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Lorentz-violating massive gauge and gravitational fields

Gregory Gabadadze, Luca Grisa

Center for Cosmology and Particle Physics, Department of Physics, New York University, New York, NY 10003, USA

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Abstract

We study nonlinear dynamics in models of Lorentz-violating massive gravity. The Boulware–Deser instability restricts severely the class of acceptable theories. We identify a model that is stable. It exhibits the following bizarre but interesting property: there are only two massive propagating degrees of freedom in the spectrum, and yet long-range instantaneous interactions are present in the theory. We discuss this property on a simpler example of a photon with a Lorentz-violating mass term where the issues of (a)causality are easier to understand. Depending on the values of the mass parameter these models can either be excluded, or become phenomenologically interesting. We discuss a similar example with more degrees of freedom, as well as a model without the long-range instantaneous interactions.

1. Introduction and summary

Models of a massive/metastable graviton could shed some light on the cosmological constant problem (see, e.g., [1]). Given the ultraviolet problems of gravity, provisionally, one would like to find a model that could be regarded as a classically and quantum mechanically consistent low-energy effective field theory (for a description of gravity as an effective theory see Ref. [2]). In searching for such a theory, that could also preserve Lorentz invariance in four dimensions, one typically encounters the following three major problems:

Problem 1 (*Linear discontinuity*). Known as the van Dam–Veltman–Zakharov (vDVZ) discontinuity [3,4]. To understand this problem let us ignore for the time being all possible nonlinear self-interactions of a spin-2 field. The Lagrangian of this theory without ghosts and tachyons was uniquely determined by Fierz and Pauli (FP) [5]. Irrespective of the details of the Lagrangian, Lorentz invariance dictates that a massive spin-2 state has to have five physical polarizations, as opposed to a massless graviton with only two polarizations. This is what gives rise to the vDVZ discontinuity: one out of the extra three degrees of freedom couples to the trace of the stress-tensor, and, no matter how small the graviton mass, gives rise to experimentally unacceptable predictions either for the light bending by the Sun or for Newtonian interac-

E-mail address: gg32@nyu.edu (G. Gabadadze).

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tions. Thus, the FP Lagrangian gives a *theoretically* consistent model of a massive spin-2 state without self-interactions, however, if this spin-2 state is declared to be a graviton, the model is in contradiction with observations.

Problem 2 (*Strong coupling*). The above arguments were based on the linearized approximation. In a theory of gravity we should take into account nonlinearities. It was first observed by Vainshtein [6] that the arguments leading to the vDVZ discontinuity fail once the nonlinear interactions are taken into account. This is because linearized approximation breaks down, and, to make predictions within the Solar System one should either solve the nonlinear equations exactly, or come up with an alternative viable approximation. The latter two approaches can restore the consistency of the predictions of classical massive gravity with observations [6,7]. The breakdown of the linearized approximation takes place because of the nonlinear but classical self-interactions of the extra polarizations [7]. This suggests a problem for quantum theory where the same nonlinear interactions appear in the loop diagrams, leading eventually to a very low ultraviolet cutoff [8] for a quantum graviton scattering on an empty background. Moreover, quantum loops are expected to generate higher-dimensional operators that are suppressed only by this observationally unacceptable low cutoff [8]. Note that in a theory of gravity a cutoff has no universal meaning. For instance, for graviton scattering in a background of classical sources the physical cutoff would depend on local curvature invariants. This in principle could be used to try to overcome the strong coupling problem (along the lines of [9,10]) if the model were consistent otherwise. This brings us to the issue of a nonlinear instability of the FP model.

Problem 3 (*Nonlinear instability*). Also known as the Boulware–Deser (BD) instability. From our standpoint, clarified below, this is the most severe problem. It emerges already at the *classical* level. To quickly sketch the essence of the BD instability one can look at a scalar field model with the Hamiltonian $H = (\partial_t\phi)^2 + (\nabla\phi)^2 + m^2\phi^2 + \sigma((\partial_t\phi)^2 + (\nabla\phi)^2) + m^3\sigma^2\phi/2$. In the quadratic approximation this describes a single free scalar field ϕ. However, because of the nonlinear (cubic in this case) terms the instabilities set in. One can simply integrate out the σ field and obtain that the Hamiltonian contains the term $-((\partial_t\phi)^2 + (\nabla\phi)^2)^2/2m^3\phi$, which is sign-indefinite and unbounded below. This shows the essence of the BD instability in an oversimplified model (for complete treatment, see Section 3). In this toy model the problem can be cured by adding new terms into the Hamiltonian. One might hope for a similar remedy in the FP theory. Indeed, a nonlinear completion of the FP gravity is not unique and, in any event, one should expect quantum loops to generate all sorts of new nonlinear terms. There are examples of nonlinear systems where certain classical instabilities are removed by quantum-loop-generated terms.[1] However, to the best of our knowledge, there is no evidence for this to be happening in the FP model. In particular: (a) the worst part of the BD instability exists because of the *nonlinear* interactions between the tensor sector of the conventional Einstein–Hilbert Lagrangian and the extra polarizations of a massive graviton that are only present in the mass term (i.e., the Nambu–Goldstone (NG) sector of massive gravity). (b) It was shown in Refs. [11,12] that the BD type instabilities persist for *an arbitrary* polynomial in *fields* completion of the FP gravity. Note that these terms include all sorts of derivatives of the Nambu–Goldstone boson. (c) One might hope that these instabilities go away once the terms that contain derivatives of fields are included. However, this would require an infinite number of fine tunings for which no symmetry principle is known in the four-dimensional context.[2] The unstable solutions found in Ref. [1] were of a cosmological type. It would be interesting to conduct similar studies for spatially inhomogeneous localized sources.

Summarizing, the BD instability is the most severe problem: unlike Problems 1 and 2, it questions the very consistency of a theoretical model itself. So far no concrete cure was proposed in the context of a Lorentz-invariant local field theory with a finite number of degrees of freedom and without an infinite num-

[1] For instance, some models exhibit chaotic behavior at the classical level while the chaos is eliminated by quantum corrections.

[2] One can also try to modify the linear part of the FP theory by introducing heavy states at the UV cutoff of the theory. Explicit calculations, similar to those of Section 3, show that the rapid BD instability persists in this case too. We thank Nima Arkani-Hamed for suggesting to study this question.

ber of fine tunings. Surely, a consistent theory of nonlinearly interacting massive spin-2 should exist, it just has not been formulated yet. Therefore, until shown otherwise, we will assume that the models with the DB instability should be avoided.

In an ideal case, one would like to have a model in which all the above three problems are absent. However, if one should compromise between Problems 1, 2, and 3, as it will be the case in one particular examples below, our approach will be to worry first of all about the Problem 3. This is because Problems 1 and 2, although unpleasant from the point of view of practical calculations, can be taken care of consistently. For instance, in the DGP model [13] these problems are solved at the classical [7] as well as at the quantum level [9,10].[3]

Recently, a new approach to massive gravity was initiated in Ref. [17]: the idea is to give up Lorentz invariance which could be spontaneously broken when graviton acquires its mass [17]. Subsequently, in Ref. [18] a general parametrization of the Lorentz-violating (LV) graviton mass term was proposed and the models evading the Problems 1 and 2 were identified. More general studies of the proposal of [18] were performed in Ref. [19]. The discussions in Ref. [18] were restricted to the linearized theory. The purpose of the present work is to study a complete nonlinear dynamics in the models with the LV mass terms, and in particular address the Problem 3. We will find that many of the LV mass models suffer from the BD instability. However, there are at least two classes of models that can evade the problem. The first class exhibit surprising properties: even though there are only massive propagating degrees of freedom, the models exhibits a long-range instantaneous interactions. This could be phenomenologically deadly or interesting depending on the value of the graviton mass. It is very likely that the properties of some of these models will not be affected by radiative corrections since they are protected by certain symmetries. (A different model but with somewhat similar properties was discussed in [19] in the linearized approximation, see also [20].) The second class of the BD stable models contains all the massive degrees of freedom and no long-range interactions. However, the issue of radiative corrections for these models remain open, without an infinite number of fine tunings, these models are likely to exhibit the instabilities at the quantum level.

The work is organized as follows. In Section 2, as an instructive example, we discuss a Lorentz-violating theory of a massive photon. This model contains only massive propagating degrees of freedom (two massive polarizations of an electromagnetic wave), and nevertheless, there are long-range instantaneous interactions in the theory. We briefly discuss whether this type of models can be consistent with observations. In Section 3 we overview the BD instability in the FP theory and show how it also appears in the LV models. In Section 4, first we discuss a model that has no BD instability and propagates only two massive degrees of freedom (transverse polarizations of gravitational waves). We study the long-range interactions in this model, showing that there is no vDVZ discontinuity. Finally we discuss two other models, one with 5 massive degrees of freedom and the long-range interactions, and another one with six degrees of freedom where all the interactions are screened.

2. Warming up with photons

As a toy but very interesting example we consider QED with Lorentz-violating mass term for a photon[4] (for convenience we call it QED′)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_j A^j - A_\mu J^\mu, \qquad (1)$$

where $\mu, \nu = 0, 1, 2, 3$, $i, j = 1, 2, 3$; J^μ is a conserved current $\partial^\mu J_\mu = 0$, and the mass term breaks explicitly the Lorentz group down to the group of three-dimensional spatial rotations $SO(3)$ (our choice of the Lorentzian signature is $[-+++]$). One could think of this model as arising from a Lorentz-invariant theory in which certain fields acquire Lorentz-violating VEV's (see, e.g., [17,22]). These VEV's set a pre-

[3] We also note that there exist models that have no Problems 2 or 3 [14,15], however they exhibit the vDVZ discontinuity. A version of [13] discussed in [16] could potentially evade all the problems in the weak coupling regime, however some nonlinear issues should still be understood in that approach (see the discussions section in [16]).

[4] Michele Papucci and Matthew Schwartz also studied this model for a photon. We thank Matthew Schwartz for communications on this.

ferred frame in the Universe in which the model (1) is defined.

The Lagrangian (1) is invariant under spatially independent gauge transformations of the fields $\delta A_0 = \partial_0\lambda(t)$, $\delta A_j = 0$. Because of this and conservation of J_μ no new terms are generated by quantum loops in (1). The equations of motion of the model are

$$\partial^\nu F_{\nu\mu} - m^2\delta^i_\mu A_i = J_\mu. \tag{2}$$

There are two key properties that follow from (2). First, the zeros component of this equation implies that the Gauss's law is identical to that of QED:

$$\partial^j E_j = -J_0, \tag{3}$$

where $E_j = F_{0j}$ is an electric field and J_0 is a charge density. Hence, in QED′, like in QED, the electric field is *not screened*! Second, taking the partial derivative of both sides of (2) we find a Lorentz-violating analog of the Proca condition

$$\partial^i A_i = 0. \tag{4}$$

As a result, there remain only two dynamical degrees of freedom in the theory: A_0 is not dynamical and one of the A_j's can be expressed though the other two using (4). Both propagating degrees of freedom are massive.

However, this is not all. We note that (4) coincides with the *Coulomb* gauge fixing condition of QED. Because of this, a free photon propagator of (1)

$$D_{00}(k) = \frac{1}{\vec{k}^2},$$
$$D_{ij}(k) = \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right)\frac{1}{-k_0^2 + \vec{k}^2 + m^2 - i\epsilon}, \tag{5}$$

resembles a causal Coulomb gauge QED propagator. The physics of QED′ (1) is rather different, however. Like in QED, there is an instantaneous Coulomb potential (the zero–zero component of the propagator). This component has no imaginary part, hence, there are no physical degrees of freedom mediating the instantaneous potential. The spatial components, on the other hand, have an imaginary part. This corresponds to the two physical degrees of freedom. Unlike in QED, in the present case both of these degrees of freedom are massive. This has a dramatic consequence: as we will see shortly, the instantaneous potential is not canceled in physical observables for time dependent sources. The remaining instantaneous field is small for transverse sources with a typical momentum/frequency scales $\gg m$, but can become essential for scales $\lesssim m$ generating *action-at-a-distance*. To examine this question in detail we follow closely massless QED in *Coulomb gauge*. In the latter case one postulates (4) as a gauge condition. As a result of this

$$A_0(r,t) = \frac{1}{4\pi}\int d^3r' \frac{J_0(r',t)}{|r-r'|}, \tag{6}$$

is an instantaneous potential. The expression (6) is identical in QED and QED′. On the other hand, the equation for the vector potential differ in the two models. The spatial part of (2) reads:

$$\left(\partial^2 - m^2\right)A_j = J_j - \partial_j\partial_0 A_0 \equiv J_j^{\rm tr}. \tag{7}$$

The mass term on the l.h.s. is present only in QED′ but not in QED. Both in QED and QED′ the vector potential A_j has an instantaneous parts. In QED this part exactly cancels (6) in physical observables such as the electric field $E_j = -\partial_j A_0 + \partial_0 A_j$. However, this cancellation is not exact in QED′. To see this we write:

$$A_j(r,t) = \int d^3r'\,dt'\,D_R(r-r';t-t')J_j^{\rm tr}(r',t'), \tag{8}$$

where the retarded Green's function

$$D_R(r;t') \equiv D_R^{\rm QED} + D_R^m$$
$$= \frac{\theta(t)}{2\pi}\delta\left(t^2 - \vec{r}^{\,2}\right)$$
$$- \frac{\theta(t-|r|)}{4\pi}\frac{m\mathcal{J}_1(m\sqrt{t^2-\vec{r}^{\,2}})}{\sqrt{t^2-\vec{r}^{\,2}}}, \tag{9}$$

is expressed through the step function θ, Dirac delta function δ, and Bessel function $\mathcal{J}$ [21]. Note that the mass dependence in the Green's functions is additive. We used this to denote the massless function by $D_R^{\rm QED}$ and the addition due to the mass term by D_R^m. Using (9) we can write

$$A_j(r,t) = A_j^{\rm QED}(r,t)$$
$$+ \int d^3r'\,dt'\,D_R^m(r-r';t-t')J_j^{\rm tr}(r',t'), \tag{10}$$

where $A_j^{\rm QED}(r,t)$ is a vector potential of massless QED in Coulomb gauge. The latter, as we mentioned before, cancels the instantaneous scalar potential (6) to produce the *retarded* electric field E_j^{QED}. Therefore,

$$E_j(r,t) = E_j^{\mathrm{QED}}(r,t) + \partial_0 \int d^3r'\, dt'\, D_R^m(r-r'; t-t') \times J_j^{\mathrm{tr}}(r',t'), \tag{11}$$

and the instantaneous part is now contained only in the last term of this expression. The latter can be calculated as follows:

$$\frac{i}{(2\pi)^4}\int d^3k\, e^{i\vec{k}\vec{r}}\, \mathrm{Re}\Big(e^{i\sqrt{k^2+m^2}t}\,\tilde{J}_j^{\mathrm{tr}}\big(\vec{k},\sqrt{k^2+m^2}\big) - e^{i|k|t}\,\tilde{J}_j^{\mathrm{tr}}\big(\vec{k},|k|\big)\Big), \tag{12}$$

where

$$\tilde{J}_j^{\mathrm{tr}}(\vec{k},\omega) \equiv \int d^3r\, dt\, e^{i\vec{k}\vec{r}-i\omega t} J_j^{\mathrm{tr}}(\vec{r},t), \tag{13}$$

is the Fourier transform of the transverse current.[5]

In general, the expression (12) is nonzero even for $t=0$. It appears that an information from an event that took place at $t=0, r=0$ can *instantaneously* be transmitted to a point that is far away from this location. This gives rise to the action-at-a-distance. The question how important this instantaneous interaction is depends on properties of sources. For a transverse source of a typical momentum k_0 and typical frequency ω_0 the effect is negligible as long as $k_0 \gg m$. For $k_0 \ll m$ the effects can be appreciable when $\omega_0 \sim m$ or $\omega_0 \sim k_0$. In this case the instantaneous electric field would decay with distance as $\sim 1/r$. If the source has no typical frequency, juts a typical momentum, then the instantaneous interactions will be important for $k_0 \ll m$, and vice versa, for a source of a typical frequency ω_0 and no typical momentum the dangerous interactions will be present for $\omega_0 \lesssim m$. In practice, to produce a low-momentum/frequency signal that could trigger the instantaneous interaction, will itself take certain characteristic time. It would be interesting to study the phenomenology of these interactions for realistic sources to put bounds on m [23]. Note also that magnetic field has no instantaneous parts in QED′: $\vec{B} = \mathrm{curl}\,\vec{A}$ and the curl eliminates the instantaneous part of $\vec{A}$.

The presence of the long-range interactions can also be understood from the Hamiltonian formulation of (1) where A_0 acts as a Lagrange multiplier:

$$\mathcal{H} = \frac{1}{2}P_j^2 + \frac{1}{2}(\epsilon_{ijk}\partial_j A_k)^2 + \frac{1}{2}m^2 A_j^2 + A_0(\partial_j P_j - J_0) + A_j J_j. \tag{14}$$

Here $P_j = F_{j0}$ is the canonical momentum. Variation w.r.t. A_0 gives rise to the Gauss's law, $\partial^j P_j = J_0$, which is identical to the Gauss's law of massless QED. This guaranties that the theory possesses the long-range interactions in spite of the fact that the two dynamical propagating degrees of freedom are massive.[6]

To summarize, there are two massive propagating degrees of freedom, however, there still exists a long-range instantaneous interaction in the theory. Depending on the value of m this can exclude a given model, or be potentially interesting for phenomenological applications.

One can of course modify the model (1) by adding a "mass term" $\alpha m^2 A_0^2$ with some nonzero positive coefficient α. In this case the long-range interactions are removed from the theory ($\alpha = 1$ corresponding to a massive Lorentz-invariant photon). However, it is worth pointing out that $\alpha = 0$ is an enhanced gauge symmetry point and unlike the $\alpha \neq 0$ cases should be stable under radiative corrections.[7]

3. Nonlinear instabilities in massive gravity

In this section we summarize how nonlinear instabilities appear in a Lorentz-invariant theory of massive gravity (the FP gravity) [11], and show that the similar instabilities exist in many of the LV mass models once the nonlinear interactions are taken into account. We will also find the conditions under which these instabilities can be removed in a Lorentz-violating massive theory. In the latter case, however, one typically ends up with long-range interactions, similar to those studied in the previous section. In this respect the theory is half-massive: all its degrees of freedom are massive nevertheless there are long-range interactions.

[5] For simplicity (12) is written for a source such that $\tilde{J}(k,-\omega) = \tilde{J}(k,\omega)$. However, a general expression can readily be obtained.

[6] The condition $\partial^j A_j = 0$ is obtained by taking a derivative of one of the Hamiltonian equations of motion.

[7] The $\alpha = 1$ case is also stable because of the restored Lorentz invariance and conservation of the current.

One atypical example that evades the instabilities and long-range instantaneous interactions will be given at the end of the next section.

We first review briefly the Hamiltonian construction of [11] to identify the terms that are responsible for the instabilities. Then, we will remove these terms in a Lorentz-violating theory. Let us start by a brief reminder of the ADM formalism [24]. This would be a natural formalism for nonlinear formulation of the LV mass gravity. Consider a foliation of space–time by hyper-surfaces Σ_t parametrized by a time variable t. The four-dimensional metric is replaced by the following three-dimensional variables:

$$\gamma_{ij} \equiv g_{ij},$$
$$N \equiv \left(-{}^{(4)}g^{00}\right)^{-1/2}, \qquad N_i \equiv {}^{(4)}g_{0i}. \tag{15}$$

In terms of these variables the invariant interval takes the form:

$$ds^2 = -\left(N^2 - N_j N^j\right) dt^2 + 2N_j\, dx^j\, dt + \gamma_{ij}\, dx^i\, dx^j, \tag{16}$$

where all the spatial indices are contracted by means of the three-dimensional metric γ_{ij}. N is called the lapse function and N_i is the shift function. With these definitions

$$\sqrt{-{}^{(4)}g} = N\sqrt{\gamma},$$
$${}^{(4)}R = {}^{(3)}R + K_{ij}K^{ij} - K^2, \tag{17}$$

where K_{ij} is the extrinsic curvature of Σ_t

$$K_{ij} = \frac{1}{2}N^{-1}[\dot{\gamma}_{ij} - \nabla_i N_j - \nabla_j N_i], \tag{18}$$

and K is its trace. The extrinsic curvature is related to the canonical momentum

$$\pi^{ij} \equiv \frac{\delta\mathcal{L}}{\delta\dot{\gamma}_{ij}} = \sqrt{\gamma}\left(K^{ij} - K\gamma^{ij}\right). \tag{19}$$

The Hamiltonian of the Einstein gravity in terms of the above variables reads

$$\mathcal{H}_{\mathrm{EH}} = \pi^{ij}\dot{\gamma}_{ij} - \mathcal{L} = \sqrt{\gamma}\left[NR^0 + N_i R^i\right], \tag{20}$$

where

$$R^0 \equiv -{}^{(3)}R + \gamma^{-1}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right),$$
$$R^i \equiv -2\nabla_j\left(\gamma^{-1/2}\pi^{ij}\right). \tag{21}$$

N and N_i appear linearly in the Hamiltonian (20). Hence they are Lagrange multipliers variation w.r.t. which gives the constraints $R^0 = 0$ and $R^i = 0$; $\mathcal{H}_{\mathrm{EH}} = 0$ on the surface of the constraints.

Let us now turn to massive FP gravity for which the mass terms is written as

$$-\frac{1}{2}m^2\left(h_{\mu\nu}^2 - h_\mu^{\mu 2}\right) = -\frac{1}{2}m^2\left[h_{ij}^2 - h^2 - 2N_i^2 + 2h\left(1 - N^2 - N_j^2\right)\right], \tag{22}$$

where the second equality is obtained by expressing $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ in terms of γ_{ij}, N and N_i (note that $h_{ij} = \gamma_{ij} - \eta_{ij}$, and $h \equiv \gamma^{ij}h_{ij}$). The key role is played by the terms in (22) which are quadratic in N and N_j. Because of these terms N and N_j ceases to be Lagrange multipliers in the massive theory. Variations w.r.t. N and N_j lead to the following equations:

$$\sqrt{\gamma}R^0 = 2m^2 hN,$$
$$\sqrt{\gamma}R^i = 2m^2\left(\eta^{ij} - h\gamma^{ij}\right)N_j. \tag{23}$$

These are not constraint equations any more but serve to determine N and N_j. Substituting these solutions into the full Hamiltonian we obtain:

$$\mathcal{H} = \frac{1}{4m^2}\left\{\frac{(\sqrt{\gamma}R^0)^2}{h} + \gamma R^i\left(\eta^{ij} - h\gamma^{ij}\right)^{-1}R^j\right\} + \frac{1}{2}m^2\left(h_{ij}^2 - h^2 + 2h\right). \tag{24}$$

This is a Hamiltonian of an ill-defined theory. The first term on the r.h.s. is unbounded below and singular in m and h. For instance, consider $\sqrt{\gamma}R^0$ fixed and $R^i = 0$; when $h \to 0^-$ the term in the Hamiltonian density $\mathcal{H} \sim (\sqrt{\gamma}R^0)^2/(m^2 h)$ is not bounded below. This demonstrates the presence of a ghost-like instability in the theory. This instability can manifest in many ways even at the *classical* level, and the time scale of the instability can be very short [12]. Such a theory is hard to make sense of.

The BD problem is associated with the terms that in the linearized theory looks as $h_{00}h_j^j$. The mass term of the model analyzed in Ref. [18] is

$$L_m = \frac{M_{\mathrm{Pl}}^2}{2}\left(m_0^2 h_{00}^2 + 2m_1^2 h_{0j}^2 - m_2^2 h_{ij}^2 + m_3^2 h^2 - 2m_4^2 h_{00}h\right), \tag{25}$$

with $m_0 = 0$ and all the other mass parameters being nonzero with a certain hierarchy between them [18]. A straightforward nonlinear completion of the mass term (25) gives rise to the BD instability. This is because $h_{00} = 1 - N^2 + N_j^2$ and N ceases to be a Lagrange multiplier, as in the FP gravity. While in the FP model this instability cannot be removed in a Lorentz-invariant fashion, in the present context Lorentz invariance is broken anyway, and nothing prevents us to eliminate the dangerous term by judicially choosing m_4 to be zero. This choice is a point of enhanced gauge symmetry and should be stable under loop corrections. However, the physics of the model with $m_0 = m_4 = 0$ is dramatically different—there appear long-range interactions. This will be discussed in detail in the next section.

4. Stable models

4.1. Half-massive gravity

We consider a simple Lorentz-violating generalization of the FP mass term:

$$\Delta\mathcal{L}_1 = -\frac{1}{2}m^2\sqrt{\gamma}N\left(h_{ij}^2 - ah^2\right), \tag{26}$$

where a is a constant. As before, all the indices are contracted by γ_{ij}. We think of this theory as being obtained from a Lorentz-invariant model through spontaneous generation of a preferred frame, similar in spirit to [17].

The total Hamiltonian in this case takes the form:

$$\mathcal{H} = \sqrt{\gamma}\left[N\left(R^0 + \frac{1}{2}m^2\left(h_{ij}^2 - ah^2\right)\right) + N_i R^i\right]. \tag{27}$$

The constraint equations that follow are:

$$R^0 = -\frac{1}{2}m^2\left(h_{ij}^2 - ah^2\right), \qquad R^i = 0. \tag{28}$$

The Hamiltonian (27) on the surface of constraints (28) is zero, just like in the Einstein theory, hence, the BD instability is gone. The model (26) is invariant in the linearized approximation under coordinate-independent gauge transformations: $\delta h_{\mu\nu} = \partial_\mu\zeta_\nu(t) + \partial_\nu\zeta_\mu(t)$, as well as under the transformations with a gauge function $\xi_\mu = (\xi_0(t,\vec{x}), \xi_j = 0)$ (at the nonlinear level there is a symmetry w.r.t. the spatially independent transformations). Due to this, we expect that the properties of this model will stay stable under quantum loop corrections.

Let us now couple this theory to matter. The $\{00\}$ component of the equation of motion takes the form (we use the units $M_{\rm Pl} = 1$):

$$R^0 + \frac{1}{2}m^2\left(h_{ij}^2 - ah^2\right) = T_{00}. \tag{29}$$

The $\{0j\}$ equations are identical to those of the Einstein theory and read as $R_j = 2T_{0j}$. Finally, the $\{ij\}$ equations are:

$$G_{ij} + \frac{1}{2}m^2(h_{ij} - a\delta_{ij}h) = T_{ij}. \tag{30}$$

We see that in the linearized theory the $\{00\}$ equation (29) is also identical to the linearized massless Einstein equation. Therefore, it is only the $\{ij\}$ equation that differentiates (26) from the Einstein gravity in the linearized approximation. Because of this one should expect the vDVZ discontinuity to be absent. This can be checked in a rigorous way. Let us follow the decomposition of Ref. [18]:

$$h_{00} = \psi, \tag{31}$$

$$h_{0i} = u_i + \partial_i v, \tag{32}$$

$$h_{ij} = \chi_{ij} + \partial_{(i}s_{j)} + \partial_i\partial_j\sigma + \delta_{ij}\tau, \tag{33}$$

where χ_{ij} is a transverse-traceless tensor, s_j is a transverse vector while the other fields are scalars. The gauge invariant combinations are: a tensor χ_{ij}, a vector $w_i = u_i - \partial_0 s_i$, and two scalars τ and $\Phi = \psi - 2\partial_0 v + \partial_0^2\sigma$. The conventional coupling to a conserved matter stress-tensor $h_{\mu\nu}T^{\mu\nu}$ can be written in terms of these invariants:

$$\chi_{ij}T_{ij} - 2w_j T_{0j} + \Phi T_{00} + \tau T_{jj}. \tag{34}$$

Solving the corresponding linearized equations for $a \neq 1$ we find:

$$\chi_{ij} = \frac{1}{-\partial_0^2 + \Delta - m^2}T_{ij}^{tt}, \tag{35}$$

$$\tau = \frac{1}{2\Delta}T_{00}, \qquad w_j = \frac{1}{\Delta}T_{0j}, \tag{36}$$

$$\Phi = \frac{1}{2\Delta}\left(T_{jj} + T_{00} - \frac{3}{\Delta}\partial_0^2 T_{00}\right) + \frac{1-3a}{2(1-a)\Delta^2}m^2 T_{00}. \tag{37}$$

(Here T^{tt}_{ij} stands for a transverse-traceless part of the tensor.) These expressions give exactly the fields of the Einstein theory in the limit $m \to 0$. Therefore, there is no vDVZ discontinuity. Note that the only propagating degrees of freedom are two polarizations of the transverse-traceless tensor χ_{ij}. The spectrum is free of ghost and tachyons. Similar properties have been found previously in a stimulating work [20], where a somewhat different model was discussed.[8]

On the other hand, the above system exhibits the same type of instantaneous interactions as QED′ discussed in Section 2. This is because the instantaneous parts cannot be exactly canceled as long as there is a mass term in the denominator of (35). However, as we discussed in the gauge field case, these instantaneous interactions can only be probed by very low-momentum/frequency sources. It would be interesting to study the phenomenology of this model [23].

4.2. More degrees of freedom

A generalization of the above model can be obtained by adding to (26) a mass term for N_j:

$$\Delta\mathcal{L}_2 = cm^2\sqrt{\gamma}N_i^2, \tag{38}$$

where c is a constant. Doing so we add three additional degrees of freedom to the theory. The Hamiltonian now takes the form:

$$\mathcal{H} = \sqrt{\gamma}\left[N\left(R^0 + \frac{1}{2}m^2\left(h_{ij}^2 - ah^2\right)\right) + N_i\left(R^i - m^2cN^i\right)\right]. \tag{39}$$

The corresponding constraint equations are:

$$R^0 = -\frac{1}{2}m^2\left(h_{ij}^2 - ah^2\right), \tag{40}$$

$$R^i = 2m^2cN^i. \tag{41}$$

As long as $c \neq 0$ the only true constraint is (40), since (41) does not restrict the number of propagating degrees of freedom but acts as an algebraic equation determining N^j. Therefore, this counting tells us that the number of propagating degrees of freedom is five.

Solving (41) for N^i and substituting the result in $\mathcal{H}$ we find the Hamiltonian

$$\mathcal{H} = \sqrt{\gamma}\frac{R_i^2}{4m^2c}, \tag{42}$$

which is positive semidefinite as long as $c > 0$. This model also has no vDVZ discontinuity. The calculations are similar to those presented above but more tedious, the spectrum contains no ghost of tachyons. The massless limit of (39) is regular and one recovers in this limit the Einstein gravity. Moreover, the theory is symmetric w.r.t. spatially independent transformations of the time variable, and, exhibits the instantaneous interactions. Further interesting properties of this model will be discussed in [23].

One can also evade the BD instability and the presence of the long-range interactions by adding into the Lagrangian yet another term

$$\Delta\mathcal{L}_3 = m^2\sqrt{\gamma}P_2(N), \tag{43}$$

where $P_2(N)$ is a polynomial in N of degree 2, namely $P_2(N) = c_0 + c_1N + c_2N^2$. We require that there are no constant and linear terms in the linearized Lagrangian. This gives the relations $c_0 + c_1 + c_2 = 0$ and $c_2 + \frac{1}{2}c_1 = 0$, with a solution $c_1 = -2c_0$ and $c_2 = c_0$. Hence, $P_2(N) = c_0(N-1)^2$. The Hamiltonians of the system is that of the previous examples plus the new term $\Delta\mathcal{H} = -m^2\sqrt{\gamma}P_2(N)$. The resulting constraint equations are:

$$R^0 + \frac{1}{2}m^2\left(h_{ij}^2 - ah^2\right) = m^2P_2'(N), \tag{44}$$

$$R^i = 2m^2cN^i. \tag{45}$$

Solving the constraints w.r.t. N and N^i we obtain

$$N = \frac{R^0 + \frac{1}{2}m^2(h_{ij}^2 - ah^2) + 2c_0m^2}{2c_0m^2}, \tag{46}$$

$$N^i = \frac{R^i}{2m^2c}, \tag{47}$$

and the Hamiltonian

$$\mathcal{H} = \sqrt{\gamma}\left[R^0_{\mathrm{mod}} + \frac{(R^0_{\mathrm{mod}})^2}{4c_0m^2} + \frac{R_i^2}{4m^2c}\right], \tag{48}$$

where $R^0_{\mathrm{mod}} \equiv R^0 + \frac{1}{2}m^2(h_{ij}^2 - ah^2)$. The above Hamiltonian is not necessarily positive semidefinite, however, it is bounded from below as long as $c, c_0 > 0$. There are six degrees of freedom propagating in this

[8] However, nonlinear stability of the model of Ref. [20], which is a suspect in the context of the discussions of our Section 3, has not been studied in Ref. [20].

model. This could exclude the model based on the Solar System data. However, it is not impossible to imagine that the nonlinear effects suppress the couplings of the extra polarizations to matter at observable distances, in analogy with [7,25]. A more serious problem of the model (43) is the absence of a symmetry principle that would guarantee the stability w.r.t. quantum corrections.

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References

[1] G. Dvali, G. Gabadadze, M. Shifman, Phys. Rev. D 67 (2003) 044020, hep-th/0202174;
G. Gabadadze, in: M. Shifman, A. Vainshtein, J. Wheater (Eds.), From Fields to Strings: Circumnavigating Theoretical Physics, Ian Kogan Memorial Volume, World Scientific, Singapore, 2005, hep-th/0408118.
[2] J.F. Donoghue, Introduction to the effective field theory description of gravity, Talk given at Advanced School on Effective Theories, Almunecar, Spain, 25 June–1 July 1995, gr-qc/9512024.
[3] H. van Dam, M.J. Veltman, Nucl. Phys. B 22 (1970) 397.
[4] V.I. Zakharov, JETP Lett. 12 (1970) 312.
[5] M. Fierz, W. Pauli, Proc. R. Soc. London, Ser. A 173 (1939) 211.
[6] A.I. Vainshtein, Phys. Lett. B 39 (1972) 393.
[7] C. Deffayet, G.R. Dvali, G. Gabadadze, A.I. Vainshtein, Phys. Rev. D 65 (2002) 044026, hep-th/0106001.
[8] N. Arkani-Hamed, H. Georgi, M.D. Schwartz, Ann. Phys. 305 (2003) 96, hep-th/0210184.
[9] G. Dvali, hep-th/0402130.
[10] A. Nicolis, R. Rattazzi, hep-th/0404159.
[11] D.G. Boulware, S. Deser, Phys. Rev. D 6 (1972) 3368.
[12] G. Gabadadze, A. Gruzinov, hep-th/0312074.
[13] G. Dvali, G. Gabadadze, M. Porrati, Phys. Lett. B 485 (2000) 208, hep-th/0005016.
[14] G. Gabadadze, M. Shifman, Phys. Rev. D 69 (2004) 124032, hep-th/0312289.
[15] M. Porrati, J.W. Rombouts, hep-th/0401211;
M. Kolanovic, M. Porrati, J.W. Rombouts, Phys. Rev. D 68 (2003) 064018, hep-th/0304148.
[16] G. Gabadadze, Phys. Rev. D 70 (2004) 064005, hep-th/0403161.
[17] N. Arkani-Hamed, H.C. Cheng, M.A. Luty, S. Mukohyama, JHEP 0405 (2004) 074, hep-th/0312099.
[18] V. Rubakov, hep-th/0407104.
[19] S.L. Dubovsky, JHEP 0410 (2004) 076, hep-th/0409124.
[20] S.L. Dubovsky, P.G. Tinyakov, I.I. Tkachev, hep-th/0411158.
[21] N.N. Bogoliubov, D.V. Shirkov, Quantum Fields, Benjamin–Cummings, Redwood City, CA, 1983.
[22] B.M. Gripaios, JHEP 0410 (2004) 069, hep-th/0408127.
[23] G. Gabadadze, L. Grisa, in preparation.
[24] R. Arnowitt, S. Deser, C.W. Misner, in: L. Witten (Ed.), Gravitation: An Introduction to Current Research, Wiley, New York, 1962, pp. 227–265, Chapter 7, gr-qc/0405109.
[25] G. Gabadadze, A. Iglesias, hep-th/0407049.