Empirical Hankel transforms and its applications to goodness-of-fit tests

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Abstract

We introduce a special Hankel transform for probability distributions on the nonnegative half-line and discuss some of its properties. Due to the uniqueness of the transform we suggest an integral type test statistic based on the empirical Hankel transform to treat simple and composite hypotheses goodness-of-fit problems. The special case of exponential distributions is studied in detail.

1. Introduction

Fourier transforms and Laplace transforms are important tools widely used in mathematics, especially in probability and statistics. Occasionally, other types of integral transforms, e.g. the Mellin transform or the Hankel transform, are needed and turn out to be more helpful in specific situations. Surprisingly, the latter one does not seem to have received much attention in stochastics. Early exceptions are the papers [1,2]. Defining the Hankel transform, we follow [3]. Let $J_\nu$ denote the Bessel function of the first kind of order $\nu \in \mathbb{R}$. From the asymptotic expansion of $J_\nu(x)$ as $x \to \infty$ (see, e.g. [4], p. 85) we deduce that the function $x^{1/2} J_\nu(x), \ x > 0$, is bounded. Let $F$ be a real function defined on the positive half-line $\mathbb{R}_+$. Assuming that the function $x^{1/2} F(x), \ x > 0$, is integrable with respect to the Lebesgue–Borel measure on the Borel sets of $\mathbb{R}_+$, the real function on $\mathbb{R}_+$ defined by

\[ G_\nu(t) = \int_0^\infty xF(x)J_\nu(tx)dx, \quad t > 0, \]

is the Hankel transform of order $\nu$ of $F$. Putting

\[ f(x) = 2x^{3/2} F(2\sqrt{x}), \quad x > 0, \quad \text{and} \quad g_\nu(t) = t^{-\nu/2} G_\nu(\sqrt{t}), \quad t > 0, \]

the Hankel transform can be stated as

\[ g_\nu(t) = \int_0^\infty (xt)^{-\nu/2} J_\nu(2\sqrt{xt})f(x)dx, \quad t > 0. \]

In what follows we exclusively deal with the special case $\nu = 0$, where $J_0(x)$ for $x \in \mathbb{R}$ is given by

\[ J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!^2} \left( \frac{x}{2} \right)^{2k} = \frac{1}{\pi} \int_0^\pi \cos(x \sin \theta) d\theta = \frac{1}{\pi} \int_0^\pi \cos(x \cos \theta) d\theta, \]
and speak of
\[ \int_0^\infty J_0(2\sqrt{tx})f(x)dx, \quad t \geq 0, \]
as a modified Hankel transform of \( f \). By noting that
\[ |J_0(x)| \leq 1 \quad \text{for each } x \geq 0, \quad (1.1) \]
the modified Hankel transform is defined for each probability density \( f \) with respect to the Lebesgue–Borel measure on the Borel sets of \( \mathbb{R}_+ \), and is equal to \( E(J_0(2\sqrt{tX})) \) for all \( t \geq 0 \), if \( X \) is a real nonnegative random variable with density function \( f \). These considerations motivate us to define the modified Hankel transform of \( X \) to be
\[ \mathcal{H}_X(t) = E(J_0(2\sqrt{tX})), \quad t \geq 0. \quad (1.2) \]

In what follows we simply speak of the Hankel transform of \( X \) or the Hankel transform of \( f \). Due to \( J_0(0) = 1 \), continuity of \( J_0 \) and \( (1.1) \) the Hankel transform of \( X \) has, just like the ordinary Fourier transform (characteristic function) of \( X \), the properties
\[ \mathcal{H}_X(0) = 1, \quad (1.3) \]
\[ |\mathcal{H}_X(t)| \leq 1 \quad \text{for each } t \geq 0, \quad (1.4) \]
\[ \mathcal{H}_X \text{ is continuous}. \quad (1.5) \]

Examples and some other important properties of Hankel transforms, e.g. a uniqueness theorem and a continuity theorem, will be derived in the next section. As main statistical applications, goodness-of-fit tests based on empirical Hankel transforms will be discussed in Section 3. The test statistics are of the Cramér–von Mises type. The special case of testing for exponentiality is studied in detail. The limit distribution of the test statistic is given in the hypothesis case and for special sequences of local alternatives. Its limiting approximate Bahadur efficiency is seen to coincide with the limiting Pitman efficiency. On the basis of this efficiency concept a comparison with the classical goodness-of-fit tests of Cramér–von Mises test and Anderson–Darling is done. Numerical results on critical values of the limit null distribution and a simulation study on the power performance of the test accompany the theoretical findings. Other applications of empirical Hankel transforms and a generalization to the multivariate case are addressed in Section 4.

2. Examples and properties of Hankel transforms

Example 2.1. Let \( X \) have the gamma distribution \( G(n+1, \lambda) \) with scale parameter \( 0 < \lambda = 1/E(X) \) and shape parameter \( n+1 \), where \( n \) is some nonnegative integer. Using formula 6.643.4 of [5] the Hankel transform of \( X \) is seen to be
\[ \mathcal{H}_X(t) = L_n \left( \frac{t}{\lambda} \right) \exp \left( -\frac{t}{\lambda} \right), \quad t \geq 0, \]
where \( L_n \) is the Laguerre polynomial of order \( n \). In the special case \( n = 0 \) the random variable \( X \) is exponentially distributed with parameter \( \lambda > 0 \). Its Hankel transform is
\[ \mathcal{H}_X(t) = \lambda \int_0^\infty J_0(2\sqrt{tx}) \exp(-\lambda x)dx = \exp \left( -\frac{t}{\lambda} \right), \quad t \geq 0. \quad (2.1) \]

Example 2.2. Let \( X \) be an exponentially distributed random variable with parameter \( 1 \), and \( Z \) be another nonnegative random variable independent of \( X \). The Hankel transform of \( ZX \) is given by
\[ \mathcal{H}_{ZX}(t) = E(J_0(2\sqrt{tzX})) = E(\exp(-tZ)) = \Psi_Z(t), \quad t \geq 0, \]
where \( \Psi_Z(t) \) represents the Laplace transform of the random variable \( Z \). This fact motivates us to extend the uniqueness theorem of Laplace transforms to that of Hankel transforms. In what follows \( \cong \) denotes equality in distribution.

Theorem 2.1. Let \( X \) and \( Y \) be two nonnegative independent random variables with corresponding Hankel transforms \( \mathcal{H}_X \) and \( \mathcal{H}_Y \). Then \( \mathcal{H}_X = \mathcal{H}_Y \) if and only if \( X \cong Y \).

Proof. If \( X \cong Y \), then it is obvious that \( \mathcal{H}_X = \mathcal{H}_Y \). For the other direction, let \( \Psi_X \) and \( \Psi_Y \) be the Laplace transforms of \( X \) and \( Y \), and suppose \( Z \) is exponentially distributed with parameter \( \lambda = 1 \) and independent of \( X \) and \( Y \). It follows from \( \mathcal{H}_X(t) = \mathcal{H}_Y(t) \) for all \( t \geq 0 \), that \( \mathcal{H}_X(tZ) = \mathcal{H}_Y(tZ) \) for all \( t \geq 0 \), and therefore
\[ \Psi_X(t) = E(J_0(2\sqrt{tZX})) = E(J_0(2\sqrt{tZY})) = \Psi_Y(t), \quad t \geq 0. \]
The result follows now from the uniqueness theorem for Laplace transforms. \( \square \)
A random variable the Fourier transform of which is of the form \( \exp(-t^2) \) for \( t \) in a neighborhood of 0 is known to have the \( N(0, 2) \) distribution. This fact can be used to obtain a corresponding result for Hankel transforms and exponential distributions.

**Theorem 2.2.** Let \( X \) be some nonnegative random variable, the Hankel transform of which being of the form \( \mathcal{H}_X(t) = \exp(-t^2) \), \( 0 \leq t \leq \varepsilon \) for some \( \varepsilon > 0 \). Then \( X \) is exponentially distributed with parameter \( \lambda = 1 \).

**Proof.** Consider a nonnegative random variable \( Z \) which has the Beta distribution \( \operatorname{B} \left( \frac{1}{2}, \frac{1}{2} \right) \) with density \( \pi^{-1}x^{-1/2}(1-x)^{-1/2} \) for \( 0 < x < 1 \) and 0 else, and which is independent of \( Y = 2\sqrt{X} \). The random variable \( B = 2Z - 1 \) is symmetrically distributed about zero and has the Fourier transform \( \Phi_B(t) = \int_0^\infty \exp(iyt) \cdot f(y) \, dy \), \( t \in \mathbb{R} \). Then we have

\[
\exp(-t^2) = \mathcal{H}_X(t^2) = E(J_0(tY)) = E(\exp(itBY)), \quad |t| \leq \varepsilon.
\]

Thus, the random variable \( BY \) has the normal \( N(0, 2) \) distribution. For \( k \in \mathbb{N} \) we get

\[
E((BY)^{2k-1}) = 0, \quad E((BY)^{2k}) = \frac{(2k)!}{2^k k!} \cdot 2^k = \frac{(2k)!}{k!}.
\]

Independence of \( B \) and \( Y \) leads to

\[
E(X^k) = \frac{1}{2^{2k}} \cdot \frac{(2k)!}{k!} \cdot 2^k = \frac{(2k)!}{k!} \cdot k \in \mathbb{N}.
\]

(2.2)

It is easy to check that \( B^2 \overset{\mathcal{D}}{=} Z \). The moments of the random variable \( B^2 \) are seen to be

\[
E((B^2)^k) = \frac{1}{2^{2k}} \cdot \frac{(2k)!}{k!} \cdot 2^k = \frac{(2k)!}{k!} \cdot k \in \mathbb{N}.
\]

Therefore (2.2) becomes \( E(X^k) = k! \cdot k \in \mathbb{N} \), which is the \( k \)th moment of the \( \text{Exp}(1) \) distribution. Since the \( \text{Exp}(1) \) distribution is uniquely determined by its moments, we conclude that \( X \sim \text{Exp}(1) \). \( \square \)

There is also a continuity theorem for Hankel transforms. Let \( \overset{\mathcal{D}}{\rightarrow} \) denote convergence in distribution.

**Theorem 2.3.** Let \( h_n, \ n \in \mathbb{N} \), be a sequence of Hankel transforms of nonnegative random variables \( X_n, \ n \in \mathbb{N} \). If there exists some nonnegative random variable \( X \) with Hankel transform \( h \) such that \( X_n \overset{\mathcal{D}}{\rightarrow} X \), then

\[
\lim_{n \to \infty} h_n(t) = h(t) \quad \text{for each } t \geq 0.
\]

(2.3)

Conversely, if there is some real function \( h \) on \( \mathbb{R}_+ \) such that (2.3) holds and \( h \) is continuous at 0 with \( h(0) = 1 \), then \( h \) is the Hankel transform of some nonnegative random variable \( X \), and \( X_n \overset{\mathcal{D}}{\rightarrow} X \).

**Proof.** The first assertion follows from the dominated convergence theorem. For the second assertion, let \( Z \geq 0 \) be independent of the \( X_1, X_2, \ldots \) and exponentially distributed with parameter 1. \( \Psi_n(t) = E(h_n(tZ)), \ t \geq 0 \), is the Laplace transform of \( X_n \). By dominated convergence, \( \lim_{n \to \infty} \Psi_n(t) = \Psi(t), \ t \geq 0 \), where \( \Psi(t) = E(h(tZ)), \ t \geq 0 \), is continuous at 0 and \( \Psi(0) = 1 \). The continuity theorem for Laplace transforms (see, e.g. [6], p. 431) implies that there is some nonnegative random variable \( X \) such that \( \Psi \) is the Laplace transform of \( X \), and \( X_n \overset{\mathcal{D}}{\rightarrow} X \). \( \square \)

### 3. Goodness-of-fit tests

Let \( X_1, X_2, \ldots \) be independent copies of a nonnegative random variable \( X \) with unknown distribution \( P^X \). Let \( \mathcal{E} \neq \emptyset \) denote some given parametric family of distributions. Let us consider testing the hypothesis \( H : P^X \in \mathcal{E} \) against the general alternative \( K : P^X \notin \mathcal{E} \). In a recent paper, Jiménez-Gamero et al. [7] present a general approach to goodness-of-fit tests based on empirical characteristic functions. Meintanis and Swanepoel [8] deals with general transform based Cramér–von Mises type test statistics for goodness-of-fit problems. Generally, these statistics are integrated squared distances between an empirical transform obtained from the given sample \( X_1, \ldots, X_n \) and the corresponding transform of the true or estimated distribution in the hypothesis case. In situations where the (limit) null distribution is unknown, bootstrap or permutation procedures can be used to obtain critical values. An overview on statistical tests using empirical Fourier transforms, empirical Laplace transforms or empirical probability generating functions is also given in [8]. Here we confine ourselves treating the hypothesis of exponentiality, i.e. \( \mathcal{E} = \{ \text{Exp}(\lambda) : \lambda > 0 \} \) where \( \text{Exp}(\lambda) \) denotes the exponential distribution with parameter \( \lambda > 0 \); the approach to other scale families \( \mathcal{E} \) of distributions is essentially the same. For a recent overview on tests for exponentiality, we refer to [9]. Let us denote by

\[
\mathcal{H}_n(t) = n^{-1} \sum_{j=1}^n J_0(2\sqrt{tY_j}), \quad t \geq 0
\]

the empirical Hankel transform of the empirically standardized variables \( Y_1 = X_1/\bar{X}_n, \ldots, Y_n = X_n/\bar{X}_n \), where \( \bar{X}_n = n^{-1} \sum_{j=1}^n X_j \) stands for the sample mean; put \( Y_j = 0 \) if \( X_n = 0 \). The Cramér–von Mises type test statistic we propose is
\[ T_n^2 = n \int_0^\infty \left( \mathcal{H}_n(t) - \exp(-t) \right)^2 \exp(-t) dt. \]  

It is motivated by the uniqueness theorem for Hankel transforms and the fact that due to the law of large numbers the almost sure pointwise limit of \( \mathcal{H}_n(\cdot) \) is \( \mathcal{H}_\infty(\cdot / \mu) \), where \( \mu = E(X) \). Carrying out the integral in (3.1), we are led to express the test statistic in form of a V-statistic of order 2,

\[ T_n^2 = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} h(Y_i, Y_j), \]  

where the kernel \( h \) is given by

\[ h(x, y) = \exp(-x - y)I_0(2\sqrt{xy}) - (\exp(-x/2) + \exp(-y/2))/2 + \frac{1}{3} \]

for \( x, y \geq 0 \). Of course, replacing the weight function \( \exp(-t) \), \( t \geq 0 \), in (3.1) by suitable other weight functions one obtains test statistics which may apply as well. The advantage of our choice is the alternative representation (3.2) with the simple kernel \( h \) ensuring that the test can be carried out easily for given data. Additionally, for the weight function chosen a complete description of the limit null distribution is possible (see Theorem 3.2). For considerations on the slightly more general weight function \( \exp(-\beta t) \), \( t \geq 0 \), we refer to [10]. See also Remark 3.1 for a result on the limit null distribution in the case of a fairly general weight function.

### 3.1. The limit null distribution

Another representation of the test statistic is

\[ T_n^2 = \int_0^\infty Z_n^2(t) dP_0(t), \]

where

\[ Z_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} [J_0(2\sqrt{tY_j}) - \exp(-t)], \quad t \geq 0, \]

and \( P_0 = \text{Exp}(1) \) denotes the exponential distribution with parameter \( \lambda = 1 \). The stochastic process

\[ Z_n := (Z_n(t), t \geq 0) \]  

(3.3)

can be regarded as a random element in the separable Hilbert space \( L^2(\mathbb{R}_+, \mathcal{B}_+, P_0) \) of squared \( P_0 \)-integrable functions \( f : \mathbb{R}_+ \to \mathbb{R} \). The statistic \( T_n^2 \) is scale invariant, i.e. \( T_n^2(X_1, \ldots, X_n) = T_n^2(cX_1, \ldots, cX_n) \) for each \( c > 0 \), which implies that the distribution of \( T_n^2 \) does not depend on the scale parameter \( \lambda \). Thus, to derive the limit distribution of \( T_n^2 \) in the case where the hypothesis is true, we can (and do) assume that \( \lambda = 1 \). Additionally, we assume without loss of generality that \( \hat{X}_n \) is positive for each sample size. In what follows, let \( J_0 \) denote the modified Bessel function of the first kind of order 0, i.e.

\[ J_0(t) = \sum_{k=0}^{\infty} \frac{1}{k!^2} \left( \frac{t}{2} \right)^{2k}, \quad t \in \mathbb{R}. \]

**Theorem 3.1.** Let \( X_1, X_2, \ldots \) be a sequence of independent \( \text{Exp}(1) \) distributed random variables and let \( Z_n \) be the stochastic process defined in (3.3). Then there exists a centered Gaussian process \( Z = (Z(t), t \geq 0) \) which can be regarded as a random element in \( L^2 \) and which has the covariance function

\[ k(s, t) = \exp(-(s + t))(J_0(2\sqrt{st}) - st - 1), \quad s, t \geq 0, \]  

(3.4)

such that

\[ Z_n \xrightarrow{D} Z \quad \text{in} \ L^2. \]

Moreover for the test statistic defined in (3.1) it holds that

\[ T_n^2 \xrightarrow{D} \int_0^\infty Z(t)^2 \exp(-t) dt. \]

**Proof.** We note that \( J_0' = -J_1 \) with \( J_1 \) being the Bessel function of first kind of order 1. A Taylor expansion gives

\[ J_0(2\sqrt{tY_j}) = J_0(2\sqrt{\hat{X}_j}) + \frac{2(\hat{X}_j - 1)}{\sqrt{\hat{X}_j}(\sqrt{\hat{X}_j} + 1)} \sqrt{\hat{X}_j} J_1(2\sqrt{\hat{X}_j}) \]

\[ + \frac{2(\hat{X}_j - 1)}{\sqrt{\hat{X}_j}(\sqrt{\hat{X}_j} + 1)} \sqrt{\hat{X}_j} \left[ J_1(2\sqrt{\hat{X}_j}(1 + \theta_{n,1}(\hat{X}_j^{-1/2} - 1))) - J_1(2\sqrt{\hat{X}_j}) \right]. \]
where $\theta_{n,j,t}^* \in [0,1]$. Introducing the auxiliary processes

$$
\tilde{Z}_n = (\tilde{Z}_n(t), t \geq 0), \quad \tilde{Z}_n = (\tilde{Z}_n(t), t \geq 0) \quad \text{and} \quad \tilde{Z}_n = (\tilde{Z}_n(t), t \geq 0),
$$

where

$$
\tilde{Z}_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left[ j_0(2\sqrt{\bar{X}_n}) + \frac{2(\tilde{X}_n - 1)}{\sqrt{\bar{X}_n}} \right] \exp(-t), \quad t \geq 0,
$$

$$
\tilde{Z}_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left[ j_0(2\sqrt{\bar{X}_n}) + 2\exp(-t) \right] \frac{\tilde{X}_n - 1}{\sqrt{\bar{X}_n}} \exp(-t), \quad t \geq 0,
$$

$$
\tilde{Z}_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left[ j_0(2\sqrt{\bar{X}_n}) + (X_j - 1)\exp(-t) \right] \exp(-t), \quad t \geq 0,
$$

we aim to verify the assertions

$$
\tilde{Z}_n \stackrel{d}{\to} Z \quad \text{in } L^2,
$$

$$
\|Z_n - \tilde{Z}_n\|_{L^2} \to 0 \quad \text{in probability},
$$

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$$

$$
\|\tilde{Z}_n - Z\|_{L^2} \to 0 \quad \text{in probability}.
$$

To prove (3.5), put $\tilde{Z}_n(t) := j_0(2\sqrt{\bar{X}_n}) + (X_j - 1)\exp(-t) - \exp(-t), \quad t \geq 0, \quad j = 1, \ldots, n$. Then $E(\tilde{Z}_n(t)) = 0$ for each $t \geq 0$. The $\tilde{Z}_n$ can be regarded as centered independent and identically distributed random elements in $L^2$. Due to $E(\|\tilde{Z}_n\|_{L^2}^2) < \infty$ the central limit theorem in the Hilbert space $L^2$ (see Section 10 of [11]) applies. Thus, there is a centered Gaussian process $Z = Z(t), \quad t \geq 0$ which has the same covariance function as $\tilde{Z}_n$ and which can be regarded as random element in $L^2$ such that $\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \tilde{Z}_n \to \tilde{Z}$. The stochastic process $\tilde{Z}_n$ (as well as the process $Z$) has the covariance function $k$ given in (3.4). To verify (3.6), note that

$$
\|Z_n - \tilde{Z}_n\|_{L^2}^2 = \left( \frac{2\sqrt{\bar{X}_n} - 1}{\sqrt{\bar{X}_n}} \right)^2 V_n,
$$

where

$$
V_n = \int \left( \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \sqrt{\bar{X}_n} \left[ j_0(2\sqrt{\bar{X}_n}(1 + \theta_{n,j\tau}^* (\tilde{X}_n^{-1/2} - 1))) - j_1(2\sqrt{\bar{X}_n}) \right] \right)^2 \, dP_0(t).
$$

$\sqrt{\bar{X}_n} \exp(-t)$ converges in distribution to a zero mean normal distribution. $V_n$ converges to zero in probability. To see this, we split the integral above into two separate pieces $V_{n1}$ and $V_{n2}$. The limit of integration for $V_{n1}$ is from zero up to some point $0 < T < \infty$ and for $V_{n2}$ is from $T$ to infinity. Defining

$$
A_\delta(x) = \sup_{0 \leq t \leq T, \delta \in [0,1]} \left| j_1(2\sqrt{\bar{X}_n}(1 + \theta_{n,j\tau}^* (\tilde{X}_n^{-1/2} - 1))) - j_1(2\sqrt{\bar{X}_n}) \right|,
$$

it is clear that for each $x \geq 0$, if $\delta \to 0$ then $A_\delta(x) \to 0$. Since $|y_1| \leq \sqrt{2}/2$, we have $A_\delta \leq \sqrt{2}$. Applying the dominated convergence theorem, for each $l \in \mathbb{N}$ we have $E(A_\delta(X_1)) \to 0$ as $\delta$ tends to zero. Now for each $\varepsilon > 0$ and $\delta > 0$ it holds that

$$
P(\{|V_n| > \varepsilon\}) \leq \varepsilon P(\{|X_1| \leq \sqrt{2}/2, \ |\tilde{X}_n^{-1/2} - 1| < \delta\}) + P(\{|\tilde{X}_n^{-1/2} - 1| \geq \delta\}).
$$

Of course, $\lim_{n \to \infty} P(\{|\tilde{X}_n^{-1/2} - 1| \geq \delta\}) = 0$ for each $\delta > 0$. Applying the inequalities of Cauchy-Schwarz and Markov we obtain using $\int t \, dP_0(t) = 1$

$$
\lim_{n \to \infty} P(\{|V_n| > \varepsilon, \ |\tilde{X}_n^{-1/2} - 1| < \delta\}) \leq \varepsilon^{-1} \left[ E(X_1^2)E(A_\delta^2(X_1)) \right]^{1/2},
$$

where $E(X_1^2) = 2$ and $E(A_\delta^2(X_1)) \to 0$ as $\delta$ tends to zero. This gives $V_{n1} \to 0$ in probability. Using the inequalities of Cauchy-Schwarz and Markov again it is seen that for each $\varepsilon > 0$,

$$
\lim_{n \to \infty} P(\{|V_{n2}| > \varepsilon\}) \leq \frac{2E(X_1)}{\varepsilon} \int_{[T,\infty)} t \, dP_0(t).
$$

(3.10)

T can be chosen so large that the value on the right side of (3.10) becomes arbitrarily small. Thus, \( V_n \to 0 \), which finishes the proof of (3.6). The assertion (3.7) follows from \( \| \hat{Z}_n - \bar{Z}_n \|_{L^2}^2 = U_n W_n \) where \( U_n = \left( \frac{2(\delta_n - 1)}{\sqrt{X_n(\sqrt{X_n} + 1)}} \right)^2 \) converges to 0 in probability and

\[
W_n = \int \left( \frac{1}{\sqrt{n}} \sum_{j=1}^{n} [\sqrt{\delta j} f_j(2\sqrt{\delta j}) - t \exp(-t)] \right)^2 d\mathcal{P}_0(t)
\]

converges in distribution by the central limit theorem in the Hilbert space \( L^2 \). The assertion (3.8) is obviously true. The assertions (3.5)–(3.8) together with Theorem 4.1 of [12] yield \( Z_n \overset{\mathcal{D}}{\to} Z \) in the Hilbert space \( L^2 \). Regarding to the continuous mapping theorem, we have \( \| Z_n \|_{L^2}^2 \overset{\mathcal{D}}{\to} \| Z \|_{L^2}^2 \), and this is precisely the last statement of the theorem. □

**Remark 3.1.** The limit null distribution of more general test statistics of the form

\[
n \int |\mathcal{H}_n(t) - \exp(-t)|^2 \, dq(t)
\]

with \( q \) being some finite measure on the Borel sets of \( \mathbb{R}_+ \) can be treated in a similar manner. Inspecting the proof of Theorem 3.1 we see that the additional condition \( \int t \mathcal{H} q(t) < \infty \) suffices to identify the limit null distribution to be that of \( \int Z^2(t) \, dq(t) \), where the centered Gaussian process \( Z \) with covariance function \( k \) given in (3.4) is now regarded as random element in \( L^2 = L^2(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+}, q) \).

It is well known that the asymptotic distribution of \( T_n^2 \) under the null hypothesis is as same as that of \( \sum_{i \geq 1} \delta_i X_i^2 \), where \( (X_i^2)_{i \geq 1} \) is a sequence of independent \( X_i^2 \)-variables, and \( (\delta_i)_{i \geq 1} \) is the sequence of positive eigenvalues of the covariance operator \( \delta \) of the random element \( Z \), see, e.g. [13], page 58. The operator \( \delta : L^2 \to L^2 \) is associated with the covariance function \( k \) and is defined by

\[
\delta \phi(t) = \int_0^\infty k(s, t) \phi(s) \exp(-s) \, ds, \quad t \geq 0, \text{ for } \phi \in L^2.
\]

\( \delta \) is a Hilbert–Schmidt operator. The countable set of eigenvalues \( \delta \) and eigenfunctions \( \phi \) of \( \delta \) is obtained as the set of solutions of the integral equation

\[
\int_0^\infty k(s, t) \phi(s) \exp(-s) \, ds = \delta \phi(t), \quad t \geq 0.
\]

The complete solution is presented in the next theorem. For a proof we refer to [10].

**Theorem 3.2.** Let \( b = \phi - 1 \) where \( \phi = (\sqrt{5} + 1)/2 \) is the golden section number. The positive eigenvalues of the integral operator associated with the kernel function introduced in (3.4) are the positive zeros of the function \( \omega(u) = A(u) B(u) - D^2(u), \quad u > 0, \) where

\[
A(u) = 1 - \sum_{n=0}^{\infty} \frac{\sqrt{5}b^{4n+4}}{b^{4n+2} - u},
\]

\[
B(u) = 1 - \sum_{n=0}^{\infty} \frac{\sqrt{5}b^{4n+4} - n\sqrt{5}b^{4n+2}}{b^{4n+2} - u},
\]

\[
D(u) = \sum_{n=0}^{\infty} \frac{\sqrt{5}b^{4n+2}(b^{4n+4} - n\sqrt{5}b^{4n+2})}{b^{4n+2} - u}.
\]

The eigenfunction corresponding to a zero \( \delta \) (which is seen to be unequal to \( b^{4n+2} \), for each \( n \in \mathbb{N}_0 \)) of \( \omega \) has the Fourier expansion

\[
\sum_{n=0}^{\infty} \alpha_n Le_n,
\]

where

\[
\alpha_n = \frac{1}{b^{4n+2} - \delta} \left[ c_1 \sqrt[4]{5}b^{4n+2} + c_2 \sqrt[4]{5}(b^{4n+4} - n\sqrt{5}b^{4n+2}) \right],
\]

and \( c_1, c_2 \) are real constants which cannot be simultaneously equal to zero and which satisfy the identities \( c_1 A(\delta) = c_2 D(\delta) \) and \( c_2 B(\delta) = c_1 D(\delta) \).

3.2 Consistency

For given significance level \( \alpha \in (0, 1) \) the test based on \( T_n^2 \) is defined by rejecting the hypothesis if \( T_n^2 > c_n, \alpha \), where

\[
c_n, \alpha = \inf \{ x \geq 0; P_{\alpha}(T_n^2 > x) < \alpha \}
\]

denotes the \( (1 - \alpha) \)-quantile of \( T_n^2 \) when the null hypothesis of exponentiality is true. The next theorem gives the consistency of the test against each fixed alternative distribution.
Theorem 3.3. If the distribution of the $X_j$ is not exponential, it holds that $\lim_{n \to \infty} P(T_n^2 > c_{n, \alpha}) = 1$.

Proof. Put $\mu = E(X)$. In view of $\lim_{n \to \infty} n^{-1} c_{n, \alpha} = 0$ it suffices to show that $n^{-1} T_n^2$ converges to
\[
\Delta = \int_0^\infty \left( E \left( J_0 \left( 2 \sqrt{t \mu} \right) \right) - \exp(-t) \right)^2 \exp(-t) \, dt \tag{3.11}
\]
in probability as $n \to \infty$, and that the limit $\Delta$ is positive. In fact, this follows from the strong law of large numbers in the Hilbert space $L^2$ and the uniqueness theorem for Hankel transforms. We omit the details. \(\square\)

3.3. Contiguous alternatives

We introduce the infinite product space $(\Omega, \mathcal{A}) = (\mathbb{R}^\infty, \mathcal{B}^\infty)$ and define the $X_i$, $i \in \mathbb{N}$, to be the coordinate projections. Let $P = P_0^\mathbb{N}$ denote the infinite product of the exponential distribution $P_0 = \text{Exp}(1)$. For some given sequence $(Q_{0,n})_{n \in \mathbb{N}}$ of probability measures on $(\mathbb{R}, \mathcal{B})$ we denote by $(Q_n, n \in \mathbb{N})$ the sequence of infinite product measures $Q_n = Q_{0,n}^\mathbb{N}, n \in \mathbb{N}$. For given $P$, the $X_1, X_2, \ldots$ are independent and exponentially distributed with parameter 1. For given $Q_n$, the $X_1, X_2, \ldots$ are independent and identically distributed with distribution $Q_{0,n}$. If $W_n, n \in \mathbb{N}$, is a sequence of random variables on $(\Omega, \mathcal{A})$ with values in some separable metric space endowed with the Borel $\sigma$-Algebra we write $W_n \xrightarrow{\mathcal{P}} W$ (under $Q_n$) and $W_n \xrightarrow{\mathcal{D}} W$ (under $P_0$) to express that there is some $S$-valued random variable $W$ such that the distribution $Q_n W_n$ of $W_n$ under $Q_n$ and the distribution $P_0 W$ of $W$ under $P_0$, respectively, converges weakly to the distribution of $W$ as $n \to \infty$. Instead of $W_n \xrightarrow{\mathcal{P}} W$ we also write $W_n \xrightarrow{\mathcal{D}} v$ if $v$ is the distribution of $W$. The sequences $(Q_{0,n}, n \in \mathbb{N})$ considered are as follows. For given $n \in \mathbb{N}$, let $Q_{0,n}$ have the Radon–Nikodym derivative $\frac{dQ_{0,n}}{dP_0} = 1 + n^{-\frac{1}{2}} h_n$ with respect to $P_0$, where $(h_n, n \in \mathbb{N})$ is some sequence of $P_0$-integrable functions $h_n$ converging to $h \in \mathcal{P} 0$-almost everywhere. Additionally, let $\sup_{n \in \mathbb{N}} E_{P_0}(|h_n|^4) < \infty$. These assumptions are weaker than those imposed by others when dealing with contiguous alternatives for goodness-of-fit tests. See, e.g. [14], where the sequence $(h_n)_{n \in \mathbb{N}}$ is supposed to be uniformly bounded. In fact, [10] our assumptions are shown to hold for the important interesting sequences of alternatives, (1) (Weibull alternatives) $Q_{0,n} = W(\lambda_n), \lambda_n = 1 + \frac{1}{n}$, where $W(\lambda_n)$ denotes the Weibull distribution with distribution function $1 - \exp(-x^{\lambda_n}), x \geq 0$; (2) (Gamma alternatives) $Q_{0,n} = C(\gamma_n, 1)$, the gamma distribution with shape parameter $\gamma_n = 1 + \frac{1}{\sqrt{n}}$ and density $dQ_{0,n}(x) = \frac{\gamma_n^{\lambda_n - 1} \exp(-x^{\lambda_n})}{\Gamma(\gamma_n)} \, dx$, $x \geq 0$; (3) (LIFR alternatives) $dQ_{0,n}(x) = (1 + \theta_n x) \exp\left(-x + \frac{\theta_n^2 x^2}{2}\right) \, dx$, $x \geq 0$, where $\theta_n = \frac{1}{\sqrt{n}}$; (4) (Contamination model) $Q_{0,n} = (1 - \theta_n) P_0 + \theta_n P_n$, where $\theta_n = \frac{1}{\sqrt{n}}$, and $P_n$ is some distribution on $\mathbb{R}_+$ absolutely continuous with respect to $P_0$, and $\int (dP_n/dP_0)^4 \, dP_0 < \infty$.

Theorem 3.4. For given $n \in \mathbb{N}$ let $Q_{0,n}$ have the Radon–Nikodym derivative $\frac{dQ_{0,n}}{dP_0} = 1 + n^{-\frac{1}{2}} h_n$ with respect to $P_0$, where $(h_n, n \in \mathbb{N})$ is some sequence of $P_0$-integrable functions $h_n$ converging to $h \in \mathcal{P} 0$-almost everywhere. Additionally, let $\sup_{n \in \mathbb{N}} E_{P_0}(|h_n|^4) < \infty$. For the $L^2$-valued random elements $Z_n = (Z_n(t), t \geq 0)$ it holds that
\[
Z_n \xrightarrow{\mathcal{D}} Z + c \quad \text{(under $Q_n$)},
\]
where $Z = (Z(t), t \geq 0)$ denotes a centered $L^2$-valued Gaussian random element with the covariance kernel given in (3.4), and where $c \in L^2$ is a shift function given by
\[
c(t) = \int_0^\infty \left( J_0(2\sqrt{t}x) + (x - 1)t \exp(-t) - \exp(-t) \right) h(x) \exp(-x) \, dx, \quad t \geq 0.
\]
For the test statistic defined in (3.1) it holds that
\[
T_n^2 \xrightarrow{\mathcal{D}} \int_0^\infty (Z(t) + c(t))^2 \exp(-t) \, dt \quad \text{(under $Q_n$)}.
\]

Proof. Let $P_0^n$ and $Q_{0,n}^n$ be the $n$-fold product measures of $P_0$ and $Q_{0,n}$, respectively. Putting $\Lambda_n = \log \frac{dQ_{0,n}}{dP_0}$ and applying a Taylor expansion for the logarithm the log-likelihood ratio becomes
\[
\Lambda_n(X_1, \ldots, X_n) = \sum_{j=1}^n \left\{ \frac{h_n(X_j)}{\sqrt{n}} - \frac{1}{2} \frac{h_n^2(X_j)}{n} \right\} + \sum_{j=1}^n R_n(X_j).
\]
Put $\sigma^2 = \int h^2 \, dP_0$. The conditions given for the sequence $(h_n, n \in \mathbb{N})$ ensure that under $P$
\[
\sum_{j=1}^n \left\{ \frac{h_n(X_j)}{\sqrt{n}} - \frac{1}{2} \frac{h_n^2(X_j)}{n} \right\} \xrightarrow{\mathcal{D}} N \left( -\frac{\sigma^2}{2} \mu, \sigma^2 \right),
\]
and \( \sum_{j=1}^{n} R_n(X_j) \to 0 \) in probability. This and LeCam’s first lemma imply the contiguity of the sequences \( (Q_{0,n}^n) \) and \( (P_n^0) \).

Recall the stochastic processes \( Z_n, \tilde{Z}_n \) and \( Z \). By means of the multivariate central limit theorem, for fixed \( \ell \in \mathbb{N} \) and \( t_1, \ldots, t_\ell \in \mathbb{R}_+ \), the joint distribution of \( \tilde{Z}_n(t_1), \ldots, \tilde{Z}_n(t_\ell) \) under \( P \) converges to a \( \ell \)-dimensional normal distribution with mean vector 0 and covariance matrix \( \Sigma = (k(t_i, t_j))_{1 \leq i, j \leq \ell} \). By means of our assumptions on the sequence \( (h_n, n \in \mathbb{N}) \) it is easily seen that under \( P \)

\[
c(t) = \lim_{n \to \infty} \text{Cov} \left( \tilde{Z}_n(t), \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left( h_n(X_j) - \frac{h_n^2(X_j)}{2\sqrt{n}} \right) \right)
= \int_{0}^{\infty} \left( J_n(2\sqrt{\lambda}x) + (x-1)t \exp(-t) - \exp(-t) \right) h(x) \exp(-x) dx, \quad t \geq 0.
\]

Using the Cramér–Wold device and the Lindeberg–Feller central limit theorem we obtain that under \( P \), the limit distribution of \( (\tilde{Z}_n(t_1), \ldots, \tilde{Z}_n(t_\ell)), \Lambda_n \) is the \( (\ell + 1) \)-variate normal distribution with mean vector \((0, \ldots, 0, -\sigma^2/2)\) and covariance matrix \( \left( \Sigma \sigma^2 \right) \), where \( \Sigma = (c(t_1), \ldots, c(t_\ell)) \). LeCam’s third lemma implies the distributional convergence of the finite dimensional distributions of \( \tilde{Z}_n \) to the finite dimensional distributions of the shifted Gaussian process \( Z + c \) under \( Q_n \). Due to the tightness of the sequence \( (P^n Z) \) and the contiguity of \( (Q_{0,n}^n) \) to \( (P^n) \) we obtain

\[ Z_n \xrightarrow{\mathcal{D}} Z + c \quad (\text{under } Q_n), \]

where the convergence is considered in the Hilbert space \( L^2 \). Now according to the continuous mapping theorem,

\[ T_n^2 \xrightarrow{\mathcal{D}} \int_{0}^{\infty} (Z(t) + c(t))^2 \exp(-t) dt \quad (\text{under } Q_n). \]

**Remark 3.2.** It follows from the uniqueness theorem of Hankel transforms that the shift function \( c \in L^2 \) appearing in Theorem 3.4 is zero if and only if the function \( h \) involved is of the form \( h(x) = \beta(x-1) \) for \( P_\beta \)-almost all \( x \geq 0 \) with \( \beta \in \mathbb{R} \). As a consequence, non-zero shift functions are obtained for the special sequences of alternatives \((1)\)–\((3)\) mentioned above. For the contamination model \((4)\) the shift function turns out to be zero if and only if the distribution \( P_\tau \) is the gamma distribution with shape parameter 2 and scale parameter 1 \((\text{density } x \exp(-x), \ x \geq 0)\).

### 3.4. Efficiency

In this section the approximate Bahadur slope of the proposed test and its approximate asymptotic relative Bahadur efficiency relative to some other tests will be investigated under a local alternatives model. Moreover the Wieand condition, under which the approximate Bahadur and the Pitman approaches to efficiency coincide, will be examined. Let the distribution of \( X \) be determined by some parameter \( \theta \) taking values in an interval \( \Theta = (-\eta, \eta) \subset \mathbb{R} \) or \( \Theta = [0, \eta) \subset \mathbb{R} \). Let \( \vartheta_0 = 0 \in \Theta \) represent the null hypothesis of exponentiality. Because \( T_n^2 \) is distribution free when the null hypothesis is true, we assume that in the hypothesis case \( \vartheta_0 = 0 \) there is a sequence of independent and identically distributed random variables \( X_1, X_2, \ldots \) with exponential distribution \( P_0 = \text{Exp}(1) \). Under the alternative represented by \( \vartheta \in \Theta_1 = \Theta \setminus \{0\} \) the independent and identically distributed random variables \( X_1, X_2, \ldots \) have the distribution \( P_\vartheta \). The probability \( \text{d}P_\vartheta = (1 + \vartheta \varphi_0) \text{d}P_0 \), where the function \( \varphi_0 \) converges in \( L^2 \) to \( h \in L^2 \) as \( \vartheta \to 0 \). Then, for each \( \vartheta \in \Theta_1 \) there is \( \int \varphi_0(x) \text{d}P_0 = 0 \). Additionally, we impose the condition \( \int x \varphi_0(x) \text{d}P_0(x) = 0 \) for each \( \vartheta \in \Theta_1 \). For the notion of standard sequences we refer to [15]. For given sample size \( n \geq 1 \), let the test statistic \( T_n \) be the nonnegative square root of \( T_n^2 \) defined in (3.1).

**Theorem 3.5.** The sequence \( (T_n)_{n \geq 1} \) is a standard sequence in the sense of Bahadur. For \( \vartheta \in \Theta_1 \) the approximate Bahadur slope is given by \( b^2(\vartheta) = a_1 \beta(\vartheta)^2 \), where the constant \( a_1 \) is the inverse of the greatest eigenvalue of the covariance operator associated with the kernel function defined in (3.4), and

\[
b^2(\vartheta) = \vartheta^2 \int \left\{ \int J_0(2\sqrt{\lambda}x) h_0(x) \text{d}P_0(x) \right\}^2 \text{d}P_0(t).
\]

The limiting approximate Bahadur slope as \( \vartheta \to 0 \) is

\[
c_\vartheta = \lim_{\vartheta \to 0} \frac{b(\vartheta)}{\vartheta^2} = a_1 \int \left\{ \int J_0(2\sqrt{\lambda}x) h(x) \text{d}P_0(x) \right\}^2 \text{d}P_0(t).
\]

**Proof.** The proof follows the lines given in [16]. We omit the details. \( \square \)

Wieand [17, 18] deals with conditions ensuring that the limiting \( (\text{as } \vartheta \to 0) \) approximate Bahadur efficiency agrees with the limiting \( (\text{as the significance level approaches } 0) \) Pitman efficiency. See [19] for a more general approach. The coincidence is seen to be true if the two sequences of test statistics satisfy Bahadur’s first two conditions in the definition of a standard sequence and a third one which for our sequence \( (T_n)_{n \geq 1} \) reads as follows.
(W) There exists some $\hat{\theta}^* > 0$ such that for each $\varepsilon > 0$ and $\delta (0, 1)$, there is a positive constant $C$ such that for each $\theta \in \Theta_1 \cap (-\hat{\theta}^*, +\hat{\theta}^*)$ and each integer $n > \frac{C}{b(\theta)^2}$ the inequality

$$P\left(\left| n^{-\frac{1}{2}}T_n - b(\theta) \right| < \varepsilon b(\theta) \right) > 1 - \delta$$

holds.

**Theorem 3.6.** Wieand's condition (W) is satisfied for the sequence $(T_n)_{n \geq 1}$ of test statistics.

**Proof.** Putting $\mathcal{H}_\theta(t) = E_\theta(J_0(2\sqrt{tX_t}))$ for $t \geq 0$ and $\theta \in \Theta$ we have

$$\left| n^{-\frac{1}{2}}T_n - b(\theta) \right| \leq \left\{ \int_0^{\infty} \left( \frac{1}{n} \sum_{j=1}^{n} J_0(2\sqrt{tX_j/X_n}) - \mathcal{H}_\theta(t) \right)^2 \exp(-t) dt \right\}^{\frac{1}{2}}$$

and

$$\left( \frac{1}{n} \sum_{j=1}^{n} J_0(2\sqrt{tX_j/X_n}) - \mathcal{H}_\theta(t) \right)^2 \leq 2 \left[ \frac{1}{n} \sum_{j=1}^{n} \left( J_0(2\sqrt{tX_j/X_n}) - J_0(2\sqrt{tX_j}) \right)^2 \right] + 2 \left( \frac{1}{n} \sum_{j=1}^{n} J_0(2\sqrt{tX_j}) - \mathcal{H}_\theta(t) \right)^2.$$ 

Expanding $J_0(2\sqrt{tX_j/X_n})$ in a Taylor series we obtain

$$J_0(2\sqrt{tX_j/X_n}) = J_0(2\sqrt{tX_j}) = 2 - \frac{1}{X_n} - 1 \right) \frac{1}{\sqrt{X_n}} \right) J_1(\xi), \quad j = 1, \ldots, n,$$

where $\xi$ is a point between $2\sqrt{tX_j/X_n}$ and $2\sqrt{tX_j}$. From this and the inequalities

$$|J_1| \leq \sqrt{2}/2, \quad \frac{1}{n} \sum_{j=1}^{n} \sqrt{X_j} \leq \sqrt{X_n}, \quad |\sqrt{X_n} - 1| \leq |\hat{X}_n - 1|$$

it follows that

$$\left| \frac{1}{n} \sum_{j=1}^{n} \left( J_0(2\sqrt{tX_j/X_n}) - J_0(2\sqrt{tX_j}) \right) \right| \leq \sqrt{2t} |\hat{X}_n - 1|.$$

By Chebyshev’s inequality,

$$P \left( \left| n^{-\frac{1}{2}}T_n - b(\theta) \right| \leq \varepsilon b(\theta) \right) \geq 1 - \frac{1}{\varepsilon^2 b^2(\theta)} \left\{ 4E_\theta(|\hat{X}_n - 1|^2) + \frac{8}{n} \right\}.$$

We further have

$$E_\theta\left( (\hat{X}_n - 1)^2 \right) = E_\theta\left( \left( n^{-1} \sum_{j=1}^{n} (X_j - 1) \right)^2 \prod_{j=1}^{n} [1 + \theta h_\theta(X_j)] \right)$$

$$= \frac{1}{n^2} \cdot n \cdot E_\theta\left( (X_1 - 1)^2 \prod_{j=1}^{n} [1 + \theta h_\theta(X_j)] \right) + \frac{1}{n^2} \cdot n(n - 1) \cdot E_\theta\left( (X_1 - 1)(X_2 - 1) \prod_{j=1}^{n} [1 + \theta h_\theta(X_j)] \right).$$

From the assumptions $\int h_\theta(x) dP_0(x) = 0$ and $\int x h_\theta(x) dP_0(x) = 0$, it follows that $E_\theta\left( (\hat{X}_n - 1)^2 \right) = \frac{1}{n} E_0\left( (X_1 - 1)^2 [1 + \theta h_\theta(X_1)] \right)$. Note that

$$E_0\left( (X_1 - 1)^2 [1 + \theta h_\theta(X_1)] \right) \leq 1 + 3|\theta| \sqrt{E_0(h_\theta^2(X_1))}.$$

$L^2$-convergence of $h_\theta$ to $h$ implies $E_0(h_\theta^2(X_1)) \to E_0(h^2(X_1))$ as $\theta \to 0$. Therefore there exists a $\varepsilon \in (0, \eta)$ such that

$$s = \sup_{\theta \in \Theta \cap (-\varepsilon, \varepsilon)} E_0\left( (X_1 - 1)^2 [1 + \theta h_\theta(X_1)] \right) < \infty.$$

So we can rewrite (3.13) as

$$P \left( \left| n^{-\frac{1}{2}}T_n - b(\theta) \right| \leq \varepsilon b(\theta) \right) \geq 1 - \frac{4(s + 2)}{n\varepsilon^2 b^2(\theta)}.$$
for \( \vartheta \in \Theta_1 \cap (\zeta, \zeta) \). Defining \( C = \frac{4(n+2)}{\delta^2} \), it follows that for each \( \vartheta \in \Theta_1 \cap (\zeta, \zeta) \) and \( n > \frac{C}{b^2(\vartheta)} \),

\[
P\left( n^{-\frac{1}{2}} T_n - b(\vartheta) \leq \epsilon b(\vartheta) \right) \geq 1 - \frac{C\delta}{n b^2(\vartheta)} > 1 - \delta,
\]

which is the desired assertion. \( \square \)

Let us consider the special case where the family of distributions \( \{P_\vartheta; \vartheta \in \Theta\} \) constitutes a contamination model, i.e. \( \vartheta = [0, 1] \). \( P_\vartheta = (1 - \vartheta)P_0 + \vartheta P_1 \) for \( \vartheta \in \Theta \) where \( P_1 \) is some distribution on the positive half-line with expectation \( \int x dP_1(x) = 1 \). Assuming that \( P_1 \) is absolutely continuous with respect to \( P_0 \) with \( P_0 \)-square integrable Radon–Nikodym density \( g = dP_1/dP_0 \), the family of \( P_0 \)-densities \( dP_1/dP_0 = 1 + \vartheta (g - 1) \) satisfies the conditions given above. The limiting approximate Bahadur slope of the sequence \( (T_n) \) is seen to be

\[
c_t = a_t \lim_{\vartheta \to 0} \frac{b^2(\vartheta)}{\vartheta^2} = a_t \int_0^\infty \left( H_{P_1}(t) - \exp(-t)^2 \right) \exp(-t) \, dt,
\]

where \( H_{P_1} \) denotes the Hankel transform of \( P_1 \), and \( a_t^{-1} = \delta \) is the largest eigenvalue of the integral operator associated with the covariance function \( \{3.4\} \). \( \delta \) was calculated by Taherizadeh [10]. Rounded to five decimal places the value is \( \delta = 0.03586 \). Denoting by \( (C_n) \) the corresponding sequences of classical competitors, the Cramér–von Mises test and the Anderson–Darling test for testing the hypothesis of exponentiality, see e.g. [20], it can be easily verified that for the given contamination model its limiting approximate Bahadur slopes exist and are seen to be equal to

\[
c_C = a_C \int_0^\infty \left( F_1(x) - (1 - \exp(-x))^2 \right) \exp(-x) \, dx
\]

and

\[
c_A = a_A \int_0^\infty \frac{(F_1(x) - (1 - \exp(-x))^2}{1 - \exp(-x)} \, dx.
\]

Thereby, \( F_1 \) is the distribution function of \( P_1 \), and \( a_C^{-1} = 0.044202 \) and \( a_A^{-1} = 0.23130 \) are the largest eigenvalues of integral operators associated with the covariance functions of certain Gaussian processes arising as limit processes of special empirical processes in the hypothesis case, see e.g. [21]. We give the limiting approximate efficiencies for three different examples of distributions \( P_1 \). The efficiencies are referenced to the Anderson–Darling test.

(1) \( P_1 \) is the uniform distribution \( U[0, 2] \) on the interval \([0, 2] \). Its Hankel transform is \( H_{U[0,2]}(t) = (2t)^{-1/2}J_1(2t2^{-1/2}) \), \( t \geq 0 \). See [5], formula 6.561.5. The limiting approximate slopes are calculated to be \( c_C = 0.2324706 \), \( c_C = 0.2638454 \), \( c_A = 0.2534433 \). This gives the limiting approximate efficiencies \( c_A/c_C = 0.9605753 \) and \( c_A/c_C = 1.090217 \).

(2) \( P_1 \) is the gamma distribution \( G(2, 2) \) with shape parameter 2 and scale parameter 2. Its Hankel transform is \( H_{G(2,2)}(t) = (1 - t/2) \exp(-t/2) \), \( t \geq 0 \). The limiting approximate slopes and limiting approximate efficiencies are computed to be \( c_C = 0.1626696 \), \( c_C = 0.1745201 \), \( c_A = 0.1832320 \), and \( c_A/c_C = 1.049909 \), \( c_A/c_C = 1.126395 \).

(3) \( P_1 \) is the half-normal distribution \( HN(\sqrt{2}/2) \), i.e. the distribution of \( |X| \) where \( X \) has the centered normal distribution with variance \( \pi/2 \). The density is \( \frac{1}{\pi} \exp(-x^2/\pi) \), \( x \geq 0 \). A closed form expression for the Hankel transform

\[
H_{HN(\sqrt{2}/2)}(t) = \int_0^\infty J_0(2 \sqrt{t} x) \frac{2}{\pi} \exp(-x^2/\pi) \, dx \quad t \geq 0,
\]

does not seem to exist. The limiting approximate slopes are calculated to be \( c_C = 0.07508487 \), \( c_C = 0.07280238 \), \( c_A = 0.07038285 \). This gives the limiting approximate efficiency \( c_A/c_C = 0.9667658 \), \( c_A/c_C = 0.9373774 \).

3.5. Numerical results

A simulation study was conducted to obtain approximations to the critical values of the null distribution of \( T_n^2 \) for some standard significance levels \( \alpha \) and different finite sample sizes. The results are shown in Table 1. Except for the values given in the last line, each entry represents 20%-trimmed means of 10 Monte Carlo estimates, each based on 10000 iterations. As stated in the previous section, the null distribution of \( T_n^2 \) is the same as that of \( \sum_{i=1}^{10} \delta_i X_{i1}^2 \), where \( \delta_1 \geq \delta_2 \geq \cdots \) are the positive eigenvalues of the operator \( \delta \), and \( X_{i1}, X_{i2}, \ldots \) are independent \( \chi_1^2 \)-distributed random variables. Approximating this limit distribution by that of \( \sum_{i=1}^{10} \delta_i X_{i1}^2 \) and using the procedure presented in [22] for computing the distribution of quadratic forms in normal variables the quantiles of \( \sum_{i=1}^{10} \delta_i X_{i1}^2 \) can be computed by straightforward calculations. The last line gives approximations to the limiting critical values obtained in this way. To compare the power of the proposed test with some alternative tests of exponentiality, a power study was conducted. The entries in columns 2–11 of Tables 2 and 3 show the percentage points of 5000 Monte Carlo samples declared significant for the significance level \( \alpha = 0.05 \) and the sample sizes \( n = 20 \) and \( n = 50 \). An asterisk denotes power 100%. The entries in columns 3–11 are borrowed from Table 4 of [23]. \( T(1) \), \( T(1) \) and \( T(10) \) denote statistical tests based on the empirical Laplace transform proposed in that article, \( M \) is related to the test of Moran [24], \( Q_1 \) stands for the test of Patwardhan [25], \( W^* \) is related to a modified version of the Shapiro–Wilk test, see [26,21]. \( C^2 \) and \( A^2 \) denote the classical Cramér–von Mises and Anderson–Darling tests. Finally, 5
represents the test of Sarkadi [27]. The notions for the alternative distributions quoted in the first column of Tables 2 and 3 are adopted from [23].

We conclude from the simulation results in Tables 2 and 3 that for Gamma alternatives the new test is slightly less powerful than Moran’s M which is seen to be the best one. For Weibull alternatives with \( \gamma > 1 \), the tests based on \( T^2 \), \( T(1) \) and \( M \) provide comparable results and have more power than all the other tests. For the half-normal and the half-Cauchy distributions, the new test has comparable power to \( T(10) \) and \( W^* \) which provide the best results. The same is valid for the family of JSHEP distributions (density \( (1 + \theta x)^{(\theta + 1)/\theta} \)). For the log-normal distribution \( T^2 \) is comparable in

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<th>Table 1</th>
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Percentage of Monte Carlo samples declared significant by the various tests of exponentiality; test size \( \alpha = 0.05 \); sample size \( n = 50 \).

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power to \( T(1) \), \( T(10) \), \( C^2 \) and \( A^2 \). For the LIFR (linear increasing failure rate) distributions, \( W^* \) and the new test dominate the other procedures. Over the whole range of alternative distributions considered, the new test is a serious competitor to the well-established goodness-of-fit tests for exponentiality. Although it does not improve significantly the existing omnibus procedures being the best one for specific alternatives, the new test is better than competitive procedures for a broader classes of alternatives. Due to the fact that a uniformly most powerful test does not exist, there is need for consistent omnibus procedures showing overall satisfactory power and desirable high power for alternatives of special importance in user-specific situations. The new test makes a contribution in this direction.

4. Concluding remarks

(1) Goodness-of-fit tests for other simple and composite hypotheses can be developed in the same way. Instead of using integral type test statistics also supremum type test statistics can be suggested. Due to the stronger uniqueness theorem, i.e. Theorem 2.2, for the hypothesis of exponentiality it is most natural to study the Kolmogorov–Smirnov type test based on the test statistic

\[
\sqrt{n} \sup_{0 \leq t \leq 1} |H_n(t) - \exp(-t)|.
\]

For theoretical and empirical results on this test we refer to [10].

(2) Empirical Hankel transforms can also be used to treat other testing problems. For example, given two independent samples \( X_1, \ldots, X_m \) and \( Y_1, \ldots, Y_n \) of nonnegative random variables \( X_j \) with common unknown distribution \( F \) and nonnegative random variables \( Y_k \) with common unknown distribution \( G \), a suitable test statistic for testing the hypothesis \( H : F = G \) against the general alternative \( K : F \neq G \) is

\[
T_{m,n} = \frac{mn}{m+n} \int_0^\infty (H_m^X(t) - H_n^Y(t))^2 \exp(-t)dt.
\]
where \( \mathcal{H}_m^X(t) = \frac{1}{m} \sum_{j=1}^m j_0(2\sqrt{tX_j}) \), \( t \geq 0 \), and \( \mathcal{H}_n^Y(t) = \frac{1}{n} \sum_{j=1}^n j_0(2\sqrt{tY_j}) \), \( t \geq 0 \), are the empirical Hankel transforms of \( X_1, \ldots, X_m \) and \( Y_1, \ldots, Y_n \), respectively. Rejection of the hypothesis is for large values of \( T_{m,n} \). Due to the fact that \( T_{m,n} \) is not distribution free when the hypothesis is true, a permutation or bootstrap procedure can be used to obtain critical values. The test obtained is clearly consistent against each fixed alternative \( F \neq G \). A generalization to the multi-sample problem is obvious.

(3) Let \( X_1, \ldots, X_n \) be a random sample from an unknown mixed distribution on the Borel sets of \( \mathbb{R}_+ \), where the mixing distribution is the exponential distribution \( \text{Exp}(1) \). The \( X_i \) can assumed to be of the from \( X_i = Z_i Y_i \) with independent \( Z_i \) and \( Y_i \) where the \( Z_i \) have the \( \text{Exp}(1) \) distribution and the (unobservable) random variables \( Y_i \) have some unknown distribution \( F \). To get an estimate of \( F \) we note that the empirical Hankel transform \( \mathcal{H}_n(t) = \frac{1}{n} \sum_{j=1}^n j_0(2\sqrt{tY_j}) \), \( t \geq 0 \), of the sample \( X_1, \ldots, X_n \) tends to the Laplace transform \( \mathcal{H}_F(t) \), \( t \geq 0 \), of \( F \) pointwise almost surely. Approximating \( \mathcal{H}_n \) by some suitable Laplace transform \( \mathcal{H}_F \) one obtains an estimator \( \mathcal{H}_n \) for \( \mathcal{H}_F \). The distribution associated with \( \mathcal{H}_n \) is an estimator for \( F \). Of course, there remains the crucial question for a method to obtain a consistent estimator \( \mathcal{H}_n \) from \( \mathcal{H}_n \).

(4) The Hankel transform can be generalized to the multivariate case. In fact, let \( \mathbb{R}_d^d \) be the set of column vector with nonnegative components \( t_j, j = 1, \ldots, d \), and let \( X = (X_1, \ldots, X_d)' \) be some random column vector with values in \( \mathbb{R}_d^d \). Then we can define the Hankel transform of \( X \) to be

\[
\mathcal{H}(t) = E j_0(2\sqrt{t'X}), \quad t \in \mathbb{R}_d^d.
\]

Of course, the uniqueness theorem and the continuity theorem for Hankel transforms carry over to the multivariate case. Introducing empirical Hankel transforms in this multivariate setting is easily done and offers a way to test problems in multivariate statistical inference.

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References