Note

Sphericity, cubicity, and edge clique covers of graphs

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Abstract

The sphericity sph\((G)\) of a graph \(G\) is the minimum dimension \(d\) for which \(G\) is the intersection graph of a family of congruent spheres in \(\mathbb{R}^d\). The edge clique cover number \(\theta(G)\) is the minimum cardinality of a set of cliques (complete subgraphs) that covers all edges of \(G\). We prove that if \(G\) has at least one edge, then \(\text{sph}(G) \leq \theta(G)\). Our upper bound remains valid for intersection graphs defined by balls in the \(L_p\)-norm for \(1 \leq p \leq \infty\).

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1. Introduction

Let \(\mathcal{F}\) be a family of subsets of a set \(S\). The intersection graph of \(\mathcal{F}\) has vertex set \(\mathcal{F}\) with distinct vertices joined by an edge provided the intersection of the corresponding sets is non-empty. When the intersection graph is isomorphic to a graph \(G\), we say that \(\mathcal{F}\) represents \(G\).

We are interested in graphs that are represented by families of balls in \(d\)-dimensional space. The sphericity of the graph \(G\), denoted by \(\text{sph}(G)\), is equal to the smallest dimension \(d\) for which \(G\) is represented by a family of open balls of the same radius in the Euclidean space \(\mathbb{R}^d\). For example, complete graphs, paths, and cycles on \(n\) vertices satisfy \(\text{sph}(K_n) = \text{sph}(P_n) = 1\) and \(\text{sph}(C_n) = 2\) for \(n \geq 4\). Graphs with sphericity 1 (unit interval graphs) possess a forbidden subgraph characterization [18]. However, the recognition problem for graphs with sphericity 2 (unit disk graphs [3,8]) is NP-hard [2]. Researchers have focused on computing the sphericity for special classes of graphs [5,6,10–12,14] and on discovering general bounds for the sphericity in terms of various graph parameters [7,13,16,17]. Maehara [9] obtained the following upper bound for the sphericity of a graph \(G\) in terms of the clique number \(\omega(G)\), i.e., the largest number of vertices in a clique (complete subgraph) of \(G\).

\textbf{Maehara’s inequality.} If \(G\) is a non-complete graph with \(n\) vertices and clique number \(\omega(G)\), then the sphericity of \(G\) satisfies

\[\text{sph}(G) \leq n - \omega(G).\]

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The inequality cannot be substantially improved; for each \( m = 1, 2, \ldots \) Maehara exhibited a graph \( G_m \) satisfying \( m = \text{sph}(G_m) = |V(G_m)| - \omega(G_m) - 1 \).

Our main theorem gives a new inequality that relates the sphericity to the cliques of a graph \( G \). Briefly, the sphericity of \( G \) cannot exceed the minimum number of cliques needed to cover all edges of \( G \). Our upper bound remains valid for intersection graphs defined by balls in the \( L_p \)-norm for \( 1 \leq p \leq \infty \).

We remark that using intersections of closed spheres instead of open spheres yields an equivalent notion of sphericity. If \( \varepsilon \) is a sufficiently small positive number, then we may replace each open sphere of radius \( r \) by a closed sphere of radius \( r - \varepsilon \) without destroying or creating any intersections; similarly a slight increase in the radii of a family of closed spheres yields a family of open spheres with the same intersection pattern.

2. Edge clique covers and sphericity

An edge clique cover of a graph \( G \) is a set of cliques \( \mathcal{Q} = \{Q_1, \ldots, Q_t\} \) that covers the edges of \( G \), i.e., every edge of \( G \) occurs among the cliques in \( \mathcal{Q} \). The edge clique cover number \( \theta(G) \) is the minimum number of cliques in an edge clique cover of \( G \). The graph \( G \) satisfies \( \theta(G) = 0 \) if and only if \( G \) has no edges; we exclude this trivial case by assuming that \( \theta(G) \) is positive throughout our work. See the surveys \([15,20]\) for a variety of results about edge clique covers and their applications.

The intersection number \( \text{int}(G) \) of the graph \( G \) is the minimum cardinality of a set \( S \) such that \( G \) has an intersection representation as a family of subsets of \( S \). A fundamental theorem of Erdős et al. \([4]\) asserts that \( \text{int}(G) = \theta(G) \) for every graph \( G \).

The sphericity \( \text{sph}(G) \) refers to an intersection representation of a graph that is minimal in a geometric sense, while the edge clique cover number \( \theta(G) \) refers to an intersection representation that is minimal in a purely combinatorial or set-theoretic sense. Our main theorem establishes a simple inequality between these two parameters.

**Theorem 1.** Let \( G \) be a graph with positive edge clique cover number \( \theta(G) \). Then

\[
\text{sph}(G) \leq \theta(G).
\]

**Proof.** Without loss of generality \( G \) has no isolated vertices. Let \( \{v_1, \ldots, v_n\} \) be the vertex set of \( G \), and let \( \mathcal{Q} = \{Q_1, \ldots, Q_t\} \) be an edge clique cover of \( G \) with cardinality \( \theta = \theta(G) \). Suppose that \( v_i \) occurs in exactly \( c_i \) cliques in \( \mathcal{Q} \) for \( i = 1, \ldots, n \), and define the components of the vector \( \mathbf{x}^{(i)} = (x_1^{(i)}, \ldots, x_t^{(i)}) \in \mathbb{R}^\theta \) by

\[
x_k^{(i)} = \begin{cases} 
(1/c_k)^{1/2} & \text{if vertex } v_i \text{ is in the clique } Q_k, \\
0 & \text{if vertex } v_i \text{ is not in the clique } Q_k.
\end{cases}
\]

If \( v_i \) and \( v_j \) are not adjacent in \( G \), then \( \|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}\| = 2^{1/2} \). On the other hand, if \( v_i \) and \( v_j \) are adjacent, then there is at least one index \( k \) such that \( x_k^{(i)} > 0 \) and \( x_k^{(j)} > 0 \), and it follows that the strict inequality \( \|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}\| < 2^{1/2} \) holds. Therefore the family of open balls with centers \( \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)} \) and radii \( \left( \frac{1}{2} \right) 2^{1/2} \) represents \( G \) as an intersection graph in \( \mathbb{R}^\theta \). \( \square \)

The complete graph \( K_n \) satisfies \( \text{sph}(K_n) = \theta(K_n) = 1 \), and thus equality sometimes holds in Theorem 1.

3. Examples, consequences, and other metric spaces

Suppose that \( c_1 = \cdots = c_n = c > 1 \) in the proof of Theorem 1. Then the components of the vector \( \mathbf{x}^{(i)} \) sum to \( c^{1/2} \) for \( i = 1, \ldots, n \), and thus the sphere centers \( \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)} \) all lie on a hyperplane in \( \mathbb{R}^\theta \). By projecting onto this hyperplane we may lower the dimension of the space by \( 1 \) and find that in this case

\[
\text{sph}(G) \leq \theta(G) - 1.
\]
Theorem 1 is sometimes stronger than Maehara’s inequality. Let the vertex subsets of $G$ consisting of $d$-tuples of integers chosen from the set $\{1, \ldots, a\}$, where two vertices are adjacent provided the corresponding $d$-tuples differ in exactly one component. Suppose that $a \geq 2$. Then $K_d^a$ has $a^d$ vertices and clique number $a$. Hence Maehara’s inequality gives $\text{sph}(K_d^a) \leq a^d - a$. It is also not difficult to see that $K_d^a$ has an edge clique cover with $da^{d-1}$ cliques and that inequality (1) gives $\text{sph}(K_d^a) \leq da^{d-1} - 1$. Therefore, Theorem 1 gives a stronger upper bound for the sphericity of $K_d^a$ than Maehara’s inequality when $d$ is much smaller than $a$.

Example 2. With a latin square $[a_{ij}]$ of order $q \geq 3$ we associate a graph $L_q$ with $q^2$ vertices as follows. The vertex set is $\{(i, j) : 1 \leq i, j \leq q\}$, and distinct vertices $(i, j)$ and $(i', j')$ are adjacent provided $i = i'$, or $j = j'$, or $a_{ij} = a_{i'j'}$. One readily shows that $\omega(L_q) = q$ and $\theta(L_q) = 3q$. Thus Maehara’s inequality gives the quadratic bound $\text{sph}(L_q) \leq q^2 - q$, while (1) gives the linear bound $\text{sph}(L_q) \leq 3q - 1$.

Theorem 1 yields a bound on the sphericity of the complement $\overline{G}$ of a bipartite graph $G$.

Corollary 2. Let $G$ be a bipartite graph with $n'$ and $n''$ vertices in the two vertex subsets, where $n' + n'' \geq 3$. Then $\text{sph}(\overline{G}) \leq \text{min}(n', n'') + 2$.

Proof. Let the vertex subsets of $G$ be $V'$ and $V''$, where $V' = \{v_1, \ldots, v_{n'}\}$. For $i = 1, \ldots, n'$ let $Q_i$ denote the clique of $\overline{G}$ consisting of $v_i$ and its neighbors in $V''$. Also, let $Q'$ and $Q''$ be the cliques of $\overline{G}$ induced by $V'$ and $V''$, respectively. Then $\{Q_1, \ldots, Q_{n''}\} \cup \{Q', Q''\}$ is an edge clique cover of $\overline{G}$, and thus $\text{sph}(\overline{G}) \leq n' + 2$ by Theorem 1. Similarly, $\text{sph}(\overline{G}) \leq n'' + 2$. □

Let $G$ be a graph with vertex set $\{v_1, \ldots, v_n\}$. The spherical dimension [16,17] of $G$, denoted by $\text{sd}(G)$, is the smallest $d$ for which there exists a set of unit vectors $v^{(1)}, \ldots, v^{(n)}$ in $\mathbb{R}^d$ with the following property: there is a real number $\tau$ such that vertices $v_i$ and $v_j$ are adjacent in $G$ if and only if the dot product of $v^{(i)}$ and $v^{(j)}$ is at least $\tau$. If we choose $\tau$ to be a sufficiently small positive number, then the same construction in our proof of Theorem 1 shows that $\text{sd}(G) \leq \theta(G)$, which is slightly stronger than our stated result, since it is known [16] that $\text{sd}(G) - 1 \leq \text{sph}(G) \leq \text{sd}(G)$.

The proof of Theorem 1 makes scant use of the Euclidean distance function; a similar representation of a graph $G$ as an intersection graph can be performed in $\mathbb{R}^{\theta(G)}$ equipped with other metrics. Recall that the $L_p$-norm of the vector $\mathbf{x} = (x_1, \ldots, x_d)$ in $\mathbb{R}^d$ equals

$$
||\mathbf{x}||_p = \begin{cases} 
(x_1^p + \cdots + x_d^p)^{1/p} & \text{if } 1 \leq p < \infty, \\
\max\{|x_1|, \ldots, |x_d|\} & \text{if } p = \infty.
\end{cases}
$$

In $\mathbb{R}^d$ an open $p$-ball with center $\mathbf{x}_0$ and radius $r$ is the set of all $\mathbf{x}$ such that $||\mathbf{x} - \mathbf{x}_0||_p < r$. Note that $p$-balls are not rotationally symmetric except in the Euclidean case $p = 2$ (or $d = 1$). We define the $p$-sphericity of the graph $G$ to be the smallest dimension $d$ for which there is a family of translates of an open $p$-ball in $\mathbb{R}^d$ that represents $G$ as an intersection graph. We denote the $p$-sphericity of $G$ by $\text{sph}_p(G)$. Of course, $\text{sph}_2(G) = \text{sph}(G)$.

When $p < \infty$, the proof of Theorem 1 remains valid for the $p$-sphericity if we replace the exponent $1/2$ by $1/p$ throughout. Thus we have the following upper bound for the $p$-sphericity of a graph.

Proposition 3. If $G$ is a graph with positive edge clique cover number, then $\text{sph}_p(G) \leq \theta(G)$ for $1 \leq p < \infty$.

4. Cubicity

If $p = \infty$, then a $p$-ball of radius $r$ in $\mathbb{R}^d$ is a $d$-dimensional cube with edge length $2r$. (All cubes in our discussion are axis-aligned.) The parameter $\text{sph}_\infty(G)$ is known as the cubicity of $G$ and is denoted by $\text{cub}(G)$. Roberts introduced cubicity in [19] and adopted the convention that the cubicity of a complete graph satisfies $\text{cub}(K_n) = 0$. A limit argument
with \( p \to \infty \) shows that Proposition 3 also holds for \( p = \infty \). Thus we have our first upper bound for the cubicity of a graph:

**Theorem 4.** Let \( G \) be a graph with positive edge clique cover number \( \theta(G) \). Then

\[
cub(G) \leq \theta(G).
\]

**Proof.** Without loss of generality \( G \) has no isolated vertices. Let \( x \) be a point in an open cube with center \( x_0 \) and radius \( r \) (i.e., edge length \( 2r \)) in \( \mathbb{R}^d \). If \( p \) is sufficiently large, then \( x \) is also in the open \( p \)-ball with the same center and radius. It follows that any representation of a graph \( G \) as an intersection graph of a family of translates of an open cube in \( \mathbb{R}^d \) yields a representation of \( G \) as an intersection graph of a family of translates of an open \( p \)-ball in \( \mathbb{R}^d \) for some sufficiently large \( p \) (depending on \( G \)). Now Proposition 3 implies that for this value of \( p \) we have \( \cub(G) \leq \sph_p(G) \leq \theta(G) \). □

A biclique is a complete bipartite graph. An edge biclique cover of a graph \( G \) is a set of bicliques \( \{B_1, \ldots, B_t\} \) that covers the edges of \( G \). The edge biclique cover number \( \eta(G) \) is the minimum number of bicliques in an edge biclique cover of \( G \). See the survey [15] for a full treatment of edge biclique covers. We now use edge biclique covers to give another upper bound for the cubicity of a graph.

**Theorem 5.** Let \( G \) be a graph whose complement has edge biclique cover number \( \eta(G) \). Then

\[
cub(G) \leq \eta(G).
\]

**Proof.** First note that the inequality holds if \( G \) is a complete graph. Now suppose that \( G \) is not complete. We use a variant of a construction employed by Boyer et al. [1] for a different problem in intersection graph theory. Let \( \{v_1, \ldots, v_n\} \) be the vertex set of \( G \), and let \( \{B_1, \ldots, B_\eta\} \) be an edge biclique cover of \( \overline{G} \) with cardinality \( \eta = \eta(G) \). Let \( V_k^+ \) and \( V_k^- \) be the vertex subsets of the biclique \( B_k \) \((k = 1, \ldots, \eta)\). Now define the components of the vector \( \mathbf{y}(i) = (y_1^{(i)}, \ldots, y_\eta^{(i)}) \) in \( \mathbb{R}^\eta \) by

\[
y_k^{(i)} = \begin{cases} 
1 & \text{if vertex } v_i \text{ is in the set } V_k^+,
-1 & \text{if vertex } v_i \text{ is in the set } V_k^-,
0 & \text{if vertex } v_i \text{ is not in the biclique } B_k.
\end{cases}
\]

On the one hand, if \( v_i \) and \( v_j \) are adjacent in \( G \), then there is no index \( k \) such that \( \{y_k^{(i)}, y_k^{(j)}\} = \{1, -1\} \), and it follows that \( \|\mathbf{y}(i) - \mathbf{y}(j)\|_\infty \leq 1 \). On the other hand, if \( v_i \) and \( v_j \) are not adjacent in \( G \), then there is an index \( k \) such that \( \{y_k^{(i)}, y_k^{(j)}\} = \{1, -1\} \), and it follows that \( \|\mathbf{y}(i) - \mathbf{y}(j)\|_\infty = 2 \). We now see that the family of open cubes with centers \( y^{(1)}, \ldots, y^{(n)} \) and edge lengths 2 represents \( G \) as an intersection graph in \( \mathbb{R}^\eta \). Therefore \( \cub(G) \leq \eta \). □

We exhibit a family of graphs for which equality holds in Theorem 5.

**Example 3.** A formula of Roberts [19] gives the cubicity of a complete multipartite graph:

\[
cub(K_{n_1, \ldots, n_q}) = \sum_{i=1}^{q} [\log_2(n_i)].
\]

Now it is not difficult to show that the edge biclique cover number of the complete graph \( K_n \) is \( \eta(K_n) = [\log_2(n)] \). It follows that the edge biclique cover number of the complement of \( K_{n_1, \ldots, n_q} \) is given by the sum in (2). Therefore \( \cub(G) = \eta(G) \) if \( G \) is a complete multipartite graph.

**References**