created by the damping mechanism has been investigated in [5, 8] under the assumption that $s(x, 0) = s_0(x)$ is a perturbation of a constant $\bar{s}$. In this case, it has been proved that (1.2) can be approximated by the simplified system

$$
\begin{align*}
\tilde{v}_t &= -\frac{1}{\alpha} p(\tilde{v}, s)_{xx}, \\
\tilde{u}_t &= -\frac{1}{\alpha} p(\tilde{v}, s)_x, \\
\tilde{s}_t &= 0,
\end{align*}
$$

(1.7)

time-asymptotically. This system is obtained from (1.2) by approximating the momentum equation in (1.2) with Darcy's law.

However, the global existence and the large time asymptotic behavior of solutions to (1.2) are not well understood in the presence of the boundaries even for the smooth solutions in the isentropic case, in view of the special boundary conditions in [13, 16]. The purpose of this paper is to investigate a typical initial boundary problem of (1.2) on bounded domain. In particular, the influence of damping mechanism with boundary effects to the asymptotic behavior of the processes in consideration. As the first step, we study smooth solutions in the present paper. The investigation on weak solutions will be discussed in the future.

We consider (1.2) on the strip $(0, 1) \times (0, \infty)$ with the following initial and boundary data

$$
\begin{align*}
v(x, 0) &= v_0(x) > 0, \\
u(x, 0) &= u_0(x), \\
s(x, 0) &= s_0(x), \\
x &\in [0, 1], (1.8)
u(0, t) = u(1, t) = 0, \\
&\quad t \geq 0, (1.9)
\end{align*}
$$

where $s_0(x) \in C^2([0, 1])$. For convenience, we assume that $s_0(x) \geq 0$. We also require the compatibility condition

$$
\begin{align*}
u_0(0) = u_0(1) = 0, \\
p_0'(0) = p_0'(1) = 0, (1.10)
\end{align*}
$$

with $p_0 = p(v_0, s_0)$. 

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In this paper, we will show that the solution of (1.2) and (1.8)–(1.9) can be described by the solution of the problem

\[
\begin{aligned}
\hat{v}_t &= -\frac{1}{\alpha} p(\hat{v}, s)_{xx} \\
\hat{u} &= -\frac{1}{\alpha} p(\hat{v}, s)_x \\
s_x &= 0, \\
\hat{p}_x(0, t) &= \hat{p}_x(1, t) = 0, \quad t \geq 0, \\
\hat{v}(x, 0) &= \hat{v}_0(x), \quad s(x, 0) = s_0(x), \quad x \in [0, 1],
\end{aligned}
\]

(1.11)

time-asymptotically, where \(\hat{p} = p(\hat{v}, s)\) and

\[
\int_0^1 \hat{v}_0(x) \, dx = \int_0^1 v_0(x) \, dx.
\]

Moreover, the \(H^1\)-norm of the difference between these two solutions tends to zero exponentially fast as time goes to infinity. This is quite different from the Cauchy problem where one only can get the algebraic decay rate (see [15]). This also shows that certain nonlinear diffusive phenomena occur for the solutions of (1.2) and (1.8)–(1.9) which is caused by the damping mechanism with boundary effects.

For simplicity of presentation, we take \(\alpha = 1\), and \(p(v, s) = (\gamma - 1) v^{-\gamma} e^s\) with \(1 < \gamma < 3\), which is the case for the polytropic gas.

It should be pointed out that the key approach in [2–5, 8] is to compare the solution of (1.3) with the similarity solution of (1.6) via energy estimates. Unfortunately, this approach does not work here, due to the boundary effects. In our problem, two main difficulties should be faced. One is the global existence and large time behavior of the solutions to the problem (1.11) is not clear. The other is the initial data and boundary effects in (1.2) and (1.8)–(1.9) will not allow us to make the higher order estimate in order to close the argument via energy estimate. We will carry out the first in Section 2 and overcome the later in Sections 3 and 4 by combining the characteristic method and energy estimates.

In Section 2, the initial boundary problem (1.11) is studied. The global existence and uniqueness for the classical solution to (1.11) are obtained, and the exponential convergence to an stationary state is also established.

In Section 3, we make a careful characteristic analysis on (1.2) and (1.8)–(1.9) to derive the uniform \(C^1\) estimates on the solutions, and the global existence for the unique \(C^1\) solution is obtained. Instead of the smallness of the initial data, we only need that the oscillations in the initial data could not be too large.
In Section 4, with the help of the results established in Section 3, we use the energy estimates method to investigate the large time behavior of the solution to (1.2) and (1.8)-(1.9). The $C^1$ bounds obtained in section 3 enable us to deduce the large time behavior with only lower order estimates. The exponential decay rate is also obtained.

Based on the results in Sections 2-4, we get some concluding remarks in section 5, which concludes that if

$$\int_0^1 v(x) \, dx = \int_0^1 \tilde{v}_0(x) \, dx,$$

then the problem (1.2) and (1.8)-(1.9) is equivalent to (1.11) time-asymptotically. The approach used in Sections 2-4 is also applied to the isentropic flow. Under milder conditions, we establish the related results.

We end this introduction by listing some notations which will be used in this paper.

We will use the usual Lebesgue measurable function space $L^p$ on the interval $[0, 1]$ with the norm $\| \cdot \|_{L^p}$ and the usual Sobolev space $H^l([0, 1])$ with the norm $\| \cdot \|_{H^l}$ and $H^0 = L^2$. In particular, $\| \cdot \| = \| \cdot \|_{L^2}$. We use the following notation for simplicity

$$\| (g_1, g_2, ..., g_k) \|_m^2 = \sum_{i=1}^k \| g_i \|_m^2.$$

We need some function spaces of Hölder continuous functions. $C^\alpha([0, 1])$ stands for the Banach space of functions on $[0, 1]$ which are uniformly Hölder continuous with exponent $\alpha$.

2. NONLINEAR DIFFUSION EQUATION

In this section, we investigate the behavior of solutions for (1.11). Consider

$$\begin{cases}
\tilde{v}_t + \tilde{p}(\tilde{v}, s)_{xx} = 0, & \text{in } (0, 1) \times (0, +\infty), \\
\tilde{p}_x(0, t) = \tilde{p}_x(1, t) = 0, & t \geq 0, \\
\tilde{v}(x, 0) = \tilde{v}_0(x) > 0, & x \in [0, 1],
\end{cases}
$$

(2.1)

where $\tilde{p}(\tilde{v}, s) = p(\tilde{v}, s), \ s = s_0(x), \ \tilde{v}_0(x) \in C^{2+\frac{l}{2}}([0, 1])$ with $l \in (0, 1)$ and $\tilde{p}'_o(0) = \tilde{p}'_o(1) = 0$, for $p_0(x) = p(\tilde{v}_0(x), s_0(x))$. 


For smooth solutions, (2.1) is equivalent to the following system on $\tilde{p}$,

$$
\begin{aligned}
\dot{\tilde{p}}_t - a(x) \tilde{p}^{1+1/\gamma} \tilde{p}_{xx} &= 0 \\
\tilde{p}_t(0, t) = \tilde{p}_t(1, t) &= 0, \\
\tilde{p}(x, 0) = \tilde{p}_0(x) &= \tilde{p}(\tilde{u}_0(x), s(x)) > 0, \\
x \in [0, 1],
\end{aligned}
$$

with $a(x) = \gamma(\gamma - 1)e^{\mu(x)}^{-1/\gamma}$.

Since the local existence and uniqueness of the solution to (2.2) is standard, we will only establish some uniform estimates on the solution of (2.2).

First of all, by the local results and the maximum principle, we have

**Lemmas 2.1**. $\tilde{p}(x, t) > 0$, for any $(x, t) \in [0, 1] \times (0, +\infty)$.

We then derive the uniform upper and lower bounds for $\tilde{p}(x, t)$.

**Lemmas 2.2**. Let $0 < \tilde{p}_1 \leq \tilde{p}_0(x) \leq \tilde{p}_2$. Then

$$
\tilde{p}_1 \leq \tilde{p}(x, t) \leq \tilde{p}_2.
$$

**Proof**. First, we note that there exist two positive constants $a_1$ and $a_2$ such that $a_1 \leq a(x) \leq a_2$.

For any given $k > 0$, we have

$$
a^{-1}(x) \tilde{p}^k \tilde{p}_t = (\tilde{p}^{k+1+1/\gamma} \tilde{p}_x)_x - \left(1 + \frac{1}{\gamma}\right) \tilde{p}^{k+1/\gamma} \tilde{p}_x^2,
$$

which implies that

$$
\left(\int_0^1 \frac{1}{k+1} a^{-1}(x) \tilde{p}^{k+1} dx\right)_{t} \leq 0.
$$

(2.3)

Thus, for any $m > 1$, it holds

$$
\|\tilde{p}(\cdot, t)\|_{L^m} \leq \left(\frac{a_2}{a_1}\right)^{1/m}\|\tilde{p}_0(x)\|_{L^m}.
$$

By taking $m \to +\infty$, we obtain

$$
\tilde{p}(x, t) \leq \tilde{p}_2.
$$

(2.4)
To bound $\tilde{p}(x, t)$ from below we consider the following problem for $p = 1/\tilde{p}$
\[
\begin{cases}
a^{-1}(x) \rho_t = (\rho^{p-2} \rho_x)_x \rho^{1-1/p} & \\
\rho_x(0, t) = \rho_x(1, t) = 0 & \\
\rho(x, 0) = \frac{1}{\tilde{p}_0(x)}
\end{cases}
\]
(2.5)

For any given $k > 0$ we have, similar to (2.3), that
\[
\left( \int_0^1 \frac{1}{k+1} a^{-1}(x) \rho^{k+1} \, dx \right) \leq 0.
\]
(2.6)

Hence, for all $m > 1$, we have
\[
\|\rho(\cdot, t)\|_{L^m} \leq \left( \frac{a_2}{a_1} \right)^{1/m} \|\rho(\cdot, 0)\|_{L^m},
\]
which yields
\[
\rho(x, t) \leq \frac{1}{\tilde{p}_1}.
\]
Namely,
\[
\tilde{p}(x, t) \geq \tilde{p}_1. \tag{2.7}
\]

We make the energy estimates on the solution of (2.2) next. In the following $C$ and $C_i (i = 1, \ldots, 14)$ denote the positive constants independent of $t$.

It is clear that
\[
\int_0^1 \tilde{v}(x, t) \, dx = \int_0^1 \tilde{v}_0 \, dx = \tilde{v}.
\]
(2.8)

Multiplying (2.2) by $a^{-1}(x) \tilde{p}$ and then integrating it over $[0, 1] \times [0, t]$, we have

\textbf{Lemma 2.3.}
\[
\frac{1}{2} \int_0^1 \tilde{p}^2(x, t) \, dx + \int_0^t \int_0^1 \tilde{p}_x^2 \, dx \, d\tau \leq C.
\]
(2.9)

For higher order estimates, we multiply (2.2) by $\tilde{p}_{xx}$ and integrate it over $[0, 1] \times [0, t]$, then integrate by parts and we arrive at
Lemma 2.4.

\[
\frac{1}{2} \int_0^1 \tilde{p}_{xx}^2(x, t) \, dx + \int_0^t \int_0^1 \tilde{p}_{xx}^2 \, dx \, dt \leq C. \tag{2.10}
\]

Differentiating (2.2) in \(x\) formally, (Here we will use the third derivatives of \(\tilde{p}\) formally. This will not cause any trouble, since we may assume \(\tilde{p} \in H^3\) first and use Fridrich’s mollifier then to deal with the original case. The same fact will be used in (4.9) below.) we have

\[
\tilde{p}_{xt} - a(x) \tilde{p}^{1+1/\gamma} \tilde{p}_{xxx} - a'(x) \tilde{p}^{1+1/\gamma} \tilde{p}_{xx} = 0. \tag{2.11}
\]

Multiplying (2.11) by \(\tilde{p}_{xxx}\) and then integrating it over \([0, 1] \times [0, t]\), integrating by parts, one has

\[
\int_0^1 \tilde{p}_{xx}^2(x, t) \, dx + \int_0^t \int_0^1 \tilde{p}_{xxx}^2 \, dx \, dt \leq C + C \left| \int_0^1 \int_0^1 \tilde{p}_{xx} \tilde{p}_{xxx} \, dx \, dt \right| + C \left| \int_0^t \int_0^1 \tilde{p}_{xx} \tilde{p}_{xxx} \, dx \, dt \right|,
\]

which implies that

\[
\int_0^1 \tilde{p}_{xx}^2(x, t) \, dx + \int_0^t \int_0^1 \tilde{p}_{xxx}^2 \, dx \, dt \leq C + C \left| \int_0^1 \int_0^1 \tilde{p}_{xx} \tilde{p}_{xxx} \, dx \, dt \right| + C \left| \int_0^t \int_0^1 \tilde{p}_{xx} \tilde{p}_{xxx} \, dx \, dt \right|.
\]

We note that

\[
\int_0^t \max_{x \in [0, 1]} \tilde{p}_{xx}^2 \, dt \leq \int_0^t \int_0^1 \tilde{p}_{xx}^2 \, dx \, dt \leq C \tag{2.13}
\]

due to boundary conditions, the Sobolev theorem and Lemma 2.4. Then the Gronwall’s inequality and (2.12)–(2.13) give

Lemma 2.5.

\[
\int_0^1 \tilde{p}_{xx}^2(x, t) \, dx + \int_0^t \int_0^1 \tilde{p}_{xxx}^2 \, dx \, dt \leq C. \tag{2.14}
\]
Now, by the standard argument we can prove the global existence and uniqueness of the classical solution to (2.2) and (2.1).

In order to obtain the large time behavior of the solution $\tilde{p}$ to (2.2) and $\tilde{v}$ to (2.1), we make the following analysis.

A routine procedure, with the help of Lemmas 2.1–2.5, implies that
\[
\lim_{t \to \infty} \| \tilde{p}_x(\cdot, t) \|_{L^\infty} = 0, \tag{2.15}
\]
which means, there is a constant $\tilde{p} \in [\tilde{p}_1, \tilde{p}_2]$ such that
\[
\lim_{t \to \infty} \| \tilde{p}(\cdot, t) - \tilde{p} \|_{L^\infty} = 0. \tag{2.16}
\]
Let us assume that $(\tilde{p}, \tilde{v})(x, t) \to (\hat{p}, \hat{v}(x))$ as $t \to \infty$. We have from (2.15) and
\[
\tilde{p}_x = \tilde{p}(-\gamma \tilde{v}^{-1} \tilde{v}_x + s_x)
\]
that
\[
-\gamma \tilde{v}^{-1} \tilde{v}_x + s_x = 0, \tag{2.17}
\]
and then
\[
\hat{v}(x) = C_1 e^{(1/\gamma) s(x)}, \tag{2.18}
\]
where $C_1$ is determined from (2.8) as
\[
C_1 = \hat{v} \left( \int_0^1 e^{(1/\gamma) s(x)} \, dx \right)^{-1}.
\]

Hence we have
\[
\begin{cases}
\hat{v}(x) = \hat{v} \left( \int_0^1 e^{(1/\gamma) s(x)} \, dx \right)^{-1} e^{(1/\gamma) s(x)} \\
\hat{p} = (\gamma - 1) \hat{v}^{-\gamma} \left( \int_0^1 e^{(1/\gamma) s(x)} \, dx \right)^{\gamma}.
\end{cases} \tag{2.19}
\]

We will see from the following theorem that $(\hat{p}, \hat{v})$ is indeed the asymptotic state for $(\tilde{p}, \tilde{v})(x, t)$ when $t$ tends to infinity.
THEOREM 2.6. For the solution $p \sim$ to (2.2) and $v \sim$ to (2.1), there exist positive constants $C_2$ and $\delta > 0$ such that

$$
\|(\tilde{p} - \hat{p})(\cdot, t)\|_{H^2} + \|(\tilde{v} - \hat{v})(\cdot, t)\|_{H^2} + \|\tilde{u}(\cdot, t)\|_{H^1}
\leq C_2 \exp\{-\delta t\}, \quad \text{as} \quad t \to +\infty,
$$

where, $\tilde{p}$ and $\tilde{v}$ are given in (2.19), and $\tilde{u} = -\tilde{p}_x$.

Proof. Due to (2.8), there exists an $x_0(t) \in [0, 1]$ for any $t \geq 0$ so that

$$
\tilde{v}(x_0(t), t) = \hat{v}(x_0(t)). \quad (2.20)
$$

Hence, it is easy to see

$$
\tilde{p}(x_0(t), t) = \hat{p}. \quad (2.21)
$$

Thus, we have

$$
\|(\tilde{p} - \hat{p})(\cdot, t)\|_{L^2} \leq \int_0^1 \hat{p}_x^2(x, t) \, dx.
$$

Similarly, we can prove that

$$
\|(\tilde{p} - \hat{p})(\cdot, t)\|_{H^2} \leq C \|\hat{p}_x(\cdot, t)\|_{H^1}^2. \quad (2.22)
$$

Now, Lemmas 2.2–2.5 give

$$
\|(\tilde{p} - \hat{p})(\cdot, t)\|_{H^2} + \int_0^t \|(\tilde{p} - \hat{p})(\cdot, \tau)\|_{H^2} \, d\tau \leq C. \quad (2.23)
$$

We multiply (2.2) by $a^{-1} \tilde{p}$ and $\hat{p}_{xx}$, respectively, and multiply (2.11) by $\hat{p}_{xxx}$, sum them up, integrate the result over $[0, 1]$ and we have

$$
\frac{d}{dt} \|(\tilde{p} - \hat{p})(\cdot, t)\|_{H^2} + C \|(\tilde{p} - \hat{p})(\cdot, t)\|_{H^2}^2 \leq 0
$$

with the help of (2.22). Thus we have

$$
\|(\tilde{p} - \hat{p})(\cdot, t)\|_{H^2} \leq C_3 \exp\{-C_4 t\}. \quad (2.24)
$$

Since

$$
\tilde{p} - \hat{p} = (\tilde{v} - \hat{v}) \int_0^1 p_x(\tilde{v} + \Theta \tilde{v} - \hat{v}) \, d\theta,
$$

we have
we have
\[
\| (\tilde{v} - \hat{v})(\cdot, t) \|_{L^\infty} \leq C \| (\tilde{\rho} - \hat{\rho})(\cdot, t) \|_{L^\infty}
\]
\[
\leq C \| \tilde{p}_x(\cdot, t) \|_{L^2}
\]
\[
\leq C \exp \{- C_5 t\}. \quad (2.26)
\]
Now, it is easy to deduce from (2.24)-(2.26) that
\[
\| (\tilde{v} - \hat{v})(\cdot, t) \|_{H^2} \leq C_7 \exp \{- C_8 t\}. \quad (2.27)
\]
Inequalities (2.24) and (2.27) complete the proof of this theorem. □

3. THE GLOBAL C^1 SOLUTION OF (1.2) AND (1.8)-(1.9)

In this section we will prove that the problem (1.2) and (1.8)-(1.9) admits a unique global \(C^1\) solution. We give the following hypotheses on the initial data \((v_0(x), u_0(x), s_0(x))\).

\[(H_1)\] Assume that \((v_0(x), u_0(x)) \in C^1([0, 1]), s_0(x) \in C^2([0, 1])\). Furthermore, there exist some positive constants \(v_*, u^*, s^*, v_1, u_1, s_1\) such that
\[
0 < v_* \leq v_0(x) \leq v^*, \quad |u_0(x)| \leq u^*, \quad 0 \leq s_0(x) \leq s^*
\]
\[
|v_0'(x)| \leq v_1, \quad |u_0'(x)| \leq u_1, \quad |s_0'(x)| \leq s_1,
\]
and
\[
v_1 < \sqrt{\frac{\gamma}{\gamma - 1}} \left( \frac{\gamma - 1}{16 \gamma} \right)^{-\gamma(\gamma + 1)/2(\gamma - 1)}
\]
\[
\times 2 e^{u^*/2 \gamma} \left( u^* + 2 \sqrt{\frac{\gamma}{\gamma - 1}} e^{u^*/2} \right)^{-\gamma(\gamma + 1)/2(\gamma - 1)}.
\]

Since the local (in time) existence and uniqueness of \(C^1\) solution can be found in [12], we only need to establish the uniform \(C^1\) estimates for the solutions of (1.2) and (1.8)-(1.9) a priori. Since \(s(x, t) = s_0(x)\), it suffices to deal with \((v(x, t), u(x, t))\).

First of all, we give the following lemma, which will play an important role in our analysis.
Lemma 3.1. Let \((F(x, t), G(x, t))\) be the solution to the problem:

\[
\begin{align*}
F_t + \lambda_1 F_x &= -f_1(F + G) \\
G_t + \lambda_2 G_x &= -f_2(G + F), \quad (x, t) \in [0, 1] \times (0, +\infty), \\
(F(x, 0), G(x, 0)) &= (F_0(x), G_0(x)), \quad x \in [0, 1], \\
|F(0, t)| &= |G(0, t)|, \quad |F(1, t)| = |G(1, t)|, \quad t \geq 0,
\end{align*}
\]

(3.1)

where \(\lambda_1 < 0 < \lambda_2\), \(f_1 > 0\) and \(f_2 > 0\) are locally bounded functions of \((x, t)\).

Then, it holds for any \(t > 0\) that

\[
\sup_{0 \leq \tau \leq t} \max \{ \|F(\cdot, \tau)\|_{L^\infty}, \|G(\cdot, \tau)\|_{L^\infty} \} \leq \max \{ \|F_0\|_{L^\infty}, \|G_0\|_{L^\infty} \}.
\]

(3.2)

Proof. Let

\[
M(t) = \sup_{0 \leq \tau \leq t} \max \{ \|F(\cdot, \tau)\|_{L^\infty}, \|G(\cdot, \tau)\|_{L^\infty} \}.
\]

For every fixed \(T > 0\), without loss of generality, we assume that \(M(T)\) is reached by \(F(x, t)\) first at some point \((x, t) \in [0, 1] \times [0, T]\). The other case can be treated similarly. In the bounded domain \([0, 1] \times [0, T]\) where the solution of (3.1) are defined, \(\lambda_1\) and \(\lambda_2\) must be bounded and away from zero. Thus, we can proceed the following characteristic analysis. From \((x, t)\), we draw a backward characteristic which interacts \(x = 1\) at \((1, t_1)\). We see from (3.1) that

\[
|F(x, t)| \leq \exp \left\{ -\int_{t_1}^t f_1 \, ds \right\} |F(1, t_1)| + \int_{t_1}^t f_1 \exp \left\{ -\int_{\tau}^t f_1 \, ds \right\} |G| \, d\tau.
\]

(3.3)

Then, from \((1, t_1)\), we draw a forward characteristic which interacts \(x = 0\) at \((0, t_2)\). Along this characteristic, similar to (3.3), we have

\[
|F(1, t_1)| = |G(1, t_1)| \leq \exp \left\{ -\int_{t_2}^{t_1} f_2 \, ds \right\} |G(0, t_2)| + \int_{t_2}^{t_1} f_2 \exp \left\{ -\int_{\tau}^{t_1} f_2 \, ds \right\} |F| \, d\tau.
\]

(3.4)
This procedure can be repeated until the characteristic interacts at \( t = 0 \) before reaching the boundaries after finite steps.

We choose a typical case to explain the procedure, the other cases can be treated similarly. Assume that the backward characteristic from \((0, t_2)\) interacts at \( t = 0 \) at \((x_0, 0)\) with \( x_0 \in [0, 1] \). Thus we have

\[
|G(0, t_2)| = |F(0, t_2)| \\
\leq \exp \left\{ - \int_0^{t_2} f_1 ds \right\} |F(x_0, 0)| + \int_0^{t_2} f_1 \exp \left\{ - \int_0^s f_1 ds \right\} |G| ds.
\]

(3.5)

Combining (3.3)–(3.5), we obtain

\[
M(T) \leq \exp \left\{ - \sum_{i=1}^3 e_i \right\} |F(x_0, 0)| \\
+ \exp \left\{ - \sum_{i=1}^2 e_i \right\} (1 - \exp \{- e_3\}) M(T) \\
+ \exp \left\{ - e_1 \right\} (1 - \exp \{- e_2\}) M(T) + (1 - \exp \{- e_1\}) M(T) \\
\leq \exp \left\{ - \sum_{i=1}^3 e_i \right\} |F(x_0, 0)| + \left( 1 - \exp \left\{ - \sum_{i=1}^3 e_i \right\} \right) M(T),
\]

(3.6)

where \( e_1 = \int_0^{t_1} f_1 ds, e_2 = \int_0^{t_2} f_2 ds \), and \( e_3 = \int_0^{t_2} f_1 ds \). Due to

\[
\exp \left\{ - \sum_{i=1}^3 e_i \right\} < 1,
\]

(3.6) implies that

\[
M(T) \leq M(0).
\]

To exploit this lemma, we introduce the characteristic speeds

\[
\mu_2 = 0, \quad -\mu_1 = \mu = \mu_3 = \sqrt{-p_s(v, s)} = \sqrt{\gamma(\gamma - 1)} v^{-(\gamma + 1)/2} e^{v^2/2},
\]

(3.7)

and the Riemann invariants

\[
\begin{align*}
\{w = u - h(v, s) \\
z = u + h(v, s)
\end{align*}
\]

(3.8)
where

\[ h(\varepsilon, s) = 2 \left( \frac{\varepsilon}{\gamma - 1} \right)^{\frac{1}{2}}. \]

Thus,

\[
\begin{align*}
\mu &= \frac{1}{2} (w + z) \\
v &= \left[ \frac{1}{4} \frac{\sqrt{\gamma - 1}}{\gamma} (z - w) \right] - 2/(\gamma - 1) e^{2/(\gamma - 1)} \\
\mu &= B(\gamma)(z - w)^{(\gamma + 1)/(\gamma - 1)} e^{-s/(\gamma - 1)},
\end{align*}
\]  

(3.9)

with \( B(\gamma) = \sqrt{\gamma - 1} ((\gamma - 1)/16\gamma)^{(\gamma + 1)/2(\gamma - 1)} \), and the problem (1.2) and (1.8)-(1.9) can be written as

\[
\begin{align*}
D^- w &= -\frac{1}{2} (z + w) + \mu s + \frac{\mu s}{4\gamma} (z - w) \\
D^+ z &= -\frac{1}{2} (z + w) + \mu s + \frac{\mu s}{4\gamma} (z - w) \\
(w + z)(0, t) &= (w + z)(1, t) = 0 \\
(w(x, 0), z(x, 0)) &= (u_0(x) - h(v_0, s_0)(x), u_0(x) + h(v_0, s_0)(x)),
\end{align*}
\]  

(3.10)

where, \( D^+ = \partial_x + \mu \partial_x^* \).

For (3.10) we have

**Theorem 3.2.** Assume that \( (H_1) \) holds. Let \((w(x, t), z(x, t))\) be the solution of (3.10) and \( M_1 = e^{\gamma^2 r} \max \| w(x, t) \|_{L^\infty}, \| z(x, t) \|_{L^\infty} \). Then it holds that

\[ |w(x, t)| \leq M_1, \quad |z(x, t)| \leq M_1. \]

**Proof.** With the help of the local result and a standard continuity argument, we assume a priori that

\[ \frac{\mu s}{4\gamma} \leq \frac{1}{4}. \]  

(3.11)

Letting

\[ \tilde{w} = -w e^{-s/2\gamma}, \quad \tilde{z} = z e^{-s/2\gamma}, \]
we have
\[
\begin{aligned}
D^- w &= - \frac{1}{2} (z + w) + \frac{\mu s_x}{4\gamma} (z - w) \\
D^+ z &= - \frac{1}{2} (z + w) + \frac{\mu s_x}{4\gamma} (z - w) \\
(\bar{w} - \bar{z})(0, t) &= (\bar{w} - \bar{z})(1, t) = 0 \\
(w(x, 0), z(x, 0)) &= e^{-\mu_s^2 t} (h(v_0, s_0)(x) - u_0(x)) + h(v_0, s_0)(x)).
\end{aligned}
\] (3.12)

Applying Lemma 3.1 to (3.12), we have
\[
|w(x, t)| e^{-\mu_s^2 t} \leq M_0, \quad |z(x, t)| e^{-\mu_s^2 t} \leq M_0,
\] (3.13)

with \( M_0 = \max\{ |w(x, 0)|_{L^\infty}, |z(x, 0)|_{L^\infty} \} \). Hence,
\[
|w(x, t)| \leq M_1, \quad |z(x, t)| \leq M_1.
\]

Then, we have
\[
\mu \leq B(\gamma)(2M_1)^{(\gamma+1)(\gamma-1)},
\]
and then
\[
\frac{\mu s_x}{4\gamma} < \frac{1}{4}
\]
which verifies the a priori assumption (3.11). This completes the proof of Theorem 3.2.

By the relations (3.9), it is easy to get the following estimates.

**Corollary 3.3.** Assume that (H1) holds. Let \((v(x, t), u(x, t), s_0(x))\) be the solution of (1.2) and (1.8)–(1.9). Then
\[
|u(x, t)| \leq M_1, \quad v(x, t) \geq \left( \frac{16\gamma}{\gamma-1} \right)^{1/\gamma} \left( 2M_1 \right)^{-2(\gamma-1)} e^{-\gamma/\gamma-1} v_-. 
\]

Now, we turn to the upper bound of \(v\).

Introducing
\[
\begin{aligned}
P &= e^{-3\gamma s_0^2 \mu} \left[ w_x + \mu^{-\gamma-1} \frac{z - w}{3 - \gamma} + \frac{1}{4\gamma} (z - w) s_x \right] \\
Q &= e^{-3\gamma s_0^2 \mu} \left[ z_x + \mu^{-\gamma-1} \frac{z - w}{3 - \gamma} - \frac{1}{4\gamma} (z - w) s_x \right] 
\end{aligned}
\] (3.14)
it is easy to see

\[
D^- P = -\frac{\gamma + 1}{\gamma - 1} (z - w)^{-1} \mu^{3/2} e^{-3\gamma/4} \left[ w_s + \frac{1}{4\gamma} (z - w) s_s \right]^2 \\
+ \left( \frac{1}{2} \frac{\mu s_x}{4\gamma} \right) (Q - P)
\]

\[
D^- Q = -\frac{\gamma + 1}{\gamma - 1} (z - w)^{-1} \mu^{3/2} e^{-3\gamma/4} \left[ z_s - \frac{1}{4\gamma} (z - w) s_s \right]^2 \\
+ \left( \frac{1}{2} \frac{\mu s_x}{4\gamma} \right) (P - Q),
\]  

(3.15)

and then

\[
\begin{cases}
D^- P \leq \left( \frac{1}{2} \frac{\mu s_x}{4\gamma} \right) (Q - P) \\
D^+ Q \leq \left( \frac{1}{2} \frac{\mu s_x}{4\gamma} \right) (P - Q).
\end{cases}
\]

(3.16)

We observe from (1.2) and (1.5) that

\[
p_x(0, t) = p_x(1, t) = 0
\]

for \( C^1 \) solutions \((v, u)(x, t)\) of (1.2) and (1.8)–(1.9). Hence

\[
(P - Q)(0, t) = (P - Q)(1, t) = 0.
\]

(3.17)

Equation (3.17) can be proved by the fact

\[
\begin{cases}
w_s + \frac{1}{4\gamma} (z - w) s_s = u_s - \frac{\gamma - 1}{2\gamma} \frac{h}{p} p_s \\
z_s - \frac{1}{4\gamma} (z - w) s_s = u_s + \frac{\gamma - 1}{2\gamma} \frac{h}{p} p_s.
\end{cases}
\]

Then, by Lemma 3.1 and the comparison principle of O.D.E., we have

**Lemma 3.4.** Assume that \((H_1)\) holds. Then

\[
P(x, t) \leq M_2, \quad Q(x, t) \leq M_2,
\]

where \( M_2 = \max \{ \|P(x, 0)\|_{L^\infty}, \|Q(x, 0)\|_{L^\infty} \} \).
With the help of the relations (3.9), we note that
\[
M_2 \leq \left( \frac{4}{3-\gamma} \nu^* (3-\gamma)^{1/4} + e^{-(\gamma+1)^{1/4}} K_1 \right)^{1/4},
\] (3.18)
with \( K_1 = u_1 + \mu(v_\nu, s^*) e_1 + ((\gamma - 1)/2\gamma) b(v_\nu, s^*) s_1 \).

Due to Lemma 3.4, we can obtain the upper bound for \( v(x, t) \).

**Theorem 3.5.** Assume that \((H_1)\) holds. Then
\[
v(x, t) \leq v_+.
\]

**Proof.** We note that
\[
\begin{align*}
D^{-}(z-w) &= -2\mu D^{-}v + \frac{2}{\gamma - 1} \mu v D^{-}s \\
D^{+}(z-w) &= -2\mu D^{+}v + \frac{2}{\gamma - 1} \mu v D^{+}s
\end{align*}
\] (3.19)
and
\[
\begin{align*}
z_v &= D^{-}v - \frac{1}{\gamma - 1} \mu v D^{-}s \\
w_v &= D^{+}v - \frac{1}{\gamma - 1} \mu v D^{+}s
\end{align*}
\] (3.20)
From Lemma 3.4, we have
\[
\begin{align*}
P &= e^{-3\gamma/4s^*} \mu^{1/2} \left( D^{+}v - \frac{1}{\gamma} v D^{+}s + \frac{4}{3-\gamma} v \right) \leq M_2 \\
Q &= e^{-3\gamma/4s^*} \mu^{1/2} \left( D^{-}v - \frac{1}{\gamma} v D^{-}s + \frac{4}{3-\gamma} v \right) \leq M_2
\end{align*}
\]
which implies that
\[
\partial_t (v^{(3-\gamma)/4} e^{-(3-\gamma)(3s^*)}) + v^{(3-\gamma)/4} e^{-(3-\gamma)(3s^*)} t \leq M_3,
\] (3.21)
with \( M_3 = ((\gamma - 3)/4)(\gamma(\gamma - 1))^{-1/4} M_2. \)

Hence, we arrive at
\[
v^{(3-\gamma)/4}(x, t) \leq v_0^{(3-\gamma)/4}(x) e^{-t} + M_3 (1 - e^{-t}) e^{(3-\gamma)s^*/4s^*},
\]
which yields that
\[
v(x, t) \leq M_3^{(3-\gamma)} e^{(1/\gamma)s^*}.
\]
Thus, by choosing
\[ v_+ = \left[ c_{\ast}^{(3-\gamma)/4} + \frac{3-\gamma}{4} c_{\ast}^{-(\gamma+1)/4} K_1 \right]^{4/(3-\gamma)} e^{(1/\gamma) s^*} \] (3.22)
the proof of Theorem 3.5 is finished.

Now, we are at the position to derive the \( L_\infty \) bounds for the first order derivatives for the solution to (1.2) and (1.4)–(1.5) or (3.7). To this end, we introduce
\[ I = \mu w_x + \frac{\mu_s x}{4} (z-w), \quad J = \mu z_x - \frac{\mu_s x}{4} (z-w), \] (3.23)
and observe that
\[
\begin{align*}
D^- I &= \left( \frac{1}{2} + \frac{\gamma + 1}{\gamma - 1} (z-w)^{-1} I + \frac{\mu_s x}{4} \right) (I + J) \\
D^+ J &= \left( \frac{1}{2} + \frac{\gamma + 1}{\gamma - 1} (z-w)^{-1} J - \frac{\mu_s x}{4} \right) (I + J)
\end{align*}
\] (3.24)
The initial data for \((I, J)\) can be easily derived from (1.2) and (1.8)–(1.9).

In addition to (H1), we need the following assumption.

\( (H_2) \) \( K_1 \) given in (3.18), it holds
\[ \frac{v_+^{-(\gamma+1)/2}}{c_{\ast}^{(3-\gamma)/4}} K_1 \leq \frac{1}{2(\gamma + 1)} e^{-(1-1/2\gamma) s^*} \]
\[ \times \left[ c_{\ast}^{(3-\gamma)/4} + \frac{3-\gamma}{4} c_{\ast}^{-(\gamma+1)/4} K_1 \right]^{-2(\gamma-1)/(3-\gamma)}. \]

**Theorem 3.6.** Assume that (H1) and (H2) hold. Then
\[ |I(x, t)| \leq M_4, \quad |J(x, t)| \leq M_4, \]
where \( M_4 = \max \{ \|I(x, 0)\|_{L^\infty}, \|J(x, 0)\|_{L^\infty} \} \).

**Proof.** With the help of the local result and a standard continuity argument, we can easily prove this theorem. We note that the (H2) and Lemma 3.1 together verify the following two facts
\[ |I(x, t)| \leq M_4, \quad |J(x, t)| \leq M_4, \]
and
\[ \frac{\gamma+1}{\gamma-1} (z-w)^{-1} M_{4} < \frac{1}{8}. \]

The details will be omitted.

With the help of the local existence and uniqueness of $C^1$ solution to (1.2) and (1.8)–(1.9), and the estimates established above, we end up with

**Theorem 3.7.** Assume that $(H_1)$ and $(H_2)$ hold. Then there exists a unique global $C^1$ solution $(v(x,t), u(x,t), s(x))$ to (1.2) and (1.8)–(1.9), which satisfies the estimates

\[
|u(x,t)| \leq M_1, \quad v_+ \geq v(x,t) \geq v_-,
\]
\[
|u_x(x,t)| \leq C M_4, \quad |v_x(x,t)| \leq C M_4,
\]

where $C$ is a suitable constant only depending on initial data and $\gamma$.

4. LARGE TIME BEHAVIOR

In this section, we will study the large time behavior of the solution $(v, u, s)(x,t)$ to (1.2) and (1.8)–(1.9) obtained in Section 3. The energy estimates will be used with the help of the $C^1$-estimates on $(v, u)(x,t)$ derived in Section 3.

It is easy to see

\[
\int_0^1 v(x,t) \, dx = \int_0^1 v_0(x) \, dx = \bar{v}. \quad (4.1)
\]

This, together with the results on (1.11) in Section 2, suggests that the asymptotic of $(v, u, s)(x,t)$ should be $(\bar{v}, 0, s_0(x))$. Since the result for $s(x,t)$ is clear, we will only deal with $(v, u)(x,t)$ in the following.

In addition to $(H_1)$ and $(H_2)$ we need the following condition.

$(H_3)$ For $v_+$ and $v_-$ given in Section 3, it holds

\[
s_1 < \frac{1}{\sqrt{2\gamma(\gamma-1)}} \left( \frac{v_+}{v_-} \right)^{-(\gamma+1)/2} v_{-}^{(\gamma+1)/2} \left( 1 + \frac{1+\gamma}{\gamma} \frac{\bar{v}}{v_-} e^{(1/\gamma)s^*} \right)^{-1} e^{-s^*}.
\]

Let

\[
\phi = v - \bar{v}, \quad \psi = u. \quad (4.2)
\]
Then (1.2) becomes to
\[
\begin{cases}
\phi_t - \psi_x = 0 \\
\psi_x + p(\phi + \hat{\epsilon}, s)_x = -\psi.
\end{cases}
\] (4.3)

Introducing \( y = \int_0^t \phi(t, \xi) \, d\xi \), we get
\[
\begin{aligned}
y_{tt} + p(y_x + \hat{\epsilon}, s)_x + y_t &= 0 \\
y(0, t) &= y(1, t) = 0 \\
y(x, 0) &= \int_0^\infty (v_q - \hat{\epsilon})(\xi) \, d\xi \\
y_t(x, 0) &= u_0(x).
\end{aligned}
\] (4.4)

We note that
\[
p(y_x + \hat{\epsilon}, s)_x = (p(y_x + \hat{\epsilon}, s) - \hat{\beta})_x \\
= (p(y_x + \hat{\epsilon}, s) - p(\hat{\epsilon}, s))_x. \quad (4.5)
\]

By the results in Section 3, it is easy to see that (4.4) has a unique \( C^2 \) solution \( y(x, t) \). We will employ the energy estimates to the solution of (4.4) and then deduce the large time behavior of \( (v(x, t), u(x, t)) \).

**Theorem 4.1.** Assume that \((H_1)-(H_3)\) hold. Then it holds that
\[
\| (y, y_t, y_{xt}, y_{xx})_x(t) \|_{L^2_x}^2 + \int_0^t \| (y, y_t, y_{xt}, y_{xx})(\tau) \|_{L^2_x}^2 \, d\tau \leq C
\]
for all \( t \geq 0 \). Hereafter, \( C \) denotes the generic constant independent of \( t \).

**Proof.** Multiplying (4.4) with \( y + 2y_t \), we have
\[
(y^2 + yy_x + \frac{1}{2} y^2 + 2q)_x + y_x^2 - y_x \int_0^\infty p'(\hat{\epsilon} + \xi, s) \, d\xi = \{ \cdots \}_x, \quad (4.6)
\]
where \( \{ \cdots \}_x \) denote the terms which disappear after integration over \([0, 1]\) with respect to \( x \), and
\[
q = -\int_0^\infty (p(\hat{\epsilon} + \xi, s) - p(\hat{\epsilon}, s)) \, d\xi. \quad (4.7)
\]

Noticing that there exist two constants \( 0 < c_1 < c_2 \) such that
\[
c_1 y_x^2 \leq q \leq c_2 y_x^2,
\]
integrating (4.6) over \([0, 1] \times [0, t]\), we obtain
\[
\| (y, y_t, y_{xx}(t))\|_2^2 + \int_0^t \| (y, y_x)(\tau)\|_2^2 \, d\tau \leq C. \tag{4.8}
\]

Differentiating (4.4) in \(x\) formally, we have
\[
y_{txt} + y_{tx} + (p(y_x + \hat{v}, s) - p(\hat{v}, s))_{xx} = 0. \tag{4.9}
\]

Multiplying (4.9) with \(y_{xx}\), we can show that
\[
(\frac{1}{2} y_x^2 + y_{xx})_x - y_{xx} - (p(\hat{v} + y_x, s) - p(\hat{v}, s))_{xx} + \{ \ldots \}_x = 0. \tag{4.10}
\]

On the other hand, multiplying (4.9) with \(y_{tx}\), we deduce that
\[
(\frac{1}{2} y_x^2)_x + y_{xx} - (p(\hat{v} + y_x, s) - p(\hat{v}, s))_{xx} + \{ \ldots \}_x = 0. \tag{4.11}
\]

We now make the following calculations,
\[
(p(\hat{v} + y_x, s) - p(\hat{v}, s))_x = p_x(v, s) y_{xx} + \left(\frac{1}{\gamma} \frac{\partial}{\partial s} A_1 + A_2\right) y_{xx} s_x, \tag{4.12}
\]

where
\[
A_1 = \int_0^1 p_{xx}(\hat{v} + \theta y_x, s) \, d\theta, \quad A_2 = \int_0^1 p_{xx}(\hat{v} + \theta y_x, s) \, d\theta. \tag{4.13}
\]

Then it is easy to see
\[
- \left[ (p(\hat{v} + y_x, s) - p(\hat{v}, s))_x y_{xx}
\right.
\]
\[
- \left. \quad = -p_x(v, s) y_{xx}^2 + \left(\frac{1}{\gamma} A_1 + A_2\right) y_{xx} s_x y_{xx}, \tag{4.14}
\right.
\]
\[
- \left. \quad = \left[ -\frac{1}{2} p_x(v, s) y_{xx}^2 - \left(\frac{1}{\gamma} A_1 + A_2\right) y_{xx} s_x y_{xx}\right]_x + \frac{1}{2} p_{xx}(v, s) y_{xx}^2_x
\right.
\]
\[
\left. \quad + \left(\frac{1}{\gamma} A_1 + A_2\right) s_x y_{xx} y_{xx} + \left(\frac{1}{\gamma} B_1 + B_2\right) s_x y_{xx} y_{xx}, \tag{4.15}
\right.
\]

and
\[
B_1 = \int_0^1 \partial p_{xx}(\hat{v} + \theta y_x, s) \, d\theta, \quad B_2 = \int_0^1 \partial p_{xx}(\hat{v} + \theta y_x, s) \, d\theta. \tag{4.16}
\]
We observe that $(H_3)$ implies that
\[(1 + \gamma) \mu^{-1} e^{-1} M_4 < \frac{1}{2}. \tag{4.17}\]

From Theorem 3.6, we have
\[|y_{tx}| = |u_x| \leq \mu^{-1} M_4. \]

We note that
\[- p_x + p_{xx} y_{tx} \geq - p_x (1 - (1 + \gamma) \mu^{-1} e^{-1} M_4) \geq - \frac{1}{2} p_x > 0. \tag{4.18}\]

With the help of $(H_3)$, we also note that there exists two positive constants $\alpha_1 < \alpha_2$ such that
\[\alpha_1 (y_{xx}^2 + y_{tx}^2) \leq \frac{1}{2} p_x y_{xx}^2 + y_{tx}^2 + 2 (\frac{1}{\gamma} A_1 + A_2) s_x y_{xx} y_{tx} \leq \alpha_2 (y_{xx}^2 + y_{tx}^2). \tag{4.19}\]

Now, integrating the equation of $[(4.10) + 2 \times (4.11)]$ over $[0, 1] \times [0, t]$, using $(4.12)-(4.19)$ and $(4.8)$, and employing the Cauchy–Schwartz inequality with suitable weight, we arrive at
\[\| (y_x, y_{xx}, y_{tx})(t) \|_{L^2}^2 + \int_0^t \| (y_{tx}, y_{xx})(\tau) \|_{L^2}^2 \, d\tau \leq C. \tag{4.20}\]

The combination of $(4.8)$ and $(4.20)$ gives the proof of Theorem 4.1.

With the estimates in Theorem 4.1, noticing that
\[\| y(\cdot, t) \|_{L^2} \leq \| y_0(\cdot, t) \|_{L^2}, \tag{4.21}\]
we have
\[\| y(\cdot, t) \|_{H^2} + \int_0^t \| y(\cdot, \tau) \|_{H^2} \, d\tau \leq C. \]

Thus we have the following results on large time behavior by integrating
\[[(4.6) + (4.10) + 2 \times (4.11)] \text{ over } [0, 1] \text{ with the help of (4.21).}

**Theorem 4.2.** Assume that $(H_1)$–$(H_3)$ hold. Then there exist some positive constants $C_9, C_{10}, \delta_1, \text{ and } \delta_2$ such that
\[\| y(\cdot, t) \|_{H^2} \leq C_9 \exp\{ - \delta_1 t \}. \]
and
\[ \|(\mathbf{v} - \mathbf{0})(\cdot, t)\|_{H^1} + \|(\mathbf{v} - \mathbf{0})(\cdot, t)\|_{H^1} \leq C_{10} \exp \{-\delta_1 t\}. \]
as \( t \to \infty \).

5. CONCLUSION

From Theorems 2.6 and 4.2 we have proved the following expectation.

Theorem 5.1. Assume that \((H_1)-(H_3)\) hold. Take \(0 < \tilde{v}_0(x)\) and \(0 < v_0(x)\) such that
\[ \int_0^1 \tilde{v}_0(x) \, dx = \int_0^1 v_0(x) \, dx. \]
Let \((\mathbf{v}, \mathbf{u}, s)(x, t)\) and \((\mathbf{v}', \mathbf{u}', s)(x, t)\) be the solutions of (1.2) and (1.8)-(1.9) and (1.11), respectively. Then we have
\[ \|(\mathbf{v} - \mathbf{v}')(\cdot, t)\|_{H^1} + \|(\mathbf{u} - \mathbf{u}')(\cdot, t)\|_{H^1} \leq C_{11} \exp \{-\delta_{12} t\}. \]
Furthermore, the large time state for both solutions is \((\mathbf{v}', 0, s)\).

Remark 1. (1) Theorem 5.1 claims that we can use the Darcy's law to approximate the momentum equations in (1.2) time asymptotically very well even with the boundary effects, and the problem (1.2) and (1.8)-(1.9) is well approximated by (1.11).
(2) It is also very interesting to investigate the initial boundary problems of (1.2) with other kinds of boundary conditions.

Remark 2. It is clear that \((H_1)-(H_3)\) hold if \(v_1, u_1\) and \(s_1\) are small. In fact, the conditions \((H_1)-(H_3)\) ask that the oscillation of the initial data should not be large. If the oscillation of the initial data is too large, singularity will develop due to the hyperbolicity and nonlinearity of (1.2).

Remark 3. Although we only deduce the results for \(p(v) = (\gamma - 1) v^{-\gamma} e^v\) with \(1 < \gamma < 3\), it is clear that our analysis can be generalized to the more general form of \(p(v, s)\) with \(p_e < 0 < p_m\).

Remark 4. In our analysis, more conditions are required in the proof of large time behavior than in the proof of global existence. Indeed, in order
to obtain the global existence for the $C^1$ solution to (1.2) and (1.8)-(1.9), we only need (H$_1$) and (H$_2$) below

$$(H_2)\quad e_*^{-(\gamma + 1)/2}K_1 < \frac{1}{(\gamma + 1)} e^{-(1-(1/2\gamma))s}$$

$$\times \left[ e^{s(3-\gamma)/4} + \frac{3-\gamma}{4} e_*^{-(\gamma + 1)/4}K_1 \right]^{-2(\gamma - 1)/(3-\gamma)}.$$

This is quite different from the case of $s_0(x) = const.$ where the same conditions guarantee both global existence and large time behavior for the solutions to the damped hyperbolic system (see the results below).

**Remark 5.** Our approach can be applied to the isentropic case under milder conditions.

For convenience, we assume that $p(v) = v^{-\gamma}$ and $\alpha = 1$. Consider

$$\begin{align*}
\begin{cases}
  \tau_x - u_x = 0 \\
  u_t + p(v)_x = -u, \\
  u(0, t) = u(1, t) = 0, \\
  (\tau(x, 0), u(x, 0)) = (\tau_0(x), u_0(x)),
\end{cases}
\end{align*}$$

(5.1)

and the related problem

$$\begin{align*}
\begin{cases}
  \hat{\tau}_x + p(\hat{\tau})_{xx} = 0 \\
  \hat{u} = -p(\hat{\tau})_x, \\
  \hat{\tau}_x(0, t) = \hat{\tau}_x(1, t) = 0, \\
  \hat{\tau}(x, 0) = \hat{\tau}_0(x).
\end{cases}
\end{align*}$$

(5.2)

We make the following hypotheses

**Condition A.** Assume that $(\tau_0(x), u_0(x)) \in C^1([0, 1])$. Furthermore, there exist some positive constants $a_1$, $a_2$, $b$, $a_3$, and $b_1$ such that

$$0 < a_1 \leq \tau_0(x) \leq a_2, \quad |u_0(x)| \leq b,$$

$$|\tau_0'(x)| \leq a_3, \quad |u_0'(x)| \leq b_1,$$

$$a_1^{-(\gamma + 1)/2}(b_1 + \sqrt{\gamma}a_1^{-(\gamma + 1)/2}a_3)$$

$$\leq \frac{2}{\gamma + 1} \left[ \frac{3 - \gamma}{4} a_1^{-(\gamma + 1)/4}b_1 + \sqrt{\gamma}a_1^{-(\gamma + 1)/2}a_3 + a_2^{(3-\gamma)/4} \right]^{2(\gamma - 1)/(3-\gamma)}.$$

(5.3)

Applying the approach in Sections 2-4, we have
Theorem 5.2. Assume that Condition A holds. Then the problems (5.1) and (5.2) have unique global classical solutions \((v, u)(x, t)\) and \((\bar{v}, \bar{u})(x, t)\), respectively, if \(0 < \check{v}_0(x) \in C^{2+\varepsilon}([0, 1])\). Furthermore, if
\[
\int_0^1 \check{v}_0(x) \, dx = \int_0^1 v_0(x) \, dx = \bar{v},
\]
then we have
\[
\| (v - \bar{v})(\cdot, t) \|_{H^1} + \| (u - \bar{u})(\cdot, t) \|_{H^1} \leq C_{13} \exp\{-C_{14}t\},
\]
and the large time asymptotic state for both solutions is the constant state \((\bar{v}, 0)\).

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