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K-HOMOLOGY OF UNIVERSAL SPACES AND LOCAL COHOMOLOGY OF THE REPRESENTATION RING

J. P. C. Greenlees

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0. PROLOGUE

In this note we calculate the K-homology of the classifying space BG of a finite group G by expressing it as the Grothendieck local cohomology of the representation ring $R(G)$ at the augmentation ideal. In symbols, we show

$$K_i(BG_+) \cong H^i_J(R(G)) \tag{0.0}$$

for $i = 0, 1$ where $J = \ker(R(G) \to \mathbb{Z})$ is the augmentation ideal. This illuminates various calculations of Wilson [22] and Knapp [17]. This result is a special case of the more general (5.2), which calculates the equivariant K-homology of other universal spaces in precisely analogous terms.

In fact G. Wilson deduces a formula for $K_*(BG_+)$ from the Atiyah–Segal completion theorem [3] and Atiyah's universal coefficient theorem for K-theory, and (0.0) is easily deduced from [22] (1.2). There is also a more recent approach to (0.0) via cohomology in [12]. By contrast, our approach is to prove the homological theorem (0.0) directly and to deduce the Atiyah–Segal theorem from it. Accordingly, we obtain a new proof of the Atiyah–Segal theorem which uses little more than equivariant Bott periodicity (see also [1]). In addition, our explicit recognition of local cohomology places many powerful algebraic techniques at our disposal. It also provides analogous statements for other theories although (for example) the statement for stable homotopy is false whilst its cohomological counterpart (i.e. the Segal conjecture) is true.

1. STRATEGY

We recall that $K_*(BG_+) \cong K^G_*(EG_+)$, and we prefer to work with equivariant K-theory [21]. There is an easily constructed G-spectrum $H_I(S^0)$ (called $M(I)$ in [10]) associated to the augmentation ideal I of the Burnside ring $A(G)$ and a map

$$c: EG_+ \to H_I(S^0). \tag{1.1}$$

It is immediate from the construction and the dimensionality of $R(G)$ that $K^G_*H_I(S^0) \cong H^*_I(R(G))$, which is the same as $H^*_J(R(G))$ since $IR(G)$ and J have the same radical; our strategy is to show (1.1) induces an isomorphism of $K^G_*(\cdot)$. We do this by showing that the mapping cone of c is K^G_*-acyclic. This in turn follows by induction on the group order using equivariant Bott periodicity [2, 21] for the inductive step.

The rest of the paper is laid out as follows. In Section 2 we recall some facts about local cohomology that we require and in Section 3 we define the spectrum $H_I(S^0)$, pointing out that the construction gives a model for various universal spaces $E\mathscr{F}_+$ using representation theory. In Section 4 we provide a spectral sequence for calculating the equivariant F-homology of $H_I(S^0)$ and observe that in the low Krull dimension case it collapses to a short exact sequence. In Section 5 we prove our main theorem, which calculates $K_*^G(E\mathscr{F}_+)$ for any family $\mathscr{F}$ of subgroups if G is finite; we then note that the Atiyah–Segal completion theorem is a corollary of (5.1). Readers only interested in the proof of our main result can stop reading there. In Section 6 we give a bivariant result combining the Atiyah–Segal theorem and (0.0) to describe K maps from BG_+ to BH_+, and in Section 7 we show how to calculate the local cohomology of the representation ring, doing so completely in certain cases, thus showing (0.0) gives explicit results. Finally, in Section 8 we compare the method of Section 5 to that of representation theoretic models from Section 3. We add an appendix on compact Lie groups of positive dimension where we give a method for proving the general analogues of (0.0) and (5.2); however it depends on conjectural coherence properties of the equivariant K spectrum.

Unless the contrary is explicitly stated G may be any compact Lie group, and we work throughout in a stable equivariant homotopy category such as that of Lewis–May [16].

2. LOCAL COHOMOLOGY

In this section we recall some commutative algebra due to Grothendieck in a form suitable for our applications ([14], [15]). Specifically our rings may be graded over $\mathbb{Z}$ or $RO(G)$, and the elements we consider will be homogeneous but not necessarily of degree zero.

Suppose A is a commutative ring with homogeneous elements $\alpha_1, \dots, \alpha_d$.

Definition (2.1).

(a) We define the Koszul complex $K^\bullet(\alpha)$ to be the cochain complex $A \xrightarrow{\alpha} A$ nonzero in codegrees 0 and 1. For a sequence we define

$$K^\bullet(\boldsymbol{\alpha}) := K^\bullet(\alpha_1, \alpha_2, \dots, \alpha_d) := K^\bullet(\alpha_1) \otimes_A K^\bullet(\alpha_2) \otimes_A \dots \otimes_A K^\bullet(\alpha_d).$$

(b) The stabilized Koszul complex $K^\bullet(\alpha^\infty)$ is defined as the cochain complex $A \to A[\frac{1}{\alpha}]$ nonzero in codegrees 0 and 1. For a sequence we take

$$K^\bullet(\boldsymbol{\alpha}^\infty) := K^\bullet(\alpha_1^\infty, \dots, \alpha_d^\infty) := K^\bullet(\alpha_1^\infty) \otimes_A \dots \otimes_A K^\bullet(\alpha_d^\infty)$$

Remark (2.2). We note that $K^\bullet(\alpha^\infty) = \varinjlim_k K^\bullet(\alpha^k)$ where the limit is taken over diagrams

$$\begin{array}{ccc} A & \xrightarrow{\alpha^k} & A \\ \downarrow 1 & & \downarrow \alpha \\ A & \xrightarrow{\alpha^{k+1}} & A \end{array}$$

and similarly for sequences.

LEMMA (2.3). *Up to quasi-isomorphism $K^\bullet(\boldsymbol{\alpha}^\infty)$ depends only on the ideal $I = (\alpha_1, \alpha_2, \dots, \alpha_d)$ in A.*

Proof. If $I = (\alpha_1, \dots, \alpha_d) = (\beta_1, \dots, \beta_e)$ we have comparison maps $K^\bullet(\boldsymbol{\alpha}^\infty) \to K^\bullet(\boldsymbol{\alpha}^\infty, \boldsymbol{\beta}^\infty) \leftarrow K^\bullet(\boldsymbol{\beta}^\infty)$ so it suffices to observe that order of generators is unimportant and that if $\beta \in (\alpha_1, \dots, \alpha_d)$ then

$$K^\bullet(\boldsymbol{\alpha}^\infty) \to K^\bullet(\beta^\infty) \otimes K^\bullet(\boldsymbol{\alpha}^\infty)$$

is a homology isomorphism.

For the latter in turn it suffices to show that $A[\frac{1}{\beta}] \otimes K^{\bullet}(\boldsymbol{\alpha}^k)$ is acyclic for each finite k. Since homology commutes with direct limits it suffices in turn to show that a suitable power of β is zero on $H^*(K^{\bullet}(\boldsymbol{\alpha}^k))$. Indeed if $\beta = \sum_j b_j \alpha_j$, then for $l \geqslant 2kd$ β^l is divisible by α_i^{2k} for some i. Now $K^{\bullet}(\boldsymbol{\alpha}^k) = K^{\bullet}(\alpha_i^k) \otimes L^*$ where $L^{\bullet} = K^{\bullet}(\alpha_1^k, \ldots, \hat{\alpha}_i^k, \ldots, \alpha_d^k)$ and so $K^{\bullet}(\boldsymbol{\alpha}^k)$ is the mapping cone of $L^{\bullet} \xrightarrow{\alpha_i^k} L^{\bullet}$. Therefore by the long exact homology sequence β^l acts as zero as required. □

Accordingly for any finitely generated ideal $I = (\alpha_1, \ldots, \alpha_d)$ and any A-module M we may define the Grothendieck local cohomology groups by

$$H_I^*(A; M) = H^*(K^{\bullet}(\boldsymbol{\alpha}^{\infty}) \otimes_A M). \tag{2.4}$$

If the ring A is understood we abbreviate this to $H_I^*(M)$.

Of course if A is an ungraded commutative ring there are more obviously invariant definitions, and we recall the relevant result.

THEOREM (Grothendieck) (2.5). *If A is an ungraded Noetherian ring then*

(i) *$H_I^n(A; M) = R^n\Gamma_{V(I)}(\mathrm{Spec} A; \tilde{M})$ where $\Gamma_Y(X; \mathscr{F})$ is the group of global sections of the sheaf $\mathscr{F}$ with support in $Y \subseteq X$.*

(ii) *There is a natural isomorphism*

$$H_I^n(A; M) = \varinjlim_k \mathrm{Ext}_A^n(A/I^k, M). \quad \square$$

From (i), since $R^n\Gamma_Y(X; \mathscr{F})$ can be calculated using flabby resolution of $\mathscr{F}$ we have the usual consequence [13]:

COROLLARY (2.6). *If A is Noetherian and of Krull dimension d then*

$$H_I^n(A; M) = 0 \text{ for } n > d. \quad \square$$

Furthermore $H_I^n(A; M)$ only depends on the radical of I; this fact and (2.6) will be critical for us.

3. THE CONSTRUCTION OF $H_I(S^0)$ AND CERTAIN UNIVERSAL SPACES.

We recall that a family $\mathscr{F}$ is a collection of subgroups of G closed under passage to conjugates and subgroups, and that a universal space $E\mathscr{F}$ is characterised by the condition that $(E\mathscr{F})^H$ is contractible if $H \in \mathscr{F}$ and empty otherwise. In particular EG is the universal space for the family $\{1\}$.

Throughout this section we suppose given a sequence $\alpha_1, \alpha_2, \ldots, \alpha_d$ of homogeneous elements of $\pi_{\bullet}^G = [S^0, S^0]_{\bullet}^G$, where $\bullet$ denotes grading over $RO(G)$. We let $I = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ be the ideal they generate.

Definition (3.1). We define the local cohomology spectra by taking S^{-1}/α^{∞} to be the fibre of $S^0 \to S^0[\frac{1}{\alpha}]$ and then let

$$H_I(S^0) := S^{-1}/\alpha_1^{\infty} \wedge S^{-1}/\alpha_2^{\infty} \wedge \ldots \wedge S^{-1}/\alpha_d^{\infty}.$$

Remark (3.2). Just as in (2.3) it can be shown that the homotopy type of $H_I(S^0)$ does not depend on the particular generators used.

Examples (3.3).

(i) If $\alpha_1, \alpha_2, \ldots, \alpha_d \in \pi_0^G \cong A(G)$ then we see that S^{-1}/α_i^∞ has cells only in dimensions -1 and 0, hence $H_I(S^0)$ has cells only between dimensions $-d$ and 0 (see also (4.4)).

(ii) On the other hand for any representation V of G we may consider the Euler class $e(V): S^0 \to S^V$ as an element of $[S^0, S^0]^G_{-V}$. In this case we find $S^0[\frac{1}{e(V)}] = S^{\infty V}$ and so $S^{-1}/e(V)^\infty$ is a universal space for the family $\mathscr{F}(V) = \{H \subseteq G \mid V^H \neq 0\}$. Accordingly if we take $\alpha_i = e(V_i)$ for representations $V_1, \ldots, V_d$ we find that

$$H_{e(\mathrm{V})}(S^0) \simeq E\mathscr{F}(\mathrm{V})_+$$

where

$$e(\mathrm{V}) = (e(V_1), \ldots, e(V_d)) \text{ and } \mathscr{F}(\mathrm{V}) = \{H \subseteq G \mid V_i^H \neq 0 \text{ for all } i\}.$$

Pre-echoes of this construction and a degenerate form of (4.1) below appear first in [7] §4. See also [8] proof of (5.10), [9] §4 and [12].

Remark (3.4). It seems very interesting to decide which families of subgroups are of form $\mathscr{F}(\mathrm{V})$. In particular we may ask for which groups G there are representations so that $\mathscr{F}(\mathrm{V}) = \{1\}$. U. Ray [19] has shown that the only possible nonabelian composition factors in such a group are the alternating groups A_5 and A_6, verifying a conjecture of J. G. Thompson. We shall content ourselves with the following easy observations.

LEMMA (3.5). *If G nilpotent or supersoluble then if $V_1, \ldots, V_d$ is a list of all simple representations $\mathscr{F}(\mathrm{V}) = \{1\}$ so that $EG_+ \simeq H_{e(\mathrm{V})}(S^0)$.*

Proof. Nilpotent or supersoluble groups can be formed by iterated extensions with kernel of prime order. Accordingly we may argue by induction on the group order, supposing G lies in an extension

$$1 \to C_p \to G \to Q \to 1.$$

Now if $x \in G$ is of prime order we either have $x \notin C_p$, in which case there is a representation V of Q with $V^x = 0$, or otherwise $x \in C_p$, and we may take $V = \mathrm{ind}_{C_p}^G \eta$ for a nontrivial simple representation η of C_p. Accordingly if $H \in \mathscr{F}(\mathrm{V})$ then $x \notin H$. □

Now we briefly consider comparison maps. Firstly if $I \subseteq J$ we may suppose $J = (\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_e)$ and smash the maps $S^{-1}/\beta^\infty \to S^0$ to obtain a map

$$H_J(S^0) \to H_I(S^0) \tag{3.6}$$

over $S^0 = H_{(0)}(S^0)$.

Another particular case is of interest. If $\mathscr{F}$ is any family of subgroups and if

$$I = I\mathscr{F} = \bigcap_{H \in \mathscr{F}} \ker\{A(G) \to A(H)\}$$

then $I\downarrow_H = (0)$ for all $H \in \mathscr{F}$ so that $H_{I\mathscr{F}}(S^0) \to S^0$ is an H-equivalence whenever $H \in \mathscr{F}$. Accordingly there is a unique map

$$c: E\mathscr{F}_+ \to H_{I\mathscr{F}}(S^0) \tag{3.7}$$

over S^0.

4. HOMOLOGY OF $H_I(S^0)$

We now suppose given an equivariant homology theory $F^G_*(\cdot)$ and give a spectral sequence for calculating $F^G_*(H_I(S^0))$. Thanks to Grothendieck vanishing (2.6), this collapses to give a short exact sequence in unexpectedly many cases.

THEOREM (4.1). *There is a spectral sequence*

$$E^2_{s,t} = H_I^{-s}(F^G_\bullet X)_t \Rightarrow F^G_{s+t}(X \wedge H_I S^0)$$

with differential $d^r\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$ *where* $F^G_\bullet X$ *denotes the* $RO(G)$*-graded form of the F-homology of* X.

Remark. (i) It is of course inconvenient to have to work with $RO(G)$-graded groups. In case I is generated in integer degrees or in case of suitable periodicity we can eliminate this difficulty.

(ii) There is an analogous spectral sequence for $F^*_G(X \wedge H_I(S^0))$ whose E_2 is expressed in terms of the local homology groups of [11]. Under Noetherian hypotheses this collapses to show the target groups are the I-completions of $F^\bullet_G(X)$.

Proof. Replacing F by $F \wedge X$ we may assume $X = S^0$.

We take $S^{-1}/\alpha^\infty = S^{-1}[\frac{1}{\alpha}] \cup CS^{-1}$ filtered by taking $S^{-1}[\frac{1}{\alpha}]$ as filtration -1 and S^{-1}/α^∞ as filtration 0; we give $H_I(S^0)$ the product filtration. The subquotient in filtration $-s$ is then the wedge of all products $A_1 \wedge A_2 \wedge \ldots \wedge A_d$ with $A_i = S^{-1}[\frac{1}{\alpha_i}]$ for s terms and $A_i = S^0$ for the rest. Since $F^G_\bullet(X \wedge S^0[\frac{1}{\alpha}]) = F^G_\bullet(X)[\frac{1}{\alpha}]$ it follows that the homology spectral sequence of the filtered spectrum has $E^1_{s,\bullet} = K^{-s}(\alpha^\infty) \otimes F^G_\bullet$, and the identification of $E^2_{s,t}$ is immediate from our definition of local cohomology. □

We remark that if $H^r_I(F^G_\bullet X) = 0$ for $r \geqslant 2$ then the spectral sequence collapses to give a short exact sequence. Two cases of interest are worth particular mention.

COROLLARY (4.2).

(a) *If I is a finitely generated ideal of $\pi^G_0 \cong A(G)$ then we obtain the short exact sequence*

$$0 \to H^1_I(F^G_{n+1}X) \to F^G_n(X \wedge H_I S^0) \to H^0_I(F^G_n X) \to 0$$

(b) *If G is a finite group and $I \subseteq \pi^G_\bullet$ is generated in degrees from $R(G)$ we obtain a short exact sequence*

$$0 \to H^1_I(K^G_\bullet X)_{n+1} \to K^G_n(X \wedge H_I S^0) \to H^0_I(K^G_\bullet X)_n \to 0.$$

Proof. (a) It is well known ([5], [6]), that the Burnside ring, $A(G)$, is of Krull dimension 1, and by Grothendieck vanishing when $A(G)$ is Noetherian, or quite generally by [10] (6.1), the higher local cohomology groups vanish.

(b) Since $\pi^G_\bullet$ acts on $K^G_\bullet X$ via the ring map $\pi^G_\bullet \to K^G_\bullet$ it is clear that $I = (\alpha_1, \ldots, \alpha_d)$ may be replaced in the E^2-term by the image ideal $\bar{I} = (\bar{\alpha}_1, \ldots, \bar{\alpha}_d)$ in $K^G_\bullet$. By Bott periodicity we may suppose $\bar{I}$ is generated in degree 0, and Segal has shown that $K^G_0 = R(G)$ is of Krull dimension 1 when G is finite ([20] (3.7)) the result follows by Grothendieck vanishing. □

COROLLARY (4.3). *If $I \subseteq \pi^G_0 \cong A(G)$ then $H_I(S^0)$ is (-2)-connected.* □

Remark (4.4). With a little more work one can show that in this case $H_I(S^0)$ may be constructed from *G-fixed* cells of dimensions -1 and 0. The analogous result for finite complexes involves the fact that $A(G)$ is one dimensional and a certain amount of stability theory for projective modules. We can get away here by using the Eilenberg swindle (i.e. if P is any projective there is a non-finitely generated free module F so that $P \oplus F \cong F$). □

5. K-FINITE DIMENSIONALITY OF UNIVERSAL SPACES

In this section G is finite, and we prove our main theorem, which concerns the map of (3.7).

THEOREM (5.1). *The map*

$$c: E\mathcal{F}_+ \to H_{I\mathcal{F}}(S^0)$$

induces an isomorphism in equivariant K-homology.

Because $H_{I\mathcal{F}}(S^0)$ is finite dimensional we may regard (5.1) as asserting the finite dimensionality of universal spaces to the eyes of K-theory. More practically, since the K-homology of $H_{I\mathcal{F}}(S^0)$ is calculated by (4.2)(a), and since $I\mathcal{F} \cdot R(G)$ has the same radical as $J\mathcal{F} = \bigcap_{H \in \mathcal{F}} \ker\{R(G) \to R(H)\}$ by (4.5) of [10], we immediately deduce the desired calculation, of which (0.0) is the special case $\mathcal{F} = \{1\}$.

COROLLARY (5.2). *The K-homology of the universal space $E\mathcal{F}_+$ is given by*

$$K_i^G(E\mathcal{F}_+) \cong H^i_{J\mathcal{F}}(R(G))$$

for $i = 0$ or 1. □

Proof of (5.1). The statement is trivial if $G \in \mathcal{F}$, so we may suppose $\mathcal{F}$ is a proper family.

To prove that c is a K-homology isomorphism it is equivalent to show that the K-homology of its mapping cone $C(c)$ is zero.

LEMMA (5.3). *The cofibre of c is equivalent to $H_{I\mathcal{F}}(S^0) \wedge \tilde{E}\mathcal{F}$.*

Proof. We take the map c and smash it with $H_{I\mathcal{F}}(S^0)$; since $H_{I\mathcal{F}}(S^0)$ is $\mathcal{F}$-equivalent to S^0 we find $E\mathcal{F}_+ \simeq E\mathcal{F}_+ \wedge H_{I\mathcal{F}}(S^0)$. The cofibre of c is therefore obtained by smashing the cofibre $\tilde{E}\mathcal{F}$ of $E\mathcal{F}_+ \to S^0$ with $H_{I\mathcal{F}}(S^0)$. □

We shall prove by induction on the group order that $C(c)$ is K-acyclic, starting with the trivial group where the result is obvious since the cofibre is contractible. Specifically we note that for every proper subgroup H of G, we have the family $\mathcal{F}\downarrow_H = \{K \subseteq H \mid K \in \mathcal{F}\}$ of subgroups of H, and an H-equivalence $(E\mathcal{F}_+)\downarrow_H \simeq E(\mathcal{F}\downarrow_H)_+$. Next, because $I(\mathcal{F}\downarrow_H)$ and $(I\mathcal{F})\downarrow_H$ have the same radical [5] the H-spectra $H_{I\mathcal{F}}(S^0)\downarrow_H = H_{(I\mathcal{F})\downarrow_H}(S^0)$ and $H_{I(\mathcal{F}\downarrow_H)}(S^0)$ are equivalent. Accordingly $C(c)\downarrow_H$ is equivalent to its H-equivariant analogue for the family $\mathcal{F}\downarrow_H$. Our inductive assumption is thus that for all proper sub-groups H of G the analogue of (5.1) holds for the family $\mathcal{F}\downarrow_H$, and so we have $K_*^H(C(c)) = 0$.

The next reduction was brought to prominence by Carlsson [4].

LEMMA (5.4). *Under the above inductive hypothesis, $C(c)$ is K_*^G-acyclic iff $H_{I\mathcal{F}}(S^0) \wedge S^{\infty V}$ is K^G-acyclic where V is the complex reduced regular representation.*

Proof. Indeed we have the cofibre sequence

$$H_{I\mathscr{F}}(S^0) \wedge \tilde{E}\mathscr{F} \to H_{I\mathscr{F}}(S^0) \wedge S^{\infty V} \to H_{I\mathscr{F}}(S^0) \wedge \tilde{E}\mathscr{F} \wedge (S^{\infty V}/S^0)$$

because $\tilde{E}\mathscr{F} \wedge S^{\infty V} \simeq S^{\infty V}$. By the induction hypothesis, the spectrum

$$H_{I\mathscr{F}}(S^0) \wedge \tilde{E}\mathscr{F} \wedge G/H_+ \simeq G_+ \wedge_H (H_{I\mathscr{F}}(S^0) \wedge \tilde{E}\mathscr{F})$$

is K-acyclic, and hence by induction on the number of cells and passage to direct limits $H_{I\mathscr{F}}(S^0) \wedge \tilde{E}\mathscr{F} \wedge Z$ is K-acyclic for any spectrum Z formed from cells $G/H_+ \wedge S^n$ for proper subgroups H. This applies in particular to $Z = S^{\infty V}/S^0$. □

Now, by equivariant Bott periodicity, if W is any complex representation, multiplication by the element $e(W)$ can be shifted into degree zero where it is multiplication by $\lambda(W)$, the alternating sum of exterior powers of W, in the sense that the diagram

$$\begin{array}{ccc} K_n^G(X) & \xrightarrow{e(W)\cdot} & K_n^G(X \wedge S^W) \\ {\scriptstyle \lambda(W)}\downarrow & & \downarrow\cong \\ K_n^G(X) & \overset{\text{Bott}}{\cong} & K_{n-W}^G(X) \end{array}$$

commutes. Hence in particular

$$K_n^G(X \wedge S^{\infty V}) \cong K_n^G(X)\left[\frac{1}{\lambda(V)}\right].$$

Applying this to the cofibre of c we find that the spectral sequence of (4.1) has E^2 term $H_{I\mathscr{F}}^*(R(G))[1/\lambda(V)]$. However, by invariance of local cohomology under base change we may replace $I\mathscr{F}$ by the ideal $I\mathscr{F} \cdot R(G)$, which has the same radical as $J\mathscr{F}$ by (4.5) of [10]. Thus we may replace $I\mathscr{F}$ by $J\mathscr{F}$ in the description of the E^2 term. Finally, $\lambda(V) \in J\mathscr{F}$ since $V^H \neq 0$ if H is proper. Thus the local cohomology groups are $\lambda(V)$-power torsion and

$$E^2_{*,*} = H_{J\mathscr{F}}^*(R(G))[1/\lambda(V)] = 0.$$

This completes the proof of (5.1). □

We have a precise analogue for real K-theory. The proof is the same except that it may be necessary to replace V by a multiple to get periodicity. As previously remarked, the coefficient groups KO_*^G are no longer concentrated in even degrees, so the spectral sequence of (4.1) only gives a short exact sequence.

THEOREM (5.5). *We have short exact sequences*

$$0 \to H_{J\mathscr{F}}^1(KO_{n+1}^G) \to KO_n^G(E\mathscr{F}_+) \to H_{J\mathscr{F}}^0(KO_n^G) \to 0$$

and these are split as sequences of abelian groups since the H^1 kernels are divisible. □

Finally we take the Atiyah–Segal theorem [3] as a corollary. Indeed, the equivariant K-spectrum is a ring spectrum, so it is a formality that since the cofibre of c is acyclic for homology it is acyclic for cohomology. Hence the map c also induces a K-cohomology isomorphism. Therefore by the cohomological analogue of (4.1) (which is [10] (3.3)) we recover the Atiyah–Segal completion theorem.

COROLLARY (5.6). *For any finite group G and any family $\mathscr{F}$ the map $c\colon E\mathscr{F} \to H_{I\mathscr{F}}(S^0)$ induces a K-cohomology isomorphism; hence*

$$K_G^0 E\mathscr{F}_+ \cong R(G)^{\wedge}_{I\mathscr{F}}$$

and $K_G^1 E\mathscr{F}_+ = 0$. □

The various extensions and generalisations of this [1] follow in the usual way.

6. THE BIVARIANT SYNTHESIS

We continue to assume G is finite in this section and we combine the Atiyah–Segal theorem with the local cohomology theorem as follows, using the notation $K^*(X; Y) = [X, Y \wedge K]$ for K-maps from X to Y. I am grateful to M. C. Crabb for suggesting this.

THEOREM (6.1). *If G and H are finite groups then there is a homology spectral sequence for $K^*(BG_+; BH_+)$ and there is a short exact sequence*

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(H_{I(G)}^{-s-1}(R(G)), H^t_{I(H)}(R(H))) \to E^2_{s,t} \to \mathrm{Hom}_{\mathbb{Z}}((H_{I(G)}^{-s}(R(G)), H^t_{I(H)}(R(H))) \to 0$$

where t (but not s) is interpreted mod 2.

Proof. We begin by making the problem equivariant since

$$K^*(BG_+; BH_+) = K_G^*(EG_+; BH_+) \qquad ([16]\ \mathrm{II}.8.4).$$

By (5.1) we may replace EG_+ by $H_{I(G)}(S^0)$, the point being that the latter is constructed from G-fixed cells S^n, unlike EG_+. We now simply filter $H_{I(G)}(S^0)$ by skeleta and apply $K_G^*(\cdot; BH_+)$. The natural map to $\mathrm{Hom}_{K_*^G}(K_*^G, K_G^* BH_+) = \mathrm{Hom}_{R(G)}(R(G), H^*_{I(H)}(R(H)) \otimes_{\mathbb{Z}} R(G))$ is isomorphic on E^1. Since $R(G) \cong \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Z})$([18](2.5)) this is $\mathrm{Hom}_{\mathbb{Z}}(R(G), H^*_{I(H)}(R(H)))$ and we obtain

$$E^2_{s,t} = H_s(\mathrm{Hom}_{\mathbb{Z}}(C_*, H^t_{I(H)}(R(H))))$$

where $0 \to C_{-d} \to C_{d-1} \to \dots \to C_0 \to 0$ is a complex of free $R(G)$-modules with $H_s(C_*) = H_{I(G)}^{-s}(R(G))$. The result follows from the algebraic universal coefficient theorem. □

COROLLARY (6.2). (a) $K^0(BG_+; BH_+) = \mathbb{Z} \oplus \bar{K}^0$ *and there is a short exact sequence*

$$0 \to \mathrm{Ext}_{\mathbb{Z}}((H^1_{I(G)}(R(G)), \mathbb{Z}) \to \bar{K}^0 \to \mathrm{Hom}_{\mathbb{Z}}(H^1_{I(G)}(R(G)), H^1_{I(H)}(R(H))) \to 0$$

and $\mathrm{Ext}_{\mathbb{Z}}(H^1_{I(G)}(R(G)), \mathbb{Z}) \cong I(G)^{\wedge}_{I(G)}$.

(b) $K^1(BG_+; BH_+) \cong H^1_{I(H)}(R(H))$.

Proof. We shall see in Section 7 that $H^0_I(R(G)) \cong \mathbb{Z}$ and that $H^1_I(R(G))$ is a sum of quasicyclic groups $\mathbb{Z}/p^\infty$ for various primes p. From this the collapse of the spectral sequence follows from Hom and Ext calculations. □

Remark (6.3). Crabb has pointed out that the original proof of the Atiyah–Segal completion theorem implies that the natural map

$$K^0(BG_+; BH_+) \xrightarrow{\cong} \mathrm{Hom}_{cts}(R(H)^{\wedge}_{I(H)}, R(G)^{\wedge}_{I(G)})$$

is an isomorphism, and has suggested that $K^1(BG_+; BH_+)$ may be interpreted as a continuous Ext group. From the relation of the topology on $R(G)$ to local cohomology (see also [11]) we see that Crabb's observation on K^0 can also be deduced from (6.2).

Remark (6.4). The analogue of these results for families $\mathscr{F}$ and $\mathscr{G}$ of G are immediate from (5.1):

$$K_G^*(E\mathscr{F}_+; E\mathscr{G}_+) = K_G^*(H_{I\mathscr{F}}(S^0); H_{I\mathscr{G}}(S^0)).$$

Combining (3.3) of [10] with (5.2) we obtain the short exact sequence

$$0 \to L_1^{I\mathscr{F}} H^1_{I\mathscr{G}}(R(G)) \to K^0_G(E\mathscr{F}_+; E\mathscr{G}_+) \to L_0^{I\mathscr{F}} H^0_{I\mathscr{G}}(R(G)) \to 0,$$

where $L_i^{I\mathscr{F}}$ is the ith left derived functor of $I\mathscr{F}$-adic completion, and similarly for $K^1_G(E\mathscr{F}_+ E\mathscr{G}_+)$ with the H^0 and H^1 transposed.

7. SOME CALCULATIONS OF $H^*_I(R(G))$

In this section we prove the analogue of the well known fact that if G is a finite p-group the augmentation ideal of $R(G)$ defines the p-adic topology on itself. Throughout this section $J(G)$ denotes the augmentation ideal, and G is finite.

PROPOSITION (7.1). *If G is a finite p-group then $H^0_{J(G)}(R(G)) \cong \mathbb{Z}$, generated by the regular representation and $H^1_{J(G)}(R(G)) \cong \bar{R}(G) \otimes \mathbb{Z}/p^\infty$ where $\bar{R}(G) = R(G)/\mathbb{Z}$.*

We first note that the calculation of $H^0_{J(G)}$ is straightforward. By definition it consists of characters χ for which $\chi(g) = 0$ if $g \neq e$. We therefore concentrate on H^1, where certain general methods are useful also for the Burnside ring.

For any ideal I, we define $\bar{R}$ by the short exact sequence

$$0 \to H^0_I(R(G)) \to R(G) \to \bar{R} \to 0. \tag{7.2}$$

It is clear from the definition that since every element of $H^0_I(M)$ is annihilated by a power of I that $H^1_I H^0_I(M) = 0$. Accordingly we see from (7.2) that $H^0_I \bar{R} = 0$, $H^1_I(R(G)) \cong H^1_I(\bar{R})$. We therefore concentrate on $\bar{R}$.

Now since $R(G)$ is additively free abelian one sees from the nature of $H^0_I(\cdot)$ that $\bar{R}$ is also. Accordingly, for any nonzero $n \in \mathbb{Z}$ we may consider

$$0 \to \bar{R} \to \bar{R}\left[\frac{1}{n}\right] \to \bar{R} \otimes \mathbb{Z}/n^\infty \to 0 \tag{7.3}$$

This gives rise to the exact sequence

$$0 \to H^0_I(\bar{R} \otimes \mathbb{Z}/n^\infty) \to H^1_I(\bar{R}) \to H^1_I\left(\bar{R}\left[\frac{1}{n}\right]\right)$$

so that if we find an n for which $H^1_I \bar{R}[\frac{1}{n}] = 0$, we have reduced the calculation of $H^1_I \bar{R}$ to an H^0_I calculation.

LEMMA (7.4). (a) *If $H^1_{(\alpha)}(M[\frac{1}{n}]) = 0$ for all $\alpha \in I$, $M \subseteq \bar{R}$ then $H^1_I(\bar{R}[\frac{1}{n}]) = 0$.*
(b) *For any I there exists a number, $n = n(I) \neq 0$ so that $H^1_I(\bar{R}[\frac{1}{n}]) = 0$.*

Proof. (a) Suppose $I = (J, \alpha)$ and argue by induction on the number of generators. We therefore suppose the result proved for J, noting that it is trivial if $J = 0$. Hence $H^1_I M = H^1_{(\alpha)}(H^0_J(M[\frac{1}{n}])) = H^1_{(\alpha)}(H^0_J(M)[\frac{1}{n}])$, and of course $M' = H^1_J(M)$ is a submodule of $\bar{R}$, so that $H^1_{(\alpha)}(M'[\frac{1}{n}]) = 0$ by hypothesis.

(b) By part (a) we may consider one element α of I at a time. Of course for $M \subseteq \bar{R}$ the cokernel of α: $M/H^0_{(\alpha)}M \to M/H^0_{(\alpha)}M$ is finite, and if we tensor with $\mathbb{Z}[\frac{1}{n}]$ where n annihilates it, we obtain an isomorphism. Hence $H^1_{(\alpha)}(M[\frac{1}{n}]) = 0$ □

Summary (7.5). For any ideal I there is an integer $n = n(I)$ so that $H^1_I(R(G)) = H^0_I(\bar{R} \otimes \mathbb{Z}/n^\infty)$ where $\bar{R} = R(G)/H^0_I(R(G))$.

Proof of (7.1). We now suppose G is a p-group and $I = J(G)$ is the augmentation ideal. It remains only to prove

(i) that we may take $n(I) = p$ and
(ii) that for each $\alpha \in I$, p divides some power of α.

Indeed it is clear from our construction of $n(I)$ that (i) follows from (ii), this is easily verified for cyclic groups and following in general by Artin's theorem that $|G|$ is a sum of characters induced from cyclic subgroups. □

8. CONCORDANCE BETWEEN THE METHODS OF SECTIONS 3 AND 5

We note that for families of the form $\mathscr{F}(\mathrm{V})$ for suitable complex representations $V_1, V_2, \ldots, V_d$ of G, we now have two methods for calculating equivariant K-theory of $E\mathscr{F}(\mathrm{V})_+$ if G is finite: one using c and one using (4.1) and Bott periodicity directly. We wish to remark here that these are compatible in the sense that the maps of (3.7)

$$E\mathscr{F}(\mathrm{V})_+ = H_{e(\mathrm{V})}(S^0) \xrightarrow{c} H_{I\mathscr{F}(\mathrm{V})}(S^0).$$

induce an isomorphism

$$H^*_{e(\mathrm{V})}(K^G_\bullet) \xrightarrow{\cong} H^*_{I\mathscr{F}(\mathrm{V})}(K^G_\bullet) \tag{8.2}$$

of E^2 terms of (4.1). By (3.5) this gives the p-group case of (0.0), from which (0.0) follows in general by an amusing transfer argument, however it does not seem possible to prove (5.2) for general families by this method.

First we consider the diagram

$$\begin{array}{ccc} & H_{e+I}(S^0) & \\ {\scriptstyle i_1}\swarrow & & \searrow{\scriptstyle i_2} \\ H_e(S^0) & \xrightarrow{\quad c \quad} & H_I(S^0) \end{array} \tag{8.3}$$

where the sloping sides are induced by inclusions of ideals as in (3.6), and where $e = e(\mathrm{V})$, $I = I\mathscr{F}(\mathrm{V})$.

LEMMA (8.4). *The diagram* (8.3) *commutes.*

Proof. Clearly all maps lie over S^0, so it remains only to observe that there is a unique map $H_{e+I}(S^0) \to H_I(S^0)$ over S^0. However $H_{e+I}(S^0) \simeq H_e(S^0) \wedge H_I(S^0)$ may be constructed from cells G/H_+ with H in $\mathscr{F}(\mathrm{V})$ since this is true for $E\mathscr{F}_+ \simeq H_e(S^0)$. Since the map $H_{I\mathscr{F}}(S^0) \to S^0$ is a H-equivalence for $H \in \mathscr{F}$ this gives the required conclusion. □

Now we take K-theory of the diagram, and use Bott periodicity to simplify the result. Indeed since $\pi^G_\bullet$ acts on $K^G_\bullet$ through a ring homomorphism $H^*_I(K^G_\bullet) = H^*_{\bar{I}}(K^G_\bullet)$ where $\bar{I} = (\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_d)$ is the ideal of $K^G_\bullet$ generated by the image of I. Next by Bott periodicity $\bar{e}(\mathrm{V}) = (\bar{e}(V_1), \ldots, \bar{e}(V_d))$ is generated by a sequence $\lambda(V_1), \ldots, \lambda(V_d)$ of elements of degree zero. Accordingly we may work entirely in degree zero.

Now we find $\lambda(V_i) \in J(\mathscr{F}(\mathrm{V}))$ since $\lambda(V)\downarrow_H = \lambda(V\downarrow_H)$ is H-null if $V^H \neq 0$. Accordingly we have the following inclusions of ideals of $R(G)$:

$$\lambda(\mathrm{V}) \subseteq J\mathscr{F} \supseteq \bar{I}(\mathscr{F}(\mathrm{V}))$$

and we may form the diagram

$$\begin{array}{ccccc} & & H^*_{\lambda+J}(R(G)) & & \\ & \swarrow^{\beta} & & \searrow^{=} & \\ & H^*_{\lambda+\bar{I}}(R(G)) & & & H^*_J(R(G)) \\ \swarrow^{i_{1*}} & & \searrow^{i_{2*}} & & \swarrow^{\alpha} \\ H^*_\lambda(R(G)) & & & H^*_{\bar{I}}(R(G)) & \\ \| & & & \| & \\ H^*_e(K^G_\bullet)_0 & & \xrightarrow{c_*} & H^*_{\bar{I}}(K^G_\bullet)_0 & \end{array} \tag{8.5}$$

in which all maps except c_* are induced by inclusions of ideals. The proof of (5.1) is completed by a lemma.

LEMMA (8.6). *The ideals λ, J and $\bar{I}$ have the same radical, and accordingly α, β and i_{1*} of* (8.5) *are isomorphisms.*

Proof of (8.6). The fact that $J(\mathscr{F})$ and $\bar{I}(\mathscr{F})$ have the same radical was (4.5) of [10]. Quite generally to show that ideals have the same radical it is enough to show they lie in the same primes, so that it remains to show that if a prime $\wp$ contains $\lambda(\mathrm{V})$ it contains $J\mathscr{F}(\mathrm{V})$. Segal has classified the primes of $R(G)$: each prime $\wp$ of $(R(G)$ is the pullback of a prime of $R(C)$ for a smallest cyclic subgroup C of G, unique up to conjugacy, called the support of $\wp$ [20].

Now we notice that the image of $\lambda(V)$ in $R(C)$ is $\lambda(V\downarrow C)$, and the image of $J(\mathscr{F})$ is contained in $J(\mathscr{F}\downarrow_C)$ (where $\mathscr{F}\downarrow_C=\{H\subseteq C\,|\,H\in\mathscr{F}\}$). It is easy to check that $\mathscr{F}(\mathrm{V})\downarrow_C=\mathscr{F}(\mathrm{V}\downarrow C)$, and so it suffices to assume G is cyclic of order n, with $R(G)=\mathbb{Z}[\eta]/(\eta^n-1)$, and that $\wp$ has support G itself.

Suppose then that $\lambda(\mathrm{V})\subseteq\wp$. We claim that in this case each V_i contains a trival summand, so that $\mathscr{F}(\mathrm{V})$ contains G and $J(\mathscr{F}(\mathrm{V}))=0$. Indeed if $V=\sum_{j=0}^{n-1}n_j\eta^j$ we have $\lambda(\mathrm{V})=\Pi_j\lambda(\eta^j)^{n_j}$ and since η^j is one dimensional we have $\lambda(\eta^j)=1-\eta^j$. Since $\wp$ is prime, if $\lambda(V)\in\wp$ we have either $n_0\neq 0$ (and $\lambda(\mathrm{V})=0$) or else the cyclotomic polynomial $\phi_k(\eta)\in\wp$ for some $0<k<n$. Since the latter implies $\wp$ has support C_k, that possibility is excluded. □

Remark (8.7). By commutativity of (8.3) and (4.1) we deduce quite generally that if the inclusions $e\subseteq e+I\supseteq I$ induce isomorphisms of the local cohomology of $F^G_\bullet$ (for instance if $F^G_\bullet$ is a module over a ring in which the images of these ideals have the same radical) then a local cohomology theorem holds for F homology and there is a short exact sequence

$$0\to H^1_{I\mathscr{F}(\mathrm{V})}(F^G_\bullet)_{n+1}\to F^G_n(E\mathscr{F}(\mathrm{V})_+)\to H^0_{I\mathscr{F}(\mathrm{V})}(F^G_\bullet)_n\to 0. \qquad □$$

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Department of Pure Mathematics
University of Sheffield
The Hicks Building
Sheffield S3 7RH
U.K.

APPENDIX: COMPACT LIE GROUPS OF POSITIVE DIMENSION

Consider the problem of calculating $K^G_*(E\mathscr{F}_+)$ for an arbitrary compact Lie group G and family $\mathscr{F}$. When G is finite $A(G)$ approximates $R(G)$ and we have been able to use $H_I(S^0) \wedge K$ as an approximation to a hypothetical spectrum "$H_J(K)$". If G is not finite $A(G)$ and $R(G)$ are not closely related, and we have had to fall back on the representation theoretic models, which (8.2) proves to be equivalent when we have a choice. Thus (4.1) still gives a spectral sequence

$$E_2^{s,t} = H^{-s}_{\lambda(V)}(R(G)) \Rightarrow K^G_{s+t}(E\mathscr{F}(V)_+)$$

and it can still be seen to collapse if the local cohomology is concentrated in two adjacent dimensions. In this appendix we outline what is really a more natural approach to the problem: we attempt to construct a spectrum $H_J(K)$ and use it directly. However we can only complete our construction of $H_J(K)$ for ideals generated by two elements: we note that this does at least cover the cases $SU(3)$ and $Sp(2)$ which we have not dealt with above. We adopt a fairly elementary approach to constructing $H_J(K)$, so even when this construction can be completed the calculations will only give $K^G_*(E\mathscr{F}_+)$ as an abelian group, and not as an $R(G)$ module. It will be clear that the obstacles to completing our construction in general are those of higher commuting homotopies of products of elements: if it were known that the equivariant K spectrum was a highly structured ring spectrum so that we might form

$L \wedge_K M$ when L and M are suitably structured module spectra our conclusions could be upgraded to isomorphisms and spectral sequences of $R(G)$ modules.

We shall begin by setting up constructions analogous to those of Section 3 in a more general context. Thus we work for the present with a commutative ring spectrum R, an ideal J in the coefficient ring R_* and a module spectrum M. We suppose $J = (\beta_1, \beta_2, \ldots, \beta_d)$ and we shall show how to construct a spectrum $H_J(M) = H_J(R; M)$ and a spectral sequence for calculating its homotopy groups. We should like to proceed by taking $H_{(\beta)}(R)$ to be the fibre of $R \to R[\frac{1}{\beta}]$ and then $H_J(M) := H_{(\beta_1)}(R) \wedge_R H_{(\beta_2)}(R) \wedge_R \ldots \wedge_R H_{(\beta_d)}(R) \wedge_R M$. However, as noted above, such a construction depends on higher coherence properties of R and M. Instead we must adopt a more indirect approach. If $d = 2$ and $J = (\alpha, \beta)$ we form a telescope diagram from the homotopy commutative squares

$$\begin{array}{ccc} M & \xrightarrow{\alpha^k} & M \\ \beta^k \downarrow & & \downarrow \beta^k \\ M & \xrightarrow{\alpha^k} & M. \end{array}$$

Thus we use identity maps in the top left, α in the top right, β in the bottom left and $\alpha\beta$ in the bottom right. Passing to telescopes we form a diagram

$$\begin{array}{ccc} M & \longrightarrow & M\left[\frac{1}{\alpha}\right] \\ \downarrow & & \downarrow \\ M\left[\frac{1}{\beta}\right] & \longrightarrow & M\left[\frac{1}{\alpha\beta}\right]. \end{array}$$

In general we consider the d-cube $\{0, 1\}^d$, and for each k we form a homotopy commutative d-cube by labelling all vertices M and all edges in which the ith coordinate changes $(\beta_i)^k$. Next we form a telescope of these diagrams by using the map $(\beta_1)^{\varepsilon_1}(\beta_2)^{\varepsilon_2} \ldots (\beta_d)^{\varepsilon_d}$ at the vertex $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d)$. The result is a homotopy commutative d-cube with $M[1/\{(\beta_1)^{\varepsilon_1}(\beta_2)^{\varepsilon_2} \ldots (\beta_d)^{\varepsilon_d}\}]$ at the vertex $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d)$ and the natural maps between them; we refer to it as the Koszul cube. Next, we want to replace the cube by an equivalent strictly commutative cube of cofibrations. If $d = 1$ this is a vacuous condition and if $d = 2$ we may use a commuting homotopy $\alpha\beta \simeq \beta\alpha$ and a natural mapping cylinder construction to do it. In general we can make an analogous construction provided there is a higher homotopy $M \wedge [0, 1]^d_+ \to M[1/\beta_1 \ldots \beta_d)]$ between the various products indexed by the increasing paths from $(0, \ldots, 0)$ to $(1, \ldots, 1)$. We intend to consider the construction of these higher homotopies in the case of equivariant K-theory elsewhere, but if $d \geqslant 3$ we continue the appendix under the assumption that the Koszul cube can be made into a strictly commutative diagram of cofibrations.

Under this assumption we define $H_J(M)$ as the desuspension of the mapping cone $M[1/(\beta_1 \ldots \beta_d)] \cup CM$ of the Koszul cube. This has an obvious filtration according to the distance from the vertex $(1, \ldots, 1)$, and the homotopy spectral sequence of this filtration gives us the means for calculating $\pi_*(H_J(M))$.

Remark (A.2). Just as in (4.1) we obtain a spectral sequence

$$E^2_{s,t} = H_J^{-s}(M_*)_t \Rightarrow \pi_{s+t}(H_J(M))$$

provided we may construct $H_J(M)$. □

We note that this gives us a means of establishing uniqueness up to homotopy of an $H_J(M)$ constructed in this way, provided the two Koszul cubes can be embedded in a larger one: in particular we may be able to prove independence of the generators of the ideal.

Now we specialise to the case of equivariant K-theory.

Remark (A.3). If J is generated in degrees from $R(G)$ it is generated in degree zero and hence the spectral sequence of (A.2) becomes

$$E^2_{s,t} = H_J^{-s}(R(G)); M_t) \Rightarrow \pi_{s+t}H_J(K; M)).$$

Just as in (3.7) we find

LEMMA (A.4). *There is a map*

$$c_K: E\mathcal{F}_+ \wedge K \to H_{J\mathcal{F}}(K)$$

over K. □

The analogue of (8.2) is now much easier by virtue of (A.3),.but as before it is only a prop for our confidence.

LEMMA (A.5). *The cofibre of* c_K *is equivalent to* $\tilde{E}\mathcal{F} \wedge H_{J\mathcal{F}}(K)$. □

We may now deduce the general case.

THEOREM (A.6). *Provided* $H_{J\mathcal{F}}(K)$ *exists, the map* $c_K: E\mathcal{F}_+ \wedge K \to H_{J\mathcal{F}}(K)$ *induces an isomorphism in homotopy.*

Proof. Precisely as before we may suppose $\mathcal{F}$ is proper and we argue by induction on the size of the group, this time using the fact that descending chains of subgroups of compact Lie groups terminate. We may therefore suppose the result proved for all proper subgroups of G, and apply Carlsson's reduction (5.4) to reduce to proving $S^{\infty V} \wedge H_{J\mathcal{F}}(K)$ is K_G^* acyclic where $S^{\infty V}$ is now a direct limit of spaces S^W where W runs through all finite dimensional G-modules with $W^G = 0$ (to make sense of this it is convenient to work with sub inner product spaces of a suitable infinite dimensional G-space in the usual way). The final step as before is to say that inverting $\lambda(W)$ for all representations W with $W^G = 0$ annihilates $H^*_{J\mathcal{F}}(R(G))$, which is the E^2 term of the spectral sequence (A.3). □

COROLLARY (A.7). *Provided* $H_{J\mathcal{F}}(K)$ *exists there is a spectral sequence*

$$E^2_{s,t} = H^{-s}_{J\mathcal{F}}(K^G_t) \Rightarrow K^G_*(E\mathcal{F}_+).$$ □

Finally we remark that for the ideals $J\mathcal{F}$ the spectral sequences of (A.7) and (8.8) can collapse for dimensional reasons despite the fact that $R(G)$ has dimension $1 + \text{rank}(G)$.

THEOREM (A.8). *The augmentation ideal* J *has depth* $\dim R(G) - 1$. □

Remark. More generally $depth_J M = \inf\{depth\, M_{\wp} \mid \wp \supseteq J\}$ and since $depth\, M_{\wp} \leqslant \dim A_{\wp}$ we see that $depth_J M \leqslant ht(J)$. Evidently we may therefore ensure that depth(J) is as small as we like by choice of J, even with J the form $J\mathcal{F}$, but it still often happens that the local cohomology is concentrated in two adjacent dimensions.