Coxeter–Petrie Complexes of Regular Maps

KEVIN ANDERSON† AND DAVID B. SUROWSKI

Coxeter–Petrie complexes naturally arise as thin diagram geometries whose rank 3 residues contain all of the dual forms of a regular algebraic map $\mathcal{M}$. Corresponding to an algebraic map is its classical dual, which is obtained simply by interchanging the vertices and faces, as well as its Petrie dual, which comes about by replacing the faces by the so-called Petrie polygons. Jones and Thornton have shown that these involutory duality operations generate the symmetric group $S_3$, giving in all six dual forms, and whose source is the outer automorphism group of the infinite triangle group generated by involutions $s_1, s_2, s_3$, subject to the additional relation $s_1 s_3 = s_3 s_1$. In fact, this outer automorphism group is parametrized by the permutations of the three commuting involutions $s_1, s_3, s_1 s_3$. These involutions together with the involution $s_2$ can be taken to define the nodes of a Coxeter diagram of shape $D_4$ (with the involution $s_2$ at the central node), and when the original map $\mathcal{M}$ is regular, there is a natural extension from $\mathcal{M}$ to a thin Coxeter complex of rank 4 all of whose rank 3 residues are isomorphic to the various dual forms of $\mathcal{M}$. These are fully explicated in case the original algebraic map is a Platonic map.

© 2002 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION: ALGEBRAIC MAPS AND (QUASI)THIN CHAMBER SYSTEMS

In this introductory section, we gather together the definitions necessary to realize an algebraic map as a quasithin chamber complex. In Section 2 we recall the duality operations of an algebraic map in the sense of Jones and Thornton [5], and construct, for a given regular algebraic map, the corresponding Coxeter–Petrie complex as a rank 4 thin chamber complex having as rank 3 residues the various dual forms of the given map. The monodromy group of such a complex is a rank 4 Coxeter group with added ‘Petrie relations’. In Section 3 the structure of the monodromy groups of the Coxeter–Petrie complexes corresponding to the Platonic solids is given, and in Section 4 presentations of the monodromy groups of the Coxeter–Petrie complexes corresponding to regular affine maps are given.

A chamber system $\mathcal{C} = (\mathcal{C}, \sim_i | i \in I)$ consists of a set $\mathcal{C}$ (of chambers), together with a family of equivalence relations $\sim_i, i \in I$. The cardinality of $I$ is called the rank of the chamber system. Arguably the most often quoted example of a rank $n$ chamber system is the following. Let $V$ be an $(n+1)$-dimensional vector space over the field $F$. Recall that a flag in $V$ is a collection $F = \{W_1, W_2, \ldots, W_l\}$ of subspaces with $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_l$. A maximal flag in $V$ is a flag $F = \{V_1, V_2, \ldots, V_n\}$, where $\dim V_j = j, j = 1, 2, \ldots, n$. We then can define a chamber system over the set $I = \{1, 2, \ldots, n\}$ to consist of the maximal flags $F$ in $V$ and where $F \sim_j F'$ when the constituent subspaces of $F$ and $F'$ are the same, except possibly in dimension $j$.

Notice that in the above example, if $F$ is the finite field of order $q$, and if $F$ is a fixed maximal flag, then $F \sim_j F'$ for precisely $q + 1$ maximal flags $F'$ in $V$. Such complexes are not the focus of the present paper; rather the complexes considered herein are quasithin inasmuch as the $\sim_i$-equivalence classes have cardinality at most 2. Of particular interest to us are the thin chamber complexes, i.e., those whose $\sim_i$-equivalence classes all have cardinality exactly 2. When the chamber system is quasithin, then an alternative, but equivalent definition can be given as follows. Namely, A quasithin chamber system $\mathcal{C} = (\mathcal{C}, a_i | i \in I)$ consists of a set $\mathcal{C}$ of chambers and a family of involutory permutations $s_i, i \in I$ on $\mathcal{C}$. Thus, in this case, the $\sim_i$-equivalence classes are the $\langle a_i \rangle$-orbits in $\mathcal{C}$. Note that the quasithin chamber

†To whom correspondence should be addressed.
system \( \mathcal{C} = (C, a_i \mid i \in I) \) is thin precisely when each of the monodromy involutions \( a_i, i \in I \) acts without fixed points on \( C \) (in general, the union of the fixed points of the monodromy involutions \( s_i, i \in I \) is called the boundary of \( \mathcal{C} \) and denoted \( \partial(\mathcal{C}) \)). As above, the cardinality of the index set \( I \) is the rank of \( \mathcal{C} \). The prototypical thin chamber system of rank \( n \) is modeled on the above example, as follows. Instead of an \((n + 1)\)-dimensional vector space \( V \), start with a set \( S \) of cardinality \( n + 1 \). Again, define flags (and maximal flags) in \( S \) in terms of subsets of \( S \), by analogy with taking subspaces of \( V \). Thus, if \( F \) is a maximal flag in \( S \), it is readily seen that other than itself, \( F \) is \( \sim \)-equivalent to exactly one other maximal flag \( F' \) in \( S \), for each \( i \in I \). Thus, the corresponding involution satisfies \( a_i(F) = F' \).

Let \( \mathcal{C} = (C, \sim_i \mid i \in I), \mathcal{D} = (D, \sim_i \mid i \in I) \) be chamber systems over the set \( I \), and let \( \phi : C \rightarrow D \) be a mapping. We say that \( \phi \) is a morphism from \( \mathcal{C} \) to \( \mathcal{D} \), and write \( \phi : \mathcal{C} \rightarrow \mathcal{D} \), if \( c \sim_i c' \) in \( \mathcal{C} \) implies that \( \phi C \sim_i \phi C' \). If \( \mathcal{C} = (C, a_i \mid i \in I), \mathcal{D} = (D, b_i \mid i \in I) \) are quasithin chamber systems over \( I \), a morphism \( \phi : \mathcal{C} \rightarrow \mathcal{D} \) of chamber systems is then simply a mapping \( \phi : C \rightarrow D \) satisfying \( (a_i c) \phi \in \{ c \phi, b_i (c \phi) \} \) for all \( i \in I \), and all \( c \in C \). An invertible morphism \( \phi : \mathcal{C} \rightarrow \mathcal{D} \) is an isomorphism, and an isomorphism \( \phi : \mathcal{C} \rightarrow \mathcal{C} \) is called an automorphism of \( \mathcal{C} \). The set of all such is clearly a group under composition and is denoted \( \text{Aut}(\mathcal{C}) \). It is easy to check that when \( \mathcal{C} = (C, a_i \mid i \in I) \) is quasithin, then

\[
\text{Aut}(\mathcal{C}) = \{ \text{bijections } \phi : C \rightarrow C \mid (a_i c) \phi = a_i (c \phi), \text{ for all } i \in I \text{ and all } c \in C \}.
\]

If \( \mathcal{C} = (C, a_i \mid i \in I) \) is a quasithin chamber system, we set \( G = \text{Mon}(\mathcal{C}) = \{ a_i \mid i \in I \} \), the group generated by the involutions \( a_i, i \in I \), and call it the monodromy group of \( \mathcal{C} \). We say that the chamber system \( \mathcal{C} = (C, \sim_i \mid i \in I) \) is connected if and only if the transitive closure of the equivalence relations \( \sim_i, i \in I \) represents \( C \) as a single equivalence class. Equivalently, if \( c, c' \) are chambers, then there is a ‘path’ from \( c \) to \( c' \) of the form

\[
c = c_0 \sim i_1 c_1 \sim i_2 c_2 \sim i_3 \cdots \sim i_k c_k = c'.
\]

In case \( \mathcal{C} \) is a quasithin chamber system, this can be stated more succinctly simply by stating that the monodromy group acts transitively on the set \( C \) of chambers.

Let \( G \) be a group with set of involutory generators \( a_i \in I \), and let \( H \) be a subgroup of \( G \). We may form the connected quasithin chamber system \( \mathcal{C}(G/H) = (G/H, a_i, i \in I) \), where the monodromy involutions \( a_i, i \in I \) act on \( G/H \) via left multiplication.

The following is easy:

**Lemma 1.1.** Let \( \mathcal{C} = (C, a_i, i \in I) \) be a connected quasithin chamber system. If \( c \in C \) is a fixed chamber, and if \( H \) is the stabilizer in \( G = \text{Mon}(\mathcal{C}) \) of \( c \), then the mapping

\[
\phi : \mathcal{C}(G/H) \rightarrow \mathcal{C}, \quad gH \mapsto g(c)
\]

is an isomorphism of quasithin chamber systems over \( I \).

Henceforth, we shall stick to quasithin chamber systems, even though the definitions and many of the results are valid more generally. Let \( \mathcal{C} = (C, a_i, i \in I) \) be a connected, quasithin chamber system over \( I \), with monodromy group \( G \). Fix a chamber \( c \in C \), and let \( H \) be the stabilizer of \( c \) in \( G \). By the above lemma, we may identify \( \mathcal{C} \) with \( \mathcal{C}(G/H) = (G/H, a_i, i \in I) \). It is trivial to verify that if \( N = N_{G}(H) \), and if \( n \in N \), then the mapping \( \mathcal{C}(G/H) \rightarrow \mathcal{C}(G/H) \) given by \( gH \mapsto gnH \) is an automorphism of \( \mathcal{C}(G/H) \). It is easy to verify that every automorphism of \( \mathcal{C}(G/H) \) arises in this way, from which it follows easily that

\[
\text{Aut}(\mathcal{C}) \cong N_{G}(H)/H.
\]
thereby elucidating the relationship between the monodromy and automorphism groups of the connected, quasithin chamber system \( C \).

Therefore, the connected quasithin chamber system \( C \) has a transitive automorphism group if and only if the monodromy group acts regularly on the chambers of \( C \), in which case the automorphism group does, as well. Therefore, we say that the quasithin chamber system \( C = (C, a_i \mid i \in I) \) is regular if and only if the monodromy group acts regularly on the set \( C \) of chambers of \( C \). In this case the automorphism group also acts regularly on the chambers \( C \) and is isomorphic with the monodromy group, as the two groups are then the right and left regular representations of the same group.

Let \( C = (C, a_i \mid i \in I) \) be a quasithin chamber complex with monodromy group \( G = \langle a_i \mid i \in I \rangle \). For any subset \( J \subseteq I \), we define the parabolic subgroup \( G_J = \langle a_j \mid j \in J \rangle \leq G \). Correspondingly, a residue of type \( J \subseteq I \) in \( C \) is simply one of the \( G_J \)-orbits in \( C \). For any \( i \in I \), the varieties of type \( i \in I \) are the residues of type \( I \setminus \{i\} \).

A quasithin rank 3 chamber system \( C = (C, a_1, a_2, a_3) \) is called a hypermap or an algebraic hypermap. If, in addition \( a_1a_3 = a_3a_1 \), we call \( C \) a map, or sometimes an algebraic map. In the context of maps and hypermaps, we typically refer to the underlying chambers as blades.

If \( G = \langle a, b, c \rangle \) is a group generated by involutions such that \( ac = ca \), and if \( H \leq G \) is a subgroup, we define the connected map \( M(G/H, a, b, c) \) to have set of blades \( G/H \) and monodromy involutions \( a, b, c \) (acting on \( G/H \)). Note that \( M(G/H, a, b, c) \) is thin precisely when the subgroup \( H \) contains no conjugate of \( a, b, \) or \( c \).

If \( \mathcal{M} = (B, a, b, c) \) is a map, we have varieties in \( \mathcal{M} \) as follows:

- **vertices**: These are the \( \langle b, c \rangle \)-orbits in \( B \);
- **edges**: These are the \( \langle a, c \rangle \)-orbits in \( B \);
- **faces**: These are the \( \langle a, b \rangle \)-orbits in \( B \).

If \( \mathcal{M} \) is a map, and if \( v \subseteq B \) is any variety, let \( G_v \) be the corresponding parabolic subgroup of the monodromy group \( G \). Thus, \( G_v = \langle x, y \rangle \) (a dihedral group), where \( x, y \in \{a, b, c\} \) and where \( v = G_v b \) (the \( G_v \)-orbit of \( B \)) for some blade \( b \in B \). We set \( G_v^+ = \langle xy \rangle \); then it is easy to see that either \( G_v^+ \) acts transitively on \( v \) (which happens in the case \( v \) contains an element of \( \partial(\mathcal{M}) \)); note that \( |v \cap \partial(\mathcal{M})| \leq 2 \), or \( G_v^+ \) acts in two orbits of the same cardinality. In either case, we say that the valency of \( v \) is the size of any \( G_v^+ \)-orbit in \( v \) (which might be infinite).

We define the vertex valency of the map \( \mathcal{M} \) to be the least common multiple of the valencies of each of the vertices. Similarly, the face valencies and edge valencies are defined. Note that if the map is thin, then the edge valency is always 2.

### 1.1. Diagrams and rank 2 residues

In this subsection, we consider the notion of Coxeter diagrams associated with thin chamber systems. To this end, let \( C = (C, a_i \mid i \in I) \) be a connected thin chamber complex of rank card(\( I \)) \( \geq 2 \), and let \( G = \langle a_i \mid i \in I \rangle \) be its monodromy group. Let \( J \subseteq I \) be a subset of cardinality 2 with corresponding parabolic subgroup \( G_J \). Since \( G_J \) is dihedral we may, exactly as above, define the valency of each type-\( J \)-residue in \( C \). Continuing to assume that \( J \) has cardinality 2, we define the \( J \)-valency of \( C \) to be the least common multiple of the valencies of varieties of type \( J \). Thus, if \( J = \{i, j\} \subseteq I \) of cardinality 2, we set \( m_{ij} \) to be the \( J \)-valency of \( C \). Correspondingly, we can associate a Coxeter diagram, whose nodes correspond with the elements \( i \in I \), and where for each pair \( i \neq j \) in \( I \) we draw an edge from node \( i \) to node \( j \) marked \( m_{ij} \). We adopt the usual convention that when \( m_{ij} = 2 \), then no edge is drawn from node \( i \) to node \( j \). We mention in passing that the notion of diagrams attached to geometries (and later adapted to chamber systems) was first formalized by Francis Buekenhout [2], in an attempt to understand non-Lie type geometries, especially those arising from sporadic simple groups.
When this process is applied to an algebraic map $\mathcal{M} = (B, a, b, c)$, then the nodes of the Coxeter diagram correspond to the involutions $a, b, c$; since $ac = ca$, one arrives at a diagram of the form

$$
\begin{array}{c}
\circ \quad k \\
\circ \quad l \\
\circ \quad a \\
\circ \quad b \\
\circ \quad c
\end{array}
$$

where $k, l$ are the face and vertex valencies, respectively, of $\mathcal{M}$. Sometimes, to emphasize the roles of vertices and faces, we exhibit the diagram thus:

$$
\begin{array}{c}
k \\
l \\
\text{vertices} \\
\text{edges} \\
\text{faces}
\end{array}
$$

More specifically, the underlying map of the tetrahedron has Coxeter diagram

$$
\begin{array}{c}
\circ \quad 3 \\
\circ \quad 3 \\
\circ \quad \text{vertices} \\
\circ \quad \text{edges} \\
\circ \quad \text{faces}
\end{array}
$$

that of the cube (hexahedron) and its dual, the octahedron, have the respective diagrams

$$
\begin{array}{c}
\circ \quad 4 \\
\circ \quad 3 \\
\circ \quad \text{vertices} \\
\circ \quad \text{edges} \\
\circ \quad \text{faces}
\end{array}
$$

2. Duality and the Coxeter–Petrie Complex

Let $\mathcal{M} = (B, a, b, c)$ be a connected map, and set

$$
\Delta = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1s_3)^2 = 1 \rangle.
$$

Clearly, there is a surjective homomorphism $\Delta \to G$, given by $s_1 \mapsto a, s_2 \mapsto b, s_3 \mapsto c$. As a result, we have that $\Delta$ acts on the left on the set $B$ of blades of $\mathcal{M}$. If we fix a blade $b \in B$, and let $H_\Delta$ be the stabilizer in $\Delta$ of $b$, then we may identify $\mathcal{M}$ as an orbit map $\mathcal{M}^{\infty}/H_\Delta$, where $\mathcal{M}^{\infty}$ is the regular map $\mathcal{M}^{\infty} = (\Delta, s_1, s_2, s_3)$. Furthermore, if $H_\Delta' \leq \Delta$ and is conjugate to $H_\Delta$, say $H_\Delta' = \gamma H_\Delta \gamma^{-1}$, then the mapping $\Delta/H_\Delta \to \Delta/H_\Delta'$ given by $xH_\Delta \mapsto x\gamma H_\Delta'$ is a well-defined isomorphism of maps. Conversely, if $H_\Delta' \leq \Delta$ is a subgroup, and if there is a bijection $\Delta/H_\Delta \to \Delta/H_\Delta'$ realizing an isomorphism of maps (i.e., is a bijection that commutes with left multiplication by elements of $\Delta$), then $H_\Delta'$ and $H_\Delta$ are conjugate in $\Delta$. Therefore, one seeks to produce new maps from old by the process $(\Delta/H_\Delta, s_1, s_2, s_3) \mapsto (\Delta/H_\Delta', s_1, s_2, s_3)$, where $\sigma : \Delta \to \Delta$ is an outer automorphism.

Jones and Thornton [5] showed that the group of outer automorphisms of $\Delta$ is isomorphic with $S_3$, the symmetric group on three symbols. In fact, in Theorem 1 of this paper, they showed that any permutation of $s_1, s_3, s_1s_3$ uniquely determines an outer automorphism of $\Delta$ that fixes $s_2$.

Thus, let $\mathcal{M}$ be a connected map, represented as $\mathcal{M} = (\Delta/H_\Delta, s_1, s_2, s_3)$ as above, let $\sigma \in \text{Out}(\Delta)$, and form the new map $\mathcal{M}^{\sigma} = (\Delta/H_\Delta', s_1, s_2, s_3)$. Note first that $\mathcal{M}^{\sigma} \cong (\Delta/H_\Delta, \sigma s_1, \sigma s_2, \sigma s_3)$, where $\sigma \gamma = \gamma \sigma \gamma^{-1}, \gamma \in \Delta$, via the mapping $\gamma H_\Delta \mapsto \sigma \gamma H_\Delta$. The map $\mathcal{M}^{\sigma}$ is called the $\sigma$-dual of $\mathcal{M}$. If $\mathcal{M}^{\sigma} \cong \mathcal{M}$ we say that $\mathcal{M}$ is $\sigma$-self-dual.
In particular, if \( \sigma \) corresponds to the transposition \( s_1 \leftrightarrow s_3 \), then \( M^\sigma \) is called the \textit{classical dual} of \( M \). If \( \sigma \) is the transposition \( s_1 \leftrightarrow s_1 s_3 \), then \( M \) is called the \textit{Petrie dual} of \( M \) and is denoted \( p(M) \).

Therefore, we see that the various dual forms of the given map \( M = (B, a, b, c) \) can be exhibited as follows:

\[
\begin{align*}
\mathcal{M} &= (B, a, b, c); \\
\mathcal{M}^* &= (B, c, b, a) \text{(Classical dual)}; \\
p(M) &= (B, ac, b, c) \text{(Petrie dual)}; \\
p(M^*) &= (B, ac, b, a); \\
(p(M))^* &= (B, c, b, ac); \\
p((p(M))^*) &= (B, a, b, ac) = (p(M^*))^*.
\end{align*}
\]

Note that the classical dual of \( M = (B, a, b, c) \), obtained by interchanging the involutory generators \( a \) and \( c \), is tantamount to reversing the roles of vertices and faces in \( \mathcal{M} \). In this way, it can be seen that the tetrahedron can be seen as self-dual and that the octahedron and cube (hexahedron) are dual to each other. However, there is an additional species of objects in the map \( \mathcal{M} \), the so-called \textit{Petrie polygons}, which occur as the orbits in \( B \) of the subgroup \( G_p = \langle b, ac \rangle \). A more geometrical description of the Petrie polygons is as follows. Any Petrie polygon is uniquely determined by two adjacent edges \( e_1, e_2 \) bounding a common face; the subsequent edges \( e_3, e_4, \ldots \) are uniquely determined by the requirement that any two consecutive edges must bound a common face, but that no three consecutive edges can bound a common face. Thus, while the tetrahedron is self-dual relative to classical duality, we can exhibit the diagram corresponding to the Petrie dual of the tetrahedron via

[Diagram of tetrahedron and Petrie polygons]

It is clear that the monodromy group of each of the dual forms of \( \mathcal{M} \) agrees with that of \( \mathcal{M} \). As a result, we see that \( \mathcal{M} \) is regular if and only if each of its duals is regular. We remark in passing that the same does not apply to orientability. Indeed, the Petrie dual of the cube is orientable of genus 1, having eight vertices, 12 edges and four faces. On the other hand, the Petrie dual of the octahedron is non-orientable of genus 4. To this end, it is an easy exercise to show that if \( \mathcal{M} \) has odd face valency, then \( p(M) \) is necessarily non-orientable. More generally, in [7] it was observed—and is easy to prove—that \( p(M) \) is orientable if and only if the underlying graph (the vertices and edges of \( \mathcal{M} \)) is bipartite.

As already noted above, the Petrie dual of the tetrahedron must be non-orientable. In fact it is doubly covered by the cube and hence tessellates the real projective plane with three 4-gonal faces. As for the duals of the cube \( \mathcal{M} \), exhibited through their diagrams, one has

[Diagrams of \( \mathcal{M} \), \( p(\mathcal{M}) \), \( p(\mathcal{M})^* \), and \( \mathcal{M}^* \)]
We turn now to the definition of the Coxeter–Petrie complex of the regular algebraic map \( M = (B, a, b, c) \). Thus, let \( M \) have vertex valency \( l \), face valency \( k \) and Petrie polygon valency \( m \). For example, the values \((k, l, m)\) for the Platonic solids (tetrahedron, octahedron, cube, icosahedron, and dodecahedron) are respectively \((3, 3, 4)\), \((4, 3, 6)\), \((3, 4, 6)\), \((5, 3, 10)\) and \((3, 5, 10)\). In general, when \( M \) is regular, we may identify \( l, k, \) and \( m \) as the orders of the elements \( bc, ab, \) and \( bac \), respectively. As a result, the Coxeter diagrams attached to \( M \) and its Petrie dual \( \mathcal{P}(M) \) are

We shall synthesize the above information into a thin, rank 4 chamber system \( \mathcal{CP}(M) \) with associated diagram

Furthermore, we require the corresponding rank 4 chamber complex to have the property that each of the rank 3 residues (obtained by omitting an outer node of the Coxeter diagram), viewed as algebraic maps, is isomorphic with one of the dual forms of \( M \). Finally, if \( \mathcal{N} = \mathcal{M}^\dagger \) is one of the dual forms of \( M \), then we will find \( \mathcal{N} \) (or \( \mathcal{N}^\ast \)) as one of the rank 3 residues of \( \mathcal{CP}(M) \).

Let \( W \) be the Coxeter group associated with the diagram

Therefore, \( W \) is generated by elements \( s_1, s_2, s_3, s_4 \), subject to the Coxeter relations, which we collectively denote by \( C \):

\[
s_1^2 = s_2^2 = s_3^2 = s_4^2 = (s_1s_3)^2 = (s_1s_4)^2 = (s_1s_2)^k = (s_2s_3)^l = (s_2s_4)^m = 1.
\]

Inside \( W \) are the rank 3 parabolic subgroups \( W_i = W_{I_i} \), \( i = 1, 2, 3, 4 \), where \( I = \{1, 2, 3, 4\} \). Furthermore, by [8, Corollary 2.4], we know that each \( W_i \) is a Coxeter group with the indicated relations. Therefore, we have surjective homomorphisms \( \theta_i : W_i \to G, \ i = 1, 3, 4 \), defined as follows:

\[
\begin{align*}
\theta_1 &: s_2 \mapsto b, s_3 \mapsto c, s_4 \mapsto ac, \\
\theta_3 &: s_1 \mapsto a, s_2 \mapsto b, s_4 \mapsto ac, \\
\theta_4 &: s_1 \mapsto a, s_2 \mapsto b, s_3 \mapsto c.
\end{align*}
\]

For \( i = 1, 3, 4 \), set \( K_i = \ker \theta_i \) and choose a subset \( R_i \subseteq W_i \) such that \( K_i \) is the normal closure in \( W_i \) of \( R_i \). Set \( R = R_1 \cup R_3 \cup R_4 \), and define the Coxeter–Petrie group of \( M \) to
be given by

\[ G = G(M) = \langle s_1, s_2, s_3, s_4 | C \cup R \rangle, \]

where \(C\) is the above set of Coxeter relations. Note that by construction, we have a surjective homomorphism

\[ G \rightarrow G, \ s_1 \mapsto a, \ s_2 \mapsto b, \ s_3 \mapsto c, \ s_4 \mapsto ac; \]

furthermore, this homomorphism restricts to isomorphisms \(W_i \rightarrow G, i = 1, 3, 4\).

Note that by construction, for any regular map \(M\) and any dual form \(M^\dagger\) of \(M\), we have that

\[ G(M) \cong G(M^\dagger). \]

For convenience, we record the following result.

**Lemma 2.1.** Let \(M\) be a regular map with monodromy group \(G = \langle a, b, c \rangle\), where \(o(ab) = k, o(bc) = l\), and let \(m = o(bac)\). If \(G(M) = \langle s_1, s_2, s_3, s_4 | C \cup R \rangle\), where \(R = R_1 \cup R_3 \cup R_4\), then the relation \((s_2 s_3 s_4)^k \cdot 1 = 1\) is implied by those contained in \(R_1\), and the relation \((s_2 s_3 s_4)^l \cdot 1 = 1\) is implied by those contained in \(R_3\).

**Proof.** Indeed, \(\theta_1(s_2 s_3 s_4) = bac = ba\), which has order \(k\) in \(G\). Similarly, \(\theta_1(s_2 s_1 s_4) = bc\), which has order \(l\) in \(G\).

**3. THE COXETER–PETRIE COMPLEXES OF THE PLATONIC SOLIDS**

In this section, we shall consider in more detail the structure of the Coxeter–Petrie group of a Platonic solid. Thus, the map in question is \(M = (B, a, b, c)\), where \(a, b, c\) satisfies the defining relations given by the Coxeter diagram:

```
  a  b  c
k   l
```

and where \(\{k, l\} = \{3, 3\}, \{3, 4\}, \{3, 5\}\), i.e., \(M\) is the tetrahedron, octahedron (cube), or the icosahedron (dodecahedron), respectively. In turn, the Coxeter–Petrie group has generators \(s_1, s_2, s_3, s_4\) satisfying the Coxeter relations depicted in the diagram

```
 s_1
   \ /
k
  \ /
  \ /
 s_3
```

and \(s_2\) together with the additional relations \(R_1, R_3, R_4\) described in Section 2.

**Theorem 3.1.** Let \(M\) be a Platonic solid with monodromy group \(G = \langle a, b, c \rangle\), vertex valency \(l\), face valency \(k\), and length of Petrie polygons \(m\). Then the corresponding Coxeter–Petrie group has the presentation

\[ G = \langle s_1, s_2, s_3, s_4 | C \cup R \rangle, \]

where \(R\) consists precisely of the two relations \((s_2 s_1 s_4)^l = (s_2 s_3 s_4)^k = 1\).
PROOF. We must determine the kernels of the surjections of the parabolic subgroups \( \theta_i : W_i \to \langle a, b, c \rangle \). Since \( \{s_1, s_2, s_3\} \to \langle a, b, c \rangle \) is already an isomorphism, we see that \( R_4 = \emptyset \). Furthermore, from Lemma 2.1, we conclude that the relations \( (s_2s_3s_4)^k = 1 \) and \( (s_2s_3s_4)^l = 1 \) necessarily hold in \( G \). We shall show that no other relations are required. Let \( N_i \) denote the normal closure in \( W_1 \) of \( (s_2s_3s_4)^k \); similarly let \( N_3 \) denote the normal closure in \( W_3 \) of \( (s_2s_3s_4)^l \). Thus we have surjective mappings \( W_i/N_i \to G \) \((\cong W_4)\), \( i = 1, 3 \). However, the mappings \( W_4 \to W_i/N_i, i = 1, 3 \) given by

\[
\begin{align*}
s_1 &\mapsto s_3s_4, s_2 \mapsto s_2, s_3 \mapsto s_3, \\
s_1 &\mapsto s_1, s_2 \mapsto s_2, s_3 \mapsto s_1s_4,
\end{align*}
\]

are easily checked to determine well-defined homomorphisms, and are inverse to the homomorphisms \( W_i/N_i \to W_4 \). The result follows. \( \square \)

Next, we calculate upper bounds on the orders of the Coxeter–Petrie groups corresponding to the Platonic solids. Note that it is sufficient to consider one solid from each dual pair.

**Theorem 3.2.** Let \( M \) be a Platonic solid, and let \( G = G(M) \) be its Coxeter–Petrie group. Then

1. \( |G| \leq 96 \) if \( M \) is the tetrahedron;
2. \( |G| \leq 384 \) if \( M \) is the octahedron;
3. \( |G| \leq 3840 \) if \( M \) is the icosahedron.

**Proof.** We consider the above results in turn.

**Tetrahedron.** Rather than carry out a coset enumeration (which is fairly easy in this case), we take a more explicit approach. The results of this approach will be important in actually constructing models for the Coxeter–Petrie group. In the present case, we have \( G = \langle s_1, s_2, s_3, s_4 \rangle \) with the Coxeter relations inferred from the diagram

![Diagram](image)

together with the additional relations \( (s_2s_3s_4)^3 = (s_2s_3s_4)^3 = 1 \). Define the elements \( x, y \in G \) by setting \( x = s_1s_3s_4, y = s_2s_3s_4 \); note that since \( s_1, s_3, s_4 \) commute, \( x^2 = y^2 = 1 \) in \( G \). Define the subgroup \( N \) of \( G \) by setting \( N = \langle x, y \rangle \). We state and prove a number of claims; bear in mind that if \( i, j, k \) is any permutation of \( 1, 2, 4 \) or of \( 2, 3, 4 \), then we have that \( (s_1s_3s_4)^3 = 1 \).

1. \( N \) is elementary abelian. Indeed, note that

\[
\begin{align*}
xy &= s_1s_3s_4s_2(s_1s_3s_4)s_2 \\
&= s_1s_3s_4s_2(s_3s_4s_1)s_2 \\
&= s_1(s_2s_4s_3s_2)s_1s_2 \\
&= s_1s_2s_4s_3(s_1s_2s_1) \\
&= s_1s_2(s_1s_3s_4)s_2s_1 \\
&= s_1s_2(s_1s_3s_4)s_2s_1 \quad \text{(since } s_1, s_3, s_4 \text{ commute)}
\end{align*}
\]

...
\[= \langle s_1 s_2 \rangle s_3 s_4 s_2 s_1 \quad \text{(since } (s_1 s_2)^3 = 1)\]
\[= s_2 s_1 (s_4 s_3 s_2 s_4 s_3) s_1 \quad \text{(since } (s_2 s_3 s_4)^3 = 1)\]
\[= s_2 s_1 (s_3 s_4 s_2 s_4 s_3 s_1) \quad \text{(since } s_3, s_4 \text{ commute)}\]
\[= s_2 s_1 s_3 s_4 s_2 (s_1 s_3 s_4) \quad \text{(since } s_1, s_3, s_4 \text{ commute)}\]
\[= s_2 s_1 s_3 s_4 (s_2 s_1 s_3 s_4) \quad \text{(since } s_2 s_3 s_4)\]
\[= \text{proving the result.}\]

(2) \(N \unlhd G\). This is carried out in steps.

(a) \(s_1 x s_1 = x, s_1 y s_1 = xy\). The first statement is, of course, obvious. As for the second,
\[s_1 y s_1 = (s_1 s_2 s_1) s_3 s_4 s_2 s_1\]
\[= (s_2 s_1 s_2) s_3 s_4 s_2 s_1\]
\[= s_2 s_1 (s_4 s_3 s_2 s_4 s_3) s_1 \quad \text{(since } (s_2 s_3 s_4)^3 = 1)\]
\[= s_2 s_1 s_3 s_4 s_2 (s_1 s_3 s_4)\]
\[= s_2 s_1 s_3 s_4 (s_2 s_1 s_3 s_4) \quad \text{since } s_2 s_3 s_4, s_3, s_4 \text{ commute}\]
\[= s_2 s_1 s_3 s_4 (s_2 s_1 s_3 s_4) \quad \text{since } s_1, s_3, s_4 \text{ commute}\]
\[= s_2 s_1 s_3 s_4 (s_2 s_1 s_3 s_4) \quad \text{since } s_2 s_3 s_4\]
\[= s_2 s_1 s_3 s_4 (s_2 s_1 s_3 s_4) \quad \text{since } s_1 s_3 s_4 = x \in N\]
\[= xy = xy.\]

(b) \(s_2 x s_2 = y, s_2 y s_2 = x\). This follows from the definitions of \(x, y\).

(c) \(s_3 x s_3 = x, s_3 y s_3 = xy\). The first statement is obvious; the proof of the second is similar to that of \(s_1 y s_1 = xy\).

(d) As a result of the above, we see that the elements \(s_1, s_2, s_3\) normalize \(N\). However, since \(s_1 s_3 s_4 = x \in N\), we conclude that \(s_4\) also normalizes \(N\), and so \(N \unlhd G\).

Furthermore, since \(s_4 \equiv s_1 s_3\) (modulo \(N\)), we see that \(G/N\) is a homomorphic image of the Coxeter group of order 24 corresponding to the diagram

\[\begin{array}{cc}
3 & 3 \\
\end{array}\]

Therefore, we conclude that \([G : N] \leq 24\); since \(N\) is elementary abelian and generated by \(x, y\), we have that \(|N| \leq 4\). From this it follows that \(|G| \leq 24 \cdot 4 = 96\).

**Octahedron.** Here, the Coxeter–Petrie group \(G = \langle s_1, s_2, s_3, s_4 \rangle\) has Coxeter relations taken from the diagram

\[\begin{array}{ccc}
s_1 & 3 & 3 \\
4 & 6 & s_2 \\
& s_3 & s_4 \\
\end{array}\]

together with the Petrie relations \((s_2 s_3 s_4)^3 = (s_2 s_1 s_4)^4 = 1\).

**Lemma 3.3.** The element \((s_2 s_3 s_4)^3\) is in the center of \(G\).
PROOF. We set \( G_i, i = 1, 3, 4 \) to be the obviously defined rank 3 parabolic subgroups in \( G \); as already noted above, these are isomorphic with the corresponding parabolic subgroups of the Coxeter group \( W: G_i \cong W_i, i = 1, 3, 4 \). We take as known the result that \( (s_2s_1s_3)^3 \) is the involution that generates the center of \( G_4 \cong W_4 \cong W(B_3) \cong S_4 \times Z_2 \), where \( W(B_3) \) is the Coxeter group of type \( B_3 \). Next we have an isomorphism \( G_4 \to G_3 \) given by

\[
s_1 \mapsto s_1, s_2 \mapsto s_2, s_3 \mapsto s_1s_4.
\]

under which \( (s_2s_1s_3)^3 \mapsto (s_2s_4)^3 \). Therefore, it follows that \( (s_2s_4)^3 \) is central in \( G_3 \). Similarly, we have an isomorphism \( G_4 \to G_1 \), given by

\[
s_1 \mapsto s_3s_4, s_2 \mapsto s_2, s_3 \mapsto s_3;
\]

under this isomorphism \( (s_2s_1s_3)^3 \mapsto (s_2s_4)^3 \). Therefore, \( (s_2s_4)^3 \) is also central in \( G_1 \), which implies that \( (s_2s_4)^3 \) is central in \( G \).

By analogy with the previous case, we define \( x, y, z \in G \) by setting \( x = s_1s_3s_4, y = s_2s_3, \) and \( z = s_4y.s_4 \). Obviously \( x^2 = y^2 = z^2 = 1 \); and define the subgroup \( N \) of \( G \) by setting \( N = \langle x, y, z \rangle \). We shall show that \( N \) is a normal elementary abelian subgroup of \( G \); since \( G = G_4N \) and \( |G_4| = 48 \), this is enough.

1. \( N \) is elementary abelian. First, note that in proving that \( xy = yx \) in the tetrahedral case, the only identities that were used were \( s_1s_2s_1 = s_2s_1s_2 \) and \( (s_2s_4s_3)^3 = 1 \), which continue to be valid in both the octahedral and icosahedral cases. Next, that \( x \) commutes with \( z \) is clear since \( z = s_4y.s_4 \) and \( s_4 \) commutes with \( x \). The proof will be complete if we can show that \( s_2z.s_2 = z \) for then \( yz = zy \) follows by conjugating \( xz = zy \) by \( s_2 \).

We have

\[
s_2z.s_2 = s_2s_4s_2(s_1s_3s_4)s_2s_4s_2
\]

\[
= (s_2s_4)^3s_4s_2s_1s_3s_2s_4s_2
\]

\[
= s_4s_2s_1s_3s_2s_4s_2(s_2s_4)^3 \quad \text{(since \( (s_2s_4)^3 \) is central)}
\]

\[
= s_4s_2s_1s_3s_4s_2s_4
\]

\[
= z.
\]

Therefore, \( N \) is elementary abelian, as claimed.

2. \( N \trianglelefteq G \). In steps:

(a) \( s_1xs_1 = x, s_1ys_1 = xy, s_1zs_1 = xz \). The first statement is trivial. Next,

\[
s_1ys_1 = s_1s_2(s_1s_3s_4)s_2s_1
\]

\[
= s_1s_2s_4s_3(s_2s_1s_2)
\]

\[
= s_1(s_3s_4s_2s_3s_4)s_1s_2 \quad \text{(since \( (s_2s_4s_3)^3 = 1 \))}
\]

\[
= s_1s_3s_4s_2(s_1s_3s_4)s_2
\]

\[
= xy.
\]

Finally,

\[
s_1zs_1 = s_1s_4s_2s_1s_3s_4s_2(s_4s_1)
\]

\[
= s_1s_4s_2s_3s_4(s_2s_1s_2)s_4
\]

\[
= s_1(s_3s_2s_4s_3)s_1s_2s_4 \quad \text{(since \( (s_4s_2s_3)^3 = 1 \))}
\]

\[
= xz.
\]
(b) $s_2x^2 = y, s_2ys_2 = x, s_2z^2 = z$. The first two statements follow by definition of $x, y$. That $s_2z^2 = z$ was already proved above.

(c) $s_3x^3 = x, s_3ys_3 = xz, s_3z^3 = xy$. Again, the first statement follows by the definition of $x$. Next,

$$s_3ys_3 = s_3s_2s_1(s_3s_4s_2s_3)$$

$$= s_3s_2s_1(s_3s_4s_2s_3)$$

(since $(s_3s_4s_2)^3 = 1$)  

$$= s_3(s_1s_2s_1)s_4s_3s_2s_4$$

$$= xz.$$  

Finally,

$$s_3z^3 = s_3s_4s_2s_1s_3s_4s_2s_3$$

$$= s_3s_4s_2s_1(s_2s_3s_4s_2)$$

(since $(s_3s_4s_2)^3 = 1$)  

$$= s_3s_4(s_1s_2s_1)s_3s_4s_2$$

$$= xy.$$  

(d) Since $s_4 \equiv s_1s_3$ (modulo $N$), we conclude that $N \leq G$, as claimed.

**ICOSAHEDRON.** Here, the Coxeter–Petrie group $G = \langle s_1, s_2, s_3, s_4 \rangle$ has Coxeter relations taken from the diagram

![Diagram of Coxeter–Petrie relations]

$$s_1s_3s_1 = s_3s_1s_3$$

$$= s_3s_1s_3$$

(since $(s_3s_1s_3)^3 = 1$)  

$$= s_3s_4,$$  

and the Petrie relations $(s_2s_3s_4)^3 = (s_2s_1s_4)^5 = 1$.

**LEMMA 3.4.** The element $(s_2s_4)^5$ is in the center of $G$.

**PROOF.** To prove that $(s_2s_4)^5$ is central in $G$, we use the known fact that $(s_2s_1s_3)^5$ is central in $G_4 \cong W_4$; argue exactly as in Lemma 3.3.

By analogy with the previous cases, define $x, y, z, w, u \in G$ by setting $x = s_1s_3s_4, y = s_2x, z = s_4ys_4, w = s_2z, u = s_4w; s_4$; again, each of these elements is either an involution or is the identity. Define the subgroup $N$ of $G$ by setting $N = \langle x, y, z, w, u \rangle$. We shall show that $N$ is a normal elementary abelian subgroup of $G$; since $G = G_4N$, and $|G_4| = 120$, this is enough.

1. $N$ is elementary abelian. The proof that $xy = yx$ is proved exactly as in the tetrahedral and octahedral cases. Thus, upon conjugating by $s_4$, it follows immediately that $xz = zy$.

Next, if we can show that $s_1u = xu$, then it will follow that $xu$ is an involution and hence $ux = xu$. Proving this assertion takes some work.

$$xu = s_1s_3s_2s_4s_2(s_3s_4s_2s_3)s_2s_4s_2s_4$$

$$= s_1s_3s_2(s_3s_2s_4s_3s_2)s_1s_2s_4s_2s_4$$

(since $(s_3s_2s_3)^3 = 1$)  

$$= s_1(s_1s_3s_2s_3)s_4s_3s_2s_4s_2s_4$$

(since $(s_1s_3s_2)^5 = 1$)
Next, if we conjugate the equation 

Note that

conclude that

in

Therefore,

\( y \)

is in the center of 

is abelian.

Since \( s_1 w s_1 = s_1 s_2 s_3 s_4 s_5 s_1 s_2 s_3 s_2 s_1 \), we will have proved that \( s_1 w s_1 = u x \) as soon as we show that

\[ s_2 s_4 s_2 s_1 s_2 s_4 s_2 s_4 = s_1 s_4 s_2 s_4 s_2 s_1. \]

Notice that this is an equation not involving the involution \( s_3 \); that is, it suffices to prove this identity within the parabolic subgroup \( G_3 \). However, we know that \( G_3 \) is mapped isomorphically onto \( G_4 \) via \( s_1 \mapsto s_1, s_2 \mapsto s_2, s_4 \mapsto s_1 s_3 \). Under this isomorphism, the above equation is equivalent to the equation

\[ s_2 s_3 s_2 s_3 s_2 s_1 s_3 s_2 = s_3 s_2 s_1 s_3 s_2 s_1 \]

in \( G_4 \). Using the fact that \((s_2 s_3)^5 = 1\) (and so \( s_2 s_3 s_2 s_3 s_2 = s_3 s_2 s_3 s_2 s_3 \)), the above equation is seen to be valid, completing the verification that \( x u = u x \), and hence the verification that \( x u = u x \). Since \( u = s_4 w s_4 \), and since \( s_4 \) commutes with \( x \), we conclude that \( x u = u x \). Therefore, \( x \) is in the center of \( N \).

Next, if we conjugate the equation \( x w = w x \) by \( s_2 \), we get \( y \) \( z \). Likewise, if we conjugate \( x z = x z \) by \( s_2 \), we get \( y w = w y \). Proving that \( y w = u y \) will follow from the assertion that \( s_2 u s_2 = u \), for then \( y w = u y \) is obtained by conjugating the equation \( x u = u x \) by \( s_2 \). To the end, we have

\[ s_2 u s_2 = s_2 s_4 s_2 s_5 s_2 s_2 (s_1 s_3 s_4) s_2 s_4 s_2 s_4 s_2 \]

\[ = (s_2 s_4)^5 s_4 s_2 s_4 s_2 s_1 s_3 s_4 s_2 s_4 s_2 \]

\[ = s_4 s_2 s_4 s_2 s_1 s_3 s_4 s_2 s_4 s_2 s_2 s_1 s_3 s_4 s_2 s_4 \]

\[ = u. \]

Therefore, \( y \) is also in the center of \( N \).

Note that \( z w = w z \) and \( z u = u z \) follow by conjugating the equations \( y u = u y \) and \( y w = w y \) by \( s_4 \). Therefore \( z \) is in the center of \( N \). Finally, since \( z u = u z \), and since \( s_2 u s_2 = u \), we may conjugate \( z u = u z \) by \( s_2 \) and get \( u w = u w \). This concludes the proof that \( N \) is abelian.

(2) \( N \triangleleft G \).

(a) \( s_1 x s_1 = x, s_1 y s_1 = y, s_1 z s_1 = z, s_1 w s_1 = x u, s_1 u s_1 = x w \). In fact the first three equations follow exactly as in the octahedral case, since only relations that continue to be satisfied in the icosahedral case were used.

Next, note that the relations \( s_1 w s_1 = x u, s_1 u s_1 = x w \) are equivalent to each other in light of the fact that \( s_1 x s_1 = x \). Since we have already noted that \( s_1 w s_1 = x u \), the second relation is valid, as well.

(b) \( s_2 x s_2 = y, s_2 y s_2 = x, s_2 z s_2 = w, s_2 w s_2 = z, s_2 u s_2 = u \). We have already proved that \( s_2 u s_2 = u \); the remaining equations are obvious.
Note first that the mapping \( G \) as above, these will be taken up individually for the tetrahedron, octahedron and icosahedron. Furthermore, we shall show that this group is a homomorphic image of the Coxeter–Petrie group corresponding Platonic solid. Therefore, we see that \( G \) determines a surjective homomorphism. Since \( G \equiv G_4 \) arising in the proof of Theorem 3.2), and hence can be regarded as a vector space over the elementary abelian 2-group (and in fact will be the elementary abelian normal subgroup \( W \) of \( G \), we conclude that \( N \cong G \).

We turn now to the determination of the structure of \( G(M) \), where \( M \) is a Platonic solid. As above, these will be taken up individually for the tetrahedron, octahedron and icosahedron. Note first that the mapping \( G \rightarrow G 
cong G_4 \) given by
\[
(\alpha, \beta) \mapsto \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}, \quad \alpha, \beta \in \mathbb{F}_2,
\]
determines a surjective homomorphism. Therefore, we see that \( G \) has the structure of a semidirect product \( N \times W \), where \( W = G_4 = (s_1, s_2, s_3) \). In each case, \( N \) will be an elementary abelian 2-group, and in fact will be the elementary abelian subgroup arising in the proof of Theorem 3.2, and hence can be regarded as a vector space over the binary field \( \mathbb{F}_2 \). Therefore, the structure of \( N \times W \) is determined by a linear representation of \( W \) on \( N \), i.e., the multiplication in \( N \times W \) is given by
\[
(n, w) \cdot (n', w') = (n + w(n'), w'), \quad n, n' \in N, \ w, w' \in W,
\]
and where \( w(n') \) is in terms of the representation of \( W \) on \( N \).

In what follows, we shall, in each case, explicitly construct a semidirect product of the form \( N \times W \), where \( N \) is an elementary abelian 2-group and \( W \) is the monodromy group of the corresponding Platonic solid. Furthermore, we shall show that this group is a homomorphic image of the Coxeter–Petrie group \( G \), thereby reversing the inequalities given in Theorem 3.2.

We now itemize the three cases.

**TETRAHEDRON.** Here, take
\[
N = \left\{ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \mid \alpha, \beta \in \mathbb{F}_2 \right\}.
\]
We take the matrix representation of \( W \) on \( N \) to be that determined by
\[
s_1 \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad s_2 \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad s_3 \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.
\]
Map \( G \rightarrow N \times W \) via
\[
s_i \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix}, i = 1, 2, 3, \quad s_4 \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}, s_1 s_3 \end{bmatrix}.
\]
It is routine to check that this determines a surjective homomorphism. Since \( |G| \leq 96 \), we conclude that this must be an isomorphism.

**OCTAHEDRON.** In this case, set
\[
N = \left\{ \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbb{F}_2 \right\}.
\]
In this case the matrix representation is given by
\[
s_1 \mapsto \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad s_2 \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad s_3 \mapsto \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.
\]
Map $G \to N \rtimes W$ via

$$s_i \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} , \quad i = 1, 2, 3, s_4 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \quad s_1 s_3 .$$

Again, this is checked to be a surjective homomorphism; comparing group orders shows it to be an isomorphism.

**ICOSAHEDRON.** In this case, set

$$N = \left\{ \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \end{bmatrix} \mid \alpha, \beta, \gamma, \delta, \epsilon \in F_2 \right\} ,$$

on which $W$ is represented by

$$s_1 \mapsto \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad s_2 \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad s_3 \mapsto \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

We map $G \to N \rtimes W$ by

$$s_i \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad i = 1, 2, 3, s_4 \mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad s_1 s_3 .$$

Again, this can be checked to be an isomorphism.

For the tetrahedron and octahedron, there are alternative descriptions of the Coxeter–Petrie groups.

**THEOREM 3.5.** Let $G$ be the Coxeter–Petrie group of the tetrahedron. Then $G \cong W(D_4)/Z$, where $W(D_4)$ is the Coxeter group of type $D_4$ and $Z$ is the center of $W(D_4)$.

**PROOF.** First of all, the Coxeter group $W$ of type $D_4$ has generators $w_1, w_2, w_3, w_4$ and relations taken from the Coxeter diagram:

```
    w1   3
      |   |
      |   3
      w2   3
      |
      |
      w3
```

It is well known that $W$ has order 192 and that its center $Z(W)$ is cyclic of order 2 [3, p. 141]. Furthermore, if $i_1, i_2, i_3, i_4$ is any permutation of 1, 2, 3, 4, then the element $(w_{i_1} w_{i_2} w_{i_3} w_{i_4})^3$ is the central involution in $Z(W)$ (see, e.g., [1, pp. 66–68]). Inside $W/Z$ define the elements
v_i = w_i, \ i = 1, 2, 3, v_4 = w_1 w_4 \ (\text{read modulo } \mathbb{Z}). \text{ Then one has that the elements } v_1, v_2, v_3, v_4 \text{ satisfy the Coxeter relations depicted by }

\begin{center}
\begin{tikzpicture}
  \node (v_1) at (0,0) [circle,draw] {$v_1$};
  \node (v_2) at (1,0) [circle,draw] {$v_2$};
  \node (v_3) at (0,-1) [circle,draw] {$v_3$};
  \node (v_4) at (1,-1) [circle,draw] {$v_4$};
  \draw (v_1) -- (v_2);
  \draw (v_1) -- (v_3);
  \draw (v_2) -- (v_4);
\end{tikzpicture}
\end{center}

\text{together with the relations (}v_1 v_2 v_4)^3 = 1 = (v_3 v_2 v_4)^3. \text{ Clearly } v_1, v_2, v_3, v_4 \text{ generate } W/\mathbb{Z} \text{ and so we get a surjective homomorphism } G \to W/\mathbb{Z}. \text{ Comparing the group orders finishes the job.} \square

**Theorem 3.6.** Let $G$ be the Coxeter–Petrie group of the octahedron. Then $G \cong W(D_4) \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ is a cyclic group of order 2.

**Proof.** As above, let $W = W(D_4)$ with relations taken from

\begin{center}
\begin{tikzpicture}
  \node (w_1) at (0,0) [circle,draw] {$w_1$};
  \node (w_2) at (1,0) [circle,draw] {$w_2$};
  \node (w_3) at (0,-1) [circle,draw] {$w_3$};
  \node (w_4) at (1,-1) [circle,draw] {$w_4$};
  \draw (w_1) -- (w_2);
  \draw (w_1) -- (w_3);
  \draw (w_2) -- (w_4);
\end{tikzpicture}
\end{center}

Let $Z_2 = \langle z \rangle$ and let $z'$ be the non-trivial central element of $W$. In $W \times Z_2$, define the elements $v_1 = w_1, v_2 = w_2, v_3 = w_1 w_2 z, v_4 = w_4 z z'$. Then one shows that the elements $v_1, \ldots, v_4$ satisfy the Coxeter relations depicted by

\begin{center}
\begin{tikzpicture}
  \node (v_1) at (0,0) [circle,draw] {$v_1$};
  \node (v_2) at (1,0) [circle,draw] {$v_2$};
  \node (v_3) at (0,-1) [circle,draw] {$v_3$};
  \node (v_4) at (1,-1) [circle,draw] {$v_4$};
  \draw (v_1) -- (v_2);
  \draw (v_1) -- (v_3);
  \draw (v_2) -- (v_4);
\end{tikzpicture}
\end{center}

and that $(v_1 v_2 v_4)^4 = (v_3 v_2 v_4)^3 = 1$. As $W \times Z_2 = \langle v_1, v_2, v_3, v_4 \rangle$, we obtain a surjective homomorphism $G \to W \times Z_2$. A comparison of group orders shows that this is an isomorphism. \square

4. **Coxeter–Petrie Complexes Corresponding to Regular Affine Maps**

The *affine* Coxeter groups of rank 3 are defined by

$$\Delta(k, l) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_3)^r = (s_1 s_2)^k = (s_2 s_3)^l = 1 \rangle,$$

where $[k, l] = [3, 3]$, (in which case $r = 3$), $[4, 4]$, or $[3, 6]$ (in which case $r = 2$). The corresponding hypermaps are called the *universal affine hypermaps*; note that except when $[k, l] = [3, 3]$, these hypermaps are maps, the *universal affine maps*. We say that the map $\mathcal{M}$ is a *regular affine map* if it is of the form $\mathcal{M} \cong \mathcal{M}(\Delta(k, l)/\text{K})$, where $\text{K} \cong \Delta(k, l)$, $[k, l] = [4, 4]$ or $[3, 6]$, and where $\text{K} \cap \text{P} = 1$ for all rank 2 parabolic subgroups $P \leq \Delta(k, l)$. This ensures that $\mathcal{M}$ also has the same face and vertex valencies as
\( \mathcal{M}(\Delta(k, l)) \); put differently, this says that the orbit mapping \( \mathcal{M}(\Delta(k, l)) \to \mathcal{M}(\Delta(k, l)/K) \) is unramified.

We begin by elucidating the structure of the above rank 3 affine Coxeter groups. First of all, we remark that in the context of Lie theory, the above groups are usually denoted as follows: 
\[ A_2 = \Delta(3, 3), B_2 = \Delta(4, 4), G_2 = \Delta(3, 6). \]
Furthermore, it is well known that the above groups can each be exhibited as the semidirect product of the corresponding rank 2 Coxeter group with the corresponding ‘coroot lattice’; see [4, (6.5)]. That the coroot lattice can be replaced by a suitable ring of algebraic integers of the form \( \mathbb{Z}[\omega] \), where \( \omega \) is a root of unity was already noticed by Jones and Singerman in [6, Section 7]. We give here an approach that is both self-contained and more concrete.

**Proposition 4.1.** There is an isomorphism
\[
\Delta(k, l) \cong \mathbb{Z}[\omega] \rtimes \langle \omega, \tau \rangle,
\]
where \((k, l) = (3, 3), (4, 4), (3, 6)\), \(\omega = e^{2\pi i/3}, \omega \) acts by left multiplication on \( \mathbb{Z}[\omega] \), and where \( \tau \) acts as conjugation (note that \( \langle \omega, \tau \rangle \cong D_3 \)).

**Proof.** In each case above we have a semidirect product of an additive group \( \mathbb{Z}[i] \), or \( \mathbb{Z}[\xi] = \mathbb{Z}[\omega], \xi = e^{2\pi i/3}, \omega = e^{2\pi i/3} \) by a dihedral group. We write elements of this group as ordered pairs \((a, g)\), remembering that the group multiplication is given by
\[(a, g) \cdot (a', g') = (a + ga', gg').\]

**Case 1.** \((k, l) = (4, 4)\). Inside the group \( W = \mathbb{Z}[i] \rtimes \langle i, \tau \rangle \), define the elements
\[ w_1 = (1, -\tau), \quad w_2 = (0, \tau i), \quad w_3 = (0, \tau). \]

**Claim.** We have

(i) \( W = \langle w_1, w_2, w_3 \rangle \);

(ii) The mapping \( s_i \mapsto w_i, \quad i = 1, 2, 3 \), determines an isomorphism
\[
\alpha : \Delta(4, 4) \cong W.
\]

**Proof.** (i) is entirely routine. For (ii), it is routine to check that \( w_1, w_2, w_3 \) satisfy the Coxeter relations satisfied by \( s_1, s_2, s_3 \) in \( \Delta(4, 4) \). Therefore, the assignment \( s_i \mapsto w_i, \quad i = 1, 2, 3 \), defines a surjective (by part (i)) homomorphism \( \Delta(4, 4) \to W \). Inside \( \Delta = \Delta(4, 4) \) is the subgroup \( K = \langle a_1 := s_1s_3s_2, a_i := s_3s_2s_1s_2 \rangle \). Note that

(i) \( a_1, a_i \) have infinite orders since their homomorphic images in \( W \) do. Indeed, a direct calculation reveals that \( a_1 \mapsto (1, 1) \) and that \( a_i \mapsto (i, 1) \) which have infinite orders in \( \mathbb{Z}[i] \rtimes \langle i, \tau \rangle \);

(ii) \( a_1, a_i \) commute (direct verification);

(iii) \( \alpha(K) = \mathbb{Z}[i] \times \{1\} \subseteq W \);

(iv) \( K \) is a rank 2 free abelian group and the restriction of \( \alpha \) to \( K \) determines an isomorphism
\[
\alpha|_K : K \to \mathbb{Z}[i];
\]

(v) \( K \preceq \Delta \). Indeed, easy calculations reveal that \( s_1a_1s_1 = a_1^{-1}, s_1a_1s_1 = a_1, s_2a_1s_2 = a_1, s_2a_1s_2 = a_1 \). Since \( s_3 \in \langle s_1, s_2, K \rangle \), this is enough.
From the above, we get a commutative diagram

\[
\begin{array}{c}
1 \longrightarrow K \longrightarrow \Delta \longrightarrow \Delta/K \cong \langle s_1, s_2 \rangle \longrightarrow 1 \\
\end{array}
\]

By the five lemma, the middle vertical arrow is an isomorphism, proving (ii).

CASE 2. \((k, l) = (3, 3)\). Inside the group \(W = \mathbb{Z}[\zeta] \rtimes \langle i, \zeta \rangle, \zeta = e^{2\pi i/3}\), define the elements

\[w_1 = (1, \tau), \quad w_2 = (0, \tau \zeta), \quad w_3 = (0, \tau \zeta^{-1}).\]

Set \(K = \langle a_1 = s_1s_2s_3s_2, a_\zeta = s_3s_1s_3s_2 \rangle\) and prove the obvious analogs of (i)–(v) of the above claim, proving the result in this case.

CASE 3. \((k, l) = (3, 6)\). Inside the group \(W = \mathbb{Z}[\omega] \rtimes \langle i, \omega \rangle, \omega = e^{2\pi i/6}\), define the elements

\[w_1 = (1, -\tau), \quad w_2 = (0, \tau \omega), \quad w_3 = (0, \tau).\]

Set \(K = \langle a_1 = s_1s_2s_3s_2s_3s_2, a_\omega = s_3s_1s_3s_2s_3s_2 \rangle\) and prove the obvious analogs of (i)–(v) of the above claim, proving the result in this case, as well.

Next, we shall determine those normal subgroups \(K \trianglelefteq \Delta(k, l)\), such that \(K\) trivially intersects each proper parabolic subgroup of \(\Delta(k, l)\). The following was already pointed out in [6, p. 304].

**Lemma 4.2.** The normal subgroups \(K \trianglelefteq \mathbb{Z}[\omega] \rtimes \langle i, \alpha \rangle, \alpha \in \{e^{2\pi i/l}, l = 3, 4, 6\}, \) not meeting any proper parabolic subgroups are of the form \(I \times \{1\}\), where \(I \subseteq \mathbb{Z}[\omega]\) is an ideal of \(\mathbb{Z}[\omega]\) which is closed under complex conjugation.

**Proof.** We prove this for \(\omega = i\), leaving the remaining (similar) cases to the reader. Let \(Z\) be the normal subgroup \(Z = \mathbb{Z}[i] \rtimes \langle 1 \rangle \subseteq W = \mathbb{Z}[i] \rtimes \langle i, \tau \rangle\). If \(K \not\subseteq Z\), then \(K\) contains an element of the form \(\alpha = \langle a, g \rangle \in \mathbb{Z}[i] \rtimes \langle i, \tau \rangle\), where \(1 \neq g \in \langle i, \tau \rangle\). Since \(K \not\subseteq W\), it follows easily that, in fact, \(K\) must contain an element of the form \(\alpha = \langle a, -1 \rangle \in W\). From this, we infer that \(Z^2 = 2\mathbb{Z}[i] \times \langle 1 \rangle = [\alpha, Z] \subseteq K\), from which it follows that \(K\) contains one of the elements \((0, -1), (1, -1), (i, -1), (1 - i, -1)\). But \((0, -1) = w_3w_2w_3, (0, i)^{-1}(i, -1)(0, i) = (1, -1) = w_1w_3,\) and \((1 - i, -1) = w_1w_2w_1w_2\), proving the result. \(\square\)

Using the above, we can determine a presentation for the monodromy group of a regular affine map of type \((4, 4)\). In the oriented case, this is already contained in [6, p. 304]. Thus, let \(M\) be a regular affine map of type \((4, 4)\), i.e., one of the form \(M(\Delta(4, 4)/K)\), where as proved above, \(K\) can be identified with an ideal \(I \subseteq \mathbb{Z}[i]\) invariant under complex conjugation. Since \(\mathbb{Z}[i]\) is Euclidean with respect to the norm map, we have that \(\mathbb{Z}[i]\) is a principal ideal domain. Therefore, we may write \(I\) in the form

\[I = (x + yi)\mathbb{Z}[i],\]

where \(x, y \in \mathbb{Z}\).

The next two results are already contained in [3, Section 8.3].

**Lemma 4.3.** The ideal \(I = (x + yi)\mathbb{Z}[i]\) is invariant under complex conjugation precisely when one of the following three conditions holds:
1. \( x = 0; \)
2. \( y = 0; \)
3. \( x = \pm y. \)

As a result of Lemma 4.3, any ideal of \( \mathbb{Z}[i] \) invariant under complex conjugation is of the form \( n\mathbb{Z}[i] \) or \( n(1 + i)\mathbb{Z}[i] \) for some non-negative integer \( n. \)

**COROLLARY 4.3.1.** The monodromy group of a finite regular affine map of type \((4,4)\) has one of the following presentations:

1. \( G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (ab)^4 = (bc)^4 = (abcb)^n = 1 \rangle, \) for some positive integer \( n; \)
2. \( G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (ab)^4 = (bc)^4 = (acbcab)^n = 1 \rangle, \) for some positive integer \( n. \)

**PROOF.** The regular affine map \( \mathcal{M} \) has the form \( \mathcal{M}(\Delta(4,4)/K), \) where, as shown above, \( K \) is one of the groups \( K_1 = (a_1^n = (s_1s_2s_3s_2)^n, a_1^n = (s_3s_2s_3s_2)^n) \) or \( K_2 = ((a_1a_1)^n, (a_1^{-1}a_1)^n). \) Therefore, \( K_1 \) is the normal closure in \( \Delta(4,4) \) of \( a_1^n \) and \( K_2 \) is the normal closure in \( \Delta(4,4) \) of \( (a_1a_1)^n = (s_1s_3s_2s_3s_2)^n. \)

We turn now to the Petrie polygons in \( \mathcal{M}. \) Recall that the length \( m \) of the Petrie polygons in \( \mathcal{M} \) is the order of the glide reflection \( w_1w_2w_3 \) (modulo \( I). \)

**PROPOSITION 4.4.** If \( I \) is either \( n\mathbb{Z}[i] \) or \( n(1 + i)\mathbb{Z}[i], n \in \mathbb{Z}, \) and if \( K = I \times \langle i, \tau \rangle \leq W, \) then

\[
m = o(w_1w_2w_3) = 2n \pmod{K}.
\]

**PROOF.** A direct calculation reveals that

\[
w_1w_2w_3 = (1, -i, \tau).
\]

From this it follows that for any non-negative integer \( k, \)

\[
(w_1w_2w_3)^{2k} = (k(1 - i), 1), \quad (w_1w_2w_3)^{2k+1} = (k + 1 - ki, -i, \tau) \notin K.
\]

The result follows easily. \( \square \)

As a result of the above, we can itemize the possible presentations for the Coxeter–Petrie group corresponding to a regular affine type \((4,4)\) map. For any fixed positive integer \( n, \) the Coxeter–Petrie group \( G \) has generators \( s_1, s_2, s_3, s_4 \) having defining relations deduced from the Coxeter diagram:

![Coxeter diagram](image)

together with the additional relations

1. \( (s_2s_1s_4)^4 = (s_2s_3s_2)^4 = (s_1s_2s_3s_2)^n = (s_1s_2s_4s_2)^n = (s_3s_4s_2s_3s_2)^n = 1, \) or
2. \( (s_2s_1s_4)^4 = (s_2s_3s_2)^4 = (s_1s_3s_2s_3s_2)^n = 1. \)
Note that in case (2) above, the remaining two relations, i.e., those obtained from \((s_1s_3s_2s_3s_1s_2)^n = 1\) by replacing \(s_1\) by \(s_3s_4\) and by replacing \(s_3\) by \(s_1s_4\) both reduce to the relation \((s_4s_2s_4s_2)^n = 1\), which is already specified among the Coxeter relations.

Finally, we summarize the relevant results for the finite regular affine maps of type \((3, 6)\). Namely, that these maps are organized into two families (corresponding to ideals of the form \(n\mathbb{Z}[\omega]\), and \(n(1 + \omega)\mathbb{Z}[\omega]\), where \(\omega = e^{2\pi i/6}\), and having monodromy groups

1. \(G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (ab)^3 = (bc)^6 = (abcabcb)^n = 1 \rangle\), for some positive integer \(n\);
2. \(G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (ab)^3 = (bc)^6 = (abcabcb)^n = 1 \rangle\), for some positive integer \(n\).

As for the Petrie polygons, we have that \(w_1w_2w_3 = (1, -\omega \tau)\), \((w_1w_2w_3)^2 = (1 - \omega, 1)\), from which it follows that if \(K = n\mathbb{Z}[\omega] \rtimes \langle \omega, \tau \rangle\) or if \(K = n(1 + \omega)\mathbb{Z}[\omega] \rtimes \langle \omega, \tau \rangle\), then

\[ o(w_1w_2w_3) = 2n \mod K. \]

Finally, for any fixed positive integer \(n\), the Coxeter–Petrie group \(G\) has generators \(s_1, s_2, s_3, s_4\) having defining relations deduced from the Coxeter diagram:

\[
\begin{array}{c}
\circ \quad s_1 \\
| \\
\circ \quad s_3 \\
| \\
\circ \quad s_2 \\
| \\
\circ \quad 6 \\
| \\
\circ \quad 2n \\
\end{array}
\]

with the additional relations

(1) \((s_2s_3s_4)^6 = (s_2s_3s_4)^3 = (s_1s_2s_3s_2s_3s_2)^n = 1\), or

(2) \((s_2s_3s_4)^6 = (s_2s_3s_4)^3 = (s_1s_3s_3s_2s_2s_2s_3s_2)^n = 1\).

In case (1) above, there are, ostensibly, two additional relations needed, i.e., those obtained from \((s_1s_2s_3s_3s_2)s^n = 1\) by replacing \(s_1\) by \(s_3s_4\) and by replacing \(s_3\) by \(s_1s_4\) reduce to the relations \(1 = (s_1s_2s_3s_3s_2s_3s_2s_2s_3s_2)^n\), and \(1 = (s_3s_3s_3s_3s_3s_3s_2s_2s_3)^n = (s_3s_3s_3s_3s_3s_2s_2s_2s_3)^n = 1\). However, as \(s_2s_1s_3s_4s_2s_4s_1s_2\) is conjugate to \(s_2s_3s_4s_2\), and as \(s_3s_3s_3s_3s_3s_3s_2s_2s_3\) is conjugate to \(s_3s_2s_3s_2s_2s_3s_2\), we see that these relations are unnecessary. Case (2) is a bit more subtle, as follows. We have

\[
\begin{align*}
(s_1s_3s_2s_3s_3s_3s_1s_2s_3s_2)^n & = a_1a_0 \\
& = (s_1s_3s_2s_3s_3s_3s_1s_3s_3) \\
& = (s_1s_3s_2s_3s_3s_3s_3s_2s_3) \\
& \xmapsto{s_1s_2s_3s_2} (s_3s_4s_2s_3s_3s_3s_2s_3s_2) \\
& = (s_3s_4s_2s_3s_3s_3s_3s_2s_3s_2)
\end{align*}
\]

Since \(a_1, a_0\) commute, do so the factors \((s_2s_3s_4s_2s_3s_3s_2)\) and \((s_4s_2s_3s_2)\). As both are conjugates of \(s_4s_2s_3s_2\), we see that the relation obtained from \((s_1s_3s_2s_3s_3s_2s_3s_1s_2s_3s_2)^n = 1\) obtained via \(s_1 \mapsto s_3\) is already implied by the Coxeter relation \((s_4s_2s_3s_2s_3s_2s_3s_3s_2)^n = 1\). Similarly, the relation obtained from \((s_1s_3s_2s_3s_3s_2s_3s_3s_2)^n = 1\) by \(s_3 \mapsto s_1s_4\) already follows from the relation \((s_4s_2s_3s_2s_3s_2)^n = 1\).

**Final Remark.** While the Coxeter–Petrie groups of the Platonic maps are finite, this question remains unsettled for the Coxeter–Petrie groups of the regular affine maps. For very small values of the parameter \(n\), the use of GAP (see [9]) gives finite group orders, but the results are far from being even suggestive.
ACKNOWLEDGEMENT

We express our most profound gratitude to the anonymous referee, whose extensive comments rendered the present version vastly improved over the earlier versions of this manuscript.

REFERENCES


Received 6 October 2001; in revised form 12 June 2002 and accepted 6 August 2002

**Kevin Anderson**
Department of Computer Science/Mathematics/Physics, 
Missouri Western State College, 
St. Joseph, MO 64507, U.S.A. 
E-mail: andersk@griffon.mwsc.edu

AND

**David B. Surowski**
Department of Mathematics, 
Cardwell Hall, Kansas State University, 
Manhattan, KS 66506-2602, U.S.A. 
E-mail: dbski@math.ksu.edu