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Asymptotic Formulas for Generalized Elliptic-Type Integrals

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Abstract—Epstein-Hubbell [1] elliptic-type integrals occur in radiation field problems. The object of the present paper is to consider a unified form of different elliptic-type integrals, defined and developed recently by several authors. We obtain asymptotic formulas for the generalized elliptic-type integrals.

Keywords—Elliptic-type integrals, Hypergeometric functions, Asymptotic formulas.

1. INTRODUCTION

Elliptic integrals occur in a number of physical problems [2–5], and frequently in the form of multiple integrals. One of the integrals being performed, it leads to an integrand which itself involves elliptic integrals.

Epstein and Hubbell [1] have treated a family of elliptic-type integrals

$$\Omega_j(k) = \int_0^{\pi} \left(1 - k^2 \cos\theta\right)^{-j-1/2} d\theta, \qquad 0 \le k < 1, \quad j = 0, 1, 2, \dots \tag{1}$$

Certain problems dealing with the computation of the radiation field off axis from a uniform circular disc radiating according to an arbitrary angular distribution law [6,7], when treated with a Legendre polynomial expansion method, give rise to integrals of form (1). For $j = 0, 1$, formula (1) reduces to

$$\Omega_0(k) = \frac{\sqrt{2}\lambda}{k} K(\lambda), \tag{2}$$

and

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$$\Omega_1(k) = \frac{\sqrt{2}\lambda}{k(1-k^2)} E(\lambda), \tag{3}$$

where $\lambda^2 = 2k^2/(1+k^2)$, and $K(\lambda)$ and $E(\lambda)$ are the complete elliptic integrals of the first and second kinds, respectively [2].

Elliptic integral (1) has been generalized and studied by several authors: Kalla [8], Kalla, Conde and Hubbell [9], Kalla and Al-Saqabi [10–12], Srivastava *et al.* [13,14], and others [15–18].

Kalla *et al.* [9,19], and Glasser and Kalla [20] have developed a systematic study of the following family of elliptic-type integrals:

$$R_\mu(k,\alpha,\gamma) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\gamma-2\alpha-1}(\theta/2)}{(1-k^2\cos\theta)^{\mu+1/2}}\,d\theta, \qquad 0 \le k < 1, \quad \Re(\gamma) > \Re(\alpha) > 0, \quad \Re(\mu) > -\frac{1}{2}. \tag{4}$$

It can be easily verified that

$$R_j\left(k, \frac{1}{2}, 1\right) = \Omega_j(k), \tag{5}$$

and in terms of hypergeometric functions

$$R_\mu(k,\alpha,\gamma) = \frac{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}{\Gamma(\gamma)\,(1+k^2)^{\mu+1/2}}\, F\left(\alpha, \mu+\frac{1}{2}; \gamma; \frac{2k^2}{1+k^2}\right). \tag{6}$$

These authors have obtained a number of recurrence formulas, asymptotic expansion for $k^2 \to 1$, computer algorithms, integrals and other useful properties.

Recently, Srivastava and Siddiqi [13] have given a unified presentation of certain families of elliptic-type integrals in the following form:

$$\Lambda_{\lambda,\mu}^{(\alpha,\beta)}(\rho;k) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\,\sin^{2\beta-1}(\theta/2)}{(1-k^2\cos\theta)^{\mu+1/2}} \left[1-\rho\sin^2\left(\frac{\theta}{2}\right)\right]^{-\lambda} d\theta, \qquad 0 \le k < 1, \quad \Re(\alpha), \Re(\beta) > 0; \quad \lambda,\mu \in C; \quad |\rho| < 1. \tag{7}$$

By comparing (6) and (7) we have

$$\Lambda_{0,\mu}^{(\alpha,\gamma-\alpha)}(\rho;k) = \Lambda_{\lambda,\mu}^{(\alpha,\gamma-\alpha)}(0;k) = R_\mu(k,\alpha,\gamma). \tag{8}$$

Here we consider an another generalization of the elliptic-type integrals

$$\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;k) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\,\sin^{2\beta-1}(\theta/2)}{(1-k^2\cos\theta)^{\mu+1/2}} \left[1-\rho\sin^2\left(\frac{\theta}{2}\right)\right]^{-\lambda} \left[1+\delta\cos^2\left(\frac{\theta}{2}\right)\right]^{-\gamma} d\theta, \qquad 0 \le k < 1, \quad \Re(\alpha), \Re(\beta) > 0; \quad \lambda,\mu,\gamma \in C; \tag{9}$$

either $|\rho|$, $|\delta| < 1$ or ρ, (or δ) $\in C$, whenever $\lambda = -m$ (or $\gamma = -m$), $m \in N_0$. We observe that

$$\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,0;k) = \Lambda_{(\lambda,0,\mu)}^{(\alpha,\beta)}(\rho,\delta;k) = \Lambda_{\lambda,\mu}^{(\alpha,\beta)}(\rho;k), \tag{10}$$

where $\Lambda_{\lambda,\mu}^{(\alpha,\beta)}(\rho;k)$ is defined by (7), and contains other families of elliptic-type integrals as its special cases.

In this paper, first we express our generalized elliptic-type integral (9) in terms of the Lauricella's hypergeometric function of three variables $F_D^{(3)}$ and then obtain its asymptotic expansion as $k^2 \to 1$. Corresponding special cases for $R_\mu(k,\alpha,\gamma)$ and $\Omega_j(k)$ are also considered.

2. ASYMPTOTIC EXPANSION FOR $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k)$

From definition (9) we have

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k) = (1+\delta)^{-\gamma}\left(1-k^2\right)^{-\mu-1/2} \int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1}(1-\rho\omega)^{-\lambda}\left(1-\frac{\delta\omega}{1+\delta}\right)^{-\gamma}\left(1-\frac{2k^2\omega}{k^2-1}\right)^{-\mu-1/2} d\omega. \tag{11}$$

Comparing the integral in (11) with the integral representation of the Lauricella's hypergeometric function of three variables $F_D^{(3)}$ (see [21]), we can express our generalized elliptic-type integral (9) in terms of the Lauricella's hypergeometric function $F_D^{(3)}$ as follows:

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}(1+\delta)^{-\gamma}\left(1-k^2\right)^{-\mu-1/2} F_D^{(3)}\left(\beta,\lambda,\gamma,\mu+\frac{1}{2};\alpha+\beta;\rho;\frac{\delta}{1+\delta},\frac{2k^2}{k^2-1}\right). \tag{12}$$

To obtain asymptotic expansion of $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k)$ as $k^2 \to 1$, we express the Lauricella's hypergeometric function $F_D^{(3)}$ in (12) as a double series of the Gauss hypergeometric functions

$$F_D^{(3)}\left(\beta,\lambda,\gamma,\mu+\frac{1}{2};\alpha+\beta;\rho;\frac{\delta}{1+\delta},\frac{2k^2}{k^2-1}\right) = \sum_{m,n=0}^{\infty}\frac{(\beta)_{m+n}(\lambda)_m(\gamma)_n}{(\alpha+\beta)_{m+n}m!n!}\rho^m\left(\frac{\delta}{1+\delta}\right)^n {}_2F_1\left(\beta+m+n,\mu+\frac{1}{2};\alpha+\beta+m+n;\frac{2k^2}{k^2-1}\right). \tag{13}$$

Using the analytic continuation formula 15.3.7 [2] for the Gauss hypergeometric functions in (13), we get

$$\begin{aligned} F_D^{(3)}&\left(\beta,\lambda,\gamma,\mu+\frac{1}{2};\alpha+\beta;\rho;\frac{\delta}{1+\delta},\frac{2k^2}{k^2-1}\right) = \frac{\Gamma(\alpha+\beta)\Gamma(\mu-\beta+1/2)}{\Gamma(\alpha)\Gamma(\mu+1/2)}\left(\frac{1-k^2}{2k^2}\right)^{\beta} \\ &\sum_{m,n=0}^{\infty}\frac{(\beta)_{m+n}(\lambda)_m(\gamma)_n}{(1/2-\mu+\beta)_{m+n}m!n!}\left(\frac{\rho(k^2-1)}{2k^2}\right)^m\left(\frac{\delta(k^2-1)}{2k^2(1+\delta)}\right)^n \\ &{}_2F_1\left(\beta+m+n,1-\alpha;\frac{1}{2}-\mu+\beta+m+n;\frac{k^2-1}{2k^2}\right) \\ &+\frac{\Gamma(\alpha+\beta)\Gamma(\beta-\mu-1/2)}{\Gamma(\beta)\Gamma(\alpha+\beta-\mu-1/2)}\left(\frac{1-k^2}{2k^2}\right)^{\mu+1/2}\sum_{m,n=0}^{\infty}\frac{(\beta-\mu-1/2)_{m+n}(\lambda)_m(\gamma)_n}{(\alpha+\beta-\mu-1/2)_{m+n}m!n!} \\ &(\rho)^m\left(\frac{\delta}{1+\delta}\right)^n {}_2F_1\left(\mu+\frac{1}{2},\mu-\alpha-\beta-m-n+\frac{3}{2};\mu-\beta-m-n+\frac{3}{2};\frac{1-k^2}{2k^2}\right), \end{aligned} \tag{14}$$

if $\mu-\beta+1/2$ is not an integer. Therefore,

$$\begin{aligned} \Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k) &= \frac{\Gamma(\beta)\Gamma(\mu-\beta+1/2)}{\Gamma(\mu+1/2)}2^{-\beta}k^{-2\beta}(1+\delta)^{-\gamma}(1-k^2)^{\beta-\mu-1/2} \\ F_D^{(3)}&\left(\beta,\lambda,\gamma,1-\alpha;\beta-\mu+\frac{1}{2};\frac{\rho(k^2-1)}{2k^2},\frac{\delta(k^2-1)}{2k^2(1+\delta)},\frac{k^2-1}{2k^2}\right)+\frac{\Gamma(\alpha)\Gamma(\beta-\mu-1/2)}{\Gamma(\alpha+\beta-\mu-1/2)} \\ &2^{-\mu-1/2}k^{-2\mu-1}(1+\delta)^{-\gamma}\sum_{n=0}^{\infty}\frac{(\mu-\alpha-\beta+3/2)_n(\mu+1/2)_n}{(\mu-\beta+3/2)_n n!}\left(\frac{1-k^2}{2k^2}\right)^n \\ &F_1\left(\beta-\mu-n-\frac{1}{2},\lambda,\gamma;\alpha+\beta-\mu-n-\frac{1}{2};\rho,\frac{\delta}{1+\delta}\right), \end{aligned} \tag{15}$$

where F_1 is the Appell hypergeometric function of two variables [21]. Formula (15) can be considered as the asymptotic series for $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k)$ as $k^2 \to 1$ if $\mu-\beta+1/2$ is not an integer.

Let now $\mu-\beta+1/2$ be an integer, say $\mu-\beta+1/2=\pm l$, $l=0,1,2,\ldots$. First, let $\mu+1/2=\beta-l$. Applying formula 15.3.14 [2]

$$\begin{aligned}
&{}_2F_1\left(\beta+m+n,\beta-l;\,\alpha+\beta+m+n;\,\frac{2k^2}{k^2-1}\right)\\
&=\frac{\Gamma(\alpha+\beta+m+n)}{\Gamma(\beta+m+n)\Gamma(\alpha+m+n+l)}\left(\frac{1-k^2}{2k^2}\right)^{\beta+m+n}\\
&\sum_{r=0}^{\infty}\frac{(\beta-l)_{r+m+n+l}(1-\alpha-m-n-l)_{r+m+n+l}}{(r+m+n+l)!r!}\left(\frac{k^2-1}{2k^2}\right)^r\\
&\left[\ln\left(2k^2\right)-\ln\left(1-k^2\right)+\Psi(1+m+n+l+r)+\Psi(1+r)-\Psi(\beta+m+n+r)-\Psi(\alpha-r)\right]\\
&+\left(\frac{1-k^2}{2k^2}\right)^{\beta-l}\frac{\Gamma(\alpha+\beta+m+n)}{\Gamma(\beta+m+n)}\sum_{r=0}^{l-1}\frac{(\beta-l)_r(m+n+l-r-1)!}{\Gamma(\alpha+m+n+l-r)r!}\left(\frac{1-k^2}{2k^2}\right)^r,
\end{aligned}\tag{16}$$

we obtain

$$\begin{aligned}
&\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\beta-l-1/2)}(\rho,\delta;k)=2^{-\beta}k^{-2\beta}(1+\delta)^{-\gamma}\left(1-k^2\right)^l(1-\beta)_l\\
&\sum_{m,n,r=0}^{\infty}\frac{(\beta)_{m+n+r}(\lambda)_m(\gamma)_n(1-\alpha)_r}{(m+n+l+r)!m!n!r!}\left(\frac{\rho(k^2-1)}{2k^2}\right)^m\left(\frac{\delta(k^2-1)}{2k^2}\right)^n\left(\frac{k^2-1}{2k^2}\right)^r\\
&\left[\ln\left(2k^2\right)-\ln\left(1-k^2\right)+\Psi(1+m+n+l+r)+\Psi(1+r)-\Psi(\beta+m+n+r)-\Psi(\alpha-r)\right]\\
&+(-1)^l2^{l-\beta}k^{2l-2\beta}(1+\delta)^{-\gamma}(1-\beta)_l\sum_{r=0}^{l-1}\frac{(1-\alpha)_{l-r}(l-r-1)!}{(1-\beta)_{l-r}r!}\left(\frac{1-k^2}{2k^2}\right)^r\\
&F_1\left(l-r,\lambda,\gamma;\,\alpha+l-r;\,\rho,\frac{\delta}{1+\delta}\right).
\end{aligned}\tag{17}$$

Let now $\mu+1/2=\beta+l$. If $m+n<l$, we have

$$\begin{aligned}
&{}_2F_1\left(\beta+m+n,\beta+l;\,\alpha+\beta+m+n;\,\frac{2k^2}{k^2-1}\right)=\frac{\Gamma(\alpha+\beta+m+n)}{\Gamma(\alpha)\Gamma(\beta+l)}\left(\frac{1-k^2}{2k^2}\right)^{\beta+l}\\
&\sum_{r=0}^{\infty}\frac{(\beta+m+n)_{r+l-m-n}(1-\alpha)_{r+l-m-n}}{(r+l-m-n)!r!}\left(\frac{k^2-1}{2k^2}\right)^r\left[\ln\left(2k^2\right)-\ln\left(1-k^2\right)\right.\\
&\left.+\Psi(1+r+l-m-n)+\Psi(1+r)-\Psi(\beta+r+l)-\Psi(\alpha+m+n-r-l)\right]\\
&+\frac{\Gamma(\alpha+\beta+m+n)}{\Gamma(\beta+l)}\left(\frac{1-k^2}{2k^2}\right)^{\beta+m+n}\sum_{r=0}^{l-m-n-1}\frac{(\beta+m+n)_r(l-m-n-r-1)!}{\Gamma(\alpha-r)r!}\left(\frac{k^2-1}{2k^2}\right)^r.
\end{aligned}\tag{18}$$

If $m+n\geq l$, then

$$\begin{aligned}
&{}_2F_1\left(\beta+m+n,\beta+l;\,\alpha+\beta+m+n;\,\frac{2k^2}{k^2-1}\right)\\
&\frac{\Gamma(\alpha+\beta+m+n)}{\Gamma(\beta+m+n)\Gamma(\alpha+m+n-l)}\left(\frac{1-k^2}{2k^2}\right)^{\beta+m+n}\\
&\sum_{r=0}^{\infty}\frac{(\beta+l)_{m+n+r-l}(1-\alpha+l-m-n)_{m+n+r-l}}{(m+n+r-l)!r!}\left(\frac{k^2-1}{2k^2}\right)^r\\
&\left[\ln\left(2k^2\right)-\ln\left(1-k^2\right)+\Psi(1+m+n+r-l)+\Psi(1+r)-\Psi(\beta+m+n+r)-\Psi(\alpha-r)\right]\\
&+\frac{\Gamma(\alpha+\beta+m+n)}{\Gamma(\beta+m+n)}\left(\frac{1-k^2}{2k^2}\right)^{\beta+l}\sum_{r=0}^{m+n-l-1}\frac{(\beta+l)_r(m+n-l-r-1)!}{\Gamma(\alpha+m+n-l-r)r!}\left(\frac{k^2-1}{2k^2}\right)^r.
\end{aligned}\tag{19}$$

Therefore,

$$
\begin{gathered}
\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\beta+l-1/2)}(\rho,\delta;k) = 2^{-\beta-l}k^{-2\beta-2l}(1+\delta)^{-\gamma} \\
\sum_{r=0}^{\infty}\sum_{m+n<l} \frac{(1-\alpha)_{r+l-m-n}(\lambda)_m(\gamma)_n(\beta+l)_r}{(r+l-m-n)!m!n!r!}\rho^m\left(\frac{\delta}{1+\delta}\right)^n\left(\frac{k^2-1}{2k^2}\right)^r \\
\left[\ln\left(2k^2\right)-\ln\left(1-k^2\right)+\Psi(1+r+l-m-n)+\Psi(1+r)-\Psi(\beta+r+l)\right. \\
\left.-\Psi(\alpha+m+n-r-l)\right]+2^{-\beta}k^{-2\beta}(1+\delta)^{-\gamma}\left(1-k^2\right)^{-l}\frac{1}{(\beta)_l} \\
\sum_{m+n<l}\sum_{r=0}^{l-m-n-1}\frac{(\beta)_{m+n+r}(\lambda)_m(\gamma)_n(1-\alpha)_r(l-m-n-r-1)!}{m!n!r!} \\
\left(\frac{\rho(1-k^2)}{2k^2}\right)^m\left(\frac{\delta(1-k^2)}{2k^2(1+\delta)}\right)^n\left(\frac{1-k^2}{2k^2}\right)^r+2^{-\beta}k^{-2\beta}(1+\delta)^{-\gamma}\left(k^2-1\right)^{-l}\frac{1}{(\beta)_l} \\
\sum_{m+n\geq l}\sum_{r=0}^{\infty}\frac{(\beta)_{m+n+r}(\lambda)_m(\gamma)_n(1-\alpha)_r}{(m+n+r-l)!m!n!r!}\left(\frac{\rho(k^2-1)}{2k^2}\right)^m\left(\frac{\delta(k^2-1)}{2k^2(1+\delta)}\right)^n\left(\frac{k^2-1}{2k^2}\right)^r \\
\left[\ln\left(2k^2\right)-\ln\left(1-k^2\right)+\Psi(1+m+n+r-l)+\Psi(1+r)-\Psi(\beta+m+n+r)-\Psi(\alpha-r)\right] \\
+2^{-\beta-l}k^{-2\beta-2l}(1+\delta)^{-\gamma}\sum_{m+n\geq l}\sum_{r=0}^{m+n-l-1}\frac{(\lambda)_m(\gamma)_n(\beta+l)_r(m+n-l-r-1)!}{(\alpha)_{m+n-r-l}!m!n!r!} \\
\rho^m\left(\frac{\delta}{1+\delta}\right)^n\left(\frac{k^2-1}{2k^2}\right)^r.
\end{gathered}
\tag{20}
$$

3. ASYMPTOTIC EXPANSION FOR $R_\mu(k,\alpha,\gamma)$

Asymptotic expansion for $R_\mu(k,\alpha,\gamma)$ can be obtained in a similar manner. Indeed, using the hypergeometric representation for $R_\mu(k,\alpha,\gamma)$

$$
R_\mu(k,\alpha,\gamma)=\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}\left(1-k^2\right)^{-\mu-1/2}{}_2F_1\left(\gamma-\alpha,\mu+\frac{1}{2};\gamma;\frac{2k^2}{k^2-1}\right), \tag{21}
$$

and formula 15.3.7 [2] to the hypergeometric function in (21), we have

$$
\begin{gathered}
R_\mu(k,\alpha,\gamma)=\frac{\Gamma(\gamma-\alpha)\Gamma(\alpha+\mu-\gamma+1/2)}{\Gamma(\mu+1/2)}2^{\alpha-\gamma}k^{2\alpha-2\gamma}\left(1-k^2\right)^{\gamma-\alpha-\mu-1/2} \\
{}_2F_1\left(\gamma-\alpha,1-\alpha;\gamma-\alpha-\mu+\frac{1}{2};\frac{k^2-1}{2k^2}\right)+\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha-\mu-1/2)}{\Gamma(\gamma-\mu-1/2)}2^{-\mu-1/2}k^{-2\mu-1} \\
{}_2F_1\left(\mu+\frac{1}{2},\mu-\gamma+\frac{3}{2};\alpha+\mu-\gamma+\frac{3}{2};\frac{k^2-1}{2k^2}\right),
\end{gathered}
\tag{22}
$$

if $\gamma-\alpha-\mu+1/2$ is not an integer.

If $\mu=\gamma-\alpha+l-1/2$, $l=0,1,2,\dots$, we have

$$
\begin{gathered}
R_{\gamma-\alpha+l-1/2}(k,\alpha,\gamma)=\frac{(2k^2)^{\alpha-\gamma-l}}{(\gamma-\alpha)_l}\sum_{n=0}^{\infty}\frac{(\gamma-\alpha)_{n+l}(1-\alpha)_{n+l}}{(n+l)!n!}\left(\frac{k^2-1}{2k^2}\right)^n \\
\left[\ln\left(2k^2\right)-\ln\left(1-k^2\right)+\Psi(1+n+l)+\Psi(1+n)-\Psi(\gamma-\alpha+n+l)-\Psi(\alpha-n-l)\right] \\
+\left(\frac{1-k^2}{2k^2}\right)^{\gamma-\alpha}\sum_{n=0}^{l-1}\frac{(\gamma-\alpha)_n(1-\alpha)_n(l-n-1)!}{(\gamma-\alpha)_l n!}\left(\frac{1-k^2}{2k^2}\right)^n.
\end{gathered}
\tag{23}
$$

If $\mu = \gamma - \alpha - l - 1/2$, $l = 1, 2, \dots$, we obtain

$$R_{\gamma-\alpha-l-1/2}(k,\alpha,\gamma) = \frac{(1+\alpha-\gamma)_l}{(\alpha)_l}\left(\frac{1-k^2}{2k^2}\right)^{\gamma-\alpha}\sum_{n=0}^{\infty}\frac{(\gamma-\alpha)_n(1-\alpha)_n}{(n+l)!n!}\left(\frac{k^2-1}{2k^2}\right)^n$$
$$\left[\ln\left(2k^2\right) - \ln\left(1-k^2\right) + \Psi(1+n+l) + \Psi(1+n) - \Psi(\gamma-\alpha+l) - \Psi(\alpha-n)\right] \quad (24)$$
$$+ \left(2k^2\right)^{\alpha-\gamma+l}\sum_{n=0}^{l-1}\frac{(\gamma-\alpha-l)_n(l-n-1)!}{(\alpha)_{l-n}n!}\left(\frac{k^2-1}{2k^2}\right)^n.$$

4. SPECIAL CASES

From the general formulas established in the previous sections for $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;\,k)$ and $R_\mu(k,\alpha,\gamma)$, one can derive the corresponding asymptotic formulas for other types of elliptic-type integrals by choosing suitable parameters.

For example, if one sets $\alpha = 1/2$, $\gamma = 1$, and $\mu = j$, in $R_\mu(k,\alpha,\gamma)$, it reduces to Epstein-Hubbell elliptic-type integral $\Omega_j(k)$, and then we have

$$\Omega_j(k) = \frac{(2k^2)^{-j-1/2}}{(1/2)_j}\sum_{n=0}^{\infty}\frac{(1/2)_{j+n}(1/2)_{j+n}}{(n+j)!n!}\left(\frac{k^2-1}{2k^2}\right)^n$$
$$\left[\ln\left(2k^2\right) - \ln\left(1-k^2\right) + \Psi(1+n+j) + \Psi(1+n) - \Psi\left(\frac{1}{2}+n+j\right) - \Psi\left(\frac{1}{2}-n-j\right)\right] \quad (25)$$
$$+ \left(\frac{1-k^2}{2k^2}\right)^{1/2}\sum_{n=0}^{j-1}\frac{(j-n-1)!(1/2)_n(1/2)_n}{(1/2)_j n!}\left(\frac{1-k^2}{2k^2}\right)^n.$$

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