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A new regularization method for solving a time-fractional inverse diffusion problem ☆

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ABSTRACT

In this paper, we consider an inverse problem for a time-fractional diffusion equation in a one-dimensional semi-infinite domain. The temperature and heat flux are sought from a measured temperature history at a fixed location inside the body. We show that such problem is severely ill-posed and further apply a new regularization method to solve it based on the solution given by the Fourier method. Convergence estimates are presented under the a priori bound assumptions for the exact solution. Finally, numerical examples are given to show that the proposed numerical method is effective.

1. Introduction

The calculus of fractional derivatives and fractional differential equations has been used recently to solve a range of problems in physics, chemistry, biology, mechanical engineering, signal processing, systems identification, electrical engineering, control theory, finance and fractional dynamics, see [10,14,17].

Time-fractional diffusion equation arises by replacing the standard time partial derivative in the diffusion equation with a time-fractional partial derivative. It is usually used to describe anomalous diffusion (superdiffusion, non-Gaussian diffusion, subdiffusion) which is not consistent with the classical Fick (or Fourier) law [2,10]. In [13], Oldham and Spanier considered the solution of a time-dependent diffusion equation for semi-infinite planar, cylindrical, or spherical geometry with common initial and asymptotic boundary conditions. It was shown that the boundary value problem may be described by a simple equation which contains only a first order spatial derivative and a half-order time-fractional derivative (i.e., time-fractional derivative of order $\alpha = \frac{1}{2}$). For that model, Murio considered a corresponding half-fractional inverse heat conduction problem in [11]:

$$\begin{aligned}
&u_x(x,t) = -a^{-\frac{1}{2}}\left({}_0^{RL}D_t^{\frac{1}{2}}u(x,t)\right) + u_\infty(\pi a t)^{-\frac{1}{2}}, \quad x>0,\ t>0,\\
&u(x_1,t) = f(t), \quad \text{with approximate data function } f^\delta(t),\ x_1>0,\ t\geqslant 0,\\
&u(0,t) = U(t), \quad \text{unknown},\ t\geqslant 0,\\
&-u_x(0,t) = q(t), \quad \text{unknown},\ t\geqslant 0,
\end{aligned}$$

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where a is a constant diffusivity coefficient,

$$u_\infty = u(x,0) = \lim_{x\to\infty} u(x,t) = \text{const}.$$

The time-fractional derivative ${}_0^{RL}D_t^\alpha u(x,t)$ is the Riemann–Liouville fractional derivative of order α $(0<\alpha\leqslant 1)$ defined by

$$ {}_0^{RL}D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{u(x,s)}{(t-s)^\alpha}\,ds, \quad 0<\alpha<1,$$

$$ {}_0^{RL}D_t^\alpha u(x,t) = \frac{\partial u(x,t)}{\partial t}, \quad \alpha=1,$$

where $\Gamma(\cdot)$ is the Gamma function.

In this paper, we study a more general ill-posed problem for the time-fractional diffusion equation in a one-dimensional semi-infinite domain as follows

$$-au_x(x,t) = {}_0D_t^\alpha u(x,t), \quad x>0,\ t>0,\ \alpha\in(0,1), \tag{1.1}$$

$$u(1,t) = f(t), \quad t\geqslant 0, \tag{1.2}$$

$$u(x,0) = \lim_{x\to\infty} u(x,t) = 0, \tag{1.3}$$

where the time-fractional derivative ${}_0D_t^\alpha u(x,t)$ is the Caputo fractional derivative of order α $(0<\alpha\leqslant 1)$ defined by [12]

$${}_0D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,s)}{\partial s}\frac{ds}{(t-s)^\alpha}, \quad 0<\alpha<1, \tag{1.4}$$

$${}_0D_t^\alpha u(x,t) = \frac{\partial u(x,t)}{\partial t}, \quad \alpha=1. \tag{1.5}$$

If the function $u(x,t)$ is continuously differentiable, the temperature field is smooth enough, and $u(x,0)=0$ (i.e., the initial state is an equilibrium state), applying integration by parts, the Riemann–Liouville fractional derivative will coincide with the Caputo fractional derivative [12]. In addition, the initial–boundary value problem for fractional differential equations with the Caputo derivatives takes on the same form as for integer order differential equations, so it is widely used in engineering, physics, chemistry, biology, etc. In particular, such time-fractional equation like (1.1) can describe the anomalous diffusion in complex systems such as random fractal media (see [1,6,15,16]). Moreover, applying the Laplace transform and the inverse Laplace transform, we know the time-fractional diffusion equation (1.1) contains a large class of higher order partial differential equations, which have a wide range of applications in flame front propagation, plasma instabilities, phase turbulence in reaction–diffusion system, spatiotemporal chaos in one space dimension, the motion of a fluid going down a vertical wall and the study of the water waves with surface tension [7,18], see Remark 5. For further details on fractional derivatives and their application, see [14].

Our main purpose is to recover the temperature and heat flux for $0\leqslant x<1$. In this article, we will present a new regularization method to construct a stable approximate solution of the time-fractional inverse diffusion problem (TFIDP) (1.1)–(1.3), which is a kind of modified equation method. Namely, we modify Eq. (1.1) as follows:

$$-au_x(x,t) = P_\mu(t) * \left[{}_0D_t^\alpha u(x,t)\right], \quad x>0,\ t>0, \tag{1.6}$$

where $P_\mu(t) = \frac{1}{2\mu}e^{-\frac{|t|}{\mu}}$, $\mu\in(0,1)$ plays a role of regularization parameter, and "$*$" denotes convolution operation.

Our paper is divided into five sections. In Section 2, we present the ill-posedness of the problem and propose our new regularization method. In Section 3, convergence estimates for temperature u and heat flux u_x are given based on the a priori assumptions for the exact solution. Numerical results are shown in Section 4. Finally, we give a conclusion in Section 5.

2. Ill-posedness of the problem and a new regularization method

In order to apply the Fourier transform, we extend all the functions to the whole line $-\infty<t<\infty$ by defining them to be zero for $t<0$.

Here, and in the following sections, $\|\cdot\|$ denotes the L_2 norm, i.e.

$$\|f\| = \left(\int_{\mathbb{R}} |f(t)|^2\,dt\right)^{\frac{1}{2}}.$$

The Fourier transform of the function $f(t)$ is written as

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt,$$

and $\|\cdot\|_p$ denotes the H_p norm, i.e.

$$\|f\|_p=\left(\int_{\mathbb{R}}\left(1+\omega^2\right)^p\left|\hat{f}(\omega)\right|^2 d\omega\right)^{\frac{1}{2}}.$$

Taking a Fourier transformation to (1.1) and (1.2) with respect to t, from [14], we have

$$\hat{u}_x(x,\omega)=-\frac{1}{a}(i\omega)^\alpha\hat{u}(x,\omega), \tag{2.1}$$

$$\hat{u}(1,\omega)=\hat{f}(\omega), \tag{2.2}$$

where

$$(i\omega)^\alpha=\begin{cases}|\omega|^\alpha(\cos\frac{\alpha\pi}{2}+i\sin\frac{\alpha\pi}{2}), & \omega\geqslant 0,\\ |\omega|^\alpha(\cos\frac{\alpha\pi}{2}-i\sin\frac{\alpha\pi}{2}), & \omega<0.\end{cases} \tag{2.3}$$

The solution to the above problem and its derivative to variable x can easily be given by

$$\hat{u}(x,\omega)=e^{\frac{1}{a}(i\omega)^\alpha(1-x)}\hat{f}(\omega), \tag{2.4}$$

$$\hat{u}_x(x,\omega)=-\frac{1}{a}(i\omega)^\alpha e^{\frac{1}{a}(i\omega)^\alpha(1-x)}\hat{f}(\omega). \tag{2.5}$$

It follows that

$$\hat{u}(0,\omega)=e^{\frac{1}{a}(i\omega)^\alpha}\hat{f}(\omega), \tag{2.6}$$

$$\hat{u}_x(0,\omega)=-\frac{1}{a}(i\omega)^\alpha e^{\frac{1}{a}(i\omega)^\alpha}\hat{f}(\omega). \tag{2.7}$$

Note that $(i\omega)^\alpha$ has a positive real part, the small error in the high frequency components will be amplified by the factor $e^{\frac{1}{a}|\omega|^\alpha(1-x)\cos\frac{\alpha\pi}{2}}$ for $0\leqslant x<1$, so recovering temperature u and heat flux u_x from a measured temperature at $x=1$ are severely ill-posed. We must use some regularization methods to deal with this problem.

In this paper, in order to obtain a stable approximation solution of the TFIDP (1.1)–(1.3), we actually consider the following problem:

$$-av_x(x,t)=P_\mu(t)*\left[{}_0D_t^\alpha v(x,t)\right],\quad x>0,\ t>0, \tag{2.8}$$

$$v(x,0)=0,\quad x\geqslant 0, \tag{2.9}$$

$$v(1,t)=f^\delta(t),\quad t\geqslant 0,\qquad v(x,t)|_{x\to\infty}\quad\text{bounded}, \tag{2.10}$$

where $P_\mu(t)=\frac{1}{2\mu}e^{-\frac{|t|}{\mu}}$, $\mu\in(0,1)$ is a regularization parameter and the measured data f^δ satisfies

$$\left\|f^\delta-f\right\|\leqslant\delta, \tag{2.11}$$

in which the constant $\delta>0$ is called an error level. We intend to recover the temperature u and heat flux u_x for $0\leqslant x<1$ from the measured data $f^\delta(t)$.

Similarly, we can get the formal solution of problem (2.8)–(2.10) by the Fourier transform as follows:

$$\hat{v}(x,\omega)=e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}(1-x)}\hat{f}^\delta(\omega), \tag{2.12}$$

$$\hat{v}_x(x,\omega)=-\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}(1-x)}\hat{f}^\delta(\omega). \tag{2.13}$$

3. Convergence estimates

In order to obtain our main results, we first give a lemma.

Lemma 1.

(1) *If $s \geqslant 0$, then we have*

$$1 - e^{-s} \leqslant s.$$

(2) *If $s \geqslant 0$, $\gamma > 0$, $0 < x < 1$ and $\alpha \in (0, 1)$, then we obtain*

$$s^{\gamma} e^{-sx\cos\frac{\alpha\pi}{2}} \leqslant \left(\frac{\gamma}{ex\cos\frac{\alpha\pi}{2}}\right)^{\gamma}.$$

Because (1) is obvious, we only prove (2).

Proof. (2) Consider the function

$$h(s) = s^{\gamma} e^{-sx\cos\frac{\alpha\pi}{2}}, \quad s \geqslant 0.$$

It is easy to check that

$$h'(s) = \left(\gamma - sx\cos\frac{\alpha\pi}{2}\right)s^{\gamma-1}e^{-sx}.$$

Thus, $h(s)$ attains its maximum at $s = \frac{\gamma}{x\cos\frac{\alpha\pi}{2}}$, that is,

$$h(s) = s^{\gamma} e^{-sx} \leqslant h\left(\frac{\gamma}{x\cos\frac{\alpha\pi}{2}}\right) = \left(\frac{\gamma}{ex\cos\frac{\alpha\pi}{2}}\right)^{\gamma}. \qquad \square$$

Theorem 2. *(Convergence estimate for temperature.) Suppose that u is the solution of problem* (1.1)–(1.3)*, whose Fourier transform is given by* (2.4)*. v is the solution of problem* (2.8)–(2.10)*, whose Fourier transform is given by* (2.12)*, and the measured data f^{δ} satisfies* (2.11)*.*

(1) *If the a priori bound $\|u(0,\cdot)\| \leqslant E$ holds, and the regularization parameter μ is selected by*

$$\mu = \frac{1}{2a\ln\frac{E}{\delta}}, \tag{3.1}$$

then for every $x \in (0, 1)$, we have a convergence estimate

$$\left\|u(x,\cdot) - v(x,\cdot)\right\| \leqslant \varepsilon_1 + C_1 \cdot \frac{E}{(\ln\frac{E}{\delta})^2}, \tag{3.2}$$

where $\varepsilon_1 = \max\{e^{\frac{1}{a}}\delta, E^{1-x}\delta^{x}\}$, $C_1 = \frac{C}{2}a^{\frac{2}{\alpha}-2}$, $C = (\frac{1+\frac{2}{\alpha}}{e\cos\frac{\alpha\pi}{2}x})^{1+\frac{2}{\alpha}}$.

(2) *If the a priori bound $\|u(0,\cdot)\|_p \leqslant E$ holds, and the regularization parameter μ is selected by*

$$\mu = \frac{1}{2a\ln(\ln\frac{E}{\delta})}, \tag{3.3}$$

then for $p > 0$, $x = 0$, we have a convergence estimate

$$\left\|u(0,\cdot) - v(0,\cdot)\right\| \leqslant \varepsilon_2 + \varepsilon_3, \tag{3.4}$$

where $\varepsilon_2 = \max\{e^{\frac{1}{a}}\delta, \delta\ln\frac{E}{\delta}\}$, $\varepsilon_3 = \max\{2\mu^{\frac{2p}{3}}, \frac{2}{a}\mu^{\frac{2}{3}(1+p-\alpha)}, \frac{2}{a}\mu^2\}$.

Proof. (1) From (2.4), (2.6), (2.12) and applying the Parseval theorem, we obtain

$$\begin{aligned}\|u(x,\cdot)-v(x,\cdot)\| &= \|\hat{u}(x,\cdot)-\hat{v}(x,\cdot)\| = \|e^{\frac{1}{a}(i\omega)^\alpha(1-x)}\hat{f}(\omega)-e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}(1-x)}\hat{f}^\delta(\omega)\| \\ &\leqslant \|(e^{\frac{1}{a}(i\omega)^\alpha(1-x)}-e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}(1-x)})\hat{f}\| + \|e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}(1-x)}(\hat{f}-\hat{f}^\delta)\| \\ &= \|(e^{\frac{1}{a}(i\omega)^\alpha(1-x)}-e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}(1-x)})e^{-\frac{1}{a}(i\omega)^\alpha}\hat{u}(0,\omega)\| + \|e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}(1-x)}(\hat{f}-\hat{f}^\delta)\| \\ &\leqslant \sup_{\omega\in\mathbb{R}} A(\omega)\cdot E + \sup_{\omega\in\mathbb{R}} B(\omega)\cdot\delta,\end{aligned}$$

where

$$A(\omega) = \left|(e^{\frac{1}{a}(i\omega)^\alpha(1-x)}-e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}(1-x)})e^{-\frac{1}{a}(i\omega)^\alpha}\right|, \qquad B(\omega) = \left|e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}(1-x)}\right|. \tag{3.5}$$

For the second term of (3.5), it is easily seen that

$$B(\omega) = e^{\frac{1}{a}\frac{|\omega|^\alpha\cos\frac{\alpha\pi}{2}}{1+\mu^2\omega^2}(1-x)} \leqslant \begin{cases} e^{\frac{1}{a}}, & |\omega|\leqslant 1, \\ e^{\frac{1-x}{2a\mu}}, & |\omega|>1. \end{cases} \tag{3.6}$$

Next, we estimate the first term of (3.5). By Lemma 3.1, we have

$$\begin{aligned}A(\omega) &= \left|e^{-\frac{x}{a}(i\omega)^\alpha}\right|\cdot\left|1-e^{-\frac{1-x}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}(i\omega)^\alpha}\right| \\ &= e^{-\frac{x}{a}|\omega|^\alpha\cos\frac{\alpha\pi}{2}}\cdot\left|1-e^{-\frac{1-x}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}(|\omega|^\alpha\cos\frac{\alpha\pi}{2}\pm|\omega|^\alpha\sin\frac{\alpha\pi}{2}i)}\right| \\ &\leqslant e^{-\frac{x}{a}|\omega|^\alpha\cos\frac{\alpha\pi}{2}}\cdot\left(\left|1-e^{\mp i\frac{1-x}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}|\omega|^\alpha\sin\frac{\alpha\pi}{2}}\right|+\left|e^{\mp i\frac{1-x}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}|\omega|^\alpha\sin\frac{\alpha\pi}{2}}\right|\cdot\left|1-e^{-\frac{1-x}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}|\omega|^\alpha\cos\frac{\alpha\pi}{2}}\right|\right) \\ &= e^{-\frac{x}{a}|\omega|^\alpha\cos\frac{\alpha\pi}{2}}\cdot\left(2\left|\sin\left(\frac{1-x}{2a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}|\omega|^\alpha\sin\frac{\alpha\pi}{2}\right)\right|+\left(1-e^{-\frac{1-x}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}|\omega|^\alpha\cos\frac{\alpha\pi}{2}}\right)\right) \\ &\leqslant e^{-\frac{x}{a}|\omega|^\alpha\cos\frac{\alpha\pi}{2}}\cdot\left(\frac{1-x}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}|\omega|^\alpha\sin\frac{\alpha\pi}{2}+\frac{1-x}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}|\omega|^\alpha\cos\frac{\alpha\pi}{2}\right) \\ &\leqslant e^{-\frac{x}{a}|\omega|^\alpha\cos\frac{\alpha\pi}{2}}\cdot\frac{2\mu^2|\omega|^{2+\alpha}}{a}\leqslant 2a^{\frac{2}{\alpha}}C\mu^2,\end{aligned}$$

where $C=(\frac{1+\frac{2}{\alpha}}{e\cos\frac{\alpha\pi}{2}x})^{1+\frac{2}{\alpha}}$.

According to the estimates of $A(\omega)$ and $B(\omega)$ above, it follows that

$$\|u(x,\cdot)-v(x,\cdot)\|\leqslant 2a^{\frac{2}{\alpha}}C\mu^2\cdot E+\max\{e^{\frac{1}{a}}\cdot\delta, e^{\frac{1-x}{2a\mu}}\cdot\delta\}.$$

Noticing that μ is given by (3.1), we get

$$\begin{aligned}\|u(x,\cdot)-v(x,\cdot)\| &\leqslant \max\{e^{\frac{1}{a}}\cdot\delta, E^{1-x}\delta^x\}+\frac{C}{2}a^{\frac{2}{\alpha}-2}\cdot\frac{E}{(\ln\frac{E}{\delta})^2} \\ &= \varepsilon_1+\frac{C}{2}a^{\frac{2}{\alpha}-2}\cdot\frac{E}{(\ln\frac{E}{\delta})^2} \\ &= \varepsilon_1+C_1\cdot\frac{E}{(\ln\frac{E}{\delta})^2}.\end{aligned}$$

(2) As in the proof of (1), and using the a priori assumption $\|u(0,\cdot)\|_p\leqslant E$, we obtain

$$\begin{aligned}\|u(0,\cdot)-v(0,\cdot)\| &\leqslant \|(e^{\frac{1}{a}(i\omega)^\alpha}-e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}})\hat{f}\|+\|e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}}(\hat{f}-\hat{f}^\delta)\| \\ &= \|(e^{\frac{1}{a}(i\omega)^\alpha}-e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}})e^{-\frac{1}{a}(i\omega)^\alpha}(1+\omega^2)^{-\frac{p}{2}}(1+\omega^2)^{\frac{p}{2}}\hat{u}(0,\cdot)\|+\|e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}}(\hat{f}-\hat{f}^\delta)\| \\ &\leqslant \sup_{\omega\in\mathbb{R}}\widetilde{A}(\omega)\cdot E+\sup_{\omega\in\mathbb{R}}\widetilde{B}(\omega)\cdot\delta,\end{aligned}$$

where

$$\widetilde{A}(\omega)=\left|(e^{\frac{1}{a}(i\omega)^\alpha}-e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}})e^{-\frac{1}{a}(i\omega)^\alpha}(1+\omega^2)^{-\frac{p}{2}}\right|, \qquad \widetilde{B}(\omega)=\left|e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}}\right|. \tag{3.7}$$

By (3.6), we have

$$\widetilde{B}(\omega) \leqslant \begin{cases} e^{\frac{1}{a}}, & |\omega| \leqslant 1, \\ e^{\frac{1}{2a\mu}}, & |\omega| > 1. \end{cases} \tag{3.8}$$

In order to estimate the first term of (3.7), we rewrite

$$\widetilde{A}(\omega) = \left|1 - e^{-\frac{1}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}(i\omega)^\alpha}\right|\left(1+\omega^2\right)^{-\frac{p}{2}}. \tag{3.9}$$

Next, we will distinguish three cases to estimate (3.9).

Case 1. $|\omega| \geqslant \omega_0 = \mu^{-\frac{2}{3}}$, and noticing that $p > 0$, we obtain

$$\widetilde{A}(\omega) \leqslant \left(1 + \left|e^{-\frac{1}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}(i\omega)^\alpha}\right|\right)\omega_0^{-p} = \left(1 + e^{-\frac{1}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}|\omega|^\alpha \cos\frac{\alpha\pi}{2}}\right)\mu^{\frac{2p}{3}} \leqslant 2\mu^{\frac{2p}{3}}. \tag{3.10}$$

Case 2. $1 < |\omega| \leqslant \omega_0$, similar to the proof of (1), we get

$$\widetilde{A}(\omega) \leqslant \left|1 - e^{-\frac{1}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}(i\omega)^\alpha}\right| \cdot |\omega|^{-p} \leqslant \frac{2}{a}|\omega|^{2+\alpha-p}\mu^2. \tag{3.11}$$

If $0 < p \leqslant 2+\alpha$, note that $|\omega| \leqslant \omega_0$, (3.33) becomes

$$\widetilde{A}(\omega) \leqslant \frac{2}{a}\mu^{\frac{2}{3}(1+p-\alpha)}. \tag{3.12}$$

If $p > 2+\alpha$, by $|\omega| > 1$, we have

$$\widetilde{A}(\omega) \leqslant \frac{2}{a}\mu^2. \tag{3.13}$$

Case 3. $|\omega| \leqslant 1$, similar to estimating (3.33), we have

$$\widetilde{A}(\omega) \leqslant \left|1 - e^{-\frac{1}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}(i\omega)^\alpha}\right| \cdot |\omega|^{-p} \leqslant \frac{2}{a}|\omega|^{2+\alpha}\mu^2 \leqslant \frac{2}{a}\mu^2. \tag{3.14}$$

Using (3.1), (3.8), (3.32)–(3.14), we obtain

$$\left\|u(0,\cdot) - v(0,\cdot)\right\| \leqslant \max\left\{e^{\frac{1}{a}} \cdot \delta, \delta \ln\frac{E}{\delta}\right\} + \varepsilon_3 \leqslant \varepsilon_2 + \varepsilon_3.$$

Therefore, (3.4) holds. □

Theorem 3. *(Convergence estimate for heat flux.) Suppose that u is the solution of problem* (1.1)–(1.3), *whose Fourier transform is given by* (2.4). *v is the solution of problem* (2.8)–(2.10), *whose Fourier transform is given by* (2.12), *and the measured data f^δ satisfies* (2.11).

(1) *If the a priori bound $\|u(0,\cdot)\| \leqslant E$ holds, and the regularization parameter μ is selected by* (3.1), *then for every $x \in (0,1)$, we have a convergence estimate*

$$\left\|u_x(x,\cdot) - v_x(x,\cdot)\right\| \leqslant \varepsilon_4 + C_3 \cdot \frac{E}{(\ln\frac{E}{\delta})^2}, \tag{3.15}$$

where $\varepsilon_4 = \max\{\frac{1}{a}e^{\frac{1}{a}}\delta, E^{1-x}\delta^x \ln\frac{E}{\delta}\}$, $C_3 = \frac{C_2}{4}a^{\frac{2}{\alpha}-2}$, $C_2 = (\frac{1+\frac{2}{\alpha}}{e\cos\frac{\alpha\pi}{2}x})^{1+\frac{2}{\alpha}} + 2(\frac{2+\frac{2}{\alpha}}{e\cos\frac{\alpha\pi}{2}x})^{2+\frac{2}{\alpha}}$.

(2) *If the a priori bound $\|u(0,\cdot)\|_p \leqslant E$ holds, and the regularization parameter μ is selected by* (3.3), *then for $p > \alpha$, $x = 0$, we have a convergence estimate*

$$\left\|u_x(0,\cdot) - v_x(0,\cdot)\right\| \leqslant \varepsilon_5 + \varepsilon_6, \tag{3.16}$$

where $\varepsilon_5 = \max\{\frac{1}{a}e^{\frac{1}{a}}\delta, \delta\ln\frac{E}{\delta}\ln(\ln\frac{E}{\delta})\}$, $\varepsilon_6 = \max\{\frac{2}{a}\mu^{\frac{2}{3}(p-\alpha)}, \frac{1}{a}\mu^{\frac{2}{3}(1+p-\alpha)} + \frac{2}{a^2}\mu^{\frac{2}{3}(1+p-2\alpha)}, \frac{1}{a}\mu^2 + \frac{2}{a^2}\mu^{\frac{2}{3}(1+p-2\alpha)}, (\frac{1}{a}+\frac{2}{a^2})\mu^2\}$.

Proof. (1) By (2.5), (2.6), (2.13) and using the Parseval theorem, we obtain

$$\begin{aligned}
\|u_x(x,\cdot)-v_x(x,\cdot)\| &= \left\| \frac{1}{a}(i\omega)^\alpha e^{\frac{1}{a}(i\omega)^\alpha(1-x)}\hat f(\omega) - \frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2} e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}(1-x)}\hat f^\delta(\omega)\right\| \\
&\leqslant \left\| \left(\frac{1}{a}(i\omega)^\alpha e^{\frac{1}{a}(i\omega)^\alpha(1-x)} - \frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2} e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}(1-x)}\right)\hat f\right\| \\
&\quad + \left\| \frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2} e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}(1-x)}\big(\hat f - \hat f^\delta\big)\right\| \\
&= \left\| \left(\frac{1}{a}(i\omega)^\alpha e^{\frac{1}{a}(i\omega)^\alpha(1-x)} - \frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2} e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}(1-x)}\right)e^{-\frac{1}{a}(i\omega)^\alpha}\hat u(0,\omega)\right\| \\
&\quad + \left\| \frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2} e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}(1-x)}\big(\hat f - \hat f^\delta\big)\right\| \\
&\leqslant \sup_{\omega\in\mathbb{R}} A_x(\omega)\cdot E + \sup_{\omega\in\mathbb{R}} B_x(\omega)\cdot\delta,
\end{aligned}$$

where

$$A_x(\omega) = \left| \left(\frac{1}{a}(i\omega)^\alpha e^{\frac{1}{a}(i\omega)^\alpha(1-x)} - \frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2} e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}(1-x)}\right)e^{-\frac{1}{a}(i\omega)^\alpha}\right|, \tag{3.17}$$

$$B_x(\omega) = \left|\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2} e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}(1-x)}\right|. \tag{3.18}$$

For (3.20), we have

$$B_x(\omega) = \frac{1}{a}\frac{|\omega|^\alpha}{1+\mu^2\omega^2} e^{\frac{1}{a}\frac{|\omega|^\alpha\cos\frac{\alpha\pi}{2}}{1+\mu^2\omega^2}(1-x)} \leqslant \begin{cases} \frac{1}{a}e^{\frac{1}{a}}, & |\omega|\leqslant 1, \\ \frac{1}{2a\mu}e^{\frac{1-x}{2a\mu}}, & |\omega|>1. \end{cases} \tag{3.19}$$

Next, we estimate (3.17). As in the proof of Theorem 3.2, we get

$$\begin{aligned}
A_x(\omega) &= \big|e^{-\frac{x}{a}(i\omega)^\alpha}\big|\cdot\left|\frac{1}{a}(i\omega)^\alpha - \frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}e^{-\frac{1-x}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}(i\omega)^\alpha}\right| \\
&\leqslant e^{-\frac{x}{a}|\omega|^\alpha\cos\frac{\alpha\pi}{2}}\cdot\left(\left|\frac{(i\omega)^\alpha}{a}\right|\cdot\big|1-e^{\mp i\frac{1-x}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}|\omega|^\alpha\sin\frac{\alpha\pi}{2}}\big| + \frac{\mu^2\omega^2}{1+\mu^2\omega^2}\left|\frac{(i\omega)^\alpha}{a}\right|\cdot\big|e^{\mp i\frac{1-x}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}|\omega|^\alpha\sin\frac{\alpha\pi}{2}}\big|\right. \\
&\quad \left. + \left|\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}\right|\cdot\big|e^{\mp i\frac{1-x}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}|\omega|^\alpha\sin\frac{\alpha\pi}{2}} - e^{-\frac{1-x}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}(|\omega|^\alpha\cos\frac{\alpha\pi}{2}\pm|\omega|^\alpha\sin\frac{\alpha\pi}{2}i)}\big|\right) \\
&\leqslant e^{-\frac{x}{a}|\omega|^\alpha\cos\frac{\alpha\pi}{2}}\left(\frac{|\omega|^\alpha}{a}\frac{1-x}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}|\omega|^\alpha\sin\frac{\alpha\pi}{2} + \frac{|\omega|^\alpha}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}\right. \\
&\quad \left. + \frac{1}{a}\frac{|\omega|^\alpha}{1+\mu^2\omega^2}\frac{1-x}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}|\omega|^\alpha\cos\frac{\alpha\pi}{2}\right) \\
&\leqslant e^{-\frac{x}{a}|\omega|^\alpha\cos\frac{\alpha\pi}{2}}\cdot\left(\frac{|\omega|^{2+\alpha}}{a} + \frac{2|\omega|^{2+2\alpha}}{a^2}\right)\mu^2 \\
&\leqslant a^{\frac{2}{\alpha}}C_2\cdot\mu^2,
\end{aligned}$$

where $C_2 = (\frac{1+\frac{2}{\alpha}}{e\cos\frac{\alpha\pi}{2}x})^{1+\frac{2}{\alpha}} + 2(\frac{2+\frac{2}{\alpha}}{e\cos\frac{\alpha\pi}{2}x})^{2+\frac{2}{\alpha}}$.

According to the above analysis, and from (3.1), it follows that

$$\begin{aligned}
\|u(x,\cdot)-v(x,\cdot)\| &\leqslant a^{\frac{2}{\alpha}}C_2\cdot\mu^2\cdot E + \max\left\{\frac{1}{a}e^{\frac{1}{a}}\cdot\delta, \frac{1}{2a\mu}e^{\frac{1-x}{2a\mu}}\cdot\delta\right\} \\
&\leqslant \max\left\{\frac{1}{a}e^{\frac{1}{a}}\cdot\delta,\ E^{1-x}\delta^x\cdot\ln\frac{E}{\delta}\right\} + \frac{C_2}{4}a^{\frac{2}{\alpha}-2}\cdot\frac{E}{(\ln\frac{E}{\delta})^2} \\
&= \varepsilon_4 + \frac{C_2}{4}a^{\frac{2}{\alpha}-2}\cdot\frac{E}{(\ln\frac{E}{\delta})^2} \\
&= \varepsilon_4 + C_3\cdot\frac{E}{(\ln\frac{E}{\delta})^2}.
\end{aligned}$$

(2) Similar to the proof of (1), and applying the a priori assumption $\|u(0,\cdot)\|_p \leqslant E$, we have

$$\begin{aligned}\|u(0,\cdot)-v(0,\cdot)\| &= \|\hat{u}(0,\omega)-\hat{v}(0,\omega)\| \\ &\leqslant \left\|\left(\frac{1}{a}(i\omega)^\alpha e^{\frac{1}{a}(i\omega)^\alpha}-\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}}\right)\hat{f}\right\| + \left\|\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}}\left(\hat{f}-\hat{f}^\delta\right)\right\| \\ &= \left\|\left(\frac{1}{a}(i\omega)^\alpha e^{\frac{1}{a}(i\omega)^\alpha}-\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}}\right)e^{-\frac{1}{a}(i\omega)^\alpha}\left(1+\omega^2\right)^{-\frac{p}{2}}\left(1+\omega^2\right)^{\frac{p}{2}}\hat{u}(0,\cdot)\right\| \\ &\quad + \left\|\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}}\left(\hat{f}-\hat{f}^\delta\right)\right\| \\ &\leqslant \sup_{\omega\in\mathbb{R}}\widetilde{A_x}(\omega)\cdot E+\sup_{\omega\in\mathbb{R}}\widetilde{B_x}(\omega)\cdot\delta,\end{aligned}$$

where

$$\widetilde{A_x}(\omega)=\left|\left(\frac{1}{a}(i\omega)^\alpha e^{\frac{1}{a}(i\omega)^\alpha}-\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}}\right)e^{-\frac{1}{a}(i\omega)^\alpha}\left(1+\omega^2\right)^{-\frac{p}{2}}\right|, \tag{3.20}$$

$$\widetilde{B_x}(\omega)=\left|\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}e^{\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}}\right|. \tag{3.21}$$

By (3.21), we have

$$\widetilde{B_x}(\omega)\leqslant\begin{cases}\frac{1}{a}e^{\frac{1}{a}}, & |\omega|\leqslant 1,\\ \frac{1}{2a\mu}e^{\frac{1}{2a\mu}}, & |\omega|>1.\end{cases} \tag{3.22}$$

In order to estimate (3.20), we rewrite

$$\widetilde{A_x}(\omega)=\left|\frac{1}{a}(i\omega)^\alpha-\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}e^{-\frac{1}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}(i\omega)^\alpha}\right|\left(1+\omega^2\right)^{-\frac{p}{2}}. \tag{3.23}$$

Next, we also distinguish three cases to estimate (3.23).

Case 1. $|\omega|\geqslant\omega_0=\mu^{-\frac{2}{3}}$, and noticing that $p>\alpha$, we obtain

$$\begin{aligned}\widetilde{A}(\omega) &\leqslant \left(\left|\frac{1}{a}(i\omega)^\alpha\right|+\left|\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}e^{-\frac{1}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}(i\omega)^\alpha}\right|\right)|\omega|^{-p} \\ &= \frac{|\omega|^\alpha}{a}\left(1+\frac{1}{1+\mu^2\omega^2}e^{-\frac{1}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}|\omega|^\alpha\cos\frac{\alpha\pi}{2}}\right)|\omega|^{-p} \\ &\leqslant \frac{2}{a}|\omega|^{\alpha-p} \\ &\leqslant \frac{2}{a}\mu^{\frac{2}{3}(p-\alpha)}.\end{aligned} \tag{3.24}$$

Case 2. $1<|\omega|\leqslant\omega_0$, as in the proof of (1), we get

$$\widetilde{A}(\omega)\leqslant\left|\frac{1}{a}(i\omega)^\alpha-\frac{1}{a}\frac{(i\omega)^\alpha}{1+\mu^2\omega^2}e^{-\frac{1}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}(i\omega)^\alpha}\right|\cdot|\omega|^{-p}\leqslant\left(\frac{|\omega|^{2+\alpha-p}}{a}+\frac{2|\omega|^{2+2\alpha-p}}{a^2}\right)\mu^2. \tag{3.25}$$

If $\alpha<p\leqslant 2+\alpha$, note that $|\omega|\leqslant\omega_0$, (3.25) becomes

$$\widetilde{A}(\omega)\leqslant\frac{1}{a}\mu^{\frac{2}{3}(1+p-\alpha)}+\frac{2}{a^2}\mu^{\frac{2}{3}(1+p-2\alpha)}. \tag{3.26}$$

If $2+\alpha<p\leqslant 2+2\alpha$, by $1<|\omega|\leqslant\omega_0$, we have

$$\widetilde{A}(\omega)\leqslant\frac{1}{a}\mu^2+\frac{2}{a^2}\mu^{\frac{2}{3}(1+p-2\alpha)}. \tag{3.27}$$

If $p>2+2\alpha$, by $|\omega|>1$, we obtain

$$\widetilde{A}(\omega)\leqslant\left(\frac{1}{a}+\frac{2}{a^2}\right)\mu^2. \tag{3.28}$$

Case 3. $|\omega| \leqslant 1$, similar to estimating (3.25), we have

$$\widetilde{A}(\omega) \leqslant \left| \frac{1}{a}(i\omega)^{\alpha} - \frac{1}{a}\frac{(i\omega)^{\alpha}}{1+\mu^2\omega^2} e^{-\frac{1}{a}\frac{\mu^2\omega^2}{1+\mu^2\omega^2}(i\omega)^{\alpha}} \right| \leqslant \left(\frac{|\omega|^{2+\alpha}}{a} + \frac{2|\omega|^{2+2\alpha}}{a^2} \right)\mu^2 \leqslant \left(\frac{1}{a} + \frac{2}{a^2} \right)\mu^2. \tag{3.29}$$

From (3.3), (3.22), (3.24)–(3.29), we obtain

$$\left\| u_x(0,\cdot) - v_x(0,\cdot) \right\| \leqslant \max\left\{ \frac{1}{a} e^{\frac{1}{a}} \delta, \delta \ln\frac{E}{\delta} \ln\left(\ln\frac{E}{\delta} \right) \right\} + \varepsilon_6, \leqslant \varepsilon_5 + \varepsilon_6. \qquad \square$$

Remark 4. If $\alpha = 1$ in (1.1), we obtain the kinematic (i.e., first order) wave equation:

$$u_t(x,t) + a u_x(x,t) = 0.$$

However, for (3.2) and (3.15), note that $C_2 \to \infty$, $C_3 \to \infty$ as $\alpha \to 1$. Therefore, the convergence of temperature and heat flux for $0 < x < 1$ may not be valid.

Remark 5. Set $\alpha = \frac{1}{n}$ in (1.1), where n is a positive integer and $n > 1$. Laplace transforming (1.1) with respect to t (see Appendix A), and noting (1.3), we obtain

$$-a\widetilde{u}_x(x,s) = s^{\frac{1}{n}} \widetilde{u}(x,s), \tag{3.30}$$

and it follows

$$\widetilde{u}(x,s) = c(s) e^{-\frac{s^{\frac{1}{n}}}{a}x}, \tag{3.31}$$

where $c(s)$ is a function of s. Thus

$$\frac{\partial^n}{\partial x^n}\widetilde{u}(x,s) = \frac{(-1)^n}{a^n} s\widetilde{u}(x,s). \tag{3.32}$$

Taking inverse Laplace transform of (3.32), we get

$$(-1)^n \frac{\partial}{\partial t} u(x,t) = a^n \frac{\partial^n}{\partial x^n} u(x,t). \tag{3.33}$$

Thus if $\alpha = \frac{1}{n}(n > 1)$, the solution of (1.1) will also be the solution of (3.33). In other words, our results are suitable for (3.33), which is a large class of higher order partial differential equations. For example:

- If $n = 2$, (3.33) is the heat conduct equation: $\frac{\partial}{\partial t}u(x,t) = a^2 \frac{\partial^2}{\partial x^2}u(x,t)$. The corresponding problem for the heat conduct equation has been considered in many papers, such as [3,4].
- If $n = 3$, (3.33) is Airy's equation [5]: $\frac{\partial}{\partial t}u(x,t) + a^3 \frac{\partial^3}{\partial x^3}u(x,t) = 0$.
- If $n = 4$, (3.33) is the simple Kuramoto–Sivashinsky equation [18]: $\frac{\partial}{\partial t}u(x,t) = a^4 \frac{\partial^4}{\partial x^4}u(x,t)$.
- If $n = 5$, (3.33) is the simple Kawahara equation or Kaup–Kupershmidt equation [7]: $\frac{\partial}{\partial t}u(x,t) + a^5 \frac{\partial^5}{\partial x^5}u(x,t) = 0$.

4. Numerical examples

In this section, we show some numerical results obtained by the new regularization method for an example. The convergence and stability of the algorithm when using noisy data can be verified.

We use the following formulae to generate the noisy data

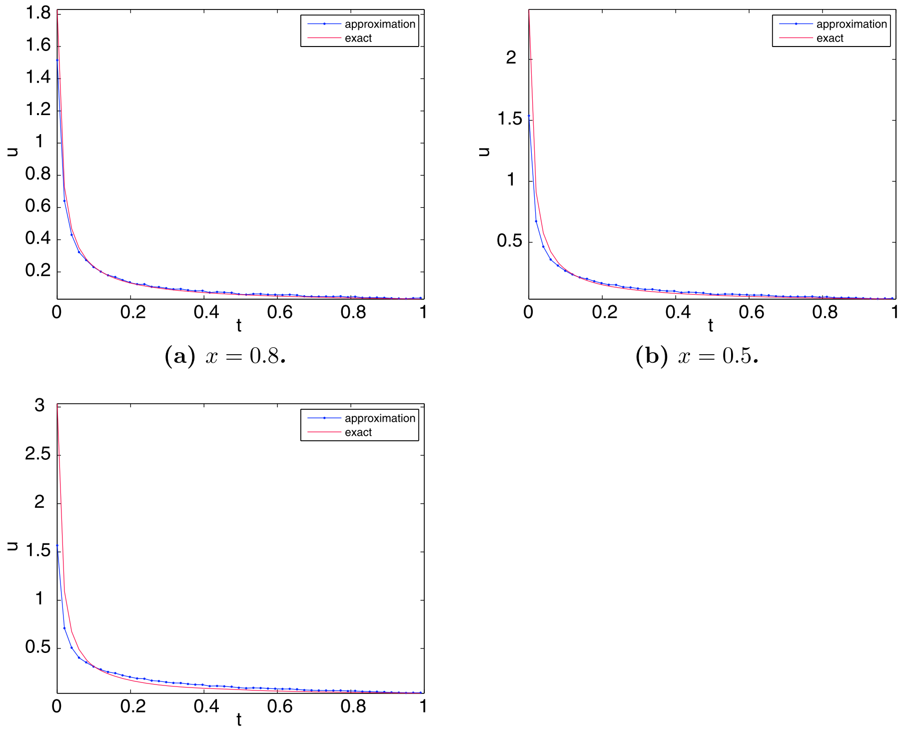
$$f^{\delta}(t_n) = f(t_n) + \varepsilon \cdot rand(n), \tag{4.1}$$

where $t_n = nk$ $(n = 1, 2, \ldots, s)$, k is a time step size, and $f(t_n)$ is the exact data, $rand(n)$ is a random number uniformly distributed in $[-1, 1]$ and the magnitude ε indicates a noise level.

In the following numerical example, we set $a = 1$ in (1.1), and choose $k = \frac{1}{101}$, $s = 100$, $\varepsilon = 0.01$ in (4.1).

Example. From Theorem 6 in Appendix A, we see that

$$u(x,t) = \begin{cases} \frac{\alpha(x+1)}{t^{\alpha+1}} M_{\alpha}\left(\frac{x+1}{t^{\alpha}}\right), & t > 0,\ x > 0, \\ 0, & t \leqslant 0, \end{cases} \tag{4.2}$$



(a) $x = 0.8$. (b) $x = 0.5$. (c) $x = 0.2$.

Fig. 1. The exact solution u and the regularized solution v with $\alpha = 0.1$, $\mu = 0.1086$.

is the exact solution of problem (1.1)–(1.3) with data

$$f(t) = \begin{cases} \frac{2\alpha}{t^{\alpha+1}} M_\alpha(\frac{2}{t^\alpha}), & t > 0,\ x > 0, \\ 0, & t \leqslant 0. \end{cases} \tag{4.3}$$

In addition,

$$u_x(x,t) = \begin{cases} \frac{\alpha}{t^{\alpha+1}} M_\alpha(\frac{x+1}{t^\alpha}) - \frac{\alpha(x+1)}{t^{2\alpha+1}} K(x,t), & t > 0,\ x > 0, \\ 0, & t \leqslant 0, \end{cases} \tag{4.4}$$

where

$$K(x,t) = \sum_{k=0}^{\infty} \frac{(-\frac{x+1}{t^\alpha})^{k-1}}{(k-1)!\Gamma(1-\alpha-\alpha k)}.$$

We apply numerical integration formula to compute the exact data. By using the noisy data $f^\delta(t_n)$ generated from (4.1), according to (2.10) and (2.11), we get the regularized solutions u_c^δ and $u_{c,x}^\delta$ computed by using the Discrete Fourier Transform (DFT).

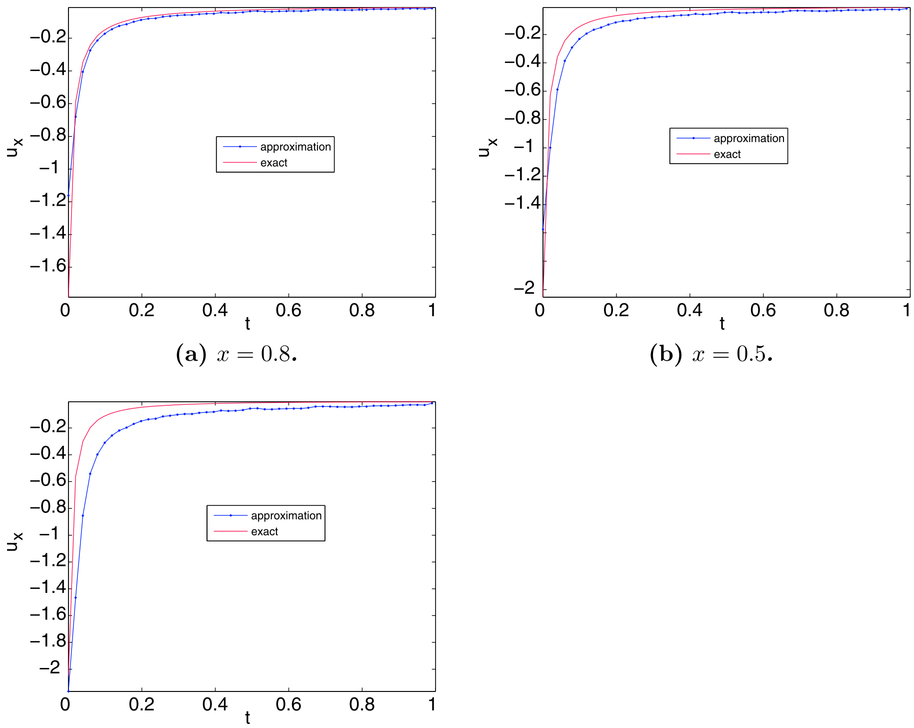
The numerical results for temperature u with $\alpha = 0.1$ at $x = 0.8, 0.5, 0.2$ are shown in Fig. 1, and heat flux u_x with $\alpha = 0.1$ at $x = 0.8, 0.5, 0.2$ are shown in Fig. 2.

The numerical results for temperature u with $\alpha = 0.1$ at $x = 0$ are shown in Fig. 3, and the numerical results for heat flux u_x with $\alpha = 0.1$ at $x = 0$ are shown in Fig. 4.

The numerical results for temperature u with $\alpha = 0.3$ at $x = 0.8, 0.5, 0.2$ are shown in Fig. 5, and heat flux u_x with $\alpha = 0.3$ at $x = 0.8, 0.5, 0.2$ are shown in Fig. 6.

The numerical results for temperature u with $\alpha = 0.3$ at $x = 0$ are shown in Fig. 7, and the numerical results for heat flux u_x with $\alpha = 0.3$ at $x = 0$ are shown in Fig. 8.

From Figs. 1–8, it can be seen that numerical results near the boundary $x = 1$ are better than the ones close to $x = 0$. Moreover, we can see numerical accuracy becomes worse as the order α of Caputo fractional derivative increases. This is consistent with our theoretical analysis (see Remark 4).

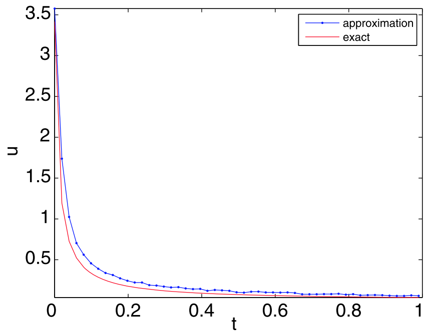


(a) $x = 0.8$.

(b) $x = 0.5$.

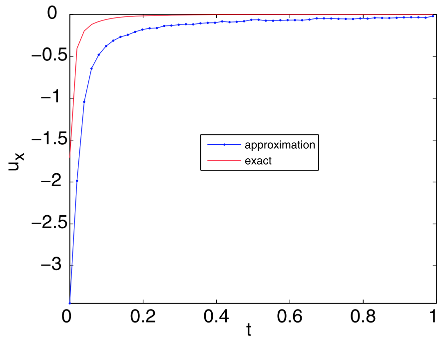
(c) $x = 0.2$.

Fig. 2. The exact solution u_x and the regularized solution v_x with $\alpha = 0.1$, $\mu = 0.01$.



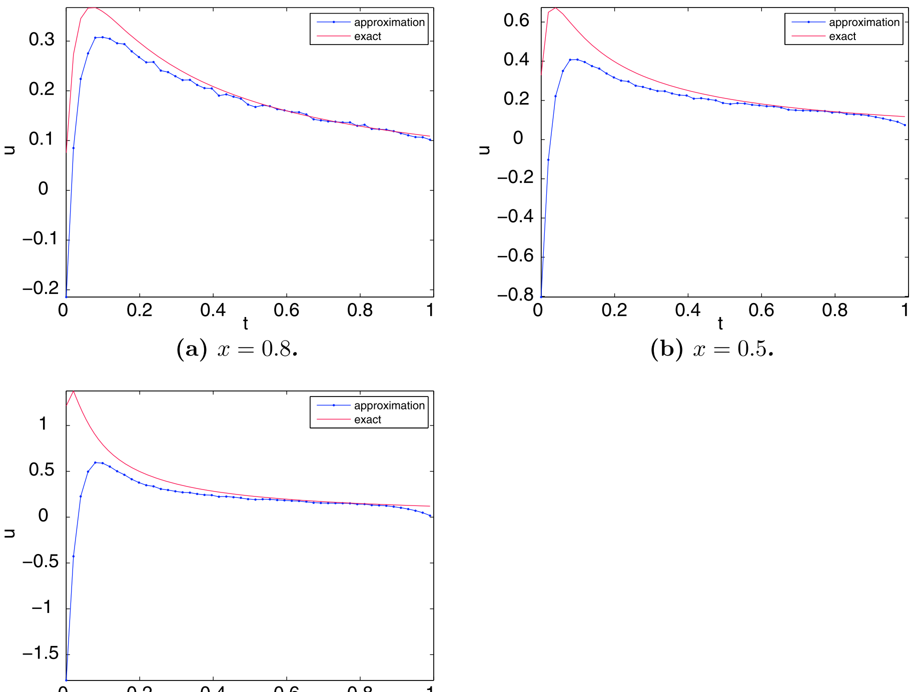
$\alpha = 0.1,\ \mu = 0.008.$

Fig. 3. The exact solution u and the regularized solution v at $x = 0$.



$\alpha = 0.1,\ \mu = 0.008.$

Fig. 4. The exact solution u_x and the regularized solution v_x at $x = 0$.

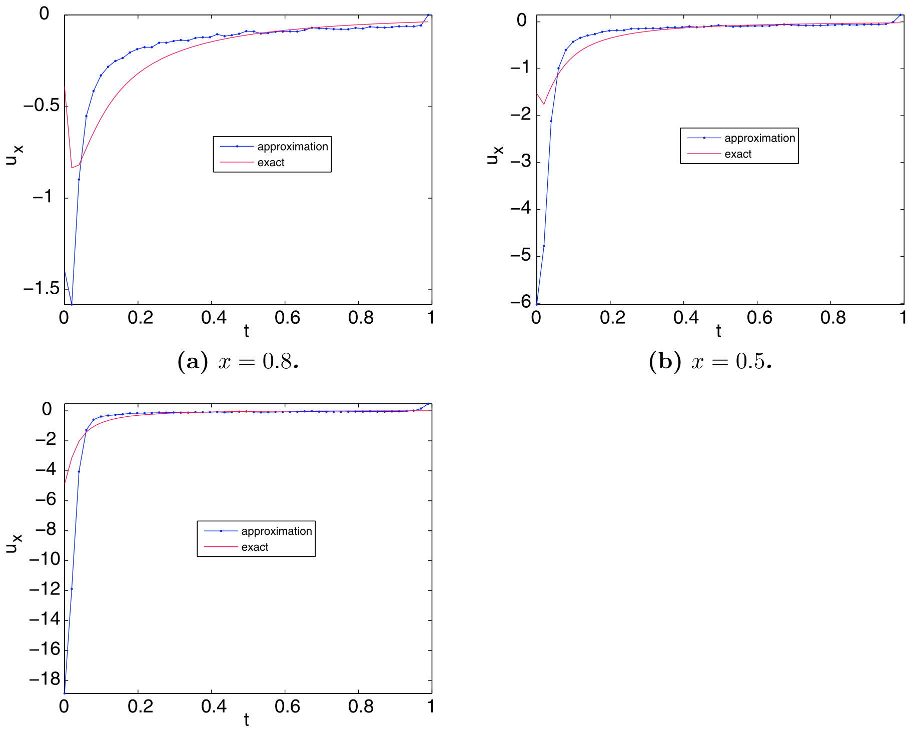


(a) $x = 0.8$.

(b) $x = 0.5$.

(c) $x = 0.2$.

Fig. 5. The exact solution u and the regularized solution v with $\alpha = 0.3$, $\mu = 0.03$.



(a) $x = 0.8$.

(b) $x = 0.5$.

(c) $x = 0.2$.

Fig. 6. The exact solution u_x and the regularized solution v_x with $\alpha = 0.3$, $\mu = 0.01$.

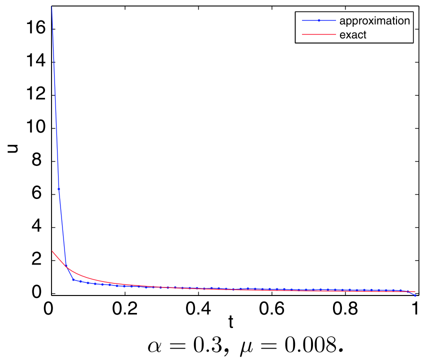


Fig. 7. The exact solution u and the regularized solution v at $x = 0$.

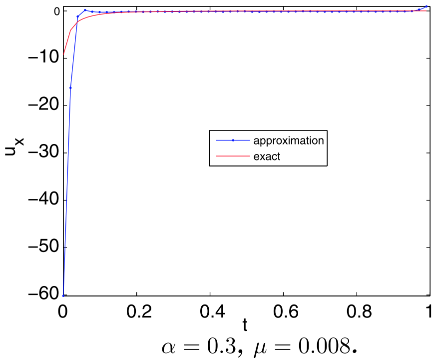


Fig. 8. The exact solution u_x and the regularized solution v_x at $x = 0$.

5. Conclusion

In this paper, we use a new regularization method to solve an inverse problem for a time-fractional diffusion equation in a one-dimensional semi-infinite domain. The convergence results have been obtained for the cases $0 \leqslant x < 1$ under the a priori bound assumptions for the exact solution and the suitable choices of the regularization parameter. Finally, the numerical results show that the proposed methods are effective and stable.

Appendix A

The Laplace transform of a function $f(t)$ is defined below [14]:

$$\widetilde{f}(s) = \mathcal{L}\{f(t); s\} = \int_0^\infty e^{-st} f(t)\, dt \tag{A.1}$$

and its inverse is given by the formula

$$f(t) = \mathcal{L}^{-1}\{\widetilde{f}(s); t\} = \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \widetilde{f}(s)\, ds, \quad \gamma = Re(s) > s_0. \tag{A.2}$$

The Laplace transform formula for the Caputo fractional derivative of $f(t)$:

$$\mathcal{L}\{{}_0D_t^\alpha f(t); s\} = s^\alpha \widetilde{f}(s) - s^{\alpha-1} f(0), \quad 0 < \alpha \leqslant 1. \tag{A.3}$$

We investigate the following time-fractional diffusion equation:

$$-u_x(x,t) = {}_0D_t^\alpha u(x,t), \quad x > 0,\ t > 0,\ \alpha \in (0,1), \tag{A.4}$$

$$u(0,t) = \delta(t), \quad t \geqslant 0, \tag{A.5}$$

$$u(x,0) = \lim_{x\to\infty} u(x,t) = 0, \tag{A.6}$$

where $\delta(t)$ denotes the Dirac delta function.

Theorem 6. *The exact solution of problem* (A.4)–(A.6) *is given by*

$$u(x,t) = \frac{\alpha x}{t^{\alpha+1}} M_\alpha\left(xt^{-\alpha}\right), \tag{A.7}$$

where $M_\alpha(z)$ *denotes Mainardi's function* [8]:

$$M_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(1-\alpha-\alpha k)}, \quad 0 < \alpha < 1. \tag{A.8}$$

Proof. Applying the Laplace transform with respect to t to (A.4) and (A.5), using (A.6), we have

$$-\widetilde{u}_x(x,s) = s^\alpha \widetilde{u}(x,s), \tag{A.9}$$

$$\widetilde{u}(0,s) = 1. \tag{A.10}$$

By (A.6), we get

$$\widetilde{u}(x,s) = e^{-s^\alpha x}. \tag{A.11}$$

Note the Laplace transform formula [9]:

$$\mathcal{L}\left\{\frac{\alpha x}{t^{\alpha+1}} M_\alpha\left(xt^{-\alpha}\right); s\right\} = e^{-s^\alpha x}. \tag{A.12}$$

Hence, (A.7) holds. □

Remark 7. Setting $\alpha = 1$ in (A.7), and noting $M_1(z) = \delta(x-1)$, we obtain the exact solution of the first order wave equation:

$$u(x,t) = \frac{x}{t}\delta(x-t). \tag{A.13}$$

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