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On the degree of regularity of some equations

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Abstract

In this paper we investigate the behaviour of the solutions of equations $\sum_{i=1}^{n} a_i x_i = b$, where $\sum_{i=1}^{n} a_i = 0$ and $b \neq 0$, with respect to colorings of the set $\mathbb{N}$ of positive integers. It turns out that for any $b \neq 0$ there exists an 8-coloring of $\mathbb{N}$, admitting no monochromatic solution of $x_3 - x_2 = x_2 - x_1 + b$. For this equation, for b odd and 2-colorings, only an odd–even coloring prevents a monochromatic solution. For b even and 2-colorings, always monochromatic solutions can be found, and bounds for the corresponding Rado numbers are given. If one imposes the ordering $x_1 < x_2 < x_3$, then there exists already a 4-coloring of $\mathbb{N}$, which prevents a monochromatic solution of $x_3 - x_2 = x_2 - x_1 + b$, where $b \in \mathbb{N}$.

1. Introduction

Let $\boldsymbol{Ax} = \boldsymbol{b}$ be a finite system of linear equations, where all entries of $\boldsymbol{A}$ and $\boldsymbol{b}$ are integers. Rado [6] called a system $\boldsymbol{Ax} = \boldsymbol{b}$ *partition regular* over $\mathbb{N}$ if for every coloring of the set $\mathbb{N}$ of positive integers with a finite number of colors there always exists a monochromatic solution of $\boldsymbol{Ax} = \boldsymbol{b}$. He characterized in [6] all such partition regular systems of equations in terms of certain linear dependencies among the column vectors of $\boldsymbol{A}$. For the special case of one homogeneous equation, he proved that $\sum_{i=1}^{n} a_i x_i = 0$ is partition regular over $\mathbb{N}$ if and only if some of the coefficients a_i sum up to zero. Moreover, Rado's results include van der Waerden's theorem on arithmetic progressions [8], i.e., one can always find monochromatic arithmetic progressions of arbitrary finite length under every coloring of $\mathbb{N}$ using a finite number of colors.

In order to show that a system $\boldsymbol{Ax} = \boldsymbol{b}$ is not partition regular, Rado used in his arguments that the number of colors depends on the entries of the matrix $\boldsymbol{A}$ and of $\boldsymbol{b}$. Following [6], we call a system $\boldsymbol{Ax} = \boldsymbol{b}$ *t-regular* if for every coloring $\Delta : \mathbb{N} \longrightarrow \{0, 1, \ldots, t-1\}$ there always exists a monochromatic solution of $\boldsymbol{Ax} = \boldsymbol{b}$. If the system $\boldsymbol{Ax} = \boldsymbol{b}$ is not partition regular, then the *degree of regularity* $dor_{\mathbb{N}}(\boldsymbol{Ax} = \boldsymbol{b})$

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of this system is defined as the largest integer t such that $\boldsymbol{Ax} = \boldsymbol{b}$ is t-regular. Rado observed that the equation $ax + by + c = 0$ is either partition regular or has degree of regularity at most 1. For homogeneous equations in three variables only the following is known:

Theorem 1.1 (Rado [6]). (i) *If $a \in \mathbb{Q}$ and $a \neq 2^k$ for every integer $k \in \mathbb{Z}$, then*

$$dor_{\mathbb{N}}\,(a \cdot (x_1 + x_2) = x_3) \leqslant 3.$$

(ii) *For every $k \in \mathbb{Z}$,*

$$\text{either } 2^k \cdot (x_1 + x_2) = x_3 \text{ is partition regular,}$$
$$\text{or } dor_{\mathbb{N}}\left(2^k \cdot (x_1 + x_2) = x_3\right) \leqslant 5.$$

(iii) *Let p be a prime number and let $a_1, a_2, a_3, \alpha \in Z$. If $\alpha \neq 0$ and $p \nmid a_1 a_2 a_3 (a_1 + a_2)$, then*

$$\text{either } a_1x_1 + a_2x_2 + p^{\alpha}a_3x_3 = 0 \text{ is partition regular}$$
$$\text{or } dor_{\mathbb{N}}\,(a_1x_1 + a_2x_2 + p^{\alpha}a_3x_3 = 0) \leqslant 5.$$

(iv) *Let p be a prime and let $a_1, a_2, a_3 \in \mathbb{Z}$, where $p \nmid a_1a_2a_3$. If $\alpha, \beta, \gamma \in \mathbb{Z}$ are pairwise distinct, then*

$$dor_{\mathbb{N}}\left(p^{\alpha}a_1x_1 + p^{\beta}a_2x_2 + p^{\gamma}a_3x_3 = 0\right) \leqslant 7.$$

This lead him to conjecture in [6] that every system $\boldsymbol{Ax} = \boldsymbol{b}$ in n variables is either partition regular or has degree of regularity at most $t = t(n)$, where t only depends on n and not on the entries of $\boldsymbol{A}$ and $\boldsymbol{b}$. This conjecture is still open. As indicated above, $n = 3$ is the smallest unknown case.

In [6] Rado showed that it suffices to prove this conjecture for systems $\boldsymbol{Ax} = \boldsymbol{b}$ consisting of one equation only. We observed in Section 2 that for an inhomogeneous equation $\sum_{i=1}^{n} a_ix_i = b$ with $\sum_{i=1}^{n} a_i = 0$ one can bound the degree of regularity from above by a constant, which is independent of b, but still depends on $a_1, a_2, \ldots, a_n$, i.e., $dor_{\mathbb{N}}\left(\sum_{i=1}^{n} a_ix_i = b\right) \leqslant 2 \cdot \sum_{i=1}^{n} |a_i| - 1$. In Section 3 we investigate in detail the equation $x_3 - x_2 = x_2 - x_1 + b$ arising from three-term arithmetic progressions. We show that if b is odd, then there exists exactly one coloring $\Delta : \mathbb{N} \longrightarrow \{0,1\}$ (up to exchanging colors) admitting no monochromatic solution of $x_3 - x_2 = x_2 - x_1 + b$. For b even, this equation turns out to be 2-regular. Also, we show that the degree of regularity of any equation of the form $x_3 - x_2 = x_2 - x_1 + b$, $b \neq 0$, is an integer between 3 and 7. Imposing the ordering restriction $x_1 < x_2 < x_3$ on the solutions, we obtain $dor_{\mathbb{N}}\,(x_3 - x_2 = x_2 - x_1 + b) \leqslant 3$ for any $b \in \mathbb{N}$.

Finally, in Section 4 we will consider the canonical situation, arising from the theorem of Erdős and Graham [3] on arithmetic progressions.

2. Nonhomogeneous equations

In this section we consider equations of the form $\sum_{i=1}^{n} a_i x_i = b$, where b is not equal to zero. For this case, Rado [6] showed that $\sum_{i=1}^{n} a_i x_i = b$ is partition regular over $\mathbb{N}$ if and only if $\sum_{i=1}^{n} a_i \neq 0$ and either $\frac{b}{\sum_{i=1}^{n} a_i}$ is a positive integer or $\frac{b}{\sum_{i=1}^{n} a_i}$ is a negative integer and the equation $\sum_{i=1}^{n} a_i x_i = 0$ is partition regular over $\mathbb{N}$.

It is useful for our purposes to consider colorings of the set $\mathbb{Q}$ of rational numbers. For this situation, we adapt the notions t-regular, partition regular over $\mathbb{Q}$ and degree of regularity $dor_{\mathbb{Q}}(\cdot)$ over $\mathbb{Q}$ correspondingly. We will restrict our attention to equations $\sum_{i=1}^{n} a_i x_i = b$ with $\sum_{i=1}^{n} a_i = 0$, as otherwise one always gets a singleton solution. In the following we will use an observation essentially made in [7]:

Lemma 2.1. *Let $a_1, a_2, \ldots, a_n$ be nonzero integers with $\sum_{i=1}^{n} a_i = 0$. Let b be any nonzero rational number. Then*

$$dor_{\mathbb{Q}}\left(\sum_{i=1}^{n} a_i \cdot x_i = b\right) < t_0,$$

where t_0 is any positive integer with

$$t_0 \geqslant \left(\left\lceil |b| + \sum_{i=1}^{n} |a_i| \right\rceil\right) \cdot \max\left\{1, \left\lceil \frac{1}{|b|} \cdot \sum_{i=1}^{n} |a_i| \right\rceil\right\}.$$

As our lower bound on t is slightly different from the one given in [7], we include a proof of Lemma 2.1 following [7]:

Proof. Take positive integers m' and m'' with $m' \geqslant |b| + \sum_{i=1}^{n} |a_i|$ and $m'' \geqslant \frac{1}{|b|} \cdot \sum_{i=1}^{n} |a_i|$. Define a coloring $\Delta : \mathbb{Q} \longrightarrow \{0, 1, \ldots, m' \cdot m'' - 1\}$ by

$$\Delta(x) \equiv \lfloor m'' \cdot x \rfloor \bmod (m'm'').$$

Now suppose that $x_1, x_2, \ldots, x_n$ is a monochromatic solution of $\sum_{i=1}^{n} a_i x_i = b$. Then, for $i = 1, 2, \ldots, n$, we have

$$\lfloor m'' x_i \rfloor \equiv \lfloor m'' x_1 \rfloor \bmod (m'm''),$$

and hence there exist integers k_i for $i = 1, 2, \ldots, n$ with

$$\lfloor m'' x_i \rfloor - \lfloor m'' x_1 \rfloor = k_i m' m''.$$

Dividing the last equation by m'', we infer that

$$\left\lfloor \frac{1}{m''} \cdot \lfloor m'' \cdot x_i \rfloor \right\rfloor - \left\lfloor \frac{1}{m''} \cdot \lfloor m'' \cdot x_1 \rfloor \right\rfloor = k_i m',$$

i.e.,

$$\lfloor x_i \rfloor - \lfloor x_1 \rfloor = k_i m' \tag{1}$$

for $i = 1, 2, \ldots, n$.

Moreover, for $i = 1, 2, \ldots, n$, there exist rational numbers r_i with $-1 < r_i < 1$, such that

$$m'' x_i - m'' x_1 = k_i m' m'' + r_i \,. \tag{2}$$

According to the value of $\sum_{i=1}^{n} a_i \cdot \lfloor x_i \rfloor$ we distinguish two cases.

If $\sum_{i=1}^{n} a_i \cdot \lfloor x_i \rfloor \neq 0$, then with $\sum_{i=1}^{n} a_i = 0$ we infer that

$$0 \neq \left| \sum_{i=1}^{n} a_i \cdot \lfloor x_i \rfloor \right| = \left| \sum_{i=1}^{n} a_i \cdot (\lfloor x_i \rfloor - \lfloor x_1 \rfloor) \right| = \left| \sum_{i=1}^{n} a_i k_i m' \right| \geqslant m',$$

and here we used that the a_i's are integers, hence,

$$\left| \sum_{i=1}^{n} a_i \cdot x_i \right| \geqslant \left| \sum_{i=1}^{n} a_i \cdot \lfloor x_i \rfloor \right| - \left| \sum_{i=1}^{n} a_i \cdot (x_i - \lfloor x_i \rfloor) \right| > m' - \sum_{i=1}^{n} |a_i| \geqslant |b|.$$

In the second case, where $\sum_{i=1}^{n} a_i \cdot \lfloor x_i \rfloor = 0$, we see with (1) and (2) that

$$\begin{aligned} \left| \sum_{i=1}^{n} a_i \cdot x_i \right| &= \left| \sum_{i=1}^{n} a_i \cdot ((x_i - x_1) - (\lfloor x_i \rfloor - \lfloor x_1 \rfloor)) \right| \\ &= \left| \sum_{i=1}^{n} \frac{a_i r_i}{m''} \right| < \frac{1}{m''} \cdot \sum_{i=1}^{n} |a_i| \leqslant |b|. \qquad \square \end{aligned}$$

Notice that the lower bound on t given in Lemma 2.1 yields for large values of b, i.e., $|b| \geqslant \sum_{i=1}^{n} |a_i|$, that

$$t \geqslant \left\lceil |b| + \sum_{i=1}^{n} |a_i| \right\rceil .$$

In particular, this shows that the equation $x_3 - x_2 = x_2 - x_1 + b$ is not $\lceil 4 + |b| \rceil$-regular over $\mathbb{Q}$ for $|b| \geqslant 4$. We will show next that the equation $x_3 - x_2 = x_2 - x_1 + b$ is not 8-regular over $\mathbb{Q}$.

First we will show the following stronger version of Lemma 2.1:

Theorem 2.2. *Let $a_1, a_2, \ldots, a_n$ be nonzero integers with $\sum_{i=1}^{n} a_i = 0$. Then there exists a least integer $t_0 \in \mathbb{N}$ with the following property: For every $b \in \mathbb{Q}$, $b \neq 0$, there exists a coloring $\Delta : \mathbb{Q} \longrightarrow \{0, 1, \ldots, t_0 - 1\}$ such that there is no monochromatic solution of the equation*

$$\sum_{i=1}^{n} a_i \cdot x_i = b.$$

We remark that t_0 might be dependent on the values $a_1, a_2, \ldots, a_n$, but it is independent of b.

Proof. Let $a_1, a_2, \ldots, a_n \in \mathbb{Z}$ with $\sum_{i=1}^n a_i = 0$ be given. Assume to the contrary that for each $t \in \mathbb{N}$ there exists $b(t) \in \mathbb{Q}$, $b(t) \neq 0$, such that for every coloring $\Delta : \mathbb{Q} \longrightarrow \{0, 1, \ldots, t-1\}$ there always exists a monochromatic solution of the equation $\sum_{i=1}^n a_i x_i = b(t)$. Take some $t \in \mathbb{N}$ and $b_0 \in \mathbb{Q}$, $b_0 \neq 0$, with

$$t \geqslant \left(\left\lceil |b_0| + \sum_{i=1}^n |a_i| \right\rceil \right) \cdot \max \left\{ \left\lceil \frac{1}{b_0} \cdot \sum_{i=1}^n |a_i| \right\rceil , 1 \right\}.$$

By Lemma 2.1 there exists a coloring $\Delta^* : \mathbb{Q} \longrightarrow \{0, 1, \ldots, t-1\}$ admitting no monochromatic solution of $\sum_{i=1}^n a_i x_i = b_0$. Define a new coloring $\Delta : \mathbb{Q} \longrightarrow \{0, 1, \ldots, t-1\}$ by

$$\Delta(x) = \Delta^* \left(\frac{x}{b(t)} \cdot b_0 \right).$$

By our assumption at the beginning of the proof, there exists a monochromatic solution of $\sum_{i=1}^n a_i x_i = b(t)$ with respect to Δ. Set $y_i = \frac{x_i}{b(t)} \cdot b_0$ for $i = 1, 2, \ldots, n$. Then $\Delta^*(y_1) = \Delta^*(y_2) = \cdots = \Delta^*(y_n)$ and $\sum_{i=1}^n a_i y_i = b_0$, which contradicts the choice of the coloring Δ^*. □

Lemma 2.3. *Let $a_1, a_2, \ldots, a_n$ be nonzero integers with $\sum_{i=1}^n a_i = 0$. Then there exists a positive integer t_1 such that for any nonzero rational number b it is*

$$dor_{\mathbb{Q}} \left(\sum_{i=1}^n a_i \cdot x_i = b \right) = t_1.$$

Proof. Let $a_1, \ldots, a_n, b$ be given as above. We show that

$$dor_{\mathbb{Q}} \left(\sum_{i=1}^n a_i \cdot x_i = b \right) = dor_{\mathbb{Q}} \left(\sum_{i=1}^n a_i \cdot x_i = 1 \right).$$

Suppose first that there exists a coloring $\Delta_1 : \mathbb{Q} \longrightarrow \{0, 1, \ldots, t-1\}$ admitting no monochromatic solution of $\sum_{i=1}^n a_i x_i = b$. Then the coloring $\Delta_2 : \mathbb{Q} \longrightarrow \{0, 1, \ldots, t-1\}$ with $\Delta_2(x) = \Delta(x \cdot b)$ allows no monochromatic solution of $\sum_{i=1}^n a_i x_i = 1$.

On the other hand, if $\sum_{i=1}^n a_i x_i = b$ is t-regular, then using arguments similar to those given above, one sees that $\sum_{i=1}^n a_i x_i = 1$ is t-regular, too. □

By Theorem 2.2 and Lemma 2.3, we infer $t_1 = t_0$. Moreover, the degree of regularity of any equation $\sum_{i=1}^n a_i x_i = b$, where $b \neq 0$ and $\sum_{i=1}^n a_i = 0$, can be bounded from above by $2 \cdot \sum_{i=1}^n |a_i| - 1$, as

$$\left(\sum_{i=1}^n |a_i| + |b| \right) \cdot \max \left\{ 1, \frac{1}{|b|} \cdot \sum_{i=1}^n |a_i| \right\} \geqslant 2 \cdot \sum_{i=1}^n |a_i| ,$$

and equality holds for $|b| = \sum_{i=1}^n |a_i|$.

Now consider the equation $\sum_{i=1}^{n} a_i x_i = b$. The proofs of Lemmas 2.1 and 2.3 actually yield the coloring $\Delta : \mathbb{Q} \longrightarrow \{0, 1, \ldots, 2 \cdot \sum_{i=1}^{n} |a_i| - 1\}$ with

$$\Delta(x) \equiv \left\lfloor \frac{x \cdot \sum_{i=1}^{n} |a_i|}{b} \right\rfloor \bmod \left(2 \cdot \sum_{i=1}^{n} |a_i|\right),$$

which shows that $\sum_{i=1}^{n} a_i x_i = b$ is not $2\sum_{i=1}^{n} |a_i|$-regular over $\mathbb{Q}$. This one could have had from the beginning of course, but the statements 2.1–2.3 are of some interest by themselves.

Clearly, the statements of Lemma 2.1 carry over for b being a nonzero integer, if we consider colorings of $\mathbb{N}$ only. As $\mathbb{N}$ is not closed under division, this does not apply immediately to Theorem 2.2 and Lemma 2.3, respectively. But as $\mathbb{N} \subset \mathbb{Q}$, it follows from the above considerations that, for example,

$$dor_{\mathbb{N}}(x_3 - x_2 = x_2 - x_1 + b) \leqslant 7$$

for every $b \in \mathbb{Z} \setminus \{0\}$.

3. The equation $x_3 - x_2 = x_2 - x_1 + b$

In this section we consider the equation $x_3 - x_2 = x_2 - x_1 + b$. For $b = 0$ and $x_1 \neq x_2$, this describes just 3-term arithmetic progressions.

Given integers b and t with $t \geqslant 1$, let $D_t(b)$ be the least positive integer $n = D_t(b)$ such that for every coloring $\Delta : \{1, 2, \ldots, n\} \longrightarrow \{0, 1, \ldots, t - 1\}$ there always exist integers x_1, x_2, x_3 with $1 \leqslant x_1 < x_2 < x_3 \leqslant n$ such that $x_3 - x_2 = x_2 - x_1 + b$ and $\Delta(x_1) = \Delta(x_2) = \Delta(x_3)$. If such a least integer $n = D_t(b)$ does not exist, we call $D_t(b)$ undefined. For the corresponding Rado numbers, where one does not have the assumption on the ordering $x_1 < x_2 < x_3$, we refer to the paper of Burr and Loo [1].

Observe that for $b = 0$ the numbers $D_t(0)$ coincide with the van der Waerden numbers $vdW_t(3)$ (cf. [8]), that is

$$D_t(0) = vdW_t(3).$$

In general, the exact determination of the van der Waerden numbers $vdW_t(3)$ for arbitrary positive integers t is a hard problem. For small values of t the exact values known are $vdW_2(3) = 9$ and $vdW_3(3) = 27$, $vdW_4(3) = 76$, cf. [2], while $vdW_5(3)$ is not known.

Recall that the degree of regularity of $x_3 - x_2 = x_2 - x_1 + b$, $b \neq 0$ is at most 7, i.e. $D_8(b)$ is undefined. However, taking into account the ordering on the solutions, we will show in this section that $dor_{\mathbb{N}}(x_3 - x_2 = x_2 - x_1 + b;\ x_1 < x_2 < x_3) \leqslant 3$.

Proposition 3.1. *If b is an odd integer, then the equation $x_3 - x_2 = x_2 - x_1 + b$ is not 2-regular.*

Proof. Let b be an odd integer. Let $\Delta : \mathbb{N} \longrightarrow \{0,1\}$ be a coloring defined by $\Delta(x) \equiv x \bmod 2$. If x_1,x_2,x_3 were colored the same, then $x_3 - x_2 \equiv 0 \bmod 2$ and $x_2 - x_1 + b \equiv 1 \bmod 2$, hence there is no monochromatic solution of $x_3 - x_2 = x_2 - x_1 + b$. □

Next we consider the case, where b is an even integer. By using exhaustive search, we found the values given in the following table.

b	0	2	4	6	8	10	12	14	16	18	20
$D_2(b)$	9	14	17	21	25	30	34	38	42	46	50

For example, the string 0 0 1 1 1 0 0 0 0 1 1 1 1 gives the lower bound $D_2(2) \geqslant 14$. Thus the numbers $D_2(b)$ are well defined for all even values of $b \leqslant 20$. Indeed, the next result shows that for arbitrary even integers b, the numbers $D_2(b)$ always exist.

Theorem 3.2. *If b is an even positive integer, then*

$$2b + 10 \leqslant D_2(b) \leqslant \frac{13}{2}b + 1,$$

where the lower bound holds only for $b \geqslant 10$.

We see from the table that the lower bound is sharp for $b \in \{2,10,12,14,16,18,20\}$.

Proof. First we show the upper bound. From the table given above we have $D_2(2) = 14$. Set $b = 2b^*$ and let $\Delta : \{1,2,\ldots,13b^* + 1\} \longrightarrow \{0,1\}$ be an arbitrary coloring. Then Δ induces another coloring $\Delta^* : \{1,2,\ldots,14\} \longrightarrow \{0,1\}$ by $\Delta^*(x) = \Delta((x-1)b^* + 1)$. As $D_2(2) = 14$, there exist x_1',x_2',x_3' with $1 \leqslant x_1' < x_2' < x_3' \leqslant 14$, where $x_3' - x_2' = x_2' - x_1' + 2$ and $\Delta^*(x_1') = \Delta^*(x_2') = \Delta^*(x_3')$. Set $x_i = (x_i' - 1) \cdot b^* + 1$ for $i = 1,2,3$. Then $\Delta(x_1) = \Delta(x_2) = \Delta(x_3)$ and $x_3 - x_2 = x_2 - x_1 + b$.

(Indeed, we have for $b = l \cdot b^*$, l even, $b^* \in \mathbb{N}$, that $D_2(b) \leqslant \frac{D_2(l)-1}{l} \cdot b + 1$. In particular, for say $b = 20 \cdot b^*$, $b^* \in \mathbb{N}$, we obtain $D_2(b) \leqslant \frac{49}{20} \cdot b + 1$.)

Let $b \geqslant 10$. Concerning the lower bound, consider the coloring $\Delta : \{1,2,\ldots,2b+10\} \longrightarrow \{0,1\}$ defined by

$$\Delta(x) = \begin{cases} 0 & \text{if } x \in \{1,2\} \cup \{6,7,\ldots,b+2\} \cup \{b+4,b+5,\ldots,b+7\}, \\ 1 & \text{if } x \in \{3,4,5\} \cup \{b+3\} \cup \{b+8,b+9,\ldots,2b+9\}. \end{cases}$$

Assume that x_1,x_2,x_3 is a monochromatic solution of

$$x_3 - x_2 = x_2 - x_1 + b \qquad (*)$$

with $1 \leqslant x_1 < x_2 < x_3$. If this solution is in color 0, then $x_2 \neq 2$, as otherwise $x_3 = 2x_2 - x_1 + b = b + 3$. But, if $x_2 = 6$, then $x_3 \geqslant b + 10$, which is impossible. Also, if $x_2 \geqslant 7$, then $x_3 \geqslant b + 8$, which again is impossible.

Assume now that we have a solution of $(*)$ in color 1. If $4 \leqslant x_2 \leqslant 5$, then $b + 7 \geqslant x_3 \geqslant b + 5$, which is impossible. If $x_2 = b + 3$, then $x_3 \geqslant 3b + 1$, again impossible for $b > 8$. Finally, x_2 cannot be greater than $b + 7$, as then $x_3 \geqslant 2b + 10$. □

Now we will show the typical structure of those colorings preventing a monochromatic solution of the equation $x_3 - x_2 = x_2 - x_1 + b$, where b is an odd integer. We call a two-coloring of the set of positive integers an *odd–even coloring* if the odd numbers are colored all in one color and the even numbers are colored in the other color.

Theorem 3.3. *Let b be an odd integer. If the coloring $\Delta \colon \mathbb{N} \longrightarrow \{0,1\}$ is not an odd–even coloring, then there exist x_1,x_2,x_3 with $1 \leqslant x_1 < x_2 < x_3$ such that $\Delta(x_1) = \Delta(x_2) = \Delta(x_3)$ and $x_3 - x_2 = x_2 - x_1 + b$.*

Proof. Let b be an odd positive integer, and let $\Delta : \mathbb{N} \longrightarrow \{0,1\}$ be not an odd–even coloring. Then there exist two consecutive integers colored the same. Suppose that there is no monochromatic solution of $x_3 - x_2 = x_2 - x_1 + b$ with $x_1 < x_2 < x_3$. In our arguments we will use the following observation:

Proposition 3.4. *If $(b+3)$ consecutive integers are colored the same, then there exists a monochromatic solution of the equation $x_3 - x_2 = x_2 - x_1 + b$, where $x_1 < x_2 < x_3$.*

Proof. If $a+1, a+2, \ldots, a+b+3$ are colored the same, then $x_1 = a+1$, $x_2 = a+2$ and $x_3 = a+b+3$ provide a monochromatic solution. □

The following lemma together with Proposition 3.4 proves the theorem:

Lemma 3.5. *If at least two consecutive integers are colored the same, then there exist $(b+3)$ consecutive integers which are colored the same.*

Proof. We will distinguish three cases according to the number of consecutive integers which are colored the same.

Case 3.1: Suppose that there exist four consecutive integers $a, a+1, a+2, a+3$ which are colored the same in, say, color 0. Then $a+b+2, a+b+3, \ldots, a+b+6$ are all colored by color 1, as otherwise we are done. Similarly, the numbers $a+2b+4$, $a+2b+5, \ldots, a+2b+10$ are all colored in color 0. Finally, after at most $c \cdot \log_2 b$ steps, where c is a positive constant, we obtain a sequence of $(b+3)$ consecutive integers, which are colored the same. Applying Proposition 3.4 yields the desired result.

Case 3.2: Suppose now that there are three consecutive integers $a, a+1, a+2$ which are colored the same. By Case 3.1 it suffices to distinguish as follows:

Case 3.2a: If the set $\{a, a+1, a+2, a+6\}$ is monochromatic, then also the set $\{a+b+2, a+b+3, a+b+4, a+b+10, a+b+11\}$. This implies that the set $\{a+2b+16, a+2b+17, a+2b+18, a+2b+19\}$ is monochromatic, which is covered in Case 3.1.

Case 3.2b: If $\{a, a+1, a+2, a+5\}$ is monochromatic, then also the set $\{a+b+2, a+b+3, a+b+4, a+b+8, a+b+9\}$. We infer that $\{a+2b+12, a+2b+13, a+2b+14, a+2b+15\}$ is monochromatic, again covered by Case 3.1.

Case 3.2c: If $\{a, a+1, a+2, a+4\}$ is monochromatic, then also $\{a+b+2, a+b+3, a+b+4, a+b+6, a+b+7\}$, and hence $\{a+2b+8, a+2b+9, a+2b+10, a+2b+11\}$, giving four consecutive integers, which are colored the same.

Case 3.3: Now assume that a and $a+1$ are colored the same. We distinguish six cases:

Case 3.3a If $\{a, a+1, a+4, a+7\}$ is monochromatic, then also $\{a+b+2, a+b+7, a+b+8, a+b+10\}$, and hence, too, $\{a+2b+12, a+2b+13, a+2b+14\}$, and Case 3.2 applies.

Case 3.3b: If $\{a, a+1, a+4, a+6\}$ is monochromatic then, also, $\{a+b+7, a+b+8, a+b+11, a+b+12\}$, hence, too, $\{a+2b+13, a+2b+14, a+2b+15\}$, and Case 3.2 applies.

Case 3.3c: If $\{a, a+1, a+4, a+5\}$ is monochromatic, then also $\{a+b+8, a+b+9, a+b+10\}$ and we are again in Case 3.2.

Case 3.3d: If $\{a, a+1, a+3, a+6\}$ is monochromatic, then also $\{a+b+5, a+b+6, a+b+9, a+b+11\}$, which is already covered by Case 3.3b.

Case 3.3e: If $\{a, a+1, a+3, a+5\}$ is monochromatic, then also $\{a+b+5, a+b+6, a+b+7\}$, which is covered by Case 3.2.

Case 3.3f: If $\{a, a+1, a+3, a+4\}$ is monochromatic, then also $\{a+b+5, a+b+6, a+b+7\}$, which is handled in Case 3.2.

This finishes the proof of Theorem 3.3. □

Next we consider colorings of the positive integers with three colors.

Proposition 3.6. *If* $b \in \mathbb{N}$ *and* $b \not\equiv 0 \bmod 6$, *then there exists a coloring* $\Delta : \mathbb{N} \longrightarrow \{0,1,2\}$ *such that no solution of* $x_3 - x_2 = x_2 - x_1 + b$ *is monochromatic.*

Proof. If $b \in \mathbb{N}$ is an odd integer, then we take for Δ an odd–even coloring. If b is even, then define $\Delta : \mathbb{N} \longrightarrow \{0,1,2\}$ by $\Delta(x) \equiv x \bmod 3$. □

For $b \equiv 0 \bmod 6$ we have some positive results. Namely, by exhaustive search we found that

$$D_3(6) = 56.$$

Indeed, there are exactly 13 nonisomorphic colorings showing $D_3(6) > 55$. Now, using the argument from Theorem 3.2 we infer

Proposition 3.7. *For each* $b \in \mathbb{N}$, $b \equiv 0 \bmod 6$,

$$D_3(b) \leqslant \frac{55}{6} b + 1 .$$

As $D_3(6)$ exists, we obtain by the techniques in Section 2 the following:

Corollary 3.8. *Let b be a nonzero rational number. Then the equation $x_3 - x_2 = x_2 - x_1 + b$ is 3-regular over $\mathbb{Q}$. Moreover, if $b > 0$, then $x_3 - x_2 = x_2 - x_1 + b$; $x_1 < x_2 < x_3$ is 3-regular over $\mathbb{Q}^+$.*

Corollary 3.9. *Let $b \in \mathbb{R}^+$ be a positive real number. Then for every coloring $\Delta : \mathbb{R}^+ \longrightarrow \{0,1,2\}$ there always exists a monochromatic solution of $x_3 - x_2 = x_2 - x_1 + b$ with $x_1 < x_2 < x_3$.*

On the other hand, the following holds:

Lemma 3.10. *There exists a coloring $\Delta : \mathbb{R} \longrightarrow \{0,1,2,3\}$ such that no solution of $x_3 - x_2 = x_2 - x_1 + 2$ is monochromatic.*

Proof. Consider the coloring $\Delta : \mathbb{R} \longrightarrow \{0,1,2,3\}$ with $\Delta(x) \equiv \lfloor x \rfloor \bmod 4$. Assume that x_1, x_2, x_3 is a monochromatic solution of $x_3 - x_2 = x_2 - x_1 + 2$.

Notice that by the choice of the coloring Δ, the difference $\lfloor x_j \rfloor - \lfloor x_i \rfloor$ is always divisible by 4.

If $\lfloor x_3 \rfloor - \lfloor x_2 \rfloor \leqslant \lfloor x_2 \rfloor - \lfloor x_1 \rfloor$, then

$$x_3 - 2x_2 + x_1 < \lfloor x_3 \rfloor - 2\lfloor x_2 \rfloor + \lfloor x_1 \rfloor + 2 \leqslant 2 .$$

Otherwise, if $\lfloor x_3 \rfloor - \lfloor x_2 \rfloor > \lfloor x_2 \rfloor - \lfloor x_1 \rfloor$, then we already have $\lfloor x_3 \rfloor - 2\lfloor x_2 \rfloor + \lfloor x_1 \rfloor \geqslant 4$, hence

$$x_3 - 2x_2 + x_1 \geqslant 4 - 2x_2 + 2\lfloor x_2 \rfloor > 2 .$$

In each case we obtain that $x_3 - 2x_2 + x_1 \neq 2$, hence Δ has the desired properties. □

We infer with Lemmas 2.3 and 3.10 and with Corollary 3.9 the following:

Corollary 3.11. *For every $b \in \mathbb{R}^+$,*

$$dor_{\mathbb{R}}\,(x_3 - x_2 = x_2 - x_1 + b;\ x_1 < x_2 < x_3) = 3 .$$

This implies immediately that $D_4(b)$ is undefined for every $b \in \mathbb{N}$. In particular, summarizing Propositions 3.1, 3.6 and 3.7, as well as Theorem 3.2 we obtain

Theorem 3.12. *For every $b \in \mathbb{N}$,*

$$dor_{\mathbb{N}}\,(x_3 - x_2 = x_2 - x_1 + b;\ x_1 < x_2 < x_3) = \begin{cases} 1 & \text{if } b \equiv 1 \bmod 2, \\ 2 & \text{if } b \equiv 0 \bmod 2 \\ & \quad \text{and } b \not\equiv 0 \bmod 6, \\ 3 & \text{if } b \equiv 0 \bmod 6. \end{cases}$$

4. The canonical case

Erdős and Graham considered in [3] arbitrary colorings of the positive integers. For arithmetic progressions they proved the following canonical partition result:

Theorem 4.1 (Erdős and Graham [3]). *For every $k \in \mathbb{N}$ there exists a least integer $n = EG(k)$ such that for every coloring $\Delta : \{1,2,\dots,n\} \longrightarrow \mathbb{N}$ there exists a k-term arithmetic progression $a, a+d, \dots, a+(k-1)d$ which is either monochromatic or totally multicolored, i.e., $\Delta(a+id) = \Delta(a+jd) \Longleftrightarrow i = j$.*

For a detailed proof we refer to [4]. As with the van der Waerden numbers not much is known about the growth of $EG(k)$ (exponential lower bound, F_4 upper bound). The canonical case for arbitrary partition regular systems of equations was given in [5]. There it was shown that, in general, two cases (monochromatic respective totally multicolored) do not suffice to describe the behaviour of the solutions of such systems, three cases are needed and sufficient. On the other hand, there are systems, whose canonical behaviour can be described by two cases:

Theorem 4.2 (Frankl et al. [4]). *Let A be a finite matrix with integer entries such that $A(1,\dots,1) = \mathbf{0}$. Assume that $A\mathbf{x} = \mathbf{0}$ admits a solution with $x_i = x_j$ if and only if $i = j$.*

Then for every coloring $\Delta : \mathbb{N} \longrightarrow \mathbb{N}$ there exists a solution $x_1, x_2, \dots, x_n$ with pairwise distinct x_i's of $A\mathbf{x} = \mathbf{0}$ such that $\{x_1, x_2, \dots, x_n\}$ is either monochromatic or totally multicolored.

For the canonical situation we consider next the equation $x_3 - x_2 = x_2 - x_1 + b$. For this case, some interesting phenomena occur:

Proposition 4.3. *If $b \in \mathbb{N}$ and $b \not\equiv 0 \bmod 6$, then there exists a coloring $\Delta : \mathbb{N} \longrightarrow \{0,1,2\}$ such that any solution of $x_3 - x_2 = x_2 - x_1 + b$ is neither monochromatic nor totally multicolored.*

Proof. By Proposition 3.6, it suffices to consider the case b even. Then the coloring $\Delta : \mathbb{N} \longrightarrow \{0,1,2\}$ with $\Delta(x) \equiv x \bmod 3$ gives no monochromatic solution of $x_3 - x_2 = x_2 - x_1 + b$. Suppose that some x_1, x_2, x_3 are totally multicolored. Then $x_1 + x_2 + x_3 \equiv 0 \bmod 3$, thus $x_3 = 2x_2 - x_1 + b$ becomes $x_1 + x_2 + x_3 = 3x_2 + b$ or $3x_2 + b \equiv 0 \bmod 3$, which is impossible. □

We remark that one can show by exhaustive search that for every coloring $\Delta : \{1,2,\dots,21\} \longrightarrow \{0,1,2\}$ there always exists a solution of $x_3 - x_2 = x_2 - x_1 + 6$, with $x_1 < x_2 < x_3$, which is either monochromatic or totally multicolored. Indeed, this is not true if we replace 21 by 20.

As $D_3(6)$ exists, one possibly would expect now that for every coloring $\Delta : \mathbb{N} \longrightarrow \{0,1,2,3\}$ there exists a solution of $x_3 - x_2 = x_2 - x_1 + 6$ which is either monochromatic or totally multicolored. Somewhat surprisingly, this is not the case. To see this take the modulo coloring $\Delta : \mathbb{N} \longrightarrow \{0,1,2,3\}$ with $\Delta(x) \equiv x \bmod 4$. Clearly, there is no monochromatic solution of $x_3 - x_2 = x_2 - x_1 + 6$. Assume that there exists a totally multicolored solution. If $x_2 \equiv 0 \bmod 4$ or $x_2 \equiv 2 \bmod 4$, then $x_1 + x_3 \equiv 2 \bmod 4$, which is a contradiction to the assumption that the solution is totally multicolored. If $x_2 \equiv 1 \bmod 4$ or $x_2 \equiv 3 \bmod 4$, then $x_1 + x_3 \equiv 0 \bmod 4$, again a contradiction. Indeed, using Proposition 3.6, it is easy to see that for every $b \not\equiv 0 \bmod 12$ there exists a coloring $\Delta : \mathbb{N} \longrightarrow \{0,1,2,3\}$ such that any solution of $x_3 - x_2 = x_2 - x_1 + b$ is neither monochromatic nor totally multicolored.

We also found that for every coloring $\Delta : \{1,2,\ldots,34\} \longrightarrow \{0,1,2,3\}$ there always exists a solution of $x_3 - x_2 = x_2 - x_1 + 12$, which is either monochromatic or totally multicolored, and the number 34 is the best possible. It might be interesting to know the behaviour of the solutions of general equations $\sum_{i=1}^{n} a_i x_i = b$, where $b \neq 0$, with respect to arbitrary colorings $\Delta : \mathbb{N} \longrightarrow C$.

References

[1] S.A. Burr and S. Loo, Numerical calculations of Rado numbers, preprint, 1992.

[2] V. Chvátal, Some unknown van der Waerden numbers, in: Calgary International Conference on Combinatorial Structures and their Applications (Gordon and Breach, New York, 1970) 31–33.

[3] P. Erdős and R.L. Graham, Old and new problems and results in combinatorial number theory, Enseign. Math. Monographie 28, Genéve (1980).

[4] P. Frankl, R.L. Graham and V. Rödl, Quantitative theorems for regular systems of equations, J. Combin. Theory, Ser. A 47 (1988) 246–261.

[5] H. Lefmann, A canonical version for partition regular systems of equations, J. Combin. Theory, Ser. A 44 (1986) 95–104.

[6] R. Rado, Studien zur Kombinatorik, Math. Z. 36 (1933) 424–480.

[7] R. Rado, Note on combinatorial analysis, Proc. London Math. Soc. Ser. II 48 (1943) 122–160.

[8] B.L. van der Waerden, Beweis einer Baudetschen Vermutung, Nieuw Arch. Wisk. 15 (1927) 212–216.