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Discrete Applied Mathematics 154 (2006) 1309 – 1313

DISCRETE APPLIED MATHEMATICS

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Note

Sphericity, cubicity, and edge clique covers of graphs

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Received 1 January 2005; received in revised form 6 January 2006; accepted 10 January 2006
Available online 15 March 2006

Abstract

The sphericity sph(G) of a graph G is the minimum dimension d for which G is the intersection graph of a family of congruent spheres in $\mathbf{R}^d$. The edge clique cover number $\theta(G)$ is the minimum cardinality of a set of cliques (complete subgraphs) that covers all edges of G. We prove that if G has at least one edge, then $\mathrm{sph}(G) \leqslant \theta(G)$. Our upper bound remains valid for intersection graphs defined by balls in the L_p-norm for $1 \leqslant p \leqslant \infty$.

Keywords: Intersection graph; Sphericity; Edge clique cover; Cubicity

1. Introduction

Let $\mathscr{F}$ be a family of subsets of a set S. The *intersection graph* of $\mathscr{F}$ has vertex set $\mathscr{F}$ with distinct vertices joined by an edge provided the intersection of the corresponding sets is non-empty. When the intersection graph is isomorphic to a graph G, we say that $\mathscr{F}$ *represents* G.

We are interested in graphs that are represented by families of balls in d-dimensional space. The *sphericity* of the graph G, denoted by sph(G), is equal to the smallest dimension d for which G is represented by a family of open balls of the same radius in the Euclidean space $\mathbf{R}^d$. For example, complete graphs, paths, and cycles on n vertices satisfy $\mathrm{sph}(K_n) = \mathrm{sph}(P_n) = 1$ and $\mathrm{sph}(C_n) = 2$ for $n \geqslant 4$. Graphs with sphericity 1 (unit interval graphs) possess a forbidden subgraph characterization [18]. However, the recognition problem for graphs with sphericity 2 (unit disk graphs [3,8]) is NP-hard [2]. Researchers have focused on computing the sphericity for special classes of graphs [5,6,10–12,14] and on discovering general bounds for the sphericity in terms of various graph parameters [7,13,16,17]. Maehara [9] obtained the following upper bound for the sphericity of a graph G in terms of the *clique number* $\omega(G)$, i.e., the largest number of vertices in a clique (complete subgraph) of G.

Maehara's inequality. If G is a non-complete graph with n vertices and clique number $\omega(G)$, then the sphericity of G satisfies

$$\mathrm{sph}(G) \leqslant n - \omega(G).$$

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0166-218X/$ - see front matter
doi:10.1016/j.dam.2006.01.004

The inequality cannot be substantially improved; for each $m = 1, 2, \ldots$ Maehara exhibited a graph G_m satisfying $m = \mathrm{sph}(G_m) = |V(G_m)| - \omega(G_m) - 1$.

Our main theorem gives a new inequality that relates the sphericity to the cliques of a graph G. Briefly, the sphericity of G cannot exceed the minimum number of cliques needed to cover all edges of G. Our upper bound remains valid for intersection graphs defined by balls in the L_p-norm for $1 \leqslant p \leqslant \infty$.

We remark that using intersections of closed spheres instead of open spheres yields an equivalent notion of sphericity. For if ε is a sufficiently small positive number, then we may replace each open sphere of radius r by a closed sphere of radius $r - \varepsilon$ without destroying or creating any intersections; similarly a slight increase in the radii of a family of closed spheres yields a family of open spheres with the same intersection pattern.

2. Edge clique covers and sphericity

An *edge clique cover* of a graph G is a set of cliques $\mathcal{Q} = \{Q_1, \ldots, Q_t\}$ that covers the edges of G, i.e., every edge of G occurs among the cliques in $\mathcal{Q}$. The *edge clique cover number* $\theta(G)$ is the minimum number of cliques in an edge clique cover of G. The graph G satisfies $\theta(G) = 0$ if and only if G has no edges; we exclude this trivial case by assuming that $\theta(G)$ is positive throughout our work. See the surveys [15,20] for a variety of results about edge clique covers and their applications.

The *intersection number* $\mathrm{int}(G)$ of the graph G is the minimum cardinality of a set S such that G has an intersection representation as a family of subsets of S. A fundamental theorem of Erdős et al. [4] asserts that $\mathrm{int}(G) = \theta(G)$ for every graph G.

The sphericity $\mathrm{sph}(G)$ refers to an intersection representation of a graph that is minimal in a geometric sense, while the edge clique cover number $\theta(G)$ refers to an intersection representation that is minimal in a purely combinatorial or set-theoretic sense. Our main theorem establishes a simple inequality between these two parameters.

Theorem 1. *Let G be a graph with positive edge clique cover number $\theta(G)$. Then*

$$\mathrm{sph}(G) \leqslant \theta(G).$$

Proof. Without loss of generality G has no isolated vertices. Let $\{v_1, \ldots, v_n\}$ be the vertex set of G, and let $\mathcal{Q} = \{Q_1, \ldots, Q_\theta\}$ be an edge clique cover of G with cardinality $\theta = \theta(G)$. Suppose that v_i occurs in exactly c_i cliques in $\mathcal{Q}$ for $i = 1, \ldots, n$, and define the components of the vector $\mathbf{x}^{(i)} = (x_1^{(i)}, \ldots, x_\theta^{(i)})$ in $\mathbf{R}^\theta$ by

$$x_k^{(i)} = \begin{cases} (1/c_i)^{1/2} & \text{if vertex } v_i \text{ is in the clique } Q_k, \\ 0 & \text{if vertex } v_i \text{ is not in the clique } Q_k. \end{cases}$$

If v_i and v_j are not adjacent in G, then $\|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}\| = 2^{1/2}$. On the other hand, if v_i and v_j are adjacent, then there is at least one index k such that $x_k^{(i)} > 0$ and $x_k^{(j)} > 0$, and it follows that the strict inequality $\|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}\| < 2^{1/2}$ holds. Therefore the family of open balls with centers $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ and radii $\left(\frac{1}{2}\right) 2^{1/2}$ represents G as an intersection graph in $\mathbf{R}^\theta$. $\square$

The complete graph K_n satisfies $\mathrm{sph}(K_n) = \theta(K_n) = 1$, and thus equality sometimes holds in Theorem 1.

3. Examples, consequences, and other metric spaces

Suppose that $c_1 = \cdots = c_n = c > 1$ in the proof of Theorem 1. Then the components of the vector $\mathbf{x}^{(i)}$ sum to $c^{1/2}$ for $i = 1, \ldots, n$, and thus the sphere centers $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ all lie on a hyperplane in $\mathbf{R}^\theta$. By projecting onto this hyperplane we may lower the dimension of the space by 1 and find that in this case

$$\mathrm{sph}(G) \leqslant \theta(G) - 1. \tag{1}$$

One may show that the graph G_m constructed by Maehara satisfies $\mathrm{sph}(G_m)=m$ and $\theta(G_m)=m+2$ for $m=1,2,\ldots,$ and thus Theorem 1 performs essentially as well as Maehara's inequality for G_m. The following examples show that Theorem 1 is sometimes stronger than Maehara's inequality.

Example 1. Let K_a^d denote the d-fold Cartesian product of the complete graph K_a with itself. Thus K_a^d has vertex set equal to the d-tuples of integers chosen from the set $\{1,\ldots,a\}$, where two vertices are adjacent provided the corresponding d-tuples differ in exactly one component. Suppose that $a\geqslant 2$. Then K_a^d has a^d vertices and clique number a. Hence Maehara's inequality gives $\mathrm{sph}(K_a^d)\leqslant a^d-a$. It is also not difficult to see that K_a^d has an edge clique cover with da^{d-1} cliques and that inequality (1) gives $\mathrm{sph}(K_a^d)\leqslant da^{d-1}-1$. Therefore, Theorem 1 gives a stronger upper bound for the sphericity of K_a^d than Maehara's inequality when d is much smaller than a.

Example 2. With a latin square $[a_{ij}]$ of order $q\geqslant 3$ we associate a graph L_q with q^2 vertices as follows. The vertex set is $\{(i,j): 1\leqslant i,j\leqslant q\}$, and distinct vertices$(i,j)$ and (i',j') are adjacent provided $i=i'$, or $j=j'$, or $a_{ij}=a_{i'j'}$. One readily shows that $\omega(L_q)=q$ and $\theta(L_q)=3q$. Thus Maehara's inequality gives the quadratic bound $\mathrm{sph}(L_q)\leqslant q^2-q$, while (1) gives the linear bound $\mathrm{sph}(L_q)\leqslant 3q-1$.

Theorem 1 yields a bound on the sphericity of the complement $\overline{G}$ of a bipartite graph G.

Corollary 2. *Let G be a bipartite graph with n' and n'' vertices in the two vertex subsets, where $n'+n''\geqslant 3$. Then* $\mathrm{sph}(\overline{G})\leqslant \min\{n',n''\}+2$.

Proof. Let the vertex subsets of G be V' and V'', where $V'=\{v_1,\ldots,v_{n'}\}$. For $i=1,\ldots,n'$ let Q_i denote the clique of $\overline{G}$ consisting of v_i and its neighbors in V''. Also, let Q' and Q'' be the cliques of $\overline{G}$ induced by V' and V'', respectively. Then $\{Q_1,\ldots,Q_{n'}\}\cup\{Q',Q''\}$is an edge clique cover of $\overline{G}$, and thus $\mathrm{sph}(\overline{G})\leqslant n'+2$ by Theorem 1. Similarly, $\mathrm{sph}(\overline{G})\leqslant n''+2$. □

Let G be a graph with vertex set $\{v_1,\ldots,v_n\}$. The *spherical dimension* [16,17] of G, denoted by $\mathrm{sd}(G)$, is the smallest d for which there exists a set of unit vectors $\mathbf{v}^{(1)},\ldots,\mathbf{v}^{(n)}$ in $\mathbf{R}^d$ with the following property: there is a real number τ such that vertices v_i and v_j are adjacent in G if and only if the dot product of $\mathbf{v}^{(i)}$ and $\mathbf{v}^{(j)}$ is at least τ. If we choose τ to be a sufficiently small positive number, then the same construction in our proof of Theorem 1 shows that $\mathrm{sd}(G)\leqslant\theta(G)$, which is slightly stronger than our stated result, since it is known [16] that $\mathrm{sd}(G)-1\leqslant\mathrm{sph}(G)\leqslant\mathrm{sd}(G)$.

The proof of Theorem 1 makes scant use of the Euclidean distance function; a similar respresentation of a graph G as an intersection graph can be performed in $\mathbf{R}^{\theta(G)}$ equipped with other metrics.

Recall that the L_p-norm of the vector $\mathbf{x}=(x_1,\ldots,x_d)$ in $\mathbf{R}^d$ equals

$$\|\mathbf{x}\|_p=\begin{cases}(|x_1|^p+\cdots+|x_d|^p)^{1/p} & \text{if } 1\leqslant p<\infty,\\ \max\{|x_1|,\ldots,|x_d|\} & \text{if } p=\infty.\end{cases}$$

In $\mathbf{R}^d$ an *open p-ball* with center $\mathbf{x}_0$ and radius r is the set of all $\mathbf{x}$ such that $\|\mathbf{x}-\mathbf{x}_0\|_p<r$. Note that p-balls are not rotationally symmetric except in the Euclidean case $p=2$ (or $d=1$). We define the *p-sphericity* of the graph G to be the smallest dimension d for which there is a family of translates of an open p-ball in $\mathbf{R}^d$ that represents G as an intersection graph. We denote the p-sphericity of G by $\mathrm{sph}_p(G)$. Of course, $\mathrm{sph}_2(G)=\mathrm{sph}(G)$.

When $p<\infty$, the proof of Theorem 1 remains valid for the p-sphericity if we replace the exponent $\frac{1}{2}$ by $1/p$ throughout. Thus we have the following upper bound for the p-sphericity of a graph.

Proposition 3. *If G is a graph with positive edge clique cover number, then* $\mathrm{sph}_p(G)\leqslant\theta(G)$ *for* $1\leqslant p<\infty$.

4. Cubicity

If $p=\infty$, then a p-ball of radius r in $\mathbf{R}^d$ is a d-dimensional cube with edge length $2r$. (All cubes in our discussion are axis-aligned.) The parameter $\mathrm{sph}_\infty(G)$ is known as the *cubicity* of G and is denoted by $\mathrm{cub}(G)$. Roberts introduced cubicity in [19] and adopted the convention that the cubicity of a complete graph satisfies $\mathrm{cub}(K_n)=0$. A limit argument with $p \longrightarrow \infty$ shows that Proposition 3 also holds for $p=\infty$. Thus we have our first upper bound for the cubicity of a graph:

Theorem 4. *Let G be a graph with positive edge clique cover number $\theta(G)$. Then*

$$\text{cub}(G) \leqslant \theta(G).$$

Proof. Without loss of generality G has no isolated vertices. Let $\mathbf{x}$ be a point in an open cube with center $\mathbf{x}_0$ and radius r (i.e., edge length $2r$) in $\mathbf{R}^d$. If p is sufficiently large, then $\mathbf{x}$ is also in the open p-ball with the same center and radius. It follows that any representation of a graph G as an intersection graph of a family of translates of an open cube in $\mathbf{R}^d$ yields a representation of G as an intersection graph of a family of translates of an open p-ball in $\mathbf{R}^d$ for some sufficiently large p (depending on G). Now Proposition 3 implies that for this value of p we have $\text{cub}(G) \leqslant \text{sph}_p(G) \leqslant \theta(G)$. □

A *biclique* is a complete bipartite graph. An *edge biclique cover* of a graph G is a set of bicliques $\{B_1, \ldots, B_t\}$ that covers the edges of G. The *edge biclique cover number* $\eta(G)$ is the minimum number of bicliques in an edge biclique cover of G. See the survey [15] for a full treatment of edge biclique covers. We now use edge biclique covers to give another upper bound for the cubicity of a graph.

Theorem 5. *Let G be a graph whose complement has edge biclique cover number $\eta(\overline{G})$. Then*

$$\text{cub}(G) \leqslant \eta(\overline{G}).$$

Proof. First note that the inequality holds if G is a complete graph. Now suppose that G is not complete. We use a variant of a construction employed by Boyer et al. [1] for a different problem in intersection graph theory. Let $\{v_1, \ldots, v_n\}$ be the vertex set of G, and let $\{B_1, \ldots, B_{\overline{\eta}}\}$ be an edge biclique cover of $\overline{G}$ with cardinality $\overline{\eta} = \eta(\overline{G})$. Let V_k^+ and V_k^- be the vertex subsets of the biclique B_k $(k = 1, \ldots, \overline{\eta})$. Now define the components of the vector $\mathbf{y}^{(i)} = (y_1^{(i)}, \ldots, y_{\overline{\eta}}^{(i)})$ in $\mathbf{R}^{\overline{\eta}}$ by

$$y_k^{(i)} = \begin{cases} 1 & \text{if vertex } v_i \text{ is in the set } V_k^+, \\ -1 & \text{if vertex } v_i \text{ is in the set } V_k^-, \\ 0 & \text{if vertex } v_i \text{ is not in the biclique } B_k. \end{cases}$$

On the one hand, if v_i and v_j are adjacent in G, then there is no index k such that $\{y_k^{(i)}, y_k^{(j)}\} = \{1, -1\}$, and it follows that $\|\mathbf{y}^{(i)} - \mathbf{y}^{(j)}\|_\infty \leqslant 1$. On the other hand, if v_i and v_j are not adjacent in G, then there is an index k such that $\{y_k^{(i)}, y_k^{(j)}\} = \{1, -1\}$, and it follows that $\|\mathbf{y}^{(i)} - \mathbf{y}^{(j)}\|_\infty = 2$. We now see that the family of open cubes with centers $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}$ and edge lengths 2 represents G as an intersection graph in $\mathbf{R}^{\overline{\eta}}$. Therefore $\text{cub}(G) \leqslant \overline{\eta}$. □

We exhibit a family of graphs for which equality holds in Theorem 5.

Example 3. A formula of Roberts [19] gives the cubicity of a complete multipartite graph:

$$\text{cub}(K_{n_1,\ldots,n_q}) = \sum_{i=1}^{q} \lceil \log_2(n_i) \rceil. \tag{2}$$

Now it is not difficult to show that the edge biclique cover number of the complete graph K_n is $\eta(K_n) = \lceil \log_2(n) \rceil$. It follows that the edge biclique cover number of the complement of $K_{n_1,\ldots,n_q}$ is given by the sum in (2). Therefore $\text{cub}(G) = \eta(\overline{G})$ if G is a complete multipartite graph.

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